

Constructing heavy-quark sum rule for $b \rightarrow c$ meson and baryon decays

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We study heavy-hadron semileptonic decays proceeding via $b \rightarrow c$ transition, such as $B \rightarrow D^{(*)}\tau\bar{\nu}_\tau$ and $\Lambda_b \rightarrow \Lambda_c\tau\bar{\nu}_\tau$. In the heavy-quark limit, where the heavy-quark symmetry holds, we provide a fundamental framework for heavy-quark sum rules among these decays based on the spin decomposition picture. The relation holds directly for the squared amplitudes without requiring phase-space integration. We then apply this relation to reproduce the sum rule among $B \rightarrow D^{(*)}\tau\bar{\nu}_\tau$ and $\Lambda_b \rightarrow \Lambda_c\tau\bar{\nu}_\tau$. Furthermore, we derive new sum rules for $\Omega_b \rightarrow \Omega_c^{(*)}$ transitions and those involving excited states, such as $B \rightarrow \{D_0^*, D_1^*\}$ and $B \rightarrow \{D_1, D_2^*\}$.

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I. INTRODUCTION

The $b \rightarrow c$ semileptonic decays are important probes for testing the Standard Model (SM) and searching for new physics (NP) beyond the SM. The anomalies observed in $B \rightarrow D^{(*)}\tau\bar{\nu}_\tau$ decays [1] have attracted a lot of attention, motivating the study of complementary decay modes with the $b \rightarrow c$ transition, such as $\Lambda_b \rightarrow \Lambda_c\tau\bar{\nu}_\tau$.

In this context, a sum rule has been proposed among the $B \rightarrow D^{(*)}\tau\bar{\nu}_\tau$ and $\Lambda_b \rightarrow \Lambda_c\tau\bar{\nu}_\tau$ decay modes [2,3]. It connects the ratio observables, $R_{H_c} = \text{BR}(H_b \rightarrow H_c\tau\bar{\nu}_\tau)/\text{BR}(H_b \rightarrow H_c l\bar{\nu}_l)$ with $l = e, \mu$, through the relation [4],

$$\frac{R_{\Lambda_c}}{R_{\Lambda_c}^{\text{SM}}} \approx \frac{1}{4} \frac{R_D}{R_D^{\text{SM}}} + \frac{3}{4} \frac{R_{D^*}}{R_{D^*}^{\text{SM}}}, \quad (1.1)$$

which holds even in the presence of NP. The sum rule provides a tool to test the consistency of the SM predictions and NP effects as well as that of experimental data of the $B \rightarrow D^{(*)}\tau\bar{\nu}_\tau$ and $\Lambda_b \rightarrow \Lambda_c\tau\bar{\nu}_\tau$ decays. The applicability of the sum rule has been investigated in detailed numerical studies in Refs. [5–7].

Besides the sum rule for the R_{H_c} observables, a more fundamental sum rule has been identified in Ref. [7], which connects the differential decay rates of $B \rightarrow D^{(*)}\tau\bar{\nu}_\tau$ and $\Lambda_b \rightarrow \Lambda_c\tau\bar{\nu}_\tau$, and is given by

$$\frac{\kappa_{\Lambda_c}}{\zeta(w)^2} = \frac{2}{1+w} \frac{\kappa_D + \kappa_{D^*}}{\xi(w)^2}, \quad (1.2)$$

where $\kappa_{H_c} = d\Gamma(H_b \rightarrow H_c\tau\bar{\nu}_\tau)/dw$ is the differential decay rate with w being the recoil parameter, and $\xi(w)$ and $\zeta(w)$ denote the leading-order (LO) Isgur-Wise (IW) functions for the transitions of ground-state mesons and those of ground-state baryons, respectively [8–11]. This sum rule holds exactly in the heavy-quark limit. Furthermore, another sum rule was derived for the double-differential decay rates [12], where κ_{H_c} in Eq. (1.2) is replaced by $d^2\Gamma(H_b \rightarrow H_c\tau\bar{\nu}_\tau)/dw d\cos\theta_\tau$ with θ_τ being the angle between H_c and τ in the H_b rest frame. The sum rule for the R_{H_c} observables, given in Eq. (1.1), can be derived from Eq. (1.2) by taking the zero-recoil limit in addition to the heavy-quark limit [4].¹

In this paper, we investigate the theoretical basis of the sum rules by analyzing the underlying structure [13,14] governed by the heavy-quark symmetry [8,9,15–19]. This analysis is carried out in the heavy-quark limit. In Ref. [7],

¹In reality, the coefficients in Eq. (1.1) are slightly modified due to corrections beyond these limits.

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the hadronic transitions are evaluated based on the Lorentz covariant formalism of form factors [11,20–22]. Alternatively, hadron states can be described in the spin decomposition picture, where wave functions are decomposed into heavy and light states with their spin/angular momenta specified [13,14]. These two approaches are equivalent in the heavy-quark limit.² Based on the spin decomposition picture, we provide a fundamental framework for sum rules among the squared transition amplitudes of hadron decays by exploiting the orthogonality of the Clebsch–Gordan coefficients.³ Since this holds even without performing the phase-space integration, it is more fundamental than those previously identified for (partially) integrated observables such as Eq. (1.2).

We then show that the above relation reproduces Eq. (1.2), namely the sum rule for ground-state transitions among $B \rightarrow D^{(*)}$ and $\Lambda_b \rightarrow \Lambda_c$. We also demonstrate that an analogous sum rule holds for other ground-state baryon decays, such as $\Omega_b \rightarrow \Omega_c \tau \bar{\nu}_\tau$. Furthermore, the relation can be applied to transitions involving excited states. We derive a new sum rule for $B \rightarrow \{D_0^*, D_1^*\}$ and $B \rightarrow \{D_1^*, D_2^*\}$ transitions.

This paper is organized as follows. In Sec. II, we present the derivation of the sum rules. We provide an expression for the decay amplitudes in the spin decomposition picture, and then, construct a framework for heavy-quark sum rules for hadron decays. Explicit examples are given in Sec. III A for the $B \rightarrow D^{(*)}$ and $\Lambda_b \rightarrow \Lambda_c$ transitions, in Sec. III B for the $\Omega_b \rightarrow \Omega_c^{(*)}$ transitions, and in Sec. III C for the excited states. Our summary and discussion are given in Sec. IV. In the Appendixes, an auxiliary formula is given in Appendix A, explicit formulas for decay amplitudes are listed in Appendix B, the definitions of the IW functions are summarized in Appendix C, and the Bjorken sum rules are discussed in Appendix D.

II. CONSTRUCTION OF HEAVY-QUARK SUM RULE

Let us consider a singly-heavy hadron H_Q , which consists of one heavy quark Q and light degrees of freedom ℓ . In the heavy-quark limit, the spin-flavor symmetry, known as the heavy-quark symmetry, holds [8,9,15–19]. The heavy quark behaves as a static color source, analogous to the proton in the hydrogen atom, with the light degrees of freedom playing the role of the electron. The spin of the heavy quark and the total angular momentum of the light

degrees of freedom are separately conserved under QCD interactions within a hadron. The heavy-quark symmetry allows hadrons to be described in the spin decomposition picture [13,14]. For given hadron spin λ_{H_Q} and heavy-quark spin λ_Q , the hadron state can be decomposed into a product of the heavy-quark spinor $u_Q(\lambda_Q)$ and the state of ℓ as

$$|H_Q(\lambda_{H_Q}, \lambda_Q)\rangle = \sum_{\lambda_\ell} \langle s_Q, \lambda_Q; s_\ell, \lambda_\ell | J_{H_Q}, \lambda_{H_Q} \rangle \times u_Q(\lambda_Q) |\text{light}(s_\ell^{P_\ell}, \lambda_\ell)\rangle. \quad (2.1)$$

This expression is valid in the heavy-quark limit. Here, J_{H_Q} , s_Q , and s_ℓ are the total angular momenta of H_Q , Q , and ℓ , respectively,⁴ the λ_{H_Q} , λ_Q and λ_ℓ are their projections, P_ℓ is the parity of ℓ , and $\langle s_Q, \lambda_Q; s_\ell, \lambda_\ell | J_{H_Q}, \lambda_{H_Q} \rangle$ is the Clebsch-Gordan coefficient. The wave functions are normalized as $\bar{u}_Q(\lambda_Q) u_Q(\lambda_Q) = 2m_Q$ and $\langle \text{light}(s_\ell^{P_\ell}, \lambda_\ell) | \text{light}(s_\ell^{P_\ell}, \lambda_\ell) \rangle = 1$, where m_Q is the heavy-quark mass. Under the heavy-quark symmetry, $s_Q = 1/2$ is fixed, while s_ℓ and P_ℓ classify the hadron state. Both the heavy-quark spinor and the light-component state implicitly depend on the four-velocity v of the heavy quark, though this dependence is not shown explicitly. In the discussion of the $H_b \rightarrow H_c$ transition below, we take all λ_i to denote projections along the decay axis, defined as the direction of the three-momentum of H_c in the H_b rest frame.

