Infrared foundations for quantum geometry. I. Catalog of totally symmetric rank-three field theories

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We systematically obtain all linear models which propagate a totally symmetric rank-three field without parity violation on a flat background. Each such model is defined exclusively by its gauge symmetry, which is usually a necessary property of effective field theories in the infrared limit. By comparison, models obtained by other means (tuning couplings and operators) may be unstable against radiative corrections. For each model, we compute the spectrum of massless and massive particles, and the no-ghost-no-tachyon constraints on the couplings. We conclude that foundational models exist which can propagate one massless particle of spin one or spin three in isolation, or both particles simultaneously, generalizing the model of Campoleoni and Francia. Our algorithm for detecting symmetric models is grounded in particle physics methods, being based directly on the Wigner decomposition of the field. Compared to our recent analysis of the totally symmetric rank-three field (whose results we confirm and extend) our new algorithm does not require an ansatz for the symmetry transformation, and is not restricted to so-called "free" symmetries.

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I. INTRODUCTION

A. Effective field theory

As explained in [1,2], attempts to perturbatively bootstrap a viable theory of some field $\mathcal{K}$ are only expected to succeed if *all* operators are included which are consistent with the symmetries of the theory. The bootstrapping procedure assumes a simultaneous expansion in powers of the canonically normalized and perturbative field $\mathcal{K}$ and its derivatives, i.e. the low-energy expansion

$$\mathcal{L} = \mathcal{L}_{n\leq 4} + \sum_{n=5}^{\infty} \sum_{i} \frac{{}^{(n)}\kappa_i \mathcal{O}^i_{(n)}}{\Lambda^{n-4}}, \tag{1}$$

where the $\{{}^{(n)}\kappa_i\}$ are dimensionless couplings for which i labels multiple operators with the same mass dimension n, and the constant Λ is a cutoff scale. When canonical normalization assumes $\mathcal{K}$ to have mass-dimension one, and no auxiliary fields are needed to eliminate higher derivatives, the starting-point of the bootstrap is the collection of relevant and marginal operators that are quadratic in $\mathcal{K}$ and have up to two derivatives, i.e. the Lagrangian $\mathcal{L}_{n\leq 4}$ in Eq. (1). If $\mathcal{K}$ is a high-rank tensor field and no extra symmetries are specified, the general ansatz for $\mathcal{L}_{n\leq 4}$ is unlikely to be unitary. What this observation tells us is that $\mathcal{K}$ alone is not a foundation for a physical theory.

B. Symmetry tuning

Good foundations may be found by additionally specifying a gauge symmetry, such that the symmetric restriction of $\mathcal{L}_{n\leq 4}$ uniquely leads to a unitary model. Such is the case for all the fundamental bosons in standard gravity and the standard model of particle physics,[1] whose free propagation is uniquely described by the Lichnerowicz and Maxwell operators. These operators are completely determined once the linear gauge symmetries are specified—unitarity then

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[1]Notably, this also includes the introduction of Higgs-type fields, which are necessary for a symmetric description of massive spin-one fields in conjunction with non-Abelian gauge fields.

follows *for free*. Beyond the free theory, the interactions of these fields are uniquely codetermined with the deformation of their gauge symmetries via the Noether procedure. For standard gravity, this procedure leads to the Einstein-Hilbert action,[2] while for vector gauge bosons it allows the Maxwell theory to be extended to Yang–Mills.

C. Unitarity tuning

This is not to say that symmetry arguments are the only road to healthy infrared foundations. Indeed, it is possible to tune the couplings in $\mathcal{L}_{n\leq 4}$ so as to obtain a unitary spectrum of propagating degrees of freedom (d.o.f.) *without* such a tuning leading to the emergence of new symmetries. The quantum fate of such models is strictly dependent on the interactions which are added to the free theory.

Unitarity tuning is a selective elimination of pathological modes. Consequently, and for all but the most trivial examples, it gives rise to constraints in the free sector, albeit not of the primary first-class type associated with gauge symmetries. Interactions must respect this constraint structure in order to avoid activating d.o.f. that were absent in the free theory. Such interactions may be hard to find, and in the absence of a protecting symmetry they may be destabilized by radiative corrections. Thus, rather than falling into the generally renormalizable or nonrenormalizable structure of Eq. (1)—which is physical and predictive—models obtained by unitarity tuning are in danger of being *not-even-nonrenormalizable* [1,3–14]. To the best of our knowledge, this radiative detuning (see Fig. 1), may be suppressed only by the emergence of serendipitous symmetries in models tuned by unitarity alone—though the nature of these symmetries may be global, rather than gauge.

To facilitate a clear comparison between symmetry tuning and unitarity tuning, we briefly review tuned models of spin one, two and three.

D. Spin one

As an alternative to the symmetric Maxwell theory, Proca theory is one of very few unitary configurations of the general vector field Lagrangian,

$$\mathcal{L}_{n\leq 4} = \overset{(4)}{\beta}_1 \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} + \overset{(2)}{\beta}_1 \Lambda^2 \mathcal{A}_\alpha \mathcal{A}^\alpha, \tag{2}$$

where $\mathcal{F}_{\alpha\beta} \equiv \partial_\alpha \mathcal{A}_\beta - \partial_\beta \mathcal{A}_\alpha$.[3] In the absence of guiding symmetries, one may add some potential, starting with the marginal interaction $\mathcal{L}_{n\leq 4} \supset \overset{(4)}{\beta}_2 \mathcal{A}_\alpha \mathcal{A}^\alpha \mathcal{A}_\beta \mathcal{A}^\beta$, and so deform the $\partial_\alpha \mathcal{A}^\alpha = 0$ constraint without introducing new modes. This leads to a loss of unitarity at the scale

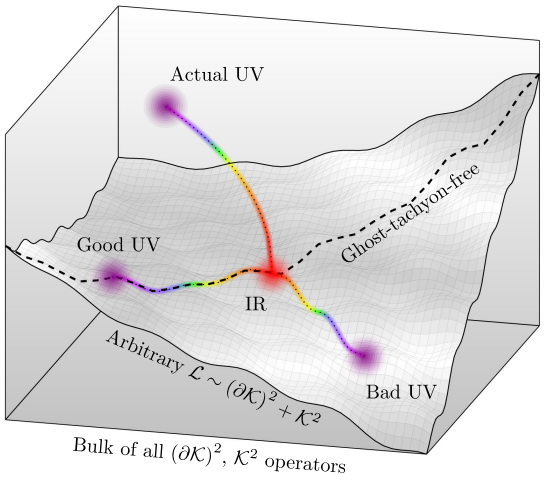


FIG. 1. Consider a low-energy theory defined by an arbitrary subset of operators. Radiative corrections will detune the no-ghost-no-tachyon constraints on this theory or, worse, require the addition of excluded operators so as to absorb uncanceled ultraviolet divergences. These effects are prevented by tuning couplings to the minimum extent needed to produce some gauge symmetry.

$\Lambda \overset{(2)}{\beta}_1^{\frac{1}{2}} / \overset{(4)}{\beta}_2^{\frac{1}{4}}$. Given that the mass of the propagating particle the EFT aims to describe is $\sim \Lambda \overset{(2)}{\beta}_1^{\frac{1}{2}}$, such theories exhibit a cutoff parametrically connected to this mass. This is in stark contrast to the symmetric EFT, where the propagating masses and the cutoff scale are decoupled. A somewhat problematic feature then emerges: either we extend the range of applicability of the theory by confining ourselves to very small effects governed by very small couplings, or we can apply the theory only below the mass of the propagating state, which is then further damped by decoupling effects.

Such cutoffs are a general risk in unitarity-tuned models, where a lack of symmetry gives rise to massive species. In cases such as Eq. (2), the mass can instead be provided by a Higgs sector, whose symmetry elevates the cutoff of the extended theory.

We may be averse to the introduction of new Higgs-like fields. Generalized Proca theory is an economical example in this sense, and provides a rich collection of derivative interactions [16]. Because these interactions are provided without symmetry tuning, it is to be expected that the cutoff is again parametrically connected to the vector mass. Moreover, the radiative stability of generalized Proca theory still hinges on emergent global symmetries, and so the symmetry principle itself is inescapable.

Ultimately, the Higgs mecahnism is known to be realized in fundamental physics; to date the generalized Proca theory is not. The latter, moreover, has deep conceptual similarities with the case of massive gravity, to which we now turn.

[2]Strictly, nonlinear diffeomorphisms are only forced to appear if matter is included.

[3]The other tuning propagates only a spin-zero mode [15].

E. Spin two

The analog of the Maxwell–Proca dichotomy for spin-two particles is provided by Fierz-Pauli theory, which forms the basis for massive gravity. Unlike for the Proca case, there is no unitary and gauge-symmetric Higgs mechanism for spin-two particles. Consequently, there have been extensive investigations into constructing generalized towers of tuned operators.

The Fierz-Pauli mass term arises uniquely from unitarity tuning, but breaks the diffeomorphism invariance essential to the conservation of the stress-energy tensor. Diffeomorphism invariance may be restored by the addition of Stueckelberg fields, which provide covariantization with respect to a reference metric and admit a space of (formally) symmetric completions.[4] Unsurprisingly, all such completions introduce the Boulware-Deser ghost via higher-derivative Stueckelberg operators [17]. Tuned completions exist in the form of the de Rham-Gabadadze-Tolley (dRGT) model [18] but—as with generalized Proca theory—these are not protected by an obvious symmetry and are likely subject to low cutoffs. In the dRGT case, this calls for careful attention because of an undesirable helicity-zero sector. There is a danger that the Vainshtein mechanism [19,20], which screens this sector, might possibly be vulnerable to radiative corrections. It has been suggested that such corrections are calculably small within the dRGT model [21], and that this has the additional benefit of lending credence to a naturally small graviton mass.

F. Spin three

For completeness, we also mention the Singh-Hagen theory [22], the tuned, unitary and massive configuration of a spin-three field which destroys the gauge symmetry of the massless Frondal model [23]. As with the Proca and Fierz-Pauli cases, tuned Singh-Hagen interactions may be constructed with the scaffold of a Stueckelberg sector [24,25], and once again the resulting model is limited by effective field theory considerations. In this case, however, the advantages of symmetry tuning over unitarity tuning are less pronounced, since the Noether procedure *fails* even in the symmetric Frondal case [26].

G. Comparison of approaches

Imagine a scenario in which the field content is first specified, and the challenge is to organize it into a consistent model of fundamental physics[5]: this is typically the case with gravitational gauge theories, such as metric-affine gravity [27]. Guided by the success of the Einstein-Hilbert, Maxwell and Yang-Mills theories, a natural step in such cases is to attempt symmetry tuning. The ensuing Noether construction, where possible, guarantees the radiative stability of the model. Arguably, failure of this procedure signals that the proposed field content is incomplete; in turn, this forces us to question the motivating framework, such as metric-affine gravity itself. The precedent for this perspective is set by Fronsdal: nonclosure of the interacting spin-three symmetry algebra has given rise to productive embeddings in string theory [28–30] and other higher-spin programs with additional fields [31,32], and indeed this has led to improved ultraviolet behavior.[6]

Once the avenues of symmetry tuning have been exhausted, recourse to unitarity tuning may still be fruitful. As with failure of the Noether procedure, one should, in this case, be prepared for an uphill battle. Precedent is set precisely by the generalized Proca and dRGT theories, which were difficult to construct even at the classical level [16,21], and whose radiative stability relies on the emergence of very particular features (nonrenormalization theorems based on Galileon symmetries in the Stueckelberg sector [35,36]). The presence of these features cannot simply be assumed in general.

While our prioritization of symmetry tuning seems practical for fundamental theories, we emphasize that unitarity tuning should be prioritized in other cases, such as when a massive low-energy spectrum is already expected and the resulting cutoffs are acceptable in some broader ultraviolet framework, which is itself known. For example, there are circumstances in which the Proca and Singh–Hagen theories find phenomenological applications within the standard model (see e.g. Refs. [37,38]).[7]

Our objection, to be clear, is not to the effective field theory interpretation itself, which can be forced upon us by both tuning approaches. Indeed, when studying gravity we are resigned to nonrenormalizable models. It might seem tempting to argue that the nonlinear Einstein-Hilbert completion, which arises from symmetry tuning, at least

[4]The uniqueness of the Fierz–Pauli mass term can itself be relaxed when Stueckelberg-type fields are present and, unlike for the simpler vector scenario, there are several ways to achieve massive poles. Not all of these quadratic limits have been explored thoroughly at the level of interactions.

[5]By *fundamental* we mean theories which are either ultraviolet complete, or those for which the ultraviolet completion remains unknown and we are forced to build an effective model with a parametrically independent cutoff.

[6]Note also that we are mostly concerned with the pathology of detuning, which is a basic concern when model-building with fields of *any* spin. For the specific case of higher-spins, there exist other theorems that greatly constrain the model-building process [33,34]. In turn, these no-go theorems can sometimes be circumvented (such as by the addition of masses, see e.g. [32]). Compared to the problem of detuning under radiative corrections, the general problems associated with higher-spin particles are a more advanced target, which become relevant only once viable infrared foundations have been established.

[7]Admittedly, the principal application of the Singh-Hagen construction is not to spin three, but to baryon resonances of half-integer spin.

defers the cutoff to the Planck scale, whereas a small graviton mass combines geometrically with this scale to produce, as discussed, undesirably low cutoffs which must be carefully negotiated [39,40].

Nor, of course, does a preference for symmetry tuning "deny or disparage" the interesting examples of generalized Proca and dRGT theories. The preference is pragmatic in nature, since we seek to avoid the complexities of these models as far as possible. Accordingly, we should avoid downplaying the relevance of a thorough investigation of ghost and tachyon free metric-affine gravity theories without linear gauge symmetries—see e.g. Refs. [41–44]. Ultimately, both symmetry tuning and unitarity tuning approaches deserve investigation, and the question of interactions and consequences for them is a challenging and wide open problem.

According to the logic set out above, we will pursue models that are exclusively defined by the minimal tunings which lead to the emergence of one or more gauge symmetries, and for which such tunings also happen to be consistent with unitarity throughout some volume of the remaining parameter space.[8] Symmetries which emerge in such cases are termed protecting symmetries. Their presence is a sufficient (yet not necessary) criterion for the model to be physical. The identification of protecting symmetries is interaction-agnostic, and represents the easiest, most systematic and most prominent path to success in the high-energy model-building landscape.

H. In this paper

We consider as $\mathcal{K}$ a totally symmetric rank-three field of the form

$$\mathcal{K}_{\alpha\beta\chi} \equiv \mathcal{K}_{(\alpha\beta\chi)}. \tag{3}$$

Such fields may have various motivations and interpretations, though these are not needed and are not necessarily productive. For example, it has been proposed that the gravitational connection may contain an independent, dynamical rank-three tensor part [45–50], for which the totally symmetric case in Eq. (3) was considered in [51]. In any case, up to boundary terms, the most general theory (denoted $\aleph$ to distinguish it from its special cases) built from the relevant and marginal quadratic operators of the field defined in Eq. (3) has Lagrangian

$$\begin{aligned}\mathcal{L}_{\aleph} \equiv {} & {}^{(2)}\kappa_1 \Lambda^2 \mathcal{K}_{\alpha\beta\chi}\mathcal{K}^{\alpha\beta\chi} + {}^{(2)}\kappa_2 \Lambda^2 \mathcal{K}^{\alpha}{}_{\alpha}{}^{\beta}\mathcal{K}_{\beta}{}^{\chi}{}_{\chi} \\ & + {}^{(4)}\kappa_1 \partial_\beta \mathcal{K}_{\chi}{}^{\delta}{}_{\delta}\partial^{\chi}\mathcal{K}^{\alpha}{}_{\alpha}{}^{\beta} + {}^{(4)}\kappa_2 \partial_\chi \mathcal{K}_{\beta}{}^{\delta}{}_{\delta}\partial^{\chi}\mathcal{K}^{\alpha}{}_{\alpha}{}^{\beta} \\ & + {}^{(4)}\kappa_3 \partial_\alpha \mathcal{K}^{\alpha\beta\chi}\partial_\delta \mathcal{K}_{\beta\chi}{}^{\delta} + {}^{(4)}\kappa_4 \partial^{\chi}\mathcal{K}^{\alpha}{}_{\alpha}{}^{\beta}\partial_\delta \mathcal{K}_{\beta\chi}{}^{\delta} \\ & + {}^{(4)}\kappa_7 \partial_\delta \mathcal{K}_{\alpha\beta\chi}\partial^{\delta}\mathcal{K}^{\alpha\beta\chi},\end{aligned} \tag{4}$$

where we require the canonical dimension of $\mathcal{K}_{\alpha\beta\chi}$ to be unity, and neglect parity-violating operators. This latter restriction is somewhat arbitrary, since parity is known already to be maximally violated in the weak sector, and it may be relaxed in an extended analysis without modifying the methods of this paper [52].

