

Comment on “Chiral symmetry restoration, the eigenvalue density of the Dirac operator, and the axial U(1) anomaly at finite temperature”

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Aoki, Fukaya, and Taniguchi claim that both the spectral density of the Dirac operator at the origin and the topological susceptibility must vanish identically for sufficiently small but nonzero quark mass m in the chirally symmetric phase of quantum chromodynamics with two light quark flavors, under certain technical assumptions on the spectrum and on the dependence of observables on m . I argue that a crucial step of their proof is not justified and the validity of these conclusions should be reassessed.

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In Ref. [1] the authors study the constraints imposed by the restoration of $N_f = 2$ chiral symmetry on the spectrum of the Dirac operator in finite-temperature quantum chromodynamics on the lattice. Their conclusions, based on certain technical assumptions on the spectral density and on the mass dependence of certain observables, include the claims that (i) the spectral density of Dirac eigenvalues, $\rho(\lambda; m)$, vanishes identically at $\lambda = 0$ for sufficiently small quark mass, m , i.e., $\rho(0; m) = 0 \forall |m| < m_0$, for some $m_0 > 0$, and (ii) the topological susceptibility, χ_t , vanishes identically for sufficiently small m . These conclusions have important consequences for the fate of the anomalous $U(1)_A$ symmetry in the chiral limit, and are used in Ref. [1] to argue that $U(1)_A$ is effectively restored in the chiral limit in the scalar and pseudoscalar sector as a necessary consequence of chiral symmetry restoration. Notably, (ii) would indicate the presence of a phase transition at small but nonzero m as one goes toward the chiral limit in the high-temperature phase.

Predictions (i) and (ii) have been extensively tested with numerical lattice simulations toward the chiral limit [2–10]. However, while Refs. [2–5] support these predictions, Refs. [6–10] reach different conclusions. On the one hand, Refs. [2–5] secure a better control of theoretical uncertainties by using Möbius domain-wall fermions, at the price of computational limitations on the available volumes and number of configurations. On the other hand, Refs. [6–10] gain access to larger volumes and statistics by using staggered

fermions, but introduce theoretical uncertainties on the chiral and topological properties of the system by doing so. Owing to these difficulties no consensus has been reached yet, and the status of these predictions remains unsettled.

The proof of (i) and (ii) given in Ref. [1] makes crucial use of the following argument. In Sec. II C of Ref. [1] [see their Eqs. (23) and (24)], the authors state that if for some mass-independent positive-definite observable $\mathcal{O}(A)$, involving only (lattice) gauge fields, A , one has for some integers $l_0 \geq 1$ and $k \geq 0$ that

$$\lim_{m \rightarrow 0} \frac{1}{m^k} \langle \mathcal{O}(A)^{l_0} \rangle = 0, \quad (1)$$

where $\langle \dots \rangle$ denotes the expectation value, then one can write

$$\langle \mathcal{O}(A)^{l_0} \rangle = m^{k_0} \int DA P(m, A) \mathcal{O}(A)^{l_0}, \quad (2)$$

with $k_0 > k$, where DA is the gauge-field integration measure and $P(m, A)$ a non-negative quantity that does not vanish identically in the chiral limit, $P(0, A) \not\equiv 0$. It is assumed that the integral in Eq. (2) is finite (i.e., not divergent) in the infinite-volume limit and, implicitly, also in the subsequent chiral limit; it is also implicitly assumed that $P(0, A)$ is well defined. According to the authors, this means that the leading m dependence of $\langle \mathcal{O}(A)^{l_0} \rangle$ arises from those gauge configurations for which $P(0, A) \neq 0$. From Eq. (2) they then infer that

$$\langle \mathcal{O}(A)^l \rangle = m^{k_0} \int DA P(m, A) \mathcal{O}(A)^l = O(m^{k_0}), \quad (3)$$

for arbitrary positive integer l [see their Eq. (25)], as long as the integral is finite (in the infinite-volume limit,

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presumably). According to the authors, this is because $\mathcal{O}(A)^{l_0}$ and $\mathcal{O}(A)^l$ are both positive and share the same support in configuration space.

It is inherent in the nature of the problem that the expectation value $\langle \dots \rangle$ in Eqs. (1)–(3) must be understood as the thermodynamic limit of the expectation value computed in a finite volume; this meaning of the notation will be understood in the following. In fact, for Eq. (1) to be a meaningful statement about the chiral limit, this should be taken only after the thermodynamic limit. Similarly, the measure $DA \mathcal{P}(m, A)$ appearing on the right-hand side of Eqs. (2) and (3) should be understood as a measure formally defined on the space of configurations of infinite volume.