The decay amplitudes for $H_b \rightarrow H_c \bar{l} \nu_l$ processes with $l = e, \mu, \tau$ are calculated using a low-energy effective Hamiltonian of the form $\mathcal{H}_{\text{eff}} = \sum_X C_X O_X$, where O_X are local operators, and C_X are the corresponding Wilson coefficients. We write the local operators as $O_X = (\bar{c} \Gamma_X b)(\bar{l} \Gamma'_X \nu_l)$. The decay amplitude for given hadron spins λ_{H_b} and λ_{H_c} is expressed as

$$\mathcal{M}_{H_c}(\lambda_{H_c}, \lambda_{H_b}) = \sum_X C_X \sum_{\lambda_c, \lambda_b} \langle H_c(\lambda_{H_c}, \lambda_c) | \bar{c} \Gamma_X b | H_b(\lambda_{H_b}, \lambda_b) \rangle \times \langle \bar{l} \nu_l | \bar{l} \Gamma'_X \nu_l | 0 \rangle, \quad (2.2)$$

where the velocities of the H_b and H_c hadrons are v and v' , respectively. The recoil parameter is given by $w = v \cdot v'$. Using Eq. (2.1), the matrix element of the hadronic current takes the form,

$$\begin{aligned} & \langle H_c(\lambda_{H_c}, \lambda_c) | \bar{c} \Gamma_X b | H_b(\lambda_{H_b}, \lambda_b) \rangle \\ &= \sum_{\lambda} \langle s_c, \lambda_c; s_{\ell'}, \lambda_{\ell'} | J_{H_c}, \lambda_{H_c} \rangle \langle s_b, \lambda_b; s_\ell, \lambda_\ell | J_{H_b}, \lambda_{H_b} \rangle \\ & \times \bar{u}_c(\lambda_c) \Gamma_X u_b(\lambda_b) \langle \text{light}'(s_{\ell'}^{P_{\ell'}}, \lambda_{\ell'}) | \text{light}(s_\ell^{P_\ell}, \lambda_\ell) \rangle, \end{aligned} \quad (2.3)$$

²Additionally, in the heavy-quark limit, all bottom (charm) hadron masses are commonly approximated by the bottom (charm) quark mass. The mass difference between the bottom and charm quarks is kept nonzero in this paper unless otherwise mentioned explicitly.

³In the textbook of Ref. [23] (Secs. 2.3 and 2.4), this spin decomposition picture has been utilized to evaluate a QCD decay process of $D^* \rightarrow D\pi$ and probabilities of charm quark hadronizations to D and D^* .

⁴ s_ℓ is the sum of the spin of ℓ and the orbital angular momentum between Q and ℓ in the nonrelativistic constituent quark model.

where ℓ' (“light’”) and ℓ (“light”) denote the light degrees of freedom in H_c and H_b , respectively. In the heavy-quark limit, the projection of the total angular momentum of the light degrees of freedom along the decay axis is conserved [24], and hence, we denote $\lambda_{\ell'} = \lambda_{\ell} \equiv \lambda$. The matrix element of the light degrees of freedom, which corresponds to the overlap of the light-component wave functions, satisfies the following relation [13,24]:

$$\begin{aligned} & \langle \text{light}'(s_{\ell'}^{P_{\ell'}}, -\lambda) | \text{light}(s_{\ell}^{P_{\ell}}, -\lambda) \rangle \\ &= P_{\ell} P_{\ell'} (-1)^{s_{\ell} - s_{\ell'}} \langle \text{light}'(s_{\ell'}^{P_{\ell'}}, \lambda) | \text{light}(s_{\ell}^{P_{\ell}}, \lambda) \rangle. \end{aligned} \quad (2.4)$$

Since λ is a common projection of $s_{\ell'}$ and s_{ℓ} , both $s_{\ell'}$ and s_{ℓ} must be either integers or half-integers. Combining Eqs. (2.2) and (2.3), the decay amplitude is expressed as

$$\begin{aligned} \mathcal{M}_{H_c}(\lambda_{H_c}, \lambda_{H_b}) &= \sum_{\lambda_c, \lambda_b} \sum_{\lambda} \langle s_c, \lambda_c; s_{\ell'}, \lambda | J_{H_c}, \lambda_{H_c} \rangle \\ &\times \langle s_b, \lambda_b; s_{\ell}, \lambda | J_{H_b}, \lambda_{H_b} \rangle \\ &\times M_{\lambda_c, \lambda_b} \langle \text{light}'(s_{\ell'}^{P_{\ell'}}, \lambda) | \text{light}(s_{\ell}^{P_{\ell}}, \lambda) \rangle, \end{aligned} \quad (2.5)$$

where M_{λ_c, λ_b} is the matrix element for the quark-level transition, including the leptonic matrix element,

$$M_{\lambda_c, \lambda_b} = \sum_X C_X \bar{u}_c(\lambda_c) \Gamma_X u_b(\lambda_b) \langle l \bar{\nu}_l | \bar{\Gamma}_X \nu_l | 0 \rangle. \quad (2.6)$$

Hereafter, we abbreviate $M_{1/2, 1/2}$ as M_{++} , $M_{1/2, -1/2}$ as M_{+-} , and so on. We stress that M_{λ_c, λ_b} is independent of the hadron species H_b and H_c , which is a key feature to construct the sum rule, while $\langle \text{light}'(s_{\ell'}^{P_{\ell'}}, \lambda) | \text{light}(s_{\ell}^{P_{\ell}}, \lambda) \rangle$ corresponds to $H_b \rightarrow H_c$ transition form factors, as we will see later explicitly. Both matrix elements of the heavy part, M_{λ_c, λ_b} , and that of the light components, $\langle \text{light}'(s_{\ell'}^{P_{\ell'}}, \lambda) | \text{light}(s_{\ell}^{P_{\ell}}, \lambda) \rangle$, depend on the recoil parameter w implicitly.

By squaring the amplitude in Eq. (2.5) and summing over λ_{H_b} and λ_{H_c} , together with the sum over possible J_{H_c} , we obtain

$$\begin{aligned} & \sum_{J_{H_c}} \frac{1}{2J_{H_b} + 1} \sum_{\lambda_{H_c}, \lambda_{H_b}} |\mathcal{M}_{H_c}(\lambda_{H_c}, \lambda_{H_b})|^2 \\ &= \frac{1}{2} \sum_{\lambda_c, \lambda_b} |M_{\lambda_c, \lambda_b}|^2 \frac{2}{2J_{H_b} + 1} \sum_{\lambda} \left| \left\langle \frac{1}{2}, \lambda_b; s_{\ell}, \lambda \middle| J_{H_b}, \lambda_b + \lambda \right\rangle \right|^2 \\ &\times |\langle \text{light}'(s_{\ell'}^{P_{\ell'}}, |\lambda|) | \text{light}(s_{\ell}^{P_{\ell}}, |\lambda|) \rangle|^2, \end{aligned} \quad (2.7)$$

where the factor $1/(2J_{H_b} + 1)$ is introduced to make an average over the initial-hadron spin. Here, we have used the selection rule $\langle j_1, m_1; j_2, m_2 | J, M \rangle \propto \delta_{m_1 + m_2, M}$ and the

orthogonality relation,

$$\begin{aligned} & \sum_{J_{H_c}} \sum_{\lambda_{H_c}} \left\langle \frac{1}{2}, \lambda_c; s_{\ell'}, \lambda \middle| J_{H_c}, \lambda_{H_c} \right\rangle \left\langle J_{H_c}, \lambda_{H_c} \middle| \frac{1}{2}, \lambda'_c; s_{\ell'}, \lambda' \right\rangle \\ &= \delta_{\lambda_c, \lambda'_c} \delta_{\lambda, \lambda'}. \end{aligned} \quad (2.8)$$

It is noticed that the summation over the final-state spin J_{H_c} allows us to use the orthogonality of the Clebsch-Gordan coefficients. If instead the summation is taken over the initial-state spin J_{H_b} , the presence of the factor $1/(2J_{H_b} + 1)$ prevents the use of the orthogonality relation.