While there have been various attempts in the past to build consistent theories from the field in Eq. (3), a systematic, recursive search for protecting symmetries which emerge out of the general Lagrangian in Eq. (4) is lacking. The case of a fully symmetric tensor as in Eq. (3) will demonstrate how consistency requirements significantly narrow down the space of viable quadratic actions. This feature is particularly advantageous, echoing the stringent consistency requirements that uniquely single out, for lower-rank cases, the theories of Maxwell and Einstein. The outcome for the current higher-rank case is both transparent and consequential: gauge symmetries, essential for excluding pathological states, strictly permit only spin-three and spin-one propagation. The letter is organized as follows. In Sec. II we describe the algorithm used to obtain physically motivated special cases of Eq. (4). In Sec. III we present the results of the algorithm and discuss their physical significance. The full catalog of models is given in Appendix A.

I. Conventions

We adopt natural units in which $\hbar \equiv c \equiv 1$, and the particle physics signature $(+,-,-,-)$.

II. THEORETICAL DEVELOPMENT

A. Particle spectroscopy

As we uncover symmetric special cases of Eq. (4), it will be important to compute their tree-level spectra. The method of computing particle spectra by means of spin projection operators has a long history [41–43,53–63]—see also more general works [64,65]. In recent years, such analyses have been performed with the assistance of computers [66–69], and in this work we particularly make use of the PSALTer software [52,70].

To study the propagator associated with some Lagrangian $\mathcal{L}_{n\leq 4}$ depending only on $\mathcal{K}_{\alpha\beta\chi}$ and its derivatives, and parametrized by the couplings $\{{}^{(n)}\kappa_i\}$ and cutoff Λ, a formal collection of test sources $\mathcal{J}^{\alpha\beta\chi}$ is introduced to the action $\mathcal{S} \equiv \int \mathrm{d}^4x[\mathcal{L}_{n\leq 4} - \mathcal{K}_{\alpha\beta\chi}\mathcal{J}^{\alpha\beta\chi}]$. In programmatical

[8] Volumes are required, rather than set-of-measure-zero surfaces. By the arguments above, unitary hypersurfaces in the remining parameter space are discarded if they are motivated by unitarity alone, and not by the emergence of yet more symmetries. Surfaces which are properly motivated by symmetries constitute additional, physically significant models: the systematic, recursive identification of such models will be our main focus.

terms we represent the independent d.o.f. in $\mathcal{K}_{\alpha\beta\chi}$ and $\mathcal{J}^{\alpha\beta\chi}$ with the column vectors J and the conjugate field K. In position space, this action is then expressed schematically as

$$\mathcal{S} = \int \mathrm{d}^4x \mathsf{K}^{\mathrm{T}}(x) \cdot [\mathsf{O}(\partial) \cdot \mathsf{K}(x) - \mathsf{J}(x)]. \tag{5}$$

The operator $\mathsf{O}(\partial)$ is the wave operator: a tensor polynomial of (up to) quadratic order in ∂_μ and parametrized by the $\{{}^{(n)}_{\kappa_i}\}$ and Λ.

The saturated propagator associated with Eq. (5) is obtained in momentum space $\partial_\mu \mapsto -ik_\mu$ by "inverting" the wave operator and sandwiching it between the sources according to

$$\Pi(k) \equiv \mathsf{J}^\dagger(k) \cdot \mathsf{O}^{-1}(k) \cdot \mathsf{J}(k). \tag{6}$$

By construction, the special cases of Eq. (4) which we consider will all have extra gauge symmetries: when such symmetries are present the "inverse" in Eq. (6) does not actually exist due to degeneracies of the wave operator.

As we will see presently, precisely those symmetries that are responsible for the degeneracies also result in (off-shell) constraints on the source currents. Accordingly, the result of the "sandwiching" in Eq. (6) is that the infinities always cancel against the zeros. According to the usual principles of pology, the positions of the poles in $\Pi(k)$ determine the masses of physical states, with real masses needed for nontachyonic particles. The residues of $\Pi(k)$ on each pole must be positive definite in order to avoid a ghost.

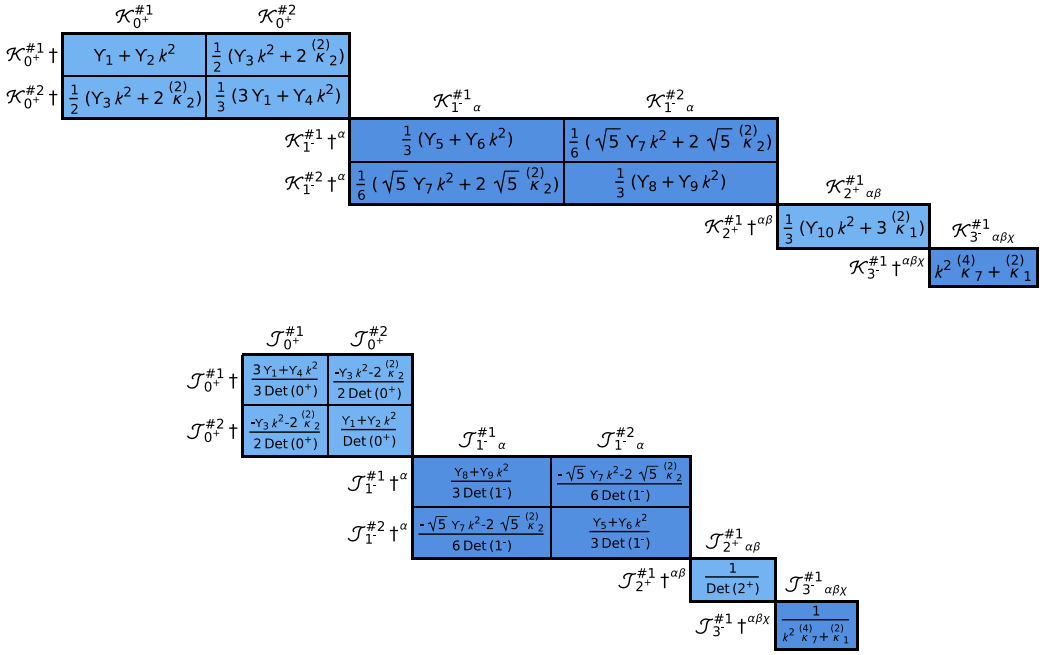
For massive states, adopting a spin-parity (J^P) basis is beneficial for casting the wave operator inversion as a linear algebra problem. In this basis, the wave operator decomposes into irreducible subspaces

$$\mathsf{O}(k) \equiv \bigoplus_{J,P} \mathsf{O}_{J^P}(k), \tag{7}$$

and the spectroscopy can be performed in each subspace separately. At the point of definition, PSALTer automatically determines the J^P sectors associated with $\mathcal{K}_{\alpha\beta\chi}$, as illustrated in Table I. Equipped with this basis, the spectrograph of the unconstrained $\aleph$ theory in Eq. (4) is shown in Fig. 2. In the unconstrained case, it turns out that *all* the poles are massive. Each of the 0^+ and 1^- sectors of $\aleph$ propagates a *pair* of particles. The expressions for the squares of the 0^+ and 1^- masses are not rational functions of the $\{{}^{(n)}_{\kappa_i}\}$, since they are obtained by solving a quadratic in $k \equiv \sqrt{k^\nu k_\nu}$. To avoid long calculations involving square roots, PSALTer sets these poles aside as "unresolved." The "resolved" poles, however, are sufficient to understand the sick nature of the $\aleph$ theory. There are isolated poles in the 2^+ and 3^- sectors of the theory whose mutual no-ghost-no-tachyon constraints are contradictory. As we have argued in Sec. I, the only path forward in such cases is to introduce gauge

TABLE I. Output generated by PSALTer. The field kinematics of the totally symmetric field defined in Eq. (3). These definitions are used in Figs. 2 and 5–9. The unit-timelike vector is $n_\mu \equiv \mathcal{K}_\mu/\sqrt{k^\nu k_\nu}$, where k_μ is the massive particle four-momentum.

Fundamental field	Symmetries	Decomposition into SO(3) irrep(s)	Source
$\mathcal{K}_{\alpha\beta\chi}$	$\mathcal{K}_{\alpha\chi\beta} == \mathcal{K}_{\alpha\beta\chi}$ && $\mathcal{K}_{\beta\alpha\chi} == \mathcal{K}_{\alpha\beta\chi}$	$\mathcal{K}^{\#1}_{3^-\,\alpha\beta\chi} + \frac{1}{15}\,\eta_{\beta\chi}\,\mathcal{K}^{\#2}_{1^-\,\alpha} + \frac{1}{15}\,\eta_{\alpha\chi}\,\mathcal{K}^{\#2}_{1^-\,\beta} + \frac{1}{15}\,\eta_{\alpha\beta}\,\mathcal{K}^{\#2}_{1^-\,\chi} + \frac{1}{9}\,\eta_{\beta\chi}\,\mathcal{K}^{\#2}_{0^+}\,n_\alpha + \frac{1}{3}\,\mathcal{K}^{\#1}_{2^+\,\beta\chi}\,n_\alpha + \frac{1}{9}\,\eta_{\alpha\chi}\,\mathcal{K}^{\#2}_{0^+}\,n_\beta + \frac{1}{3}\,\mathcal{K}^{\#1}_{2^+\,\alpha\chi}\,n_\beta - \frac{1}{15}\,\mathcal{K}^{\#2}_{1^-\,\chi}\,n_\alpha\,n_\beta + \frac{1}{3}\,\mathcal{K}^{\#1}_{1^-\,\chi}\,n_\alpha\,n_\beta + \frac{1}{9}\,\eta_{\alpha\beta}\,\mathcal{K}^{\#2}_{0^+}\,n_\chi + \frac{1}{3}\,\mathcal{K}^{\#1}_{2^+\,\alpha\beta}\,n_\chi - \frac{1}{15}\,\mathcal{K}^{\#2}_{1^-\,\beta}\,n_\alpha\,n_\chi + \frac{1}{3}\,\mathcal{K}^{\#1}_{1^-\,\beta}\,n_\alpha\,n_\chi - \frac{1}{15}\,\mathcal{K}^{\#2}_{1^-\,\alpha}\,n_\beta\,n_\chi + \frac{1}{3}\,\mathcal{K}^{\#1}_{1^-\,\alpha}\,n_\beta\,n_\chi - \frac{1}{3}\,\mathcal{K}^{\#2}_{0^+}\,n_\alpha\,n_\beta\,n_\chi + \mathcal{K}^{\#1}_{0^+}\,n_\alpha\,n_\beta\,n_\chi$	$\mathcal{J}_{\alpha\beta\chi}$
SO(3) irrep	Symmetries	Expansion in terms of the fundamental field	Source SO(3) irrep
$\mathcal{K}^{\#1}_{0^+}$	(None)	$\mathcal{K}_{\alpha\beta\chi}\,n^\alpha\,n^\beta\,n^\chi$	$\mathcal{J}^{\#1}_{0^+}$
$\mathcal{K}^{\#2}_{0^+}$	(None)	$3\,\mathcal{K}_\alpha{}^\beta{}_\beta\,n^\alpha - 3\,\mathcal{K}_{\alpha\beta\chi}\,n^\alpha\,n^\beta\,n^\chi$	$\mathcal{J}^{\#2}_{0^+}$
$\mathcal{K}^{\#1}_{1^-\,\alpha}$	(None)	$3\,\mathcal{K}_{\alpha\beta\chi}\,n^\beta\,n^\chi - 3\,\mathcal{K}_{\beta\chi\delta}\,n_\alpha\,n^\beta\,n^\chi\,n^\delta$	$\mathcal{J}^{\#1}_{1^-\,\alpha}$
$\mathcal{K}^{\#2}_{1^-\,\alpha}$	(None)	$3\,\mathcal{K}_\alpha{}^\beta{}_\beta - 3\,\mathcal{K}_\beta{}^\chi{}_\chi\,n_\alpha\,n^\beta - 3\,\mathcal{K}_{\alpha\beta\chi}\,n^\beta\,n^\chi + 3\,\mathcal{K}_{\beta\chi\delta}\,n_\alpha\,n^\beta\,n^\chi\,n^\delta$	$\mathcal{J}^{\#2}_{1^-\,\alpha}$
$\mathcal{K}^{\#1}_{2^+\,\alpha\beta}$	$\mathcal{K}^{\#1}_{2^+\,\beta\alpha} == \mathcal{K}^{\#1}_{2^+\,\alpha\beta}$	$3\,\mathcal{K}_{\alpha\beta\chi}\,n^\chi - \eta_{\alpha\beta}\,\mathcal{K}_\chi{}^\delta{}_\delta\,n^\chi + \mathcal{K}_\chi{}^\delta{}_\delta\,n_\alpha\,n_\beta\,n^\chi - 3\,\mathcal{K}_{\beta\chi\delta}\,n_\alpha\,n^\chi\,n^\delta - 3\,\mathcal{K}_{\alpha\chi\delta}\,n_\beta\,n^\chi\,n^\delta + \eta_{\alpha\beta}\,\mathcal{K}_{\chi\delta\epsilon}\,n^\chi\,n^\delta\,n^\epsilon + 2\,\mathcal{K}_{\chi\delta\epsilon}\,n_\alpha\,n_\beta\,n^\chi\,n^\delta\,n^\epsilon$	$\mathcal{J}^{\#1}_{2^+\,\alpha\beta}$
$\mathcal{K}^{\#1}_{3^-\,\alpha\beta\chi}$	$\mathcal{K}^{\#1}_{3^-\,\alpha\chi\beta} == \mathcal{K}^{\#1}_{3^-\,\alpha\beta\chi}$ && $\mathcal{K}^{\#1}_{3^-\,\beta\alpha\chi} == \mathcal{K}^{\#1}_{3^-\,\alpha\beta\chi}$	$\mathcal{K}_{\alpha\beta\chi} - \frac{1}{5}\,\eta_{\beta\chi}\,\mathcal{K}_\alpha{}^\delta{}_\delta - \frac{1}{5}\,\eta_{\alpha\chi}\,\mathcal{K}_\beta{}^\delta{}_\delta - \frac{1}{5}\,\eta_{\alpha\beta}\,\mathcal{K}_\chi{}^\delta{}_\delta + \frac{1}{5}\,\mathcal{K}_\chi{}^\delta{}_\delta\,n_\alpha\,n_\beta + \frac{1}{5}\,\mathcal{K}_\beta{}^\delta{}_\delta\,n_\alpha\,n_\chi + \frac{1}{5}\,\mathcal{K}_\alpha{}^\delta{}_\delta\,n_\beta\,n_\chi - \mathcal{K}_{\beta\chi\delta}\,n_\alpha\,n^\delta + \frac{1}{5}\,\eta_{\beta\chi}\,\mathcal{K}_\delta{}^\epsilon{}_\epsilon\,n_\alpha\,n^\delta - \mathcal{K}_{\alpha\chi\delta}\,n_\beta\,n^\delta + \frac{1}{5}\,\eta_{\alpha\chi}\,\mathcal{K}_\delta{}^\epsilon{}_\epsilon\,n_\beta\,n^\delta - \mathcal{K}_{\alpha\beta\delta}\,n_\chi\,n^\delta + \frac{1}{5}\,\eta_{\alpha\beta}\,\mathcal{K}_\delta{}^\epsilon{}_\epsilon\,n_\chi\,n^\delta - \frac{3}{5}\,\mathcal{K}_\delta{}^\epsilon{}_\epsilon\,n_\alpha\,n_\beta\,n_\chi\,n^\delta + \frac{1}{5}\,\eta_{\beta\chi}\,\mathcal{K}_{\alpha\delta\epsilon}\,n^\delta\,n^\epsilon + \frac{1}{5}\,\eta_{\alpha\chi}\,\mathcal{K}_{\beta\delta\epsilon}\,n^\delta\,n^\epsilon + \frac{1}{5}\,\eta_{\alpha\beta}\,\mathcal{K}_{\chi\delta\epsilon}\,n^\delta\,n^\epsilon + \frac{4}{5}\,\mathcal{K}_{\chi\delta\epsilon}\,n_\alpha\,n_\beta\,n^\delta\,n^\epsilon + \frac{4}{5}\,\mathcal{K}_{\beta\delta\epsilon}\,n_\alpha\,n_\chi\,n^\delta\,n^\epsilon + \frac{4}{5}\,\mathcal{K}_{\alpha\delta\epsilon}\,n_\beta\,n_\chi\,n^\delta\,n^\epsilon - \frac{1}{5}\,\eta_{\beta\chi}\,\mathcal{K}_{\delta\epsilon\phi}\,n_\alpha\,n^\delta\,n^\epsilon\,n^\phi - \frac{1}{5}\,\eta_{\alpha\chi}\,\mathcal{K}_{\delta\epsilon\phi}\,n_\beta\,n^\delta\,n^\epsilon\,n^\phi - \frac{1}{5}\,\eta_{\alpha\beta}\,\mathcal{K}_{\delta\epsilon\phi}\,n_\chi\,n^\delta\,n^\epsilon\,n^\phi - \frac{2}{5}\,\mathcal{K}_{\delta\epsilon\phi}\,n_\alpha\,n_\beta\,n_\chi\,n^\delta\,n^\epsilon\,n^\phi$	$\mathcal{J}^{\#1}_{3^-\,\alpha\beta\chi}$