By referring to Eq. (2) as providing the leading m dependence of $\langle \mathcal{O}(A)^{l_0} \rangle$ [see after their Eq. (24)], the authors of Ref. [1] are implicitly assuming that the integral does neither diverge nor vanish in the chiral limit, and so not only that $P(0, A)$ is well defined and not identically zero, but also that the measure of the support of $\mathcal{O}(A)^{l_0}$ according to $DA \mathcal{P}(0, A)$ is nonzero. Nonetheless, if this measure were zero one would simply find that $\langle \mathcal{O}(A)^{l_0} \rangle$ vanishes faster than m^{k_0} , and their argument would not change substantially.

Generally, Eq. (1) does not imply $\langle \mathcal{O}(A)^{l_0} \rangle = O(m^{k_0})$ with $k_0 > k$, since $m^{-k} \langle \mathcal{O}(A)^{l_0} \rangle$ may be vanishing more slowly than a power. However, in Ref. [1] this is justified by the assumption that in the symmetric phase the expectation value in the thermodynamic limit (if it exists at $m \neq 0$) of a mass-independent observable involving only gauge fields is an analytic function of m^2 in a neighborhood of $m = 0$ (referred to as “ m^2 -analyticity assumption” in the following). This also allows the authors to write $k_0 = 2(\lfloor k/2 \rfloor + 1)$ and $P(m, A) = \hat{P}(m^2, A)$, which is, however, immaterial for what follows. Since the m^2 -analyticity assumption has far-reaching and unexpected consequences, it is better to proceed at first without imposing it. One should then understand a factor m^{k_0} as denoting the leading, $o(m^k)$ dependence of $\langle \mathcal{O}(A)^{l_0} \rangle$ (changing the notation is straightforward but would add to the clutter, without any change in the arguments).

Finally, since the authors of Ref. [1] apply Eq. (3) to the topological charge squared divided by the volume (see their Sec. III G), when they require that $\mathcal{O}(A)$ be positive they most likely mean $\mathcal{O}(A) \geq 0$. Here I will understand positivity of an observable in the same way.

The authors of Ref. [1] do not provide details on how they obtain Eq. (2) from Eq. (1) and Eq. (3) from Eq. (2). As one can always write an $O(m^{k_0})$ expectation value in the form of Eq. (2) simply by factoring m^{k_0} out from the usual path integral, this equation is empty unless one requires that $P(0, A)$ is well defined; and to show that it is so, one needs to characterize $P(m, A)$ in more detail. Indeed, just factoring out m^{k_0} from the usual path-integral measure would

generally lead to a divergent $P(m, A)$ in the chiral limit. This can be seen explicitly for a one-dimensional configuration space with measure $dA \theta(m^2 - |A|)/(2m^2)$, with $\theta(x)$ the Heaviside step function, taking $\mathcal{O}(A) = |A|$ as the observable. In this example, after observing that $\langle \mathcal{O}(A)^{l_0} \rangle = O(m^{k_0})$ with $k_0 = 2l_0$ one would tentatively set $P(m, A) = \theta(m^2 - |A|)/(2m^{2(l_0+1)})$. However, in the chiral limit $P(0, A)$ would not be well defined, not even as a distribution on the space of polynomials of $\mathcal{O}(A)$ that vanish at zero, and one would not be able to conclude that $\langle \mathcal{O}(A)^l \rangle = O(m^{k_0})$, $\forall l > 0$ (except if the initial observation of a vanishing expectation value in the chiral limit had been made for $l_0 = 1$). In fact, $\langle \mathcal{O}(A)^l \rangle = O(m^{2l})$, which is not $O(m^{k_0})$ for $l < l_0$.

Moreover, $\mathcal{O}(A)^{l_0}$ and $\mathcal{O}(A)^l$ sharing the same support in configuration space is by itself not sufficient to obtain Eq. (3) from Eq. (2), as in general the configurations in this support that are relevant respectively to $\langle \mathcal{O}(A)^{l_0} \rangle$ and $\langle \mathcal{O}(A)^l \rangle$ for $l \neq l_0$ have no reason to provide contributions of the same order in m . The l independence of how fast $\langle \mathcal{O}(A)^l \rangle$ vanishes, if it does so for some $l = l_0$, is an interesting technical claim on its own, so it is worth discussing this point in detail.