Let us describe the squared hadron decay amplitude normalized with the squared form factor as

$$\begin{aligned} \Delta(H_b \rightarrow H_c l \bar{\nu}_l) &= \frac{\sum_{J_{H_c}} \frac{1}{2J_{H_b} + 1} \sum_{\lambda_{H_c}, \lambda_{H_b}} |\mathcal{M}_{H_c}(\lambda_{H_c}, \lambda_{H_b})|^2}{\frac{1}{2s_{\ell} + 1} \sum_{\lambda} |\langle \text{light}'(s_{\ell'}^{P_{\ell'}}, |\lambda|) | \text{light}(s_{\ell}^{P_{\ell}}, |\lambda|) \rangle|^2}, \end{aligned} \quad (2.9)$$

where the summation over J_{H_c} corresponds to that over all possible hadrons H_c which can be formed for given $s_{\ell'}^{P_{\ell'}}$. From Eq. (2.7) by using an auxiliary formula in Eq. (A3), we arrive at the relation,

$$\Delta(H_b \rightarrow H_c l \bar{\nu}_l) = \frac{1}{2} \sum_{\lambda_c, \lambda_b} |M_{\lambda_c, \lambda_b}|^2. \quad (2.10)$$

The right-hand side corresponds to the squared amplitude of the partonic $b \rightarrow c l \bar{\nu}_l$ decay averaged over the initial-quark spin. Hence, based on the spin decomposition picture, the hadron decay amplitude, which is given by the numerator of $\Delta(H_b \rightarrow H_c l \bar{\nu}_l)$, is expressed by the partonic decay amplitude multiplied by a form factor in the heavy-quark limit.

Since M_{λ_c, λ_b} is independent of the hadron species H_b and H_c as stressed above, Eq. (2.10) leads to a framework for sum rules among various heavy-hadron decay channels,⁵

$$\begin{aligned} \Delta(H_b \rightarrow H_c l \bar{\nu}_l) &= \Delta(H'_b \rightarrow H'_c l \bar{\nu}_l) = \dots \\ &= \frac{1}{2} \sum_{\lambda_c, \lambda_b} |M_{\lambda_c, \lambda_b}|^2. \end{aligned} \quad (2.11)$$

This relation holds directly for the squared amplitudes even without requiring phase-space integration. From this relation, we can construct sum rules for decay rates as demonstrated in the next section. We note that the sum rules hold separately for each spin amplitude, namely for the

⁵In Ref. [25], a sum rule was derived for the squared decay amplitudes of $\bar{B}_s^0 \rightarrow D_s^{(*)+} l \bar{\nu}_l$ and $\Lambda_b \rightarrow \Lambda_c l \bar{\nu}_l$, under the conjecture of a dynamical supersymmetry between the $\bar{s}$ constituent quark and the ud scalar diquark.

terms $|M_{++}|^2$, $|M_{--}|^2$, $|M_{-+}|^2$ and $|M_{+-}|^2$. Furthermore, the sum rule holds irrespective of the presence of NP contributions, which enter through M_{λ_c, λ_b} .

In addition, the sum rule derived above can be extended beyond semileptonic transitions. In fact, by replacing the lepton current in Eq. (2.5) with a light-quark current, one can derive a sum rule for hadronic decays. However, this requires that the matrix element of the light-quark current is factorized from that of the light component inside the heavy hadron. This condition is approximately satisfied in Class-I B -meson decays into heavy-light final states, such as $\bar{B}^0 \rightarrow D^{(*)+} K^-$ and $\bar{B}_s^0 \rightarrow D_s^{(*)+} \pi^-$.

III. EXPLICIT EXAMPLES

A. Sum rule for $B \rightarrow D^{(*)}$ and $\Lambda_b \rightarrow \Lambda_c$

Let us consider the $B \rightarrow D^{(*)}$ and $\Lambda_b \rightarrow \Lambda_c$ transitions, where $J_{H_b} = 0$, $J_{H_c} = \{0, 1\}$ and $s_{\ell}^{P_{\ell}} = s_{\ell'}^{P_{\ell'}} = \frac{1}{2}^-$ for the former, and $J_{H_b} = J_{H_c} = \frac{1}{2}$ and $s_{\ell}^{P_{\ell}} = s_{\ell'}^{P_{\ell'}} = 0^+$ for the latter. The quantum numbers of the singly-heavy ground-state hadrons are listed in Table I.

Using Eq. (2.11), we obtain the sum rule,

$$\Delta(B \rightarrow D^{(*)} l \bar{\nu}_l) = \Delta(\Lambda_b \rightarrow \Lambda_c l \bar{\nu}_l) = \frac{1}{2} \sum_{\lambda_c, \lambda_b} |M_{\lambda_c, \lambda_b}|^2, \quad (3.1)$$

or it is explicitly shown as

TABLE I. Ground states of bottom ($Q = b$) and charm ($Q = c$) hadrons, where q denotes u or d . The ‘‘Content’’ column refers to the quark content in the constituent quark model, where the brackets $[]$ indicate a ‘‘good’’ diquark, while the braces $\{\}$ indicate a ‘‘bad’’ diquark. I and J^P are the isospin and the spin-parity of the hadron, respectively, and $s_{\ell}^{P_{\ell}}$ is the spin-parity of the light component. Quarks (antiquarks) are assigned positive (negative) parity by convention.

Content	$I(J^P)$	$s_{\ell}^{P_{\ell}}$	b hadron	c hadron
$Q\bar{q}$	$\frac{1}{2}(0^-)$ $\frac{1}{2}(1^-)$	$\frac{1}{2}^-$	$B^-, \bar{B}^0$ $B^{*-}, \bar{B}^{*0}$	D^0, D^+ D^{*0}, D^{*+}
$Q\bar{s}$	$0(0^-)$ $0(1^-)$	$\frac{1}{2}^-$	$\bar{B}_s^0$ $\bar{B}_s^{*0}$	D_s^+ D_s^{*+}
$Q[ud]$	$0(\frac{1}{2}^+)$	0^+	Λ_b^0	Λ_c^+
$Q[qs]$	$\frac{1}{2}(\frac{1}{2}^+)$	0^+	Ξ_b^0, Ξ_b^-	Ξ_c^+, Ξ_c^0
$Q\{qs\}$	$\frac{1}{2}(\frac{1}{2}^+)$ $\frac{1}{2}(\frac{3}{2}^+)$	1^+	Ξ_b^{*0}, Ξ_b^{*-} Ξ_b^{*0}, Ξ_b^{*-}	Ξ_c^{*+}, Ξ_c^{*0} Ξ_c^{*+}, Ξ_c^{*0}
$Q\{qq\}$	$1(\frac{1}{2}^+)$ $1(\frac{3}{2}^+)$	1^+	$\Sigma_b^+, \Sigma_b^0, \Sigma_b^-$ $\Sigma_b^{*+}, \Sigma_b^{*0}, \Sigma_b^{*-}$	$\Sigma_c^{++}, \Sigma_c^+, \Sigma_c^0$ $\Sigma_c^{*++}, \Sigma_c^{*+}, \Sigma_c^{*0}$
$Q\{ss\}$	$0(\frac{1}{2}^+)$ $0(\frac{3}{2}^+)$	1^+	Ω_b^- Ω_b^{*-}	Ω_c^0 Ω_c^{*0}

$$\frac{|\mathcal{M}_D|^2 + \sum_{\lambda_{D^*}} |\mathcal{M}_{D^*}(\lambda_{D^*})|^2}{\langle \bar{q}(\frac{1}{2}^-, \frac{1}{2}) | \bar{q}(\frac{1}{2}^-, \frac{1}{2}) \rangle^2} = \frac{\frac{1}{2} \sum_{\lambda_{\Lambda_c}, \lambda_{\Lambda_b}} |\mathcal{M}_{\Lambda_c}(\lambda_{\Lambda_c}, \lambda_{\Lambda_b})|^2}{\langle qq(0^+, 0) | qq(0^+, 0) \rangle^2}. \quad (3.2)$$

As in the constituent quark model, the light component on the left-hand side is effectively described by an antiquark $\bar{q} = \bar{u}$ or $\bar{d}$, while that on the right-hand side is by a spinless and parity-even diquark, the so-called ‘‘good’’ diquark. The full expressions of the decay amplitudes are given in Appendix B. Since the only difference between the $B \rightarrow D^{(*)}$ and $\Lambda_b \rightarrow \Lambda_c$ transitions lies in the structure of the light components, the equality in Eq. (3.2) reflects that the underlying heavy-quark transitions are identical. Furthermore, due to the orthogonality of the Clebsch-Gordan coefficients, the contributions from the $B \rightarrow D$ and $B \rightarrow D^*$ modes must be summed on the left-hand side, reflecting the fact that D and D^* form a heavy-quark spin doublet.