Abbreviations used in matrices

$$\mathrm{Y}_1 == {}^{(2)}\kappa_1 + {}^{(2)}\kappa_2 \;\&\&\; \mathrm{Y}_2 == {}^{(4)}\kappa_1 + {}^{(4)}\kappa_2 + {}^{(4)}\kappa_3 + {}^{(4)}\kappa_4 + {}^{(4)}\kappa_7 \;\&\&\; \mathrm{Y}_3 == 2\,({}^{(4)}\kappa_1 + {}^{(4)}\kappa_2) + {}^{(4)}\kappa_4 \;\&\&$$

$$\mathrm{Y}_4 == 3\,{}^{(4)}\kappa_1 + 3\,{}^{(4)}\kappa_2 + {}^{(4)}\kappa_3 + 3\,{}^{(4)}\kappa_7 \;\&\&\; \mathrm{Y}_5 == 3\,{}^{(2)}\kappa_1 + {}^{(2)}\kappa_2 \;\&\&\; \mathrm{Y}_6 == {}^{(4)}\kappa_2 + 2\,{}^{(4)}\kappa_3 + {}^{(4)}\kappa_4 + 3\,{}^{(4)}\kappa_7 \;\&\&$$

$$\mathrm{Y}_7 == 2\,{}^{(4)}\kappa_2 + {}^{(4)}\kappa_4 \;\&\&\; \mathrm{Y}_8 == 3\,{}^{(2)}\kappa_1 + 5\,{}^{(2)}\kappa_2 \;\&\&\; \mathrm{Y}_9 == 5\,{}^{(4)}\kappa_2 + 3\,{}^{(4)}\kappa_7 \;\&\&\; \mathrm{Y}_{10} == {}^{(4)}\kappa_3 + 3\,{}^{(4)}\kappa_7 \;\&\&\; \mathrm{Det}\,(0^+) ==$$

$$\tfrac{1}{12}k^4(16\,{}^{(4)}\kappa_1\,{}^{(4)}\kappa_3 + 16\,{}^{(4)}\kappa_2\,{}^{(4)}\kappa_3 + 4\,{}^{(4)}\kappa_3{}^2 + 4\,{}^{(4)}\kappa_3\,{}^{(4)}\kappa_4 - 3\,{}^{(4)}\kappa_4{}^2 + 4(6\,{}^{(4)}\kappa_1 + 6\,{}^{(4)}\kappa_2 + 4\,{}^{(4)}\kappa_3 + 3\,{}^{(4)}\kappa_4)\,{}^{(4)}\kappa_7 + 12\,{}^{(4)}\kappa_7{}^2) +$$

$${}^{(2)}\kappa_1({}^{(2)}\kappa_1 + 2\,{}^{(2)}\kappa_2) + \tfrac{1}{3}k^2(6\,{}^{(4)}\kappa_1\,{}^{(2)}\kappa_1 + 6\,{}^{(4)}\kappa_2\,{}^{(2)}\kappa_1 + 4\,{}^{(4)}\kappa_3\,{}^{(2)}\kappa_1 + 3\,{}^{(4)}\kappa_4\,{}^{(2)}\kappa_1 + 6\,{}^{(4)}\kappa_7\,{}^{(2)}\kappa_1 + 4\,{}^{(4)}\kappa_3\,{}^{(2)}\kappa_2 + 6\,{}^{(4)}\kappa_7\,{}^{(2)}\kappa_2) \;\&\&$$

$$\mathrm{Det}\,(1^-) == \tfrac{1}{36}k^4(-5\,{}^{(4)}\kappa_4{}^2 + 12\,{}^{(4)}\kappa_7(2\,{}^{(4)}\kappa_3 + {}^{(4)}\kappa_4 + 3\,{}^{(4)}\kappa_7) + 8\,{}^{(4)}\kappa_2(5\,{}^{(4)}\kappa_3 + 9\,{}^{(4)}\kappa_7)) + {}^{(2)}\kappa_1({}^{(2)}\kappa_1 + 2\,{}^{(2)}\kappa_2) +$$

$$\tfrac{1}{9}k^2(3(6\,{}^{(4)}\kappa_2 + 2\,{}^{(4)}\kappa_3 + {}^{(4)}\kappa_4 + 6\,{}^{(4)}\kappa_7)\,{}^{(2)}\kappa_1 + 2(5\,{}^{(4)}\kappa_3 + 9\,{}^{(4)}\kappa_7)\,{}^{(2)}\kappa_2) \;\&\&\; \mathrm{Det}\,(2^+) == k^2({}^{(4)}\tfrac{\kappa_3}{3} + {}^{(4)}\kappa_7) + {}^{(2)}\kappa_1$$

Lagrangian

$$\begin{aligned}&{}^{(2)}\kappa_1\,\mathcal{K}_{\alpha\beta\chi}\,\mathcal{K}^{\alpha\beta\chi} + {}^{(2)}\kappa_2\,\mathcal{K}^{\alpha}{}_{\alpha}{}^{\beta}\,\mathcal{K}_{\beta}{}^{\chi}{}_{\chi} + {}^{(4)}\kappa_1\,\partial_\beta\mathcal{K}_{\chi}{}^{\delta}{}_{\delta}\,\partial^{\chi}\mathcal{K}^{\alpha}{}_{\alpha}{}^{\beta} +\\ &{}^{(4)}\kappa_2\,\partial_{\chi}\mathcal{K}_{\beta}{}^{\delta}{}_{\delta}\,\partial^{\chi}\mathcal{K}^{\alpha}{}_{\alpha}{}^{\beta} + {}^{(4)}\kappa_3\,\partial_\alpha\mathcal{K}^{\alpha\beta\chi}\,\partial_\delta\mathcal{K}_{\beta\chi}{}^{\delta} + {}^{(4)}\kappa_4\,\partial^{\chi}\mathcal{K}^{\alpha}{}_{\alpha}{}^{\beta}\,\partial_\delta\mathcal{K}_{\beta\chi}{}^{\delta} + {}^{(4)}\kappa_7\,\partial_\delta\mathcal{K}_{\alpha\beta\chi}\,\partial^\delta\mathcal{K}^{\alpha\beta\chi}\end{aligned}$$

Added source term(s):	$\mathcal{K}^{\alpha\beta\chi}\;\mathcal{J}_{\alpha\beta\chi}$		
Unresolved pole(s)	No. of d.o.f.	Pole structure(s)	
$J^P = 0^+$	2	$k^4(16\,{}^{(4)}\kappa_1\,{}^{(4)}\kappa_3 + 16\,{}^{(4)}\kappa_2\,{}^{(4)}\kappa_3 + 4\,{}^{(4)}\kappa_3{}^2 + 4\,{}^{(4)}\kappa_3\,{}^{(4)}\kappa_4 - 3\,{}^{(4)}\kappa_4{}^2 + 4(6\,{}^{(4)}\kappa_1 + 6\,{}^{(4)}\kappa_2 + 4\,{}^{(4)}\kappa_3 + 3\,{}^{(4)}\kappa_4)\,{}^{(4)}\kappa_7 + 12\,{}^{(4)}\kappa_7{}^2) +$ $12\,{}^{(2)}\kappa_1({}^{(2)}\kappa_1 + 2\,{}^{(2)}\kappa_2) + 4k^2(6\,{}^{(4)}\kappa_1\,{}^{(2)}\kappa_1 + 6\,{}^{(4)}\kappa_2\,{}^{(2)}\kappa_1 + 4\,{}^{(4)}\kappa_3\,{}^{(2)}\kappa_1 + 3\,{}^{(4)}\kappa_4\,{}^{(2)}\kappa_1 + 6\,{}^{(4)}\kappa_7\,{}^{(2)}\kappa_1 + 4\,{}^{(4)}\kappa_3\,{}^{(2)}\kappa_2 + 6\,{}^{(4)}\kappa_7\,{}^{(2)}\kappa_2)$	
$J^P = 1^-$	6	$k^4(-5\,{}^{(4)}\kappa_4{}^2 + 12\,{}^{(4)}\kappa_7(2\,{}^{(4)}\kappa_3 + {}^{(4)}\kappa_4 + 3\,{}^{(4)}\kappa_7) + 8\,{}^{(4)}\kappa_2(5\,{}^{(4)}\kappa_3 + 9\,{}^{(4)}\kappa_7)) +$ $36\,{}^{(2)}\kappa_1({}^{(2)}\kappa_1 + 2\,{}^{(2)}\kappa_2) + 4k^2(3(6\,{}^{(4)}\kappa_2 + 2\,{}^{(4)}\kappa_3 + {}^{(4)}\kappa_4 + 6\,{}^{(4)}\kappa_7)\,{}^{(2)}\kappa_1 + 2(5\,{}^{(4)}\kappa_3 + 9\,{}^{(4)}\kappa_7)\,{}^{(2)}\kappa_2)$		
Resolved pole(s)	No. of polarization(s)	Square mass	Residue
$J^P = 2^+$	5	$-\frac{3\,{}^{(2)}\kappa_1}{{}^{(4)}\kappa_3 + 3\,{}^{(4)}\kappa_7}$	$\frac{3}{{}^{(4)}\kappa_3 + 3\,{}^{(4)}\kappa_7}$
$J^P = 3^-$	7	$-\frac{{}^{(2)}\kappa_1}{{}^{(4)}\kappa_7}$	$-\frac{1}{{}^{(4)}\kappa_7}$
Resolved unitarity condition(s):	(Demonstrably impossible)		

FIG. 2. Output generated by PSALTer. The spectrograph of $\aleph$, as defined in Eq. (4). All notation is defined in Table I. Note that when presenting spectrographs, we absorb the factor Λ^2 into the $\{{}^{(2)}\kappa_i\}$. The upper matrices denote the blocks O_{J^P} in Eq. (7), and the lower matrices denote the pseudoinverses $\mathsf{O}^+_{J^P}$. The model is not unitary.

symmetries. This process will naturally eliminate masses, as is the case with the Fierz-Pauli and Proca-Maxwell models, and so we will have to contend with the massless pology.

B. Helicity extension

While massive states can be fully understood using the J^P basis, in the case of massless states spin J is replaced by helicity h. The existing implementation in PSALTer proceeds component-wise with the massless computation in the fiducial frame $[k^\mu] = (\omega, 0, 0, \omega)$. This approach is guaranteed to determine the number of massless polarizations, and their unitarity conditions, but it does not inform about the J or h quantum numbers.

For the first time in this paper, we extend beyond the theory developed in [52,70] by interpreting the component-wise results from PSALTer in terms of the physical helicity decomposition, i.e. in terms of states of definite angular momentum. As a basic example, we consider the case of a conjugate source $\mathcal{J}^\mu$ to some vector field $\mathcal{K}_\mu$. In the fiducial lightlike frame the source has components $[\mathcal{J}^\mu(k)] = (\mathcal{J}^0(k), \mathcal{J}^1(k), \mathcal{J}^2(k), \mathcal{J}^3(k))$, and the well-known combinations $\varepsilon_{\pm 3}(k) \equiv \mathcal{J}^1(k) \pm i\mathcal{J}^2(k)$ clearly possess definite helicities of value ± 1.

Extending this example to the present case, straightforward decomposition rules for angular momentum in the reducible tensorial representation $\mathcal{J}^{\alpha\beta\gamma}(k)$ yield the analogous formulas

$$\varepsilon_{\pm 3} \equiv \mathcal{J}^{222} - 3\mathcal{J}^{112} \pm i(3\mathcal{J}^{122} - \mathcal{J}^{111}), \tag{8a}$$

$$\varepsilon^{\#1}_{\pm 2} \equiv \mathcal{J}^{223} - \mathcal{J}^{113} \pm 2i\mathcal{J}^{123}, \tag{8b}$$

$$\varepsilon^{\#2}_{\pm 2} \equiv \mathcal{J}^{022} - \mathcal{J}^{011} \pm 2i\mathcal{J}^{012}, \tag{8c}$$

$$\varepsilon^{\#1}_{\pm 1} \equiv \mathcal{J}^{233} \pm i\mathcal{J}^{133}, \tag{8d}$$

$$\varepsilon^{\#2}_{\pm 1} \equiv \mathcal{J}^{112} + \mathcal{J}^{222} \pm i(\mathcal{J}^{111} + \mathcal{J}^{122}), \tag{8e}$$

$$\varepsilon^{\#3}_{\pm 1} \equiv \mathcal{J}^{023} \pm i\mathcal{J}^{013}, \tag{8f}$$

$$\varepsilon^{\#4}_{\pm 1} \equiv \mathcal{J}^{002} \pm i\mathcal{J}^{001}, \tag{8g}$$

$$\varepsilon^{\#1}_{0} \equiv \mathcal{J}^{333}, \tag{8h}$$

$$\varepsilon^{\#2}_{0} \equiv \mathcal{J}^{113} + \mathcal{J}^{223}, \tag{8i}$$

$$\varepsilon^{\#3}_{0} \equiv \mathcal{J}^{033}, \tag{8j}$$

$$\varepsilon^{\#4}_{0} \equiv \mathcal{J}^{011} + \mathcal{J}^{022}, \tag{8k}$$

$$\varepsilon^{\#5}_{0} \equiv \mathcal{J}^{003}, \tag{8l}$$

$$\varepsilon^{\#6}_{0} \equiv \mathcal{J}^{000}. \tag{8m}$$

For the 20 independent components of the totally symmetric source $\mathcal{J}^{\alpha\beta\chi}(k)$, the invariant combinations corresponding to definite helicity values are denoted by the symbols $\varepsilon^{\#d}_h$. The subscript h indicates the helicity value, while the superscript distinguishes between multiple degenerate eigenvectors, where present.

In our case, these formulas must further account for reductions in d.o.f. imposed by the gauge constraints specific to each model we find. Information about gauge constraints is already managed internally by the component-wise procedure in PSALTer, and this allows us to construct simultaneously helicity- and gauge-invariant combinations of source components. Applying the reverse transformation from Eqs. (8a) to (8m) to the saturated propagator $\Pi(k)$ yields a simplified expression, conclusively resolving the identification of the massless propagating states. This method is not yet systematized in PSALTer, and so the massless spectrographs in Figs. 5–9 do not refer to the helicity. Rather, we apply the method manually to confirm our claims in Sec. III about the J-sector of origin for each massless mode.

Note that, spurious poles appearing in the massless cases due to the presence of powers of $1/q^2$ in the projection operators, automatically vanish when correctly considering source constraints to calculate the saturated propagator. This is a key feature of the PSALTer implementation, which is explained in detail in [52,70]. Note also that in Figs. 5–9 the constraints on the source currents are provided both in terms of irreducible source components and in position-space differential form—the latter are obtained by multiplying with the appropriate powers of k^2, though it is possible for information to be lost in this process. Since the source constraints are relevant mostly for the identification of gauge symmetries, we provide these explicitly in Sec. III.

C. Symmetric recursion

As explained in Sec. I, our goal is to obtain all special cases of Eq. (4) which are defined exclusively by the emergence of one or more gauge symmetries, and which are unitary throughout some volume of the remaining parameter space. To achieve this, it is sufficient to first identify all the symmetric special cases, and then to check the unitarity of each as an independent task.

A very naïve way in which to do this would be to consider the expressions for the square masses which appear in the general spectrograph in Fig. 2, and then to set masses to zero. The process could be repeated for subsequent spectrographs. The apparent weakness of this approach is its reliance on the "lore" that vanishing masses correspond to gauge symmetries.

A more thorough procedure, that does not rely on shortcuts, is to let rank reductions be systematically and exhaustively enumerated. The distinction between these methods may turn out to be physically significant, though

this is a question to be addressed in further work. The key question is whether rank reductions of the spin projection operator matrices correspond to symmetries which offer protection against radiative corrections when the Noether procedure is used to derive their corresponding interactions. In principle, it is possible to "puff-up" the spin-projector operators of any theory to arbitrarily high dimension without altering their rank, by the introduction of auxiliary fields. This is seen, for example, in the case of gravity where the metric can be replaced by the square of a tetrad field. The latter has more *a priori* field d.o.f., which are exactly compensated for by a local Lorentz symmetry in such a way that the physical content of the theory remains unchanged. Such a symmetry is spurious: it does not conform to van Nieuwenhuizen's rule, that "gauge fixing counts twice."