The LO mesonic IW function $\xi(w)$ is expressed in terms of the matrix element of the light component as [14]

$$\xi(w) = \sqrt{\frac{2}{w+1}} \left\langle \bar{q}\left(\frac{1}{2}^-, \frac{1}{2}\right) \left| \bar{q}\left(\frac{1}{2}^-, \frac{1}{2}\right) \right. \right\rangle. \quad (3.3)$$

Similarly, the baryonic IW function $\zeta(w)$ is written as

$$\zeta(w) = \langle qq(0^+, 0) | qq(0^+, 0) \rangle. \quad (3.4)$$

More details of these relations are given in Appendix C. Using Eqs. (3.3) and (3.4), the sum rule in Eq. (3.2) can be rewritten as

$$\begin{aligned} & \frac{2}{1+w} \frac{|\mathcal{M}_D|^2 + \sum_{\lambda_{D^*}} |\mathcal{M}_{D^*}(\lambda_{D^*})|^2}{\xi(w)^2} \\ &= \frac{\frac{1}{2} \sum_{\lambda_{\Lambda_c}, \lambda_{\Lambda_b}} |\mathcal{M}_{\Lambda_c}(\lambda_{\Lambda_c}, \lambda_{\Lambda_b})|^2}{\zeta(w)^2}. \end{aligned} \quad (3.5)$$

From this relation, Eq. (1.2) is reproduced by multiplying a kinematic factor necessary to obtain the decay rates from the squared amplitudes appearing in the numerators of Eq. (2.9). Additionally, the phase-space integration is performed over the angular distributions of the final-state particles, while w is kept unintegrated. It is noticed that, in the heavy-quark limit, since the hadron masses are commonly approximated by the corresponding heavy-quark masses, these factors are universal in all transitions. Also, the form factors are functions of w but independent of the angular distributions. We stress that the sum rule derived here is more fundamental than Eq. (1.2), because the latter is obtained by performing an integration over the phase space.

B. Sum rule for $\Omega_b \rightarrow \Omega_c^{(*)}$

In the previous subsection, we discussed the decay of the ground-state bottom baryon Λ_b to the ground-state charm baryon Λ_c , both characterized by $J^P = \frac{1}{2}^+$. In this case, the light component in the baryons is effectively described by a spinless “good” diquark. We now turn to the case where the light component is described by a spin-one and parity-even diquark, the so-called “bad” diquark. Here, we derive a sum rule for the $\Omega_b \rightarrow \Omega_c^{(*)}$ transitions, where Ω_b and Ω_c have $J^P = \frac{1}{2}^+$, while Ω_c^* has $J^P = \frac{3}{2}^+$. The derived formula can be applied to other transitions involving the “bad” diquark, *i.e.*, $\Xi'_b \rightarrow \Xi'_c(\Xi_c^*)$ and $\Sigma_b \rightarrow \Sigma_c^{(*)}$.

In the case of the $\Omega_b \rightarrow \Omega_c^{(*)}$ transitions, there are two LO IW functions, $\xi_1(w)$ and $\xi_2(w)$ [10,11]. We find that they are related to the matrix elements of the spin-one diquark as

$$\xi_1(w) = \langle ss(1^+, 1) | ss(1^+, 1) \rangle, \quad (3.6)$$

$$\xi_2(w) = \frac{1}{w^2 - 1} (w \langle ss(1^+, 1) | ss(1^+, 1) \rangle - \langle ss(1^+, 0) | ss(1^+, 0) \rangle), \quad (3.7)$$

where $\xi_1(w)$ is normalized as $\xi_1(1) = 1$, and the derivation of these relations is given in Appendix C. The denominator in Eq. (2.9) is expressed as⁶

$$\begin{aligned} & \frac{1}{3} (\langle ss(1^+, 0) | ss(1^+, 0) \rangle^2 + 2 \langle ss(1^+, 1) | ss(1^+, 1) \rangle^2) \\ &= \frac{2 + w^2}{3} \xi_1(w)^2 + \frac{(w^2 - 1)^2}{3} \xi_2(w)^2 \\ & \quad - \frac{2w(w^2 - 1)}{3} \xi_1(w) \xi_2(w). \end{aligned} \quad (3.8)$$

If the equality $\langle ss(1^+, 1) | ss(1^+, 1) \rangle = \langle ss(1^+, 0) | ss(1^+, 0) \rangle$ holds, we have the relation $\xi_2(w) = \xi_1(w)/(w + 1)$, which was derived in the spectator-quark model [27], in the large N_c limit [28], and in the relativistic quark model [29].

From Eq. (2.11), we find the sum rule for the $\Omega_b \rightarrow \Omega_c^{(*)}$ transitions $\Delta(\Omega_b \rightarrow \Omega_c^{(*)} l \bar{\nu}_l)$ as

$$\begin{aligned} & \Delta(\Omega_b \rightarrow \Omega_c^{(*)} l \bar{\nu}_l) \\ &= \frac{\sum_{H_c = \Omega_c, \Omega_c^*} \frac{1}{2} \sum_{\lambda_{H_c}, \lambda_{\Omega_b}} |\mathcal{M}_{H_c}(\lambda_{H_c}, \lambda_{\Omega_b})|^2}{\frac{2 + w^2}{3} \xi_1(w)^2 + \frac{(w^2 - 1)^2}{3} \xi_2(w)^2 - \frac{2w(w^2 - 1)}{3} \xi_1(w) \xi_2(w)}, \\ &= \frac{1}{2} \sum_{\lambda_c, \lambda_b} |M_{\lambda_c, \lambda_b}|^2. \end{aligned} \quad (3.9)$$

⁶The combination of the IW functions in Eq. (3.8) is the same as that appearing in the ground-state contribution to the Bjorken sum rule for the semileptonic Ω_b decays [26].

TABLE II. Four lightest excited states of charm mesons.

Content	$I(J^P)$	$s_{\ell'}^{P_{\ell'}}$	c hadron
$c\bar{q}$	$\frac{1}{2}(0^+)$	$\frac{1}{2}^+$	D_0^{*0}, D_0^{*+}
	$\frac{1}{2}(1^+)$		D_1^{*0}, D_1^{*+}
	$\frac{1}{2}(1^+)$	$\frac{3}{2}^+$	D_1^0, D_1^+
	$\frac{1}{2}(2^+)$		D_2^{*0}, D_2^{*+}

C. Sum rule for excited states

Equation (2.11) also provides sum rules for transitions involving excited states. For example, the $B \rightarrow D_0^*$ and $B \rightarrow D_1^*$ transitions correspond to $J_{H_b} = 0$, $s_{\ell'}^{P_{\ell'}} = \frac{1}{2}^-$ and $s_{\ell'}^{P_{\ell'}} = \frac{1}{2}^+$, while the $B \rightarrow D_1$ and $B \rightarrow D_2^*$ correspond to $J_{H_b} = 0$, $s_{\ell'}^{P_{\ell'}} = \frac{1}{2}^-$ and $s_{\ell'}^{P_{\ell'}} = \frac{3}{2}^+$, where the quantum numbers of the excited charm mesons are summarized in Table II. The IW functions for these transitions, $\tau_{1/2}(w)$ and $\tau_{3/2}(w)$ [30], are given by [31]⁷

$$\tau_{1/2}(w) = \frac{1}{\sqrt{2(w-1)}} \left\langle \bar{q} \left(\frac{1^+}{2}, \frac{1}{2} \right) \left| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right. \right\rangle, \quad (3.10)$$

$$\tau_{3/2}(w) = -\frac{1}{(w+1)} \frac{1}{\sqrt{w-1}} \left\langle \bar{q} \left(\frac{3^+}{2}, \frac{1}{2} \right) \left| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right. \right\rangle. \quad (3.11)$$

As in the case of the ground-state transitions, we obtain the sum rule for $\Delta(B \rightarrow \{D_0^*, D_1^*\} l \bar{\nu}_l)$ and $\Delta(B \rightarrow \{D_1, D_2^*\} l \bar{\nu}_l)$ as

$$\begin{aligned} \Delta(B \rightarrow \{D_0^*, D_1^*\} l \bar{\nu}_l) &= \Delta(B \rightarrow \{D_1, D_2^*\} l \bar{\nu}_l) \\ &= \frac{1}{2} \sum_{\lambda_c, \lambda_b} |M_{\lambda_c, \lambda_b}|^2, \end{aligned} \quad (3.12)$$

where

$$\Delta(B \rightarrow \{D_0^*, D_1^*\} l \bar{\nu}_l) = \frac{\sum_{H_c = D_0^*, D_1^*} \sum_{\lambda_{H_c}} |\mathcal{M}_{H_c}(\lambda_{H_c})|^2}{2(w-1)\tau_{1/2}(w)^2}, \quad (3.13)$$

$$\Delta(B \rightarrow \{D_1, D_2^*\} l \bar{\nu}_l) = \frac{\sum_{H_c = D_1, D_2^*} \sum_{\lambda_{H_c}} |\mathcal{M}_{H_c}(\lambda_{H_c})|^2}{(w+1)^2(w-1)\tau_{3/2}(w)^2}. \quad (3.14)$$

⁷The matrix element $\langle \text{light}'(s_{\ell'}^{P_{\ell'}}, \lambda) | \text{light}(s_{\ell'}^{P_{\ell'}}, \lambda) \rangle$ vanishes for $s_{\ell'}^{P_{\ell'}} \neq s_{\ell'}^{P_{\ell'}}$ at zero recoil $w \rightarrow 1$, because H_b and H_c belong to different multiplets of the heavy-quark spin symmetry.

IV. SUMMARY AND DISCUSSION

In this paper, we clarified the theoretical structure underlying the sum rules for the $b \rightarrow c$ semileptonic hadron decays such as those among $B \rightarrow D^{(*)} \tau \bar{\nu}_\tau$ and $\Lambda_b \rightarrow \Lambda_c \tau \bar{\nu}_\tau$. The analysis was carried out in the heavy-quark limit, where the heavy-quark symmetry holds. In this limit, the spin/angular-momentum structure of the decay amplitudes becomes transparent by using the spin decomposition picture. Adopting this expression, we constructed a fundamental framework for heavy-quark sum rules for heavy-hadron decays by exploiting the orthogonality of the Clebsch-Gordan coefficients.

The sum rules arise from the fact that the differences among the transitions are encoded solely in the matrix element of the light degrees of freedom, namely in the form factors, whereas the quark-level structure of the heavy-quark transition is universal. Therefore, by normalizing the squared amplitudes with the form factors, they become independent of the hadron species involved in the decay, thereby yielding the sum rules. Moreover, the relation holds directly among the squared amplitudes, i.e., without performing the phase-space integration.

We then applied the above relation to reproduce those for $B \rightarrow D^{(*)} \tau \bar{\nu}_\tau$ and $\Lambda_b \rightarrow \Lambda_c \tau \bar{\nu}_\tau$. From the spin structure of the sum rule, we clarified that the $B \rightarrow D$ contribution alone is not sufficient, and the $B \rightarrow D^*$ channel must also be included. Furthermore, we extended the analysis to the Ω_b transitions and derived a new sum rule relating $\Omega_b \rightarrow \Omega_c^{(*)}$ to $B \rightarrow D^{(*)}$ and $\Lambda_b \rightarrow \Lambda_c$. Analogous to the $B \rightarrow D$ and $B \rightarrow D^*$ case, both $\Omega_b \rightarrow \Omega_c$ and $\Omega_b \rightarrow \Omega_c^*$ channels are included in the sum rule. Although precise experimental data are not yet available, it would be of interest to investigate violations of the sum rule beyond the heavy-quark limit and to predict the $\Omega_b \rightarrow \Omega_c^{(*)}$ transitions from other channels using the sum rule. As the third example, we also constructed a sum rule for transitions involving excited states, $B \rightarrow \{D_0^*, D_1^*\}$ and $B \rightarrow \{D_1, D_2^*\}$.

It should be emphasized that the equality of the sum rules holds only in the heavy-quark limit, and deviations arise because the limit is broken for realistic hadrons. Although the corrections have been evaluated for the sum rule among $B \rightarrow D^{(*)}$ and $\Lambda_b \rightarrow \Lambda_c$ [4,7], they remain unexplored for others. Their detailed analyses are beyond the scope of this paper and will be performed elsewhere.

There are a variety of further applications. For instance, assuming the flavor $SU(3)$ symmetry, one can construct analogous sum rules involving $B_s \rightarrow D_s^{(*)}$ and so on. Naively, the $SU(3)$ symmetry breaking amounts to $\mathcal{O}(10\%)$ corrections [32], and hence, could not be negligible. However, a detailed analysis is necessary to evaluate its effects on the sum rules. Moreover, the sum rules can be extended beyond semileptonic transitions, e.g., to Class-I hadronic two-body decays such as $\bar{B}^0 \rightarrow D^{(*)+} K^-$ and $\bar{B}_s^0 \rightarrow D_s^{(*)+} \pi^-$. A detailed study for these cases is also left for future work.

On the experimental side, precise data are not yet available for many decay channels. The current and future experiments such as Belle II, LHCb, and Tera-Z are expected to measure them with high precisions (see, e.g., Refs. [33–38]). For instance, LHCb is projected to determine $R_{D_s^{(*)}}$ and R_{D_1} at percent levels [36], while the precision for $R_{D_s^{(*)}}$ at Tera-Z could reach $\mathcal{O}(0.1)\%$, although systematic uncertainties have not been taken into account [37,38]. These experimental results would be applied to the sum rules in future. Since many decay channels remain unexplored, further studies are required.

Finally, as an application of the spin decomposition picture of the decay amplitudes, we reproduced the Bjorken sum rules based on this picture in Appendix D.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: AUXILIARY RELATION

In this appendix, we derive the auxiliary relation that is used to obtain Eq. (2.10).

The square of the Clebsch-Gordan coefficients appearing in Eq. (2.7) takes the following form:

$$\left| \left\langle \frac{1}{2}, \lambda_b; s_\ell, \lambda \left| J_{H_b}, \lambda_b + \lambda \right. \right\rangle \right|^2 = \begin{cases} \frac{s_\ell + \lambda + 1}{2s_\ell + 1} & \text{for } J_{H_b} = s_\ell + \frac{1}{2}, \quad \lambda_b = \pm \frac{1}{2}, \\ \frac{s_\ell - \lambda}{2s_\ell + 1} & \text{for } J_{H_b} = s_\ell - \frac{1}{2}, \quad \lambda_b = \pm \frac{1}{2}. \end{cases} \quad (\text{A1})$$

In particular, for $s_\ell = 0$, this expression reduces to

$$\left| \left\langle \frac{1}{2}, \lambda_b; s_\ell, \lambda \middle| J_{H_b}, \lambda_b + \lambda \right\rangle \right|^2 = \left| \left\langle \frac{1}{2}, \lambda_b; 0, 0 \middle| \frac{1}{2}, \lambda_b \right\rangle \right|^2 = 1. \quad (\text{A2})$$

Using these expressions, we obtain

$$\begin{aligned} & \frac{2}{2J_{H_b} + 1} \sum_{\lambda=-J}^J \left| \left\langle \frac{1}{2}, \lambda_b; s_\ell, \lambda \middle| J_{H_b}, \lambda_b + \lambda \right\rangle \right|^2 |\langle \text{light}'(s_{\ell'}^{P_{\ell'}}, |\lambda|) | \text{light}(s_{\ell}^{P_{\ell}}, |\lambda|) \rangle|^2 \\ &= \frac{1}{2s_\ell + 1} \sum_{\lambda=-J}^J |\langle \text{light}'(s_{\ell'}^{P_{\ell'}}, |\lambda|) | \text{light}(s_{\ell}^{P_{\ell}}, |\lambda|) \rangle|^2, \end{aligned} \quad (\text{A3})$$

where $J = \min(s_{\ell'}, s_\ell)$, and the final expression is independent of J_{H_b} and λ_b .