In practice, this thorough approach of rank reduction requires a careful implementation, and we now explain how it can be done in detail, by augmenting the method of particle spectroscopy with a recursive search algorithm. Working from Eq. (5), the Fourier space shell of classical field equations is

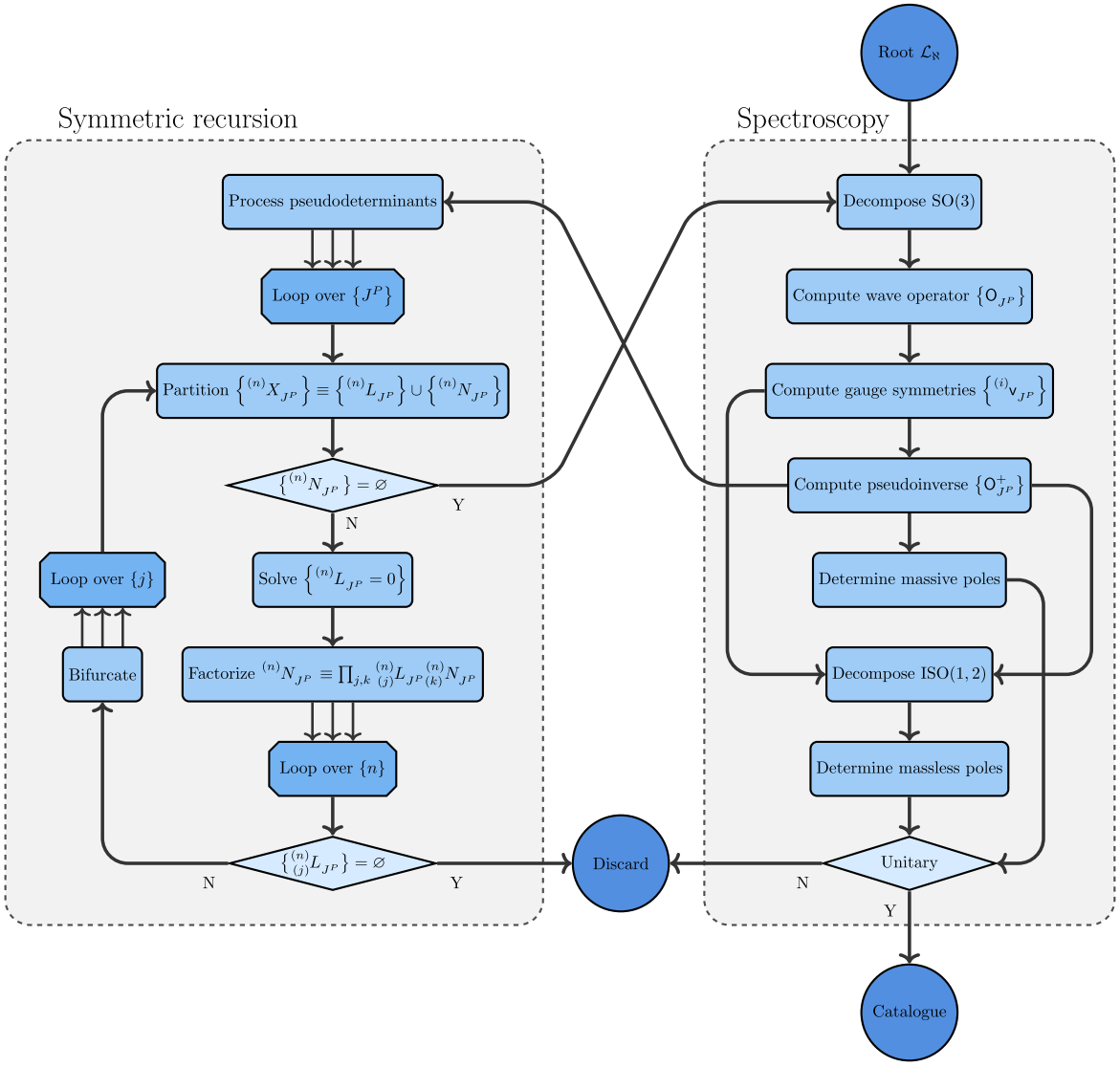


FIG. 3. The algorithm takes the general theory $\mathcal{L}_{\aleph}$ as input, as defined in Eq. (4). The PSALTer software is used to perform the spectroscopy of the model (right), and this is integrated with a new routine for obtaining special cases with enhanced gauge symmetries (left). The existing implementation in PSALTer simultaneously assesses the unitarity of each model as a final output, and performs the spin-parity decomposition of the wave operator needed by the new routine as raw input for inducing new symmetries. The final science product is an exhaustive catalog of physically motivated special cases of the general theory (see Fig. 4 and Table II). By physically motivated, we mean precisely the criteria set out in Sec. I—the tunings implemented are exactly the minimal ones needed for the emergence of gauge symmetries, and the resulting models are unitary throughout some volume of the remaining parameter space.

$$\mathsf{O}(k)\cdot\mathsf{K}(k)=\frac{1}{2}\mathsf{J}(k). \tag{9}$$

If $\mathsf{K}(k)$ is a solution to Eq. (9), then so is $\mathsf{K}(k)+\delta\mathsf{K}(k)$, where $\delta\mathsf{K}\equiv\sum_i c_i(k)\,{}^{(i)}\mathsf{v}(k)$ is spanned by the collection of all null right eigenvectors $\mathsf{O}(k)\cdot{}^{(i)}\mathsf{v}(k)\equiv 0$. Such shifts evidently describe the gauge symmetries of the theory, for which $c_i(x)$ are the independent local generators. By physicality, the wave operator $\mathsf{O}(k)$ is assumed to be Hermitian, so that the right eigenvectors ${}^{(i)}\mathsf{v}(k)$ are also left eigenvectors ${}^{(i)}\mathsf{v}^\dagger(k)\cdot\mathsf{O}(k)\equiv 0$. Left-multiplying Eq. (9) by ${}^{(i)}\mathsf{v}^\dagger(k)$ gives ${}^{(i)}\mathsf{v}^\dagger(k)\cdot\mathsf{J}(k)=0$, which defines the constraints on the sources.

It is the existence of the collection $\{{}^{(i)}\mathsf{v}(k)\}$ and, accordingly, the existence of one or more zero eigenvalues of $\mathsf{O}(k)$ that makes the inverse in Eq. (6) ill-defined; it is the source constraints which prevent this from being a problem. Accordingly, it follows that the poles in the saturated propagator may be identified by the pseudodeterminant $\Pi(k)\propto\det\tilde{\mathsf{O}}^{-1}(k)$ where the invertible matrix $\tilde{\mathsf{O}}$ is defined as

$$\tilde{\mathsf{O}}(k)\equiv\mathsf{O}(k)+\sum_i{}^{(i)}\mathsf{v}(k)\cdot{}^{(i)}\mathsf{v}^\dagger(k). \tag{10}$$

In general, we have the polynomial structure $\det\tilde{\mathsf{O}}(k)=\sum_n{}^{(n)}Xk^n$ for $n\geq 0$, and some collection $\{{}^{(n)}X\}$ of coefficients ${}^{(n)}X$ which are themselves polynomials in the couplings. Starting from the most general possible ansatz of quadratic operators, every solution to the simultaneous system $\{{}^{(n)}X=0\}$ defines a distinct theory whose only motivating principle is the emergence of one or more new gauge symmetry.

There is no reason why the $\{{}^{(n)}X\}$ should be linear in the couplings. In those nonlinear cases which factorize, the solutions to $\{{}^{(n)}X=0\}$ will bifurcate into multiple branches, each of which corresponds to a distinct theory. To study all the branches, it is prudent to work with any linear equations for as long as possible, since all equations in the system must anyway be satisfied in the end. One may partition the various expressions into linear and nonlinear parts

$$\{{}^{(n)}X\}\equiv\{{}^{(n)}L\}\cup\{{}^{(n)}N\}. \tag{11}$$

The system $\{{}^{(n)}L=0\}$ can be uniquely solved, and new bifurcation points will become apparent if the unique solution allows any of the $\{{}^{(n)}N\}$ to factorize algebraically. If any of the new factors are linear, this process repeats. For a given $\mathsf{O}(k)$ one is ideally left with a collection of models defined *entirely* by linear constraints on the couplings: these constraints are not arbitrary, because they are independent of coupling redefinitions.

The algorithm may also leave residual cases which resist factorization, and which are defined by inherently nonlinear constraints: each nonlinear constraint can be "sliced" into two or more linear constraints, but the infinite choice of slicings makes this arbitrary. Fortunately, the inherently nonlinear models are relatively few in the system following from Eq. (4).

In practice, as with the spectroscopy itself, the symmetric recursion algorithm is easiest to implement in the J^P basis. With reference to Eq. (7), the search proceeds independently in each subspace $\mathsf{O}_{J^P}(k)$, with null vectors $\{{}^{(i)}\mathsf{v}_{J^P}(k)\}$ giving rise to $\{{}^{(n)}X_{J^P}\}$, $\{{}^{(n)}L_{J^P}\}$ and $\{{}^{(n)}N_{J^P}\}$. We illustrate this algorithm in Fig. 3, and provide an example of its use in Appendix B.

III. RESULTS AND DISCUSSION

A. Summary

Using the algorithm presented in Sec. II we obtain 23 symmetric models between the root theory $\aleph$ and the trivial empty theory $\varnothing$, for which $\mathcal{L}_\varnothing\equiv 0$. These are illustrated in Fig. 4. The notation allocates a letter to signify the number of constraints (where $\mathfrak{B}$ signifies two constraints, $\mathfrak{C}$

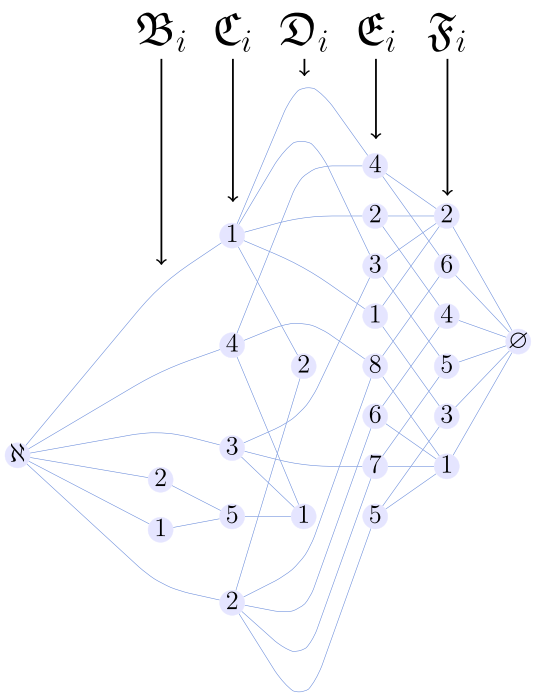


FIG. 4. Catalog of ways to propagate a totally symmetric rank-three field. Each node in the graph represents a distinct theory, defined by imposing constraints (from left to right) on the unconstrained root theory $\aleph$ defined in Eq. (4). The final node $\varnothing$ is the trivial theory $\mathcal{L}_\varnothing\equiv 0$. Constraints that connect a given daughter theory to its nearest parent are represented by edges. All these models are listed in Table II. Among them, the cases $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_6$, $\mathfrak{F}_2$, and $\mathfrak{F}_4$ are unitary, see Figs. 5–9. Note that the degree of rank reduction, and the J^P sectors affected, can be read off from the spectrographs from the number of source constraints, with which they are in one-to-one correspondence.

signifies three constraints, and so on) and a subscript to distinguish different models.[9] The full details of these models are provided in Appendix A and Table II. Among these models, we find five unitary cases $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_6$, $\mathfrak{F}_2$, and $\mathfrak{F}_4$, whose full spectrographs are provided in Figs. 5–9, respectively.

B. Spin one

An important consistency check emerges in the identification of model $\mathfrak{F}_2$ in Fig. 8, which confirms our earlier analysis based on a natural ansatz for the defining gauge symmetry as discussed in [2]. This model exhibits a gauge symmetry under three independent transformations of the form

$$\begin{aligned}\delta\mathcal{K}^{\alpha\beta\gamma} =& -2\partial_\zeta\partial_\epsilon\partial_\delta\partial^\gamma\partial^\beta\partial^\alpha\xi^{\delta\epsilon\zeta} - 3\Box\partial_\epsilon\partial^\gamma\partial^\beta\partial^\alpha\xi^\delta{}_\delta{}^\epsilon \\ &+ 12\Box\partial_\epsilon\partial_\delta\partial^{(\gamma}\partial^\beta\xi^{\alpha)\delta\epsilon} + 3\Box^2\partial^{(\gamma}\partial^\beta\xi^{\alpha)\delta}{}_\delta \\ &- 15\Box^2\partial_\delta\partial^{(\gamma}\xi^{\alpha\beta)\delta} + 5\Box^3\xi^{\alpha\beta\gamma} \\ &- 3\eta^{(\alpha\beta|}\Box\partial_\zeta\partial_\epsilon\partial_\delta\partial^{|\gamma)}\xi^{\delta\epsilon\zeta} + 3\eta^{(\alpha\beta|}\Box^2\partial_\epsilon\partial^{|\gamma)}\xi^\delta{}_\delta{}^\epsilon \\ &+ 3\eta^{(\alpha\beta|}\Box^2\partial_\epsilon\partial_\delta\xi^{|\gamma)\delta\epsilon} - 3\eta^{(\alpha\beta|}\Box^3\xi^{|\gamma)\delta}{}_\delta, \end{aligned} \tag{12a}$$

$$\begin{aligned}\delta\mathcal{K}^{\alpha\beta\gamma} =& 2\partial_\zeta\partial_\epsilon\partial_\delta\partial^\gamma\partial^\beta\partial^\alpha\xi^{\delta\epsilon\zeta} + \Box\partial_\epsilon\partial^\gamma\partial^\beta\partial^\alpha\xi^\delta{}_\delta{}^\epsilon \\ &- 6\Box\partial_\epsilon\partial_\delta\partial^{(\gamma}\partial^\beta\xi^{\alpha)\delta\epsilon} + 3\Box^2\partial_\delta\partial^{(\gamma}\xi^{\alpha\beta)\delta} \\ &+ \eta^{(\alpha\beta|}\Box\partial_\zeta\partial_\epsilon\partial_\delta\partial^{|\gamma)}\xi^{\delta\epsilon\zeta} \\ &- \eta^{(\alpha\beta|}\Box^2\partial_\epsilon\partial^{|\gamma)}\xi^\delta{}_\delta{}^\epsilon, \end{aligned} \tag{12b}$$

$$\delta\mathcal{K}^{\alpha\beta\gamma} = -\partial_\zeta\partial_\epsilon\partial_\delta\partial^\gamma\partial^\beta\partial^\alpha\xi^{\delta\epsilon\zeta} + \Box\partial_\epsilon\partial^\gamma\partial^\beta\partial^\alpha\xi^\delta{}_\delta{}^\epsilon. \tag{12c}$$

where $\xi_{\alpha\beta\gamma} \equiv \xi_{(\alpha\beta\gamma)}$ is a totally symmetric rank-three gauge parameter.

In contrast, the newly found model $\mathfrak{E}_1$ displays an intriguing structure characterized by multiple parameters while remaining the most general form permitted by the given symmetry. The presence of k^2 terms in the $J = 2$ sector of Fig. 5 might naively suggest particle propagation in that channel. However, a helicity decomposition definitively establishes that only $J = 1$ particles propagate within this model. The model's gauge symmetries are generated by two independent transformations of the form

$$\begin{aligned}\delta\mathcal{K}^{\alpha\beta\gamma} =& -2\partial_\zeta\partial_\epsilon\partial_\delta\partial^\gamma\partial^\beta\partial^\alpha\xi^{\delta\epsilon\zeta} - 3\Box\partial_\epsilon\partial^\gamma\partial^\beta\partial^\alpha\xi^\delta{}_\delta{}^\epsilon \\ &+ 12\Box\partial_\epsilon\partial_\delta\partial^{(\gamma}\partial^\beta\xi^{\alpha)\delta\epsilon} + 3\Box^2\partial^{(\gamma}\partial^\beta\xi^{\alpha)\delta}{}_\delta \\ &- 15\Box^2\partial_\delta\partial^{(\gamma}\xi^{\alpha\beta)\delta} + 5\Box^3\xi^{\alpha\beta\gamma} \\ &- 3\eta^{(\alpha\beta|}\Box\partial_\zeta\partial_\epsilon\partial_\delta\partial^{|\gamma)}\xi^{\delta\epsilon\zeta} + 3\eta^{(\alpha\beta|}\Box^2\partial_\epsilon\partial^{|\gamma)}\xi^\delta{}_\delta{}^\epsilon \\ &+ 3\eta^{(\alpha\beta|}\Box^2\partial_\epsilon\partial_\delta\xi^{|\gamma)\delta\epsilon} - 3\eta^{(\alpha\beta|}\Box^3\xi^{|\gamma)\delta}{}_\delta, \end{aligned} \tag{13a}$$

$$\begin{aligned}\delta\mathcal{K}^{\alpha\beta\gamma} =& -\eta^{(\alpha\beta|}\Box\partial_\zeta\partial_\epsilon\partial_\delta\partial^{|\gamma)}\xi^{\delta\epsilon\zeta} \\ &+ \eta^{(\alpha\beta|}\Box^2\partial_\epsilon\partial^{|\gamma)}\xi^\delta{}_\delta{}^\epsilon. \end{aligned} \tag{13b}$$

C. Spin three

A further valuable validation of our methodology is the identification of model $\mathfrak{F}_4$ in Fig. 9, which aligns exactly with Fronsdal's formulation for massless $J = 3$ particle propagation [23]. We will not repeat here the well-known Fronsdal gauge transformation.