APPENDIX B: DECAY AMPLITUDES

In this appendix, we present the expressions of the decay amplitudes in Eq. (2.5) for each ground-state transition. For the $B \rightarrow D$ transition, the amplitude is written as

$$\begin{aligned} \mathcal{M}_D &= \left\langle \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \middle| 0, 0 \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \middle| 0, 0 \right\rangle M_{++} \left\langle \bar{q} \left(\frac{1^-}{2}, -\frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, -\frac{1}{2} \right) \right\rangle \\ &+ \left\langle \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2} \middle| 0, 0 \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2} \middle| 0, 0 \right\rangle M_{--} \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle, \\ &= \frac{1}{2} (M_{++} + M_{--}) \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle. \end{aligned} \quad (\text{B1})$$

In the case of $B \rightarrow D^*$, the amplitudes for each D^* spin are given by

$$\begin{aligned} \mathcal{M}_{D^*}(\lambda_{D^*}) &= \begin{cases} \left\langle \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \middle| 1, 0 \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \middle| 0, 0 \right\rangle M_{++} \left\langle \bar{q} \left(\frac{1^-}{2}, -\frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, -\frac{1}{2} \right) \right\rangle \\ + \left\langle \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2} \middle| 1, 0 \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2} \middle| 0, 0 \right\rangle M_{--} \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle, \\ \left\langle \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2} \middle| 1, 1 \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2} \middle| 0, 0 \right\rangle M_{+-} \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle, \\ \left\langle \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \middle| 1, -1 \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \middle| 0, 0 \right\rangle M_{-+} \left\langle \bar{q} \left(\frac{1^-}{2}, -\frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, -\frac{1}{2} \right) \right\rangle, \\ \frac{1}{2} (M_{++} - M_{--}) \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle, \\ -\frac{1}{\sqrt{2}} M_{+-} \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle, \\ \frac{1}{\sqrt{2}} M_{-+} \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle, \end{cases} \\ &= \begin{cases} \frac{1}{2} (M_{++} - M_{--}) \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle, \\ -\frac{1}{\sqrt{2}} M_{+-} \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle, \\ \frac{1}{\sqrt{2}} M_{-+} \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle, \end{cases} \end{aligned} \quad (\text{B2})$$

which correspond to $\lambda_{D^*} = 0, 1$, and -1 , respectively. Squaring the amplitudes and summing over $B \rightarrow D$ and $B \rightarrow D^*$, we obtain

$$|\mathcal{M}_D|^2 + \sum_{\lambda_{D^*}} |\mathcal{M}_{D^*}(\lambda_{D^*})|^2 = \frac{1}{2} (|M_{++}|^2 + |M_{--}|^2 + |M_{+-}|^2 + |M_{-+}|^2) \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle^2, \quad (\text{B3})$$

which is consistent with Eq. (2.10).

In the $\Lambda_b \rightarrow \Lambda_c$ case, the spin of the heavy quark coincides with that of the heavy baryon, yielding

$$\begin{aligned} \mathcal{M}_{\Lambda_c}(\lambda_{\Lambda_c}, \lambda_{\Lambda_b}) &= \left\langle \frac{1}{2}, \lambda_{\Lambda_c}; 0, 0 \left| \frac{1}{2}, \lambda_{\Lambda_c} \right\rangle \left\langle \frac{1}{2}, \lambda_{\Lambda_b}; 0, 0 \left| \frac{1}{2}, \lambda_{\Lambda_b} \right\rangle M_{\lambda_{\Lambda_c}, \lambda_{\Lambda_b}} \langle qq(0^+, 0) | qq(0^+, 0) \rangle, \right. \\ &= M_{\lambda_{\Lambda_c}, \lambda_{\Lambda_b}} \langle qq(0^+, 0) | qq(0^+, 0) \rangle. \end{aligned} \quad (\text{B4})$$

The squared amplitude is given by

$$\frac{1}{2} \sum_{\lambda_{\Lambda_c}, \lambda_{\Lambda_b}} |\mathcal{M}_{\Lambda_c}(\lambda_{\Lambda_c}, \lambda_{\Lambda_b})|^2 = \frac{1}{2} (|M_{++}|^2 + |M_{--}|^2 + |M_{+-}|^2 + |M_{-+}|^2) \langle qq(0^+, 0) | qq(0^+, 0) \rangle^2, \quad (\text{B5})$$

where the factor 1/2 reflects the average over the spin of the initial baryon.

The amplitudes for the $\Omega_b \rightarrow \Omega_c$ transition are given by

$$\begin{aligned} \mathcal{M}_{\Omega_c}(\lambda_{\Omega_c}, \lambda_{\Omega_b}) &= \begin{cases} \left\langle \frac{1}{2}, \frac{1}{2}; 1, 0 \left| \frac{1}{2}, \frac{1}{2} \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}; 1, 0 \left| \frac{1}{2}, \frac{1}{2} \right\rangle M_{++} \langle ss(1^+, 0) | ss(1^+, 0) \rangle \right. \\ \quad + \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 1 \left| \frac{1}{2}, \frac{1}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 1 \left| \frac{1}{2}, \frac{1}{2} \right\rangle M_{--} \langle ss(1^+, 1) | ss(1^+, 1) \rangle, \\ \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 0 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 0 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle M_{--} \langle ss(1^+, 0) | ss(1^+, 0) \rangle \right. \\ \quad + \left\langle \frac{1}{2}, \frac{1}{2}; 1, -1 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}; 1, -1 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle M_{++} \langle ss(1^+, -1) | ss(1^+, -1) \rangle, \\ \left\langle \frac{1}{2}, \frac{1}{2}; 1, 0 \left| \frac{1}{2}, \frac{1}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 0 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle M_{+-} \langle ss(1^+, 0) | ss(1^+, 0) \rangle, \\ \left. \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 0 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}; 1, 0 \left| \frac{1}{2}, \frac{1}{2} \right\rangle M_{-+} \langle ss(1^+, 0) | ss(1^+, 0) \rangle, \right. \right. \\ \frac{1}{3} (M_{++} \langle ss(1^+, 0) | ss(1^+, 0) \rangle + 2M_{--} \langle ss(1^+, 1) | ss(1^+, 1) \rangle), \\ \frac{1}{3} (M_{--} \langle ss(1^+, 0) | ss(1^+, 0) \rangle + 2M_{++} \langle ss(1^+, 1) | ss(1^+, 1) \rangle), \\ -\frac{1}{3} M_{+-} \langle ss(1^+, 0) | ss(1^+, 0) \rangle, \\ \left. -\frac{1}{3} M_{-+} \langle ss(1^+, 0) | ss(1^+, 0) \rangle, \right\} \\ = \begin{cases} \frac{1}{3} (M_{++} \langle ss(1^+, 0) | ss(1^+, 0) \rangle + 2M_{--} \langle ss(1^+, 1) | ss(1^+, 1) \rangle), \\ \frac{1}{3} (M_{--} \langle ss(1^+, 0) | ss(1^+, 0) \rangle + 2M_{++} \langle ss(1^+, 1) | ss(1^+, 1) \rangle), \\ -\frac{1}{3} M_{+-} \langle ss(1^+, 0) | ss(1^+, 0) \rangle, \\ -\frac{1}{3} M_{-+} \langle ss(1^+, 0) | ss(1^+, 0) \rangle, \end{cases} \end{aligned} \quad (\text{B6})$$

for $(\lambda_{\Omega_c}, \lambda_{\Omega_b}) = (\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{1}{2})$, respectively, while those for the $\Omega_b \rightarrow \Omega_c^*$ transition are given by

$$\begin{aligned}
\mathcal{M}_{\Omega_c^*}(\lambda_{\Omega_c^*}, \lambda_{\Omega_b}) = & \left\{ \begin{aligned} & \left\langle \frac{1}{2}, \frac{1}{2}; 1, 0 \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}; 1, 0 \left| \frac{1}{2}, \frac{1}{2} \right\rangle M_{++} \langle ss(1^+, 0) | ss(1^+, 0) \rangle \right. \\ & + \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 1 \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 1 \left| \frac{1}{2}, \frac{1}{2} \right\rangle M_{--} \langle ss(1^+, 1) | ss(1^+, 1) \rangle, \right. \\ & \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 0 \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 0 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle M_{--} \langle ss(1^+, 0) | ss(1^+, 0) \rangle \right. \\ & + \left\langle \frac{1}{2}, \frac{1}{2}; 1, -1 \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}; 1, -1 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle M_{++} \langle ss(1^+, -1) | ss(1^+, -1) \rangle, \right. \\ & \left\langle \frac{1}{2}, \frac{1}{2}; 1, 0 \left| \frac{3}{2}, \frac{1}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 0 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle M_{+-} \langle ss(1^+, 0) | ss(1^+, 0) \rangle, \right. \\ & \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 0 \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}; 1, 0 \left| \frac{1}{2}, \frac{1}{2} \right\rangle M_{-+} \langle ss(1^+, 0) | ss(1^+, 0) \rangle, \right. \\ & \left\langle \frac{1}{2}, \frac{1}{2}; 1, 1 \left| \frac{3}{2}, \frac{3}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}; 1, 1 \left| \frac{1}{2}, \frac{1}{2} \right\rangle M_{+-} \langle ss(1^+, 1) | ss(1^+, 1) \rangle, \right. \\ & \left. \left\langle \frac{1}{2}, -\frac{1}{2}; 1, -1 \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}; 1, -1 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle M_{-+} \langle ss(1^+, -1) | ss(1^+, -1) \rangle, \right. \right. \\ & \left. \left. \begin{aligned} & \frac{\sqrt{2}}{3} (M_{++} \langle ss(1^+, 0) | ss(1^+, 0) \rangle - M_{--} \langle ss(1^+, 1) | ss(1^+, 1) \rangle), \\ & -\frac{\sqrt{2}}{3} (M_{--} \langle ss(1^+, 0) | ss(1^+, 0) \rangle - M_{++} \langle ss(1^+, 1) | ss(1^+, 1) \rangle), \\ & -\frac{\sqrt{2}}{3} M_{+-} \langle ss(1^+, 0) | ss(1^+, 0) \rangle, \\ & \frac{\sqrt{2}}{3} M_{-+} \langle ss(1^+, 0) | ss(1^+, 0) \rangle, \\ & -\sqrt{\frac{2}{3}} M_{+-} \langle ss(1^+, 1) | ss(1^+, 1) \rangle, \\ & \sqrt{\frac{2}{3}} M_{-+} \langle ss(1^+, 1) | ss(1^+, 1) \rangle, \end{aligned} \right. \right\} \quad (\text{B7})
\end{aligned}$$

for $(\lambda_{\Omega_c^*}, \lambda_{\Omega_b}) = (\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (\frac{3}{2}, \frac{1}{2})$ and $(-\frac{3}{2}, -\frac{1}{2})$, respectively, Averaging over the spin of the initial-state baryon and summing over that of the final-state baryon, the sum of the squared amplitudes becomes

$$\begin{aligned}
\frac{1}{2} \sum_{\lambda_{\Omega_c^*}, \lambda_{\Omega_b}} |\mathcal{M}_{\Omega_c^*}(\lambda_{\Omega_c^*}, \lambda_{\Omega_b})|^2 + \frac{1}{2} \sum_{\lambda_{\Omega_c^*}, \lambda_{\Omega_b}} |\mathcal{M}_{\Omega_c^*}(\lambda_{\Omega_c^*}, \lambda_{\Omega_b})|^2 = & \frac{1}{2} (|M_{++}|^2 + |M_{--}|^2 + |M_{+-}|^2 + |M_{-+}|^2) \\
& \times \frac{1}{3} (\langle ss(1^+, 0) | ss(1^+, 0) \rangle^2 + 2 \langle ss(1^+, 1) | ss(1^+, 1) \rangle^2), \quad (\text{B8})
\end{aligned}$$

which has the same structure as Eqs. (B3) and (B5) apart from the matrix element of the light component.

APPENDIX C: ISGUR-WISE FUNCTIONS

For completeness, we summarize the definitions of the IW functions and their relation to the light-component matrix elements in the spin decomposition picture. This clarifies the equivalence between the usual form-factor parametrization and the spin decomposition picture.

The $B \rightarrow D$ transition matrix element of the vector current is expressed in terms of the IW function $\xi(w)$ as [8,9]

$$\langle D | \bar{c} \gamma^\mu b | B \rangle = \sqrt{m_B m_D} \xi(w) (v + v')^\mu, \quad (\text{C1})$$

where m_B and m_D denote the masses of the B and D mesons, v and v' denote their four-velocities, respectively, and $w = v \cdot v'$. The normalization of $\xi(w)$ is given by $\xi(1) = 1$. Using Eq. (2.3), the same matrix element can be rewritten in the spin decomposition picture with the Clebsch-Gordan coefficients, the heavy-quark spinors and the overlap of the light degrees of freedom as

$$\begin{aligned} \langle D | \bar{c} \gamma^\mu b | B \rangle &= \left\langle \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \middle| 0, 0 \right\rangle^2 \bar{u}_c \left(\frac{1}{2} \right) \gamma^\mu u_b \left(\frac{1}{2} \right) \left\langle \bar{q} \left(\frac{1^-}{2}, -\frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, -\frac{1}{2} \right) \right\rangle \\ &\quad + \left\langle \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2} \middle| 0, 0 \right\rangle^2 \bar{u}_c \left(-\frac{1}{2} \right) \gamma^\mu u_b \left(-\frac{1}{2} \right) \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle, \\ &= \frac{1}{2} \left[\bar{u}_c \left(\frac{1}{2} \right) \gamma^\mu u_b \left(\frac{1}{2} \right) + \bar{u}_c \left(-\frac{1}{2} \right) \gamma^\mu u_b \left(-\frac{1}{2} \right) \right] \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle, \\ &= \sqrt{\frac{2m_B m_D}{w+1}} (v + v')^\mu \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle. \end{aligned} \quad (C2)$$

In the last equality, we used $\bar{u}_c(\pm\frac{1}{2})\gamma^\mu u_b(\pm\frac{1}{2}) = \sqrt{2m_B m_D/(w+1)}(v+v')^\mu$. Comparing Eqs. (C2) and (C1), we obtain the relation,

$$\xi(w) = \sqrt{\frac{2}{w+1}} \left\langle \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \middle| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle. \quad (C3)$$

This is given in Eq. (3.3). The same IW function is obtained by the other currents related to the vector one under the heavy-quark symmetry. This result is also obtained correspondingly in the spin decomposition picture. The same argument also holds for the $B \rightarrow D^*$ transition.

For the $\Lambda_b \rightarrow \Lambda_c$ transition, the IW function $\zeta(w)$ is defined as [10,11]

$$\langle \Lambda_c(\lambda_{\Lambda_c}) | \bar{c} \Gamma b | \Lambda_b(\lambda_{\Lambda_b}) \rangle = \zeta(w) \bar{u}_{\Lambda_c}(\lambda_{\Lambda_c}) \Gamma u_{\Lambda_b}(\lambda_{\Lambda_b}), \quad (C4)$$

where Γ denotes a general γ matrix, and $\zeta(1) = 1$. In the spin decomposition picture, this matrix element takes the form,

$$\begin{aligned} \langle \Lambda_c(\lambda_{\Lambda_c}) | \bar{c} \Gamma b | \Lambda_b(\lambda_{\Lambda_b}) \rangle &= \left\langle \frac{1}{2}, \lambda_{\Lambda_c}; 0, 0 \middle| \frac{1}{2}, \lambda_{\Lambda_c} \right\rangle \left\langle \frac{1}{2}, \lambda_{\Lambda_b}; 0, 0 \middle| \frac{1}{2}, \lambda_{\Lambda_b} \right\rangle \bar{u}_c(\lambda_{\Lambda_c}) \Gamma u_b(\lambda_{\Lambda_b}) \langle qq(0^+, 0) | qq(0^+, 0) \rangle \\ &= \bar{u}_c(\lambda_{\Lambda_c}) \Gamma u_b(\lambda_{\Lambda_b}) \langle qq(0^+, 0) | qq(0^+, 0) \rangle. \end{aligned} \quad (C5)$$

Since the spinors of the heavy quarks coincide with those of the baryons in the heavy-quark limit, we obtain the relation,

$$\zeta(w) = \langle qq(0^+, 0) | qq(0^+, 0) \rangle. \quad (C6)$$

This is given in Eq. (3.4).