Closely related is model $\mathfrak{E}_6$ in Fig. 7, initially appearing to contain multiple propagating sectors and coupling constants. This model exhibits gauge symmetries generated by two independent transformations of the form

$$\begin{aligned}\delta\mathcal{K}^{\alpha\beta\gamma} =& 2\partial_\zeta\partial_\epsilon\partial_\delta\partial^\gamma\partial^\beta\partial^\alpha\xi^{\delta\epsilon\zeta} + \Box\partial_\epsilon\partial^\gamma\partial^\beta\partial^\alpha\xi^\delta{}_\delta{}^\epsilon \\ &- 6\Box\partial_\epsilon\partial_\delta\partial^{(\gamma}\partial^\beta\xi^{\alpha)\delta\epsilon} + 3\Box^2\partial_\delta\partial^{(\gamma}\xi^{\alpha\beta)\delta} \\ &+ \eta^{(\alpha\beta|}\Box\partial_\zeta\partial_\epsilon\partial_\delta\partial^{|\gamma)}\xi^{\delta\epsilon\zeta} \\ &- \eta^{(\alpha\beta|}\Box^2\partial_\epsilon\partial^{|\gamma)}\xi^\delta{}_\delta{}^\epsilon, \end{aligned} \tag{14a}$$

$$\begin{aligned}\delta\mathcal{K}^{\alpha\beta\gamma} =& \eta^{(\alpha\beta|}\Box\partial_\zeta\partial_\epsilon\partial_\delta\partial^{|\gamma)}\xi^{\delta\epsilon\zeta} - \eta^{(\alpha\beta|}\Box^2\partial_\epsilon\partial^{|\gamma)}\xi^\delta{}_\delta{}^\epsilon \\ &- \eta^{(\alpha\beta|}\Box^2\partial_\epsilon\partial_\delta\xi^{|\gamma)\delta\epsilon} + \eta^{(\alpha\beta|}\Box^3\xi^{|\gamma)\delta}{}_\delta. \end{aligned} \tag{14b}$$

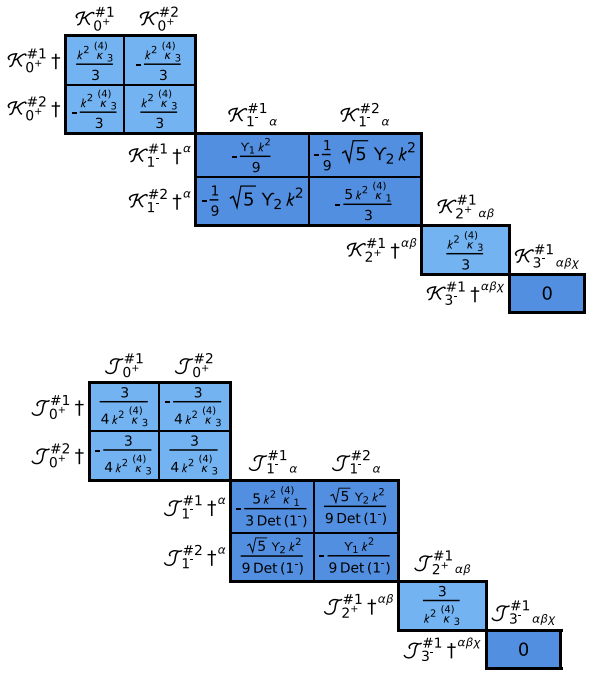
Nevertheless, spectral analysis conclusively demonstrates that only two physical states propagate, indicating the involvement of a single physical sector.

Furthermore, only one coupling parameter influences the unitarity conditions, suggesting potential redundancy among couplings. A deeper examination confirms this intuition, revealing that this model decomposes precisely into Fronsdal's $J = 3$ structure (as found in model $\mathfrak{F}_4$) combined with model $\mathfrak{F}_1$. As established previously, the latter model contributes no physical states to the spectrum (see Table II).

D. Spin one and spin three

Finally, we identify model $\mathfrak{E}_2$ as the unique instance of a symmetric framework capable of simultaneous propagation, as illustrated in Fig. 6. To our knowledge, this particular characteristic has previously appeared only within the Maxwell-like theories presented by Campoleoni and Francia in [71]. A key prerequisite for achieving their model is the adoption of traceless fields, a significant distinction from our unconstrained approach in Eq. (3). Imposition of the traceless condition $\mathcal{K}^\alpha{}_\alpha{}^\beta \equiv 0$ on $\mathcal{L}_{\mathfrak{E}_2}$ establishes an equivalence with their approach through a

[9]Note that we choose to categorize the models in the Hasse diagram and in the tables according to the numbers of free parameters in the quadratic action: this is somewhat arbitrary, and an alternative scheme would categorize according to the rank reduction (kernel size) of the matrices. Regardless of the way in which the Hasse diagram is organized, we show the complete spectrographs for all models of interest. The spectrographs provide full details about the rank reduction and the propagating particles. The effect of rank reduction on the actual particle spectrum is highly nontrivial: it cannot simply be "read of" from the highest-rank matrices shown in Fig. 2, and must be computed afresh for each model using the methods described in Sec. II.



Abbreviations used in matrices

$$Y_1 == 3\,{}^{(4)}\kappa_1 - 4\,{}^{(4)}\kappa_3 \;\&\&\; Y_2 == 3\,{}^{(4)}\kappa_1 + {}^{(4)}\kappa_3 \;\&\&\; \text{Det}(1^-) == -\frac{5}{81}k^4\,{}^{(4)}\kappa_3\,(18\,{}^{(4)}\kappa_1 + {}^{(4)}\kappa_3)$$

Lagrangian

$$\mathcal{L} = {}^{(4)}\kappa_1\,\partial_\beta\mathcal{K}_\chi{}^\delta{}_\delta\partial^\chi\mathcal{K}^{\alpha\ \beta}_{\ \alpha} - {}^{(4)}\kappa_1\,\partial_\chi\mathcal{K}_\beta{}^\delta{}_\delta\partial^\chi\mathcal{K}^{\alpha\ \beta}_{\ \alpha} + {}^{(4)}\kappa_3\,\partial_\alpha\mathcal{K}^{\alpha\beta\chi}\partial_\delta\mathcal{K}_{\beta\chi}{}^\delta - \frac{2}{3}\,{}^{(4)}\kappa_3\,\partial^\chi\mathcal{K}^{\alpha\ \beta}_{\ \alpha}\partial_\delta\mathcal{K}_{\beta\chi}{}^\delta$$

Added source term(s):	$\mathcal{K}^{\alpha\beta\chi}\ \mathcal{J}_{\alpha\beta\chi}$	
Source constraint(s)	No. of constraint(s)	Covariant form
$\mathcal{J}_{0^+}^{\#2} + 3\,\mathcal{J}_{0^+}^{\#1} == 0$	1	$\partial_\beta\mathcal{J}^{\alpha\ \beta}_{\ \alpha} == 0$
$\mathcal{J}_{3^-}^{\#1\,\alpha\beta\chi} == 0$	7	$k^\delta\,k^\epsilon\,(3\,\partial_\epsilon\partial^\chi\partial^\beta\partial^\alpha\mathcal{J}_\delta{}^\phi{}_\phi - \partial_\epsilon\partial_\delta\partial^\beta\partial^\alpha\mathcal{J}^{\chi\phi}{}_\phi - \partial_\epsilon\partial_\delta\partial^\chi\partial^\beta\mathcal{J}^{\alpha\phi}{}_\phi + 2\,\partial_\phi\partial^\chi\partial^\beta\partial^\alpha\mathcal{J}_{\delta\epsilon}{}^\phi - 4\,\partial_\phi\partial_\epsilon\partial^\beta\partial^\alpha\mathcal{J}^\chi{}_\delta{}^\phi - 4\,\partial_\phi\partial_\epsilon\partial^\chi\partial^\alpha\mathcal{J}^\beta{}_\delta{}^\phi - 4\,\partial_\phi\partial_\epsilon\partial^\chi\partial^\beta\mathcal{J}^\alpha{}_\delta{}^\phi + 5\,\partial_\phi\partial_\epsilon\partial_\delta\partial^\alpha\mathcal{J}^{\beta\chi\phi} + 5\,\partial_\phi\partial_\epsilon\partial_\delta\partial^\beta\mathcal{J}^{\alpha\chi\phi} + 5\,\partial_\phi\partial_\epsilon\partial_\delta\partial^\chi\mathcal{J}^{\alpha\beta\phi} - 5\,\partial_\phi\partial^\phi\partial_\epsilon\partial_\delta\mathcal{J}^{\alpha\beta\chi} - \eta^{\beta\chi}\,\partial_\gamma\partial_\epsilon\partial_\delta\partial^\alpha\mathcal{J}^\phi{}_\phi{}^\gamma - \eta^{\alpha\chi}\,\partial_\gamma\partial_\epsilon\partial_\delta\partial^\beta\mathcal{J}^\phi{}_\phi{}^\gamma - \eta^{\alpha\beta}\,\partial_\gamma\partial_\epsilon\partial_\delta\partial^\chi\mathcal{J}^\phi{}_\phi{}^\gamma + \eta^{\beta\chi}\,\partial_\gamma\partial_\phi\partial_\epsilon\partial^\alpha\mathcal{J}_\delta{}^{\phi\gamma} + \eta^{\alpha\chi}\,\partial_\gamma\partial_\phi\partial_\epsilon\partial^\beta\mathcal{J}_\delta{}^{\phi\gamma} + \eta^{\alpha\beta}\,\partial_\gamma\partial_\phi\partial_\epsilon\partial^\chi\mathcal{J}_\delta{}^{\phi\gamma} - \eta^{\beta\chi}\,\partial_\gamma\partial_\phi\partial_\epsilon\partial_\delta\mathcal{J}^{\alpha\phi\gamma} - \eta^{\alpha\chi}\,\partial_\gamma\partial_\phi\partial_\epsilon\partial_\delta\mathcal{J}^{\beta\phi\gamma} - \eta^{\alpha\beta}\,\partial_\gamma\partial_\phi\partial_\epsilon\partial_\delta\mathcal{J}^{\chi\phi\gamma} + \eta^{\beta\chi}\,\partial_\gamma\partial^\gamma\partial_\epsilon\partial_\delta\mathcal{J}^{\alpha\phi}{}_\phi + \eta^{\alpha\chi}\,\partial_\gamma\partial^\gamma\partial_\epsilon\partial_\delta\mathcal{J}^{\beta\phi}{}_\phi + \eta^{\alpha\beta}\,\partial_\gamma\partial^\gamma\partial_\epsilon\partial_\delta\mathcal{J}^{\chi\phi}{}_\phi) == 0$
Total no. of constraint(s):	8	

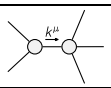
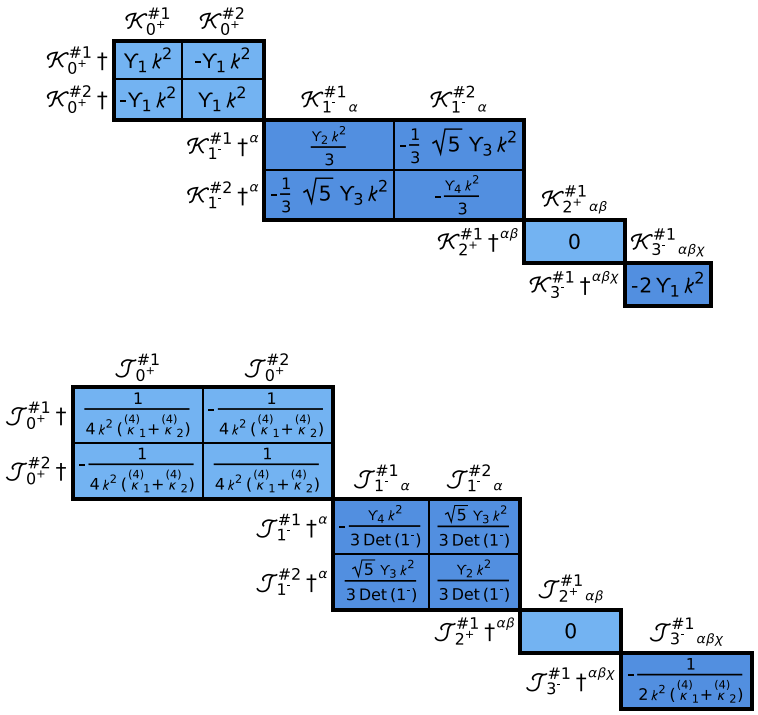
Resolved pole(s)	No. of polarization(s)	Square mass	Residue
	2	0	$\frac{1}{18\,{}^{(4)}\kappa_1 + {}^{(4)}\kappa_3}$
Resolved unitarity condition(s):			${}^{(4)}\kappa_1 > -\frac{{}^{(4)}\kappa_3}{18}$

FIG. 5. Output generated by PSALTer. The spectrograph of $\mathfrak{C}_1$, as defined in Table II, with the gauge symmetries defined in Eqs. (13a) and (13b). All notation is defined in Table I and Fig. 2. In common with $\mathfrak{F}_2$ in Fig. 8 this is a $J = 1$ model.

straightforward field redefinition. Thus, it is both surprising and noteworthy to observe that simultaneous propagation, typically reliant on traceless tensors, can also be realized using unconstrained tensors and gauge symmetries.

The gauge symmetries of this model are generated by two independent transformations of the form

$$\begin{aligned}\delta\mathcal{K}^{\alpha\beta\gamma} = {}& 2\partial_\zeta\partial_\epsilon\partial_\delta\partial^\gamma\partial^\beta\partial^\alpha\xi^{\delta\epsilon\zeta} + \Box\partial_\epsilon\partial^\gamma\partial^\beta\partial^\alpha\xi^\delta{}_\delta{}^\epsilon \\ & - 6\Box\partial_\epsilon\partial_\delta\partial^{(\gamma}\partial^\beta\xi^{\alpha)\delta\epsilon} + 3\Box^2\partial_\delta\partial^{(\gamma}\xi^{\alpha\beta)\delta} \\ & + \eta^{(\alpha\beta|}\Box\partial_\zeta\partial_\epsilon\partial_\delta\partial^{|\gamma)}\xi^{\delta\epsilon\zeta} \\ & - \eta^{(\alpha\beta|}\Box^2\partial_\epsilon\partial^{|\gamma)}\xi^\delta{}_\delta{}^\epsilon,\end{aligned} \tag{15a}$$



Abbreviations used in matrices

$$Y_1 == \overset{(4)}{\kappa}_1 + \overset{(4)}{\kappa}_2 \;\&\&\; Y_2 == 2\overset{(4)}{\kappa}_1 + 3\overset{(4)}{\kappa}_2 \;\&\&\; Y_3 == 2\overset{(4)}{\kappa}_1 + \overset{(4)}{\kappa}_2 \;\&\&$$
$$Y_4 == 6\overset{(4)}{\kappa}_1 + \overset{(4)}{\kappa}_2 \;\&\&\; \mathrm{Det}(1^-) == -\frac{8}{9}k^4(\overset{(4)}{\kappa}_1 + \overset{(4)}{\kappa}_2)(4\overset{(4)}{\kappa}_1 + \overset{(4)}{\kappa}_2)$$

Lagrangian

$$\overset{(4)}{\kappa}_1 \partial_\beta \mathcal{K}_\chi{}^\delta{}_\delta \partial^\chi \mathcal{K}^\alpha{}_\alpha{}^\beta + \overset{(4)}{\kappa}_2 \partial_\chi \mathcal{K}_\beta{}^\delta{}_\delta \partial^\chi \mathcal{K}^\alpha{}_\alpha{}^\beta + 6\overset{(4)}{\kappa}_1 \partial_\alpha \mathcal{K}^{\alpha\beta\chi} \partial_\delta \mathcal{K}_{\beta\chi}{}^\delta + 6\overset{(4)}{\kappa}_2 \partial_\alpha \mathcal{K}^{\alpha\beta\chi} \partial_\delta \mathcal{K}_{\beta\chi}{}^\delta -$$
$$4\overset{(4)}{\kappa}_1 \partial^\chi \mathcal{K}^\alpha{}_\alpha{}^\beta \partial_\delta \mathcal{K}_{\beta\chi}{}^\delta - 4\overset{(4)}{\kappa}_2 \partial^\chi \mathcal{K}^\alpha{}_\alpha{}^\beta \partial_\delta \mathcal{K}_{\beta\chi}{}^\delta - 2\overset{(4)}{\kappa}_1 \partial_\delta \mathcal{K}_{\alpha\beta\chi} \partial^\delta \mathcal{K}^{\alpha\beta\chi} - 2\overset{(4)}{\kappa}_2 \partial_\delta \mathcal{K}_{\alpha\beta\chi} \partial^\delta \mathcal{K}^{\alpha\beta\chi}$$

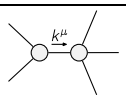
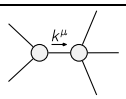
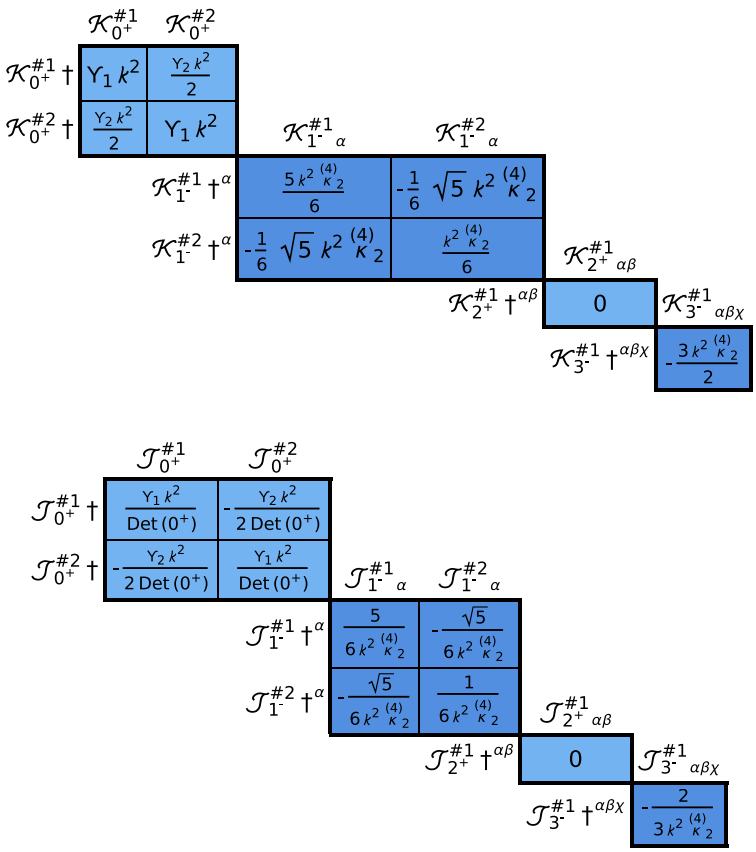
Added source term(s):	$\mathcal{K}^{\alpha\beta\chi}\ \mathcal{J}_{\alpha\beta\chi}$		
Source constraint(s)	No. of constraint(s)	Covariant form	
$\mathcal{J}_{0^+}^{\#2} + 3\mathcal{J}_{0^+}^{\#1} == 0$	1	$\partial_\beta \mathcal{J}^\alpha{}_\alpha{}^\beta == 0$	
$\mathcal{J}_{2^+}^{\#1\,\alpha\beta} == 0$	5	$k^\chi(\partial_\epsilon\partial_\chi\partial^\beta\partial^\alpha\mathcal{J}^\delta{}_\delta{}^\epsilon + 2\partial_\epsilon\partial_\delta\partial^\beta\partial^\alpha\mathcal{J}_\chi{}^{\delta\epsilon} - 3\partial_\epsilon\partial_\delta\partial_\chi\partial^\alpha\mathcal{J}^{\beta\delta\epsilon} -$ $3\partial_\epsilon\partial_\delta\partial_\chi\partial^\beta\mathcal{J}^{\alpha\delta\epsilon} + 3\partial_\epsilon\partial^\epsilon\partial_\delta\partial_\chi\mathcal{J}^{\alpha\beta\delta} + \eta^{\alpha\beta}\partial_\phi\partial_\epsilon\partial_\delta\partial_\chi\mathcal{J}^{\delta\epsilon\phi} - \eta^{\alpha\beta}\partial_\phi\partial^\phi\partial_\epsilon\partial_\chi\mathcal{J}^\delta{}_\delta{}^\epsilon) == 0$	
Total no. of constraint(s):	6		
Resolved pole(s)	No. of polarization(s)	Square mass	Residue
	2	0	$\frac{50\overset{(4)}{\kappa}_1 + 35\overset{(4)}{\kappa}_2 - \sqrt{196\overset{(4)}{\kappa}_1{}^2 + 620\overset{(4)}{\kappa}_1\overset{(4)}{\kappa}_2 + 649\overset{(4)}{\kappa}_2{}^2}}{4\overset{(4)}{\kappa}_1{}^2 + 5\overset{(4)}{\kappa}_1\overset{(4)}{\kappa}_2 + \overset{(4)}{\kappa}_2{}^2}$
	2	0	$\frac{50\overset{(4)}{\kappa}_1 + 35\overset{(4)}{\kappa}_2 + \sqrt{196\overset{(4)}{\kappa}_1{}^2 + 620\overset{(4)}{\kappa}_1\overset{(4)}{\kappa}_2 + 649\overset{(4)}{\kappa}_2{}^2}}{4\overset{(4)}{\kappa}_1{}^2 + 5\overset{(4)}{\kappa}_1\overset{(4)}{\kappa}_2 + \overset{(4)}{\kappa}_2{}^2}$
Resolved unitarity condition(s):	$(\overset{(4)}{\kappa}_2 \le 0 \;\&\&\; \overset{(4)}{\kappa}_1 > -\overset{(4)}{\kappa}_2) \,||\, (\overset{(4)}{\kappa}_2 > 0 \;\&\&\; \overset{(4)}{\kappa}_1 > -\frac{\overset{(4)}{\kappa}_2}{4})$		

FIG. 6. Output generated by PSALTer. The spectrograph of $\mathfrak{E}_2$, as defined in Table II, with the gauge symmetries defined in Eqs. (15a) and (15b). All notation is defined in Table I and Fig. 2. This is a generalization of the model found in [71], to which it reduces under the assumption $\mathcal{K}^\alpha{}_\alpha{}^\beta \equiv 0$. The model propagates $J = 1$ and $J = 3$ simultaneously.



Abbreviations used in matrices

$$Y_1 == {}^{(4)}\kappa_1 + {}^{(4)}\kappa_2 \;\&\&\; Y_2 == 2\,{}^{(4)}\kappa_1 - {}^{(4)}\kappa_2 \;\&\&\; \mathrm{Det}(0^+) == \frac{3}{4}k^4\,{}^{(4)}\kappa_2\,(4\,{}^{(4)}\kappa_1 + {}^{(4)}\kappa_2)$$

Lagrangian

$$\begin{aligned}&{}^{(4)}\kappa_1\,\partial_\beta\mathcal{K}_\chi{}^\delta{}_\delta\,\partial^\chi\mathcal{K}^{\alpha\beta}{}_\alpha + {}^{(4)}\kappa_2\,\partial_\chi\mathcal{K}_\beta{}^\delta{}_\delta\,\partial^\chi\mathcal{K}^{\alpha\beta}{}_\alpha + \frac{9}{2}\,{}^{(4)}\kappa_2\,\partial_\alpha\mathcal{K}^{\alpha\beta\chi}\,\partial_\delta\mathcal{K}_{\beta\chi}{}^\delta - 3\,{}^{(4)}\kappa_2\,\partial^\chi\mathcal{K}^{\alpha\beta}{}_\alpha\,\partial_\delta\mathcal{K}_{\beta\chi}{}^\delta - \frac{3}{2}\,{}^{(4)}\kappa_2\,\partial_\delta\mathcal{K}_{\alpha\beta\chi}\,\partial^\delta\mathcal{K}^{\alpha\beta\chi}\end{aligned}$$

Added source term(s):	$\mathcal{K}^{\alpha\beta\chi}\;\mathcal{J}_{\alpha\beta\chi}$	
Source constraint(s)	No. of constraint(s)	Covariant form
$\mathcal{J}_{1^-}^{\#2\,\alpha} + \mathcal{J}_{1^-}^{\#1\,\alpha} == 0$	3	$\partial_\chi\partial^\alpha\mathcal{J}^{\beta}{}_{\beta}{}^{\chi} == \partial_\chi\partial^\chi\mathcal{J}^{\alpha\beta}{}_{\beta}$
$\mathcal{J}_{2^+}^{\#1\,\alpha\beta} == 0$	5	$k^\chi\,(\partial_\epsilon\partial_\chi\partial^\beta\partial^\alpha\mathcal{J}^{\delta}{}_{\delta}{}^{\epsilon} + 2\,\partial_\epsilon\partial_\delta\partial^\beta\partial^\alpha\mathcal{J}_\chi{}^{\delta\epsilon} - 3\,\partial_\epsilon\partial_\delta\partial_\chi\partial^\alpha\mathcal{J}^{\beta\delta\epsilon} - 3\,\partial_\epsilon\partial_\delta\partial_\chi\partial^\beta\mathcal{J}^{\alpha\delta\epsilon} + 3\,\partial_\epsilon\partial^\epsilon\partial_\delta\partial_\chi\mathcal{J}^{\alpha\beta\delta} + \eta^{\alpha\beta}\,\partial_\phi\partial_\epsilon\partial_\delta\partial_\chi\mathcal{J}^{\delta\epsilon\phi} - \eta^{\alpha\beta}\,\partial_\phi\partial^\phi\partial_\epsilon\partial_\chi\mathcal{J}^{\delta}{}_{\delta}{}^{\epsilon}) == 0$
Total no. of constraint(s):	8	

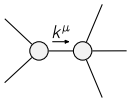
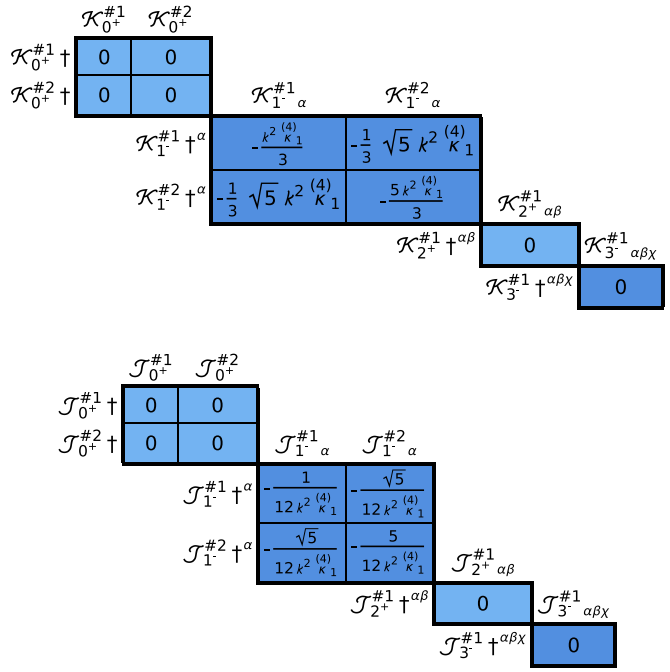
Resolved pole(s)	No. of polarization(s)	Square mass	Residue
	2	0	$\frac{1}{{}^{(4)}\kappa_2}$
Resolved unitarity condition(s):		${}^{(4)}\kappa_2 > 0$	

FIG. 7. Output generated by PSALTer. The spectrograph of $\mathfrak{E}_6$, as defined in Table II, with the gauge symmetries defined in Eqs. (14a) and (14b). All notation is defined in Table I and Fig. 2. This model is a superposition of the Fronsdal theory $\mathfrak{F}_4$ in Fig. 9 and the nonpropagating model $\mathfrak{F}_1$.



Lagrangian

$${}^{(4)}\kappa_1\,\partial_\beta\mathcal{K}_\chi{}^\delta{}_\delta\partial^\chi\mathcal{K}^\alpha{}_\alpha{}^\beta - {}^{(4)}\kappa_1\,\partial_\chi\mathcal{K}_\beta{}^\delta{}_\delta\partial^\chi\mathcal{K}^\alpha{}_\alpha{}^\beta$$

Added source term(s):	$\mathcal{K}^{\alpha\beta\chi}\,\mathcal{J}_{\alpha\beta\chi}$	
Source constraint(s)	No. of constraint(s)	Covariant form
$\mathcal{J}_{0^+}^{\#2} == 0$	1	$\partial_\chi\partial_\beta\partial_\alpha\mathcal{J}^{\alpha\beta\chi} == \partial_\chi\partial^\chi\partial_\beta\mathcal{J}^\alpha{}_\alpha{}^\beta$
$\mathcal{J}_{0^+}^{\#1} == 0$	1	$\partial_\chi\partial_\beta\partial_\alpha\mathcal{J}^{\alpha\beta\chi} == 0$
$\mathcal{J}_{1^-}^{\#2\,\alpha} == 5\,\mathcal{J}_{1^-}^{\#1\,\alpha}$	3	$6\,\partial_\delta\partial_\chi\partial_\beta\partial^\alpha\mathcal{J}^{\beta\chi\delta} + \partial_\delta\partial^\delta\partial_\chi\partial^\chi\mathcal{J}^{\alpha\beta}{}_\beta == \partial_\delta\partial^\delta\partial_\chi\partial^\alpha\mathcal{J}^\beta{}_\beta{}^\chi + 6\,\partial_\delta\partial^\delta\partial_\chi\partial_\beta\mathcal{J}^{\alpha\beta\chi}$
$\mathcal{J}_{2^+}^{\#1\,\alpha\beta} == 0$	5	$k^\chi\,(\partial_\epsilon\partial_\chi\partial^\beta\partial^\alpha\mathcal{J}^\delta{}_\delta{}^\epsilon + 2\,\partial_\epsilon\partial_\delta\partial^\beta\partial^\alpha\mathcal{J}_\chi{}^{\delta\epsilon} - 3\,\partial_\epsilon\partial_\delta\partial_\chi\partial^\alpha\mathcal{J}^{\beta\delta\epsilon} - 3\,\partial_\epsilon\partial_\delta\partial_\chi\partial^\beta\mathcal{J}^{\alpha\delta\epsilon} + 3\,\partial_\epsilon\partial^\epsilon\partial_\delta\partial_\chi\mathcal{J}^{\alpha\beta\delta} + \eta^{\alpha\beta}\,\partial_\phi\partial_\epsilon\partial_\delta\partial_\chi\mathcal{J}^{\delta\epsilon\phi} - \eta^{\alpha\beta}\,\partial_\phi\partial^\phi\partial_\epsilon\partial_\chi\mathcal{J}^\delta{}_\delta{}^\epsilon) == 0$
$\mathcal{J}_{3^-}^{\#1\,\alpha\beta\chi} == 0$	7	$k^\delta\,k^\epsilon\,(3\,\partial_\epsilon\partial^\chi\partial^\beta\partial^\alpha\mathcal{J}_\delta{}^\phi{}_\phi - \partial_\epsilon\partial_\delta\partial^\beta\partial^\alpha\mathcal{J}^{\chi\phi}{}_\phi - \partial_\epsilon\partial_\delta\partial^\chi\partial^\alpha\mathcal{J}^{\beta\phi}{}_\phi - \partial_\epsilon\partial_\delta\partial^\chi\partial^\beta\mathcal{J}^{\alpha\phi}{}_\phi + 2\,\partial_\phi\partial^\chi\partial^\beta\partial^\alpha\mathcal{J}_\delta{}^\phi{}_\epsilon - 4\,\partial_\phi\partial_\epsilon\partial^\beta\partial^\alpha\mathcal{J}^\chi{}_\delta{}^\phi - 4\,\partial_\phi\partial_\epsilon\partial^\chi\partial^\alpha\mathcal{J}^\beta{}_\delta{}^\phi - 4\,\partial_\phi\partial_\epsilon\partial^\chi\partial^\beta\mathcal{J}^\alpha{}_\delta{}^\phi + 5\,\partial_\phi\partial_\epsilon\partial_\delta\partial^\alpha\mathcal{J}^{\beta\chi\phi} + 5\,\partial_\phi\partial_\epsilon\partial_\delta\partial^\beta\mathcal{J}^{\alpha\chi\phi} + 5\,\partial_\phi\partial_\epsilon\partial_\delta\partial^\chi\mathcal{J}^{\alpha\beta\phi} - 5\,\partial_\phi\partial^\phi\partial_\epsilon\partial_\delta\mathcal{J}^{\alpha\beta\chi} - \eta^{\beta\chi}\,\partial_\gamma\partial_\epsilon\partial_\delta\partial^\alpha\mathcal{J}^\phi{}_\phi{}^\gamma - \eta^{\alpha\chi}\,\partial_\gamma\partial_\epsilon\partial_\delta\partial^\beta\mathcal{J}^\phi{}_\phi{}^\gamma - \eta^{\alpha\beta}\,\partial_\gamma\partial_\epsilon\partial_\delta\partial^\chi\mathcal{J}^\phi{}_\phi{}^\gamma + \eta^{\beta\chi}\,\partial_\gamma\partial_\phi\partial_\epsilon\partial^\alpha\mathcal{J}_\delta{}^{\phi\gamma} + \eta^{\alpha\chi}\,\partial_\gamma\partial_\phi\partial_\epsilon\partial^\beta\mathcal{J}_\delta{}^{\phi\gamma} + \eta^{\alpha\beta}\,\partial_\gamma\partial_\phi\partial_\epsilon\partial^\chi\mathcal{J}_\delta{}^{\phi\gamma} - \eta^{\beta\chi}\,\partial_\gamma\partial_\phi\partial_\epsilon\partial_\delta\mathcal{J}^{\alpha\phi\gamma} - \eta^{\alpha\chi}\,\partial_\gamma\partial_\phi\partial_\epsilon\partial_\delta\mathcal{J}^{\beta\phi\gamma} - \eta^{\alpha\beta}\,\partial_\gamma\partial_\phi\partial_\epsilon\partial_\delta\mathcal{J}^{\chi\phi\gamma} + \eta^{\beta\chi}\,\partial_\gamma\partial^\gamma\partial_\epsilon\partial_\delta\mathcal{J}^{\alpha\phi}{}_\phi + \eta^{\alpha\chi}\,\partial_\gamma\partial^\gamma\partial_\epsilon\partial_\delta\mathcal{J}^{\beta\phi}{}_\phi + \eta^{\alpha\beta}\,\partial_\gamma\partial^\gamma\partial_\epsilon\partial_\delta\mathcal{J}^{\chi\phi}{}_\phi) == 0$
Total no. of constraint(s):	17	