On the other hand, the $\Omega_b \rightarrow \Omega_c$ transition is described by two IW functions $\xi_1(w)$ and $\xi_2(w)$, defined as [10,11]

$$\langle \Omega_c(\lambda_{\Omega_c}) | \bar{c} \Gamma b | \Omega_b(\lambda_{\Omega_b}) \rangle = [-g_{\mu\nu} \xi_1(w) + v_\mu v'_\nu \xi_2(w)] \bar{\Omega}_c^\mu(\lambda_{\Omega_c}) \Gamma \Omega_b^\nu(\lambda_{\Omega_b}), \quad (C7)$$

where $\xi_1(1) = 1$, and $\Omega_b^\nu(\lambda_{\Omega_b})$ is given in terms of the Dirac spinor $u_{\Omega_b}(\lambda_{\Omega_b})$ as

$$\Omega_b^\nu(\lambda_{\Omega_b}) = \frac{1}{\sqrt{3}} (\gamma^\nu + v^\nu) \gamma_5 u_{\Omega_b}(\lambda_{\Omega_b}). \quad (C8)$$

For example, in the case of $\Gamma = \gamma^\rho$ we obtain,

$$\left\langle \Omega_c \left(\pm \frac{1}{2} \right) \middle| \bar{c} \gamma^\rho b \middle| \Omega_b \left(\pm \frac{1}{2} \right) \right\rangle = \frac{1}{3} [(w+2)\xi_1(w) - (w^2-1)\xi_2(w)] \bar{u}_{\Omega_c} \left(\pm \frac{1}{2} \right) \gamma^\rho u_{\Omega_b} \left(\pm \frac{1}{2} \right), \quad (C9)$$

$$\left\langle \Omega_c \left(\pm \frac{1}{2} \right) \left| \bar{c} \gamma^\rho b \right| \Omega_b \left(\mp \frac{1}{2} \right) \right\rangle = \frac{1}{3} [-w \xi_1(w) + (w^2 - 1) \xi_2(w)] \bar{u}_c \left(\pm \frac{1}{2} \right) \gamma^\rho u_b \left(\mp \frac{1}{2} \right), \quad (\text{C10})$$

The corresponding matrix elements in the spin decomposition picture are given by

$$\begin{aligned} \left\langle \Omega_c \left(\pm \frac{1}{2} \right) \left| \bar{c} \gamma^\rho b \right| \Omega_b \left(\pm \frac{1}{2} \right) \right\rangle &= \left\langle \frac{1}{2}, \pm \frac{1}{2}; 1, 0 \left| \frac{1}{2}, \pm \frac{1}{2} \right\rangle^2 \bar{u}_c \left(\pm \frac{1}{2} \right) \gamma^\rho u_b \left(\pm \frac{1}{2} \right) \langle ss(1^+, 0) | ss(1^+, 0) \rangle \right. \\ &\quad + \left\langle \frac{1}{2}, \mp \frac{1}{2}; 1, \pm 1 \left| \frac{1}{2}, \pm \frac{1}{2} \right\rangle^2 \bar{u}_c \left(\mp \frac{1}{2} \right) \gamma^\rho u_b \left(\mp \frac{1}{2} \right) \langle ss(1^+, \pm 1) | ss(1^+, \pm 1) \rangle \right. \\ &= \frac{1}{3} \bar{u}_c \left(\pm \frac{1}{2} \right) \gamma^\rho u_b \left(\pm \frac{1}{2} \right) [\langle ss(1^+, 0) | ss(1^+, 0) \rangle + 2 \langle ss(1^+, \pm 1) | ss(1^+, \pm 1) \rangle], \end{aligned} \quad (\text{C11})$$

$$\begin{aligned} \left\langle \Omega_c \left(\pm \frac{1}{2} \right) \left| \bar{c} \gamma^\rho b \right| \Omega_b \left(\mp \frac{1}{2} \right) \right\rangle &= \left\langle \frac{1}{2}, \pm \frac{1}{2}; 1, 0 \left| \frac{1}{2}, \pm \frac{1}{2} \right\rangle \left\langle \frac{1}{2}, \mp \frac{1}{2}; 1, 0 \left| \frac{1}{2}, \mp \frac{1}{2} \right\rangle \bar{u}_c \left(\pm \frac{1}{2} \right) \gamma^\rho u_b \left(\mp \frac{1}{2} \right) \langle ss(1^+, 0) | ss(1^+, 0) \rangle \right. \\ &= -\frac{1}{3} \bar{u}_c \left(\pm \frac{1}{2} \right) \gamma^\rho u_b \left(\mp \frac{1}{2} \right) \langle ss(1^+, 0) | ss(1^+, 0) \rangle. \end{aligned} \quad (\text{C12})$$

From Eqs. (C9)–(C12), we find that the IW functions are written in terms of the matrix elements of the light degrees of freedom as

$$\xi_1(w) = \langle ss(1^+, 1) | ss(1^+, 1) \rangle, \quad (\text{C13})$$

$$\xi_2(w) = \frac{1}{w^2 - 1} (w \langle ss(1^+, 1) | ss(1^+, 1) \rangle - \langle ss(1^+, 0) | ss(1^+, 0) \rangle). \quad (\text{C14})$$

They are the same as Eqs. (3.6) and (3.7).

Finally, the IW functions $\tau_{1/2}(w)$ and $\tau_{3/2}(w)$ for the $B \rightarrow \{D_0^*, D_1^*\}$ and $B \rightarrow \{D_1, D_2^*\}$ transitions are defined, for instance, through the vector-current matrix elements as [30]

$$\langle D_1^*(\lambda_{D_1^*}) | \bar{c} \gamma^\mu b | B \rangle = 2 \sqrt{m_B m_{D_1^*}} [(w - 1) \epsilon^{*\mu} - (\epsilon^* \cdot v) v'^\mu] \tau_{1/2}(w), \quad (\text{C15})$$

$$\langle D_1(\lambda_{D_1}) | \bar{c} \gamma^\mu b | B \rangle = \sqrt{\frac{m_B m_{D_1}}{2}} [(1 - w^2) \epsilon^{*\mu} + (\epsilon^* \cdot v) (-3v^\mu + (w - 2)v'^\mu)] \tau_{3/2}(w), \quad (\text{C16})$$

where $m_{D_1^*}$ and m_{D_1} are the masses of the D_1^* and D_1 mesons, respectively, and $\epsilon^{*\mu}$ denotes the polarization vector of the charm mesons. As in the previous cases, by comparing these expressions with those obtained in the spin decomposition picture, we obtain the relations given in Eqs. (3.10) and (3.11) as

$$\tau_{1/2}(w) = \frac{1}{\sqrt{2(w-1)}} \left\langle \bar{q} \left(\frac{1^+}{2}, \frac{1}{2} \right) \left| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle, \quad (\text{C17})$$

$$\tau_{3/2}(w) = -\frac{1}{(w+1)} \frac{1}{\sqrt{w-1}} \left\langle \bar{q} \left(\frac{3^+}{2}, \frac{1}{2} \right) \left| \bar{q} \left(\frac{1^-}{2}, \frac{1}{2} \right) \right\rangle. \quad (\text{C18})$$

The same relations also hold for other matrix elements in the $B \rightarrow \{D_0^*, D_1^*\}$ and $B \rightarrow \{D_1, D_2^*\}$ transitions.

For each transition, the hadronic matrix elements can be expressed either in terms of the IW functions or as overlaps of the light components in the spin decomposition picture, leading to identical results. We thus conclude that the spin decomposition picture provides an equivalent description of the hadronic matrix elements as that given by the IW functions.

APPENDIX D: BJORKEN SUM RULES

Here, we briefly comment on the derivation of the Bjorken sum rules [30,39] in terms of the hadron decay amplitude obtained in Eq. (2.10).

For the B -meson semileptonic decays $B \rightarrow H_c l \bar{\nu}_l$, summing the squared decay amplitudes in the spin decomposition picture over all possible final-state mesons H_c and integrating over the phase space, we obtain the inclusive differential decay rate

$$\begin{aligned} \frac{d\Gamma(B \rightarrow X_c l \bar{\nu}_l)}{dw} &= \sum_{H_c} \frac{d\Gamma(B \rightarrow H_c l \bar{\nu}_l)}{dw} \\ &= \frac{d\Gamma(b \rightarrow c l \bar{\nu}_l)}{dw} \left[\frac{1+w}{2} \xi(w)^2 + (w-1)(2\tau_{1/2}(w)^2 + (w+1)^2 \tau_{3/2}(w)^2) + \dots \right], \end{aligned} \quad (\text{D1})$$

where $d\Gamma(b \rightarrow c l \bar{\nu}_l)/dw$ is the partonic differential rate, and the ellipsis denotes contributions from higher excited states. Here, we applied Eq. (2.10) in the second equality. By assuming the quark-hadron duality, Eq. (D1) yields the Bjorken sum rule [30,39],

$$1 = \frac{1+w}{2} \xi(w)^2 + (w-1)(2\tau_{1/2}(w)^2 + (w+1)^2 \tau_{3/2}(w)^2) + \dots \quad (\text{D2})$$

Analogous Bjorken sum rules also hold for the semileptonic Λ_b decays [40] and the semileptonic Ω_b decays [26,41].

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