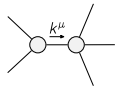
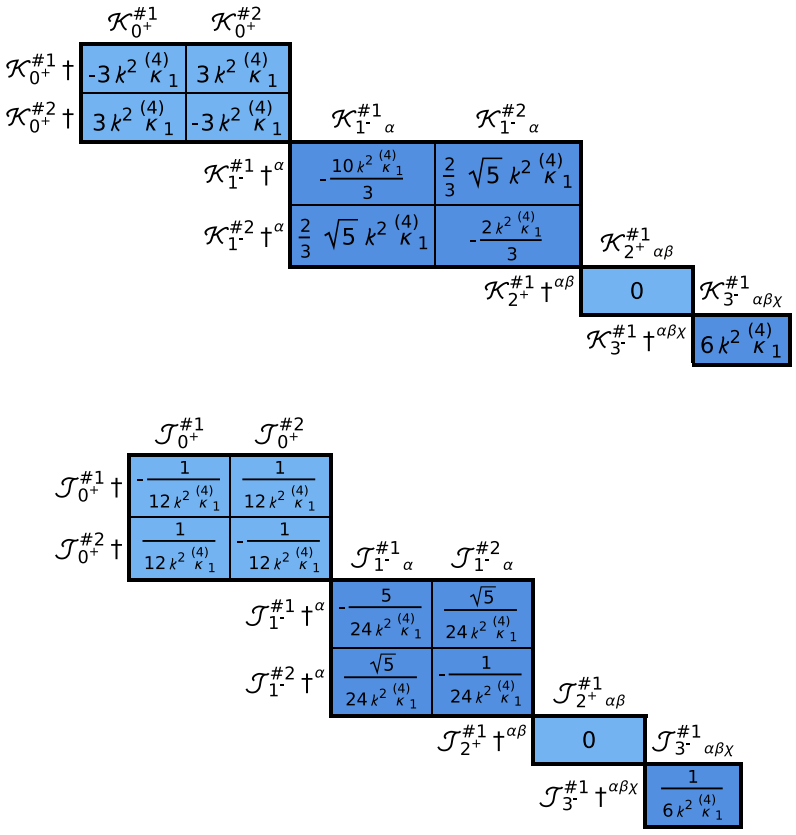
Resolved pole(s)	No. of polarization(s)	Square mass	Residue
	2	0	$\frac{1}{{}^{(4)}\kappa_1}$
Resolved unitarity condition(s):		${}^{(4)}\kappa_1 > 0$	

FIG. 8. Output generated by PSALTer. The spectrograph of $\mathfrak{F}_2$, as defined in Table II, with the gauge symmetries defined in Eqs. (12a) and (12c). All notation is defined in Table I and Fig. 2. This Maxwell-type model was found previously in [2].



Lagrangian

$$
\begin{aligned}
&{}^{(4)}\kappa_1\,\partial_\beta\mathcal{K}_\chi{}^\delta{}_\delta\,\partial^\chi\mathcal{K}^{\alpha\ \beta}_{\ \alpha} - 4\,{}^{(4)}\kappa_1\,\partial_\chi\mathcal{K}_\beta{}^\delta{}_\delta\,\partial^\chi\mathcal{K}^{\alpha\ \beta}_{\ \alpha} - \\
&18\,{}^{(4)}\kappa_1\,\partial_\alpha\mathcal{K}^{\alpha\beta\chi}\,\partial_\delta\mathcal{K}_{\beta\chi}{}^\delta + 12\,{}^{(4)}\kappa_1\,\partial^\chi\mathcal{K}^{\alpha\ \beta}_{\ \alpha}\,\partial_\delta\mathcal{K}_{\beta\chi}{}^\delta + 6\,{}^{(4)}\kappa_1\,\partial_\delta\mathcal{K}_{\alpha\beta\chi}\,\partial^\delta\mathcal{K}^{\alpha\beta\chi}
\end{aligned}
$$

Added source term(s):	$\mathcal{K}^{\alpha\beta\chi}\ \mathcal{J}_{\alpha\beta\chi}$	
Source constraint(s)	No. of constraint(s)	Covariant form
$\mathcal{J}_{0^+}^{\#2} + 3\,\mathcal{J}_{0^+}^{\#1} == 0$	1	$\partial_\beta\mathcal{J}^{\alpha\ \beta}_{\ \alpha} == 0$
$\mathcal{J}_{1^-}^{\#2\,\alpha} + \mathcal{J}_{1^-}^{\#1\,\alpha} == 0$	3	$\partial_\chi\partial^\alpha\mathcal{J}^{\beta\ \chi}_{\ \beta} == \partial_\chi\partial^\chi\mathcal{J}^{\alpha\beta}_{\ \ \beta}$
$\mathcal{J}_{2^+}^{\#1\,\alpha\beta} == 0$	5	$k^\chi\,(\partial_\epsilon\partial_\chi\partial^\beta\partial^\alpha\mathcal{J}^{\delta\ \epsilon}_{\ \delta} + 2\,\partial_\epsilon\partial_\delta\partial^\beta\partial^\alpha\mathcal{J}_\chi{}^{\delta\epsilon} - 3\,\partial_\epsilon\partial_\delta\partial_\chi\partial^\alpha\mathcal{J}^{\beta\delta\epsilon} - 3\,\partial_\epsilon\partial_\delta\partial_\chi\partial^\beta\mathcal{J}^{\alpha\delta\epsilon} + 3\,\partial_\epsilon\partial^\epsilon\partial_\delta\partial_\chi\mathcal{J}^{\alpha\beta\delta} + \eta^{\alpha\beta}\,\partial_\phi\partial_\epsilon\partial_\delta\partial_\chi\mathcal{J}^{\delta\epsilon\phi} - \eta^{\alpha\beta}\,\partial_\phi\partial^\phi\partial_\epsilon\partial_\chi\mathcal{J}^{\delta\ \epsilon}_{\ \delta}) == 0$
Total no. of constraint(s):	9	

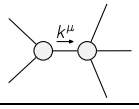
Resolved pole(s)	No. of polarization(s)	Square mass	Residue
	2	0	$-\frac{1}{{}^{(4)}\kappa_1}$
Resolved unitarity condition(s):		$\kappa_1 < 0$	

FIG. 9. Output generated by PSALTer. The spectrograph of $\mathfrak{F}_4$, as defined in Table II. All notation is defined in Table I and Fig. 2. This is Fronsdal theory, which propagates the $J = 3$ sector [23].

$$\delta\mathcal{K}^{\alpha\beta\gamma} = -\eta^{(\alpha\beta|}\Box\partial_\zeta\partial_\epsilon\partial_\delta\partial^{|\gamma)}\xi^{\delta\epsilon\zeta} + \eta^{(\alpha\beta|}\Box^2\partial_\epsilon\partial^{|\gamma)}\xi^{\delta}{}_{\delta}{}^{\epsilon}. \quad (15b)$$

E. Further work

The apparatus presented in Fig. 3 can be applied to any class of models accepted by PSALTer, i.e. quadratic field theories involving arbitrary numbers of arbitrarily symmetrized tensor fields up to rank three, with or without parity violation. Since PSALTer is parallelized, the algorithm may also be scaled using high-performance computing resources.

Regarding the current science products, we caution that unitarity of the most general model for a given gauge symmetry is only a necessary condition for the nonlinear completion of the model in Eq. (1) to be found. Indeed, consistent completion of the symmetry at higher orders is already known to be problematic for the Fronsdal case [72,73]. Finally, notice that, since our algorithm only deals with conditions linear in the coefficients, symmetric models deriving from quadratic deconfliction constraints are not considered here. However, such special cases are only relevant for the reduction steps concerning the models $\mathfrak{B}_1$, $\mathfrak{B}_2$ and $\aleph$ itself.

Note added during proof. The companion article [74], published alongside this one, presents the analogous catalogue for the pair-antisymmetric rank-three field, whose geometric interpretation is the torsion of spacetime.

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Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or European Research Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

DATA AVAILABILITY

The code and data that support the findings of this article are openly available [77].

APPENDIX A: CATALOG

In this appendix we provide Table II, which contains full definitions of all the models in Fig. 4.

TABLE II. Complete definitions of all models in Fig. 4. The second column lists all linear combinations of the parameters in Eq. (4) which must vanish in order to reach the model in question, starting from the root theory Eq. (4). The third column lists all linear combinations of the parameters in Eq. (4) which, when vanishing, lead to further special cases. All these conditions emerge by factorizing and imposing linear constraints from the spin matrices determinants and recursively iterating the process, as explained in the main text. The spectrum is inconsistent (✖), consistent (✔) or empty (○). The consistent spectra are illustrated in Figs. 5–9.

Model	Definition	Deconfliction	Unitary
$\aleph$	...	$[{}^{(4)}\kappa_7 = 0 \wedge {}^{(2)}\kappa_1 = 0] \vee [{}^{(4)}\kappa_3 + 3\,{}^{(4)}\kappa_7 = 0 \wedge {}^{(2)}\kappa_1 = 0] \vee [{}^{(4)}\kappa_3 + \tfrac{3}{2}\,{}^{(4)}\kappa_4 = 0 \wedge 2\,{}^{(4)}\kappa_1 + 2\,{}^{(4)}\kappa_2 + {}^{(4)}\kappa_7 = 0 \wedge {}^{(2)}\kappa_1 + 2\,{}^{(2)}\kappa_2 = 0] \vee [{}^{(4)}\kappa_3 + \tfrac{3}{2}\,{}^{(4)}\kappa_4 = 0 \wedge 12\,{}^{(4)}\kappa_2 + {}^{(4)}\kappa_4 + 6\,{}^{(4)}\kappa_7 = 0 \wedge {}^{(2)}\kappa_1 + 2\,{}^{(2)}\kappa_2 = 0] \vee [5\,{}^{(4)}\kappa_4 - 6\,{}^{(4)}\kappa_7 = 0 \wedge {}^{(4)}\kappa_3 + \tfrac{9}{5}\,{}^{(4)}\kappa_7 = 0 \wedge {}^{(2)}\kappa_1 = 0] \vee [{}^{(4)}\kappa_4 - {}^{(4)}\kappa_7 = 0 \wedge {}^{(4)}\kappa_3 + \tfrac{3}{2}\,{}^{(4)}\kappa_7 = 0 \wedge {}^{(2)}\kappa_1 = 0]$	✖
$\mathfrak{B}_1$	${}^{(4)}\kappa_7 = 0 \wedge {}^{(2)}\kappa_1 = 0$	${}^{(4)}\kappa_3 = 0 \vee [{}^{(4)}\kappa_1 + {}^{(4)}\kappa_2 = 0 \wedge \tfrac{2}{3}\,{}^{(4)}\kappa_3 + {}^{(4)}\kappa_4 = 0 \wedge {}^{(2)}\kappa_2 = 0] \vee [18\,{}^{(4)}\kappa_2 - {}^{(4)}\kappa_3 = 0 \wedge 12\,{}^{(4)}\kappa_2 + {}^{(4)}\kappa_4 = 0 \wedge {}^{(2)}\kappa_2 = 0]$	✖

(Table continued)

TABLE II. *(Continued)*

Model	Definition	Deconfliction	Unitary
$\mathfrak{B}_2$	$\frac{1}{3}\overset{(4)}{\kappa}_3+\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0$	$\overset{(4)}{\kappa}_3=0\vee[6\overset{(4)}{\kappa}_1+6\overset{(4)}{\kappa}_2-\overset{(4)}{\kappa}_3=0\wedge 4\overset{(4)}{\kappa}_1+4\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_4=0\wedge\overset{(2)}{\kappa}_2=0]\vee[9\overset{(4)}{\kappa}_2-2\overset{(4)}{\kappa}_3=0\wedge 3\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_4=0\wedge\overset{(2)}{\kappa}_2=0]$	✖
$\mathfrak{C}_1$	$\frac{2}{3}\overset{(4)}{\kappa}_3+\overset{(4)}{\kappa}_4=0\wedge 2\overset{(4)}{\kappa}_1+2\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_7=0\wedge\frac{1}{2}\overset{(2)}{\kappa}_1+\overset{(2)}{\kappa}_2=0$	$18\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_3=0\vee[\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\wedge\overset{(2)}{\kappa}_1=0]\vee[6\overset{(4)}{\kappa}_1+6\overset{(4)}{\kappa}_2-\overset{(4)}{\kappa}_3=0\wedge\overset{(2)}{\kappa}_1=0]\vee[18\overset{(4)}{\kappa}_1+18\overset{(4)}{\kappa}_2-5\overset{(4)}{\kappa}_3=0\wedge\overset{(2)}{\kappa}_1=0]\vee[3\overset{(4)}{\kappa}_1+3\overset{(4)}{\kappa}_2-\overset{(4)}{\kappa}_3=0\wedge\overset{(2)}{\kappa}_1=0]$	✖
$\mathfrak{C}_2$	$\frac{2}{3}\overset{(4)}{\kappa}_3+\overset{(4)}{\kappa}_4=0\wedge 2\overset{(4)}{\kappa}_2-\frac{1}{9}\overset{(4)}{\kappa}_3+\overset{(4)}{\kappa}_7=0\wedge\frac{1}{2}\overset{(2)}{\kappa}_1+\overset{(2)}{\kappa}_2=0$	$18\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_3=0\vee[18\overset{(4)}{\kappa}_2-\overset{(4)}{\kappa}_3=0\wedge\overset{(2)}{\kappa}_1=0]\vee[9\overset{(4)}{\kappa}_2-2\overset{(4)}{\kappa}_3=0\wedge\overset{(2)}{\kappa}_1=0]\vee[3\overset{(4)}{\kappa}_2-\overset{(4)}{\kappa}_3=0\wedge\overset{(2)}{\kappa}_1=0]\vee[18\overset{(4)}{\kappa}_2-7\overset{(4)}{\kappa}_3=0\wedge\overset{(2)}{\kappa}_1=0]$	✖
$\mathfrak{C}_3$	$\frac{2}{3}\overset{(4)}{\kappa}_3+\overset{(4)}{\kappa}_4=0\wedge\frac{5}{9}\overset{(4)}{\kappa}_3+\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0$	$\overset{(4)}{\kappa}_3=0\vee[18\overset{(4)}{\kappa}_1+18\overset{(4)}{\kappa}_2-5\overset{(4)}{\kappa}_3=0\wedge\overset{(2)}{\kappa}_2=0]\vee[3\overset{(4)}{\kappa}_2-\overset{(4)}{\kappa}_3=0\wedge\overset{(2)}{\kappa}_2=0]$	✖
$\mathfrak{C}_4$	$\frac{2}{3}\overset{(4)}{\kappa}_3+\overset{(4)}{\kappa}_4=0\wedge\frac{2}{3}\overset{(4)}{\kappa}_3+\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0$	$\overset{(4)}{\kappa}_3=0\vee[3\overset{(4)}{\kappa}_1+3\overset{(4)}{\kappa}_2-\overset{(4)}{\kappa}_3=0\wedge\overset{(2)}{\kappa}_2=0]\vee[18\overset{(4)}{\kappa}_2-7\overset{(4)}{\kappa}_3=0\wedge\overset{(2)}{\kappa}_2=0]$	✖
$\mathfrak{C}_5$	$\overset{(4)}{\kappa}_3=0\wedge\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0$	$\overset{(4)}{\kappa}_4=0$	○
$\mathfrak{D}_1$	$\overset{(4)}{\kappa}_3=0\wedge\overset{(4)}{\kappa}_4=0\wedge\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0$	$[\overset{(4)}{\kappa}_2=0\wedge\overset{(2)}{\kappa}_2=0]\vee[\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\wedge\overset{(2)}{\kappa}_2=0]$	✖
$\mathfrak{D}_2$	$18\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_3=0\wedge-12\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_4=0\wedge 2\overset{(4)}{\kappa}_1+2\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_7=0\wedge\frac{1}{2}\overset{(2)}{\kappa}_1+\overset{(2)}{\kappa}_2=0$	$[\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\wedge\overset{(2)}{\kappa}_1=0]\vee[4\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\wedge\overset{(2)}{\kappa}_1=0]\vee[6\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\wedge\overset{(2)}{\kappa}_1=0]\vee[7\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\wedge\overset{(2)}{\kappa}_1=0]$	✖
$\mathfrak{E}_1$	$\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\wedge\frac{2}{3}\overset{(4)}{\kappa}_3+\overset{(4)}{\kappa}_4=0\wedge\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0\wedge\overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_3=0\vee 18\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_3=0$	✔
$\mathfrak{E}_2$	$-6\overset{(4)}{\kappa}_1-6\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_3=0\wedge 4\overset{(4)}{\kappa}_1+4\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_4=0\wedge 2\overset{(4)}{\kappa}_1+2\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0\wedge\overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\vee 4\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0$	✔
$\mathfrak{E}_3$	$-\frac{18}{5}\overset{(4)}{\kappa}_1-\frac{18}{5}\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_3=0\wedge\frac{12}{5}\overset{(4)}{\kappa}_1+\frac{12}{5}\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_4=0\wedge 2\overset{(4)}{\kappa}_1+2\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0\wedge\overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\vee 6\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0$	✖
$\mathfrak{E}_4$	$-3\overset{(4)}{\kappa}_1-3\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_3=0\wedge 2\overset{(4)}{\kappa}_1+2\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_4=0\wedge 2\overset{(4)}{\kappa}_1+2\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0\wedge\overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\vee 7\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0$	✖
$\mathfrak{E}_5$	$-18\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_3=0\wedge 12\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_4=0\wedge\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0\wedge\overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_2=0\vee\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0$	○
$\mathfrak{E}_6$	$-\frac{9}{2}\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_3=0\wedge 3\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_4=0\wedge\frac{3}{2}\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0\wedge\overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_2=0\vee 4\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0$	✔
$\mathfrak{E}_7$	$-3\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_3=0\wedge 2\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_4=0\wedge\frac{5}{3}\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0\wedge\overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_2=0\vee 6\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0$	✖
$\mathfrak{E}_8$	$-\frac{18}{7}\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_3=0\wedge\frac{12}{7}\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_4=0\wedge\frac{12}{7}\overset{(4)}{\kappa}_2+\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0\wedge\overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_2=0\vee 7\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0$	✖
$\mathfrak{F}_1$	$\overset{(4)}{\kappa}_2=0\wedge\overset{(4)}{\kappa}_3=0\wedge\overset{(4)}{\kappa}_4=0\wedge\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0\wedge\overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_1=0$	○
$\mathfrak{F}_2$	$\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\wedge\overset{(4)}{\kappa}_3=0\wedge\overset{(4)}{\kappa}_4=0\wedge\overset{(4)}{\kappa}_7=0\wedge\overset{(2)}{\kappa}_1=0\wedge\overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_1=0$	✔

(Table continued)

TABLE II. *(Continued)*

Model	Definition	Deconfliction	Unitary
$\mathfrak{F}_3$	$\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\wedge 18\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_3=0\wedge -12\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_4=0\wedge \overset{(4)}{\kappa}_7=0\wedge \overset{(2)}{\kappa}_1=0\wedge \overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_1=0$	○
$\mathfrak{F}_4$	$4\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\wedge 18\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_3=0\wedge -12\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_4=0\wedge -6\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_7=0\wedge \overset{(2)}{\kappa}_1=0\wedge \overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_1=0$	✔
$\mathfrak{F}_5$	$6\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\wedge 18\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_3=0\wedge -12\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_4=0\wedge -10\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_7=0\wedge \overset{(2)}{\kappa}_1=0\wedge \overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_1=0$	✖
$\mathfrak{F}_6$	$7\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_2=0\wedge 18\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_3=0\wedge -12\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_4=0\wedge -12\overset{(4)}{\kappa}_1+\overset{(4)}{\kappa}_7=0\wedge \overset{(2)}{\kappa}_1=0\wedge \overset{(2)}{\kappa}_2=0$	$\overset{(4)}{\kappa}_1=0$	✖
∅	$\overset{(4)}{\kappa}_1=0\wedge \overset{(4)}{\kappa}_2=0\wedge \overset{(4)}{\kappa}_3=0\wedge \overset{(4)}{\kappa}_4=0\wedge \overset{(4)}{\kappa}_7=0\wedge \overset{(2)}{\kappa}_1=0\wedge \overset{(2)}{\kappa}_2=0$	⋯	○

APPENDIX B: SYMMETRIC RECURSION

1. Tensor-vector-scalar theory

In this appendix, we illustrate with an explicit example the symmetric recursion algorithm introduced in Sec. II and Fig. 3. Rather than working with the rank-three theory of Eq. (4), we take a richer example in the form of the most general parity-preserving, free-field mixing of a tensor $\mathcal{H}_{\alpha\beta}\equiv\mathcal{H}_{(\alpha\beta)}$ (with trace $\mathcal{H}\equiv\mathcal{H}^{\alpha}{}_{\alpha}$), a vector $\mathcal{A}_\alpha$, and a scalar ϕ, which (up to boundary terms) is

$$\begin{aligned}\mathcal{L}_{\aleph}\equiv{}&\overset{(4)}{\gamma}_1\partial_\beta\mathcal{H}\partial^\beta\mathcal{H}+\overset{(4)}{\gamma}_2\partial_\alpha\mathcal{H}^{\alpha\beta}\partial_\chi\mathcal{H}_\beta{}^\chi+\overset{(4)}{\gamma}_3\partial^\beta\mathcal{H}\partial_\chi\mathcal{H}_\beta{}^\chi+\overset{(4)}{\gamma}_4\partial_\chi\mathcal{H}_{\alpha\beta}\partial^\chi\mathcal{H}^{\alpha\beta}+\overset{(2)}{\gamma}_1\mathcal{H}_{\alpha\beta}\mathcal{H}^{\alpha\beta}+\overset{(2)}{\gamma}_2\mathcal{H}^2+\overset{(3)}{\gamma}_1\phi\partial_\alpha\mathcal{A}^\alpha\\&+\overset{(4)}{\beta}_1\partial_\alpha\mathcal{A}^\alpha\partial_\beta\mathcal{A}^\beta+\overset{(4)}{\beta}_2\partial_\beta\mathcal{A}_\alpha\partial^\beta\mathcal{A}^\alpha+\overset{(2)}{\beta}_1\mathcal{A}_\alpha\mathcal{A}^\alpha+\overset{(2)}{\beta}_2\mathcal{H}\phi+\overset{(4)}{\beta}_3\mathcal{H}^{\alpha\beta}\partial_\beta\partial_\alpha\phi+\overset{(4)}{\beta}_4\mathcal{H}\partial_\beta\partial^\beta\phi\\&+\overset{(4)}{\alpha}_1\partial_\alpha\phi\partial^\alpha\phi+\overset{(2)}{\alpha}_1\phi^2+\overset{(3)}{\alpha}_1\mathcal{H}\partial_\alpha\mathcal{A}^\alpha+\overset{(3)}{\alpha}_2\mathcal{H}_{\alpha\beta}\partial^\beta\mathcal{A}^\alpha.\end{aligned}\tag{B1}$$

As with Eq. (4), the presumed mass-dimensions of the couplings are indicated.[10] The advantage of working with Eq. (B1) over Eq. (4) is that it allows a range of steps in the recursive search procedure to be quickly illustrated, without proceeding to great depth into the recursion tree.

2. Spin-one

The wave operator matrix $\mathsf{O}_{1^-}(k)$ is found (in a suitable basis of the fields) to be

$$\mathsf{O}_{0^+}=\begin{pmatrix}\overset{(2)}{\alpha}_1+\overset{(4)}{\alpha}_1k^2 & \frac{\sqrt{3}}{2}\left(\overset{(2)}{\beta}_2-\overset{(4)}{\beta}_4k^2\right) & \frac{1}{2}\left[\overset{(2)}{\beta}_2-(\overset{(4)}{\beta}_3+\overset{(4)}{\beta}_4)k^2\right] & \frac{\mathrm{i}}{2}\overset{(3)}{\gamma}_1k\\ \frac{\sqrt{3}}{2}\left(\overset{(2)}{\beta}_2-\overset{(4)}{\beta}_4k^2\right) & \overset{(2)}{\gamma}_1+3\overset{(2)}{\gamma}_2+\left(3\overset{(4)}{\gamma}_1+\overset{(4)}{\gamma}_4\right)k^2 & \frac{\sqrt{3}}{2}\left[2\overset{(2)}{\gamma}_2+(2\overset{(4)}{\gamma}_1+\overset{(4)}{\gamma}_3)k^2\right] & \frac{\mathrm{i}\sqrt{3}}{2}\overset{(3)}{\alpha}_1k\\ \frac{1}{2}\left[\overset{(2)}{\beta}_2-(\overset{(4)}{\beta}_3+\overset{(4)}{\beta}_4)k^2\right] & \frac{\sqrt{3}}{2}\left[2\overset{(2)}{\gamma}_2+(2\overset{(4)}{\gamma}_1+\overset{(4)}{\gamma}_3)k^2\right] & \overset{(2)}{\gamma}_1+\overset{(2)}{\gamma}_2+\left(\overset{(4)}{\gamma}_1+\overset{(4)}{\gamma}_2+\overset{(4)}{\gamma}_3+\overset{(4)}{\gamma}_4\right)k^2 & \frac{\mathrm{i}}{2}\left(\overset{(3)}{\alpha}_1+\overset{(3)}{\alpha}_2\right)k\\ -\frac{\mathrm{i}}{2}\overset{(3)}{\gamma}_1k & -\frac{\mathrm{i}\sqrt{3}}{2}\overset{(3)}{\alpha}_1k & -\frac{\mathrm{i}}{2}\left(\overset{(3)}{\alpha}_1+\overset{(3)}{\alpha}_2\right)k & \overset{(2)}{\beta}_1+\left(\overset{(4)}{\beta}_1+\overset{(4)}{\beta}_2\right)k^2\end{pmatrix}.\tag{B2}$$

[10] In the physically interesting special cases of Eq. (B1), these mass dimensions turn out not to be appropriate because the engineering dimensions of the $\mathcal{A}_\alpha$ and ϕ fields depart from unity. These are cases in which those fields are actually Stueckelberg fields, while the true propagating content of the theory stems from the tensor field $\mathcal{H}_{\alpha\beta}$.

which is *a priori* full-rank. Following the simple notation of the body of the text in Sec. II we drop the J^P subscript; from Eq. (B2) the determinant can be expanded in powers of momentum k as

$$\det\tilde{\mathsf{O}} \propto {}^{(2)}\beta_1\,{}^{(2)}\gamma_1 + \frac{1}{8}\Big[4\,{}^{(2)}\beta_1\,{}^{(4)}\gamma_2 + 8\,{}^{(2)}\beta_1\,{}^{(4)}\gamma_4 + 8\,{}^{(4)}\beta_2\,{}^{(2)}\gamma_1 - {}^{(3)}\alpha_2{}^2\Big]k^2 + \frac{{}^{(4)}\beta_2}{2}\Big[{}^{(4)}\gamma_2 + 2\,{}^{(4)}\gamma_4\Big]k^4. \tag{B3}$$

From Eqs. (11) and (B3) we initially read off $\{{}^{(n)}X\} \equiv \{{}^{(0)}N, {}^{(2)}N, {}^{(4)}N\}$ as all being quadratic in the couplings. It is obvious, however, that ${}^{(0)}N$ and ${}^{(4)}N$ immediately force two independent bifurcations of the search algorithm

$${}^{(2)}\beta_1 = 0 \vee {}^{(2)}\gamma_1 = 0, \tag{B4a}$$

$${}^{(4)}\beta_2 = 0 \vee {}^{(4)}\gamma_2 + 2\,{}^{(4)}\gamma_4 = 0. \tag{B4b}$$

In the more complete notation of Fig. 3, we see that Eqs. (B4a) and (B4b) correspond to ${}^{(0)}N_{1^-} \equiv {}^{(0)}_{(1)}L_{1^-}{}^{(0)}_{(2)}L_{1^-}$ and ${}^{(4)}N_{1^-} \equiv {}^{(4)}_{(1)}L_{1^-}{}^{(4)}_{(2)}L_{1^-}$, respectively. If we initially pursue the first branch in Eq. (B4a), we find that

$${}^{(2)}N_{1^-} = {}^{(4)}\beta_2\,{}^{(2)}\gamma_1 - {}^{(3)}\alpha_2{}^2/8, \tag{B5}$$

remains essentially quadratic. This is one way in which what we term *inherently nonlinear* models can arise in the recursion tree, since the solutions to ${}^{(2)}N = 0$ can be parametrized.[11] As stated in the main text, such models arise infrequently in the study of Eq. (4), and are neglected in our analysis.

3. Spin-zero

It is also shown in Fig. 3 that the procedure must be looped over all the J^P sectors. Indeed, for the theory in Eq. (B1) the wave operator matrix in Eq. (B2) is accompanied by a much larger matrix in the 0^+ sector, namely

$$\mathsf{O}_{0^+} = \begin{pmatrix} {}^{(2)}\alpha_1 + {}^{(4)}\alpha_1 k^2 & \frac{\sqrt{3}}{2}\left({}^{(2)}\beta_2 - {}^{(4)}\beta_4 k^2\right) & \frac{1}{2}\left[{}^{(2)}\beta_2 - ({}^{(4)}\beta_3 + {}^{(4)}\beta_4)k^2\right] & \frac{\mathrm{i}}{2}\,{}^{(3)}\gamma_1 k \\ \frac{\sqrt{3}}{2}\left({}^{(2)}\beta_2 - {}^{(4)}\beta_4 k^2\right) & {}^{(2)}\gamma_1 + 3\,{}^{(2)}\gamma_2 + \left(3\,{}^{(4)}\gamma_1 + {}^{(4)}\gamma_4\right)k^2 & \frac{\sqrt{3}}{2}\left[2\,{}^{(2)}\gamma_2 + (2\,{}^{(4)}\gamma_1 + {}^{(4)}\gamma_3)k^2\right] & \frac{\mathrm{i}\sqrt{3}}{2}\,{}^{(3)}\alpha_1 k \\ \frac{1}{2}\left[{}^{(2)}\beta_2 - ({}^{(4)}\beta_3 + {}^{(4)}\beta_4)k^2\right] & \frac{\sqrt{3}}{2}\left[2\,{}^{(2)}\gamma_2 + (2\,{}^{(4)}\gamma_1 + {}^{(4)}\gamma_3)k^2\right] & {}^{(2)}\gamma_1 + {}^{(2)}\gamma_2 + \left({}^{(4)}\gamma_1 + {}^{(4)}\gamma_2 + {}^{(4)}\gamma_3 + {}^{(4)}\gamma_4\right)k^2 & \frac{\mathrm{i}}{2}\left({}^{(3)}\alpha_1 + {}^{(3)}\alpha_2\right)k \\ -\frac{\mathrm{i}}{2}\,{}^{(3)}\gamma_1 k & -\frac{\mathrm{i}\sqrt{3}}{2}\,{}^{(3)}\alpha_1 k & -\frac{\mathrm{i}}{2}\left({}^{(3)}\alpha_1 + {}^{(3)}\alpha_2\right)k & {}^{(2)}\beta_1 + \left({}^{(4)}\beta_1 + {}^{(4)}\beta_2\right)k^2 \end{pmatrix}. \tag{B6}$$

The determinant of Eq. (B6) is not a small expression. Yet, for a systematic approach, Eq. (B6) must be considered carefully, as being on the same level as Eq. (B2), otherwise theories will be missed. The algorithm described in Sec. II is intended to work for all freely mixing tensorial field theories, regardless of rank and field content, and the complexity already seen in Eq. (B1) illustrates why a notation such as in Fig. 3 is helpful, however cumbersome it may seem. It is to be noted that, when extending beyond Eq. (4), the proportion of inherently nonlinear models increases significantly. Methods for dealing with such models, and the complete analysis of Eq. (B1), will be presented elsewhere.

[11] Quadratic forms can be diagonalized by alternative choice of couplings, but this procedure does not generalize to cubics or quartics (which arise in the $J^P = 0^+$ sector)—moreover the vanishing of all terms in the diagonal quadratic form has no obvious physical significance as a basis for effective field theory.

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