A characterization of $P(m, A)$ that I believe is in the spirit of Ref. [1], and that allows one to derive Eq. (3) from Eq. (2), is the following. The arguments are heuristic, and any issue of rigor in the construction of the various measures involved is ignored. One starts by writing the thermodynamic limit of the expectation value of a generic observable $\mathcal{G}(A)$ depending only on gauge fields as $\langle \mathcal{G}(A) \rangle = \int DA \mathcal{P}(m, A) \mathcal{G}(A)$, for a suitable measure $DA \mathcal{P}(m, A)$ on the space of configurations of infinite volume, assumed to be well behaved in the chiral limit. This measure typically vanishes in most of configuration space, and has support only on the configurations that remain relevant in the thermodynamic limit. If one now assumes that the (mass-dependent) support $S_m(\mathcal{P})$ of $\mathcal{P}(m, A)$ and the (mass-independent) support $S(\mathcal{O})$ of $\mathcal{O}(A)^{l_0}$, which is the same as the support of $\mathcal{O}(A)$ (for $l_0 \geq 1$), overlap in a region of measure $O(m^{k_0})$ according to $DA \mathcal{P}(m, A)$, and that in the chiral limit $\mathcal{O}(A)$ remains bounded in this region, it is possible to define in a meaningful way the measure $DA \mathcal{P}(m, A)$ appearing in Eq. (2), leading in turn to Eq. (3). Indeed, denoting with $\chi_R(A)$ the indicator function of the set of configurations R , and setting $P_{\mathcal{O}}(m, A) \equiv m^{-k_0} \mathcal{P}(m, A) \chi_{S(\mathcal{O})}(A)$, one has

$$\langle \mathcal{O}(A)^l \rangle = m^{k_0} \int DA P_{\mathcal{O}}(m, A) \mathcal{O}(A)^l = O(m^{k_0}), \quad (4)$$

for any $l \geq 1$ (this construction of course does not apply to $l = 0$). In fact, since the effective support of $\mathcal{O}(A)$, i.e., the support $S_m(\mathcal{P}) \cap S(\mathcal{O})$ of $\chi_{S_m(\mathcal{P})}(A) \chi_{S(\mathcal{O})}(A)$, is assumed to be of measure $O(m^{k_0})$ according to $DA \mathcal{P}(m, A)$, after

rescaling by m^{k_0} one obtains for the shrinking region of overlap at most a finite nonzero measure according to $DA P_O(m, A)$; since $\mathcal{O}(A)$ is assumed to be bounded there, the integral is finite.

Summarizing, if the effective support $S_m(\mathcal{P}) \cap S(\mathcal{O})$ of $\mathcal{O}(A)$ is of measure $O(m^{k_0})$ according to $DA \mathcal{P}(m, A)$, then Eqs. (2) and (3) hold with the same meaningful $P(m, A)$ [i.e., $P_O(m, A)$ defined above], nondivergent in the chiral limit. If $\mathcal{O}(A)$ is bounded (at least on its effective support) this region gives at most $O(1)$ contributions to the integrals in Eqs. (2) and (3), that cannot weaken the $O(m^{k_0})$ suppression given by the prefactor, independently of l , and $\langle \mathcal{O}(A)^l \rangle = O(m^{k_0})$ follows, $\forall l \geq 1$.

Even though Eq. (3) is a necessary consequence of Eq. (2) if one accepts the assumptions and the heuristic argument given above, Eq. (2) with $P(m, A)$ having the required properties is not a necessary consequence of Eq. (1). In fact, the relation $\langle \mathcal{O}(A)^{l_0} \rangle = O(m^{k_0})$ does not necessarily require that the expectation value receives contributions only from a region of configuration space of vanishing measure $O(m^{k_0})$ [according to $DA \mathcal{P}(m, A)$] in the chiral limit, as assumed above, not even for positive observables; but it could alternatively be due to $\mathcal{P}(m, A)$ being supported where $\mathcal{O}(A)^{l_0}$ is of magnitude comparable with m^{k_0} , so that $\chi_{S_m(\mathcal{P})}(A) \mathcal{O}(A)^{l_0} = O(m^{k_0})$. This applies, e.g., to self-averaging observables, that in the thermodynamic limit equal their expectation value on the support of $\mathcal{P}(m, A)$. Under this alternative assumption, $P_O(m, A)$ given above Eq. (4) is not well defined in the chiral limit, so Eq. (3) is not guaranteed to hold; in fact, one finds instead that $\langle \mathcal{O}(A)^l \rangle = O(m^{(k_0/l_0)l})$ if $\langle \mathcal{O}(A)^{l_0} \rangle = O(m^{k_0})$.

It is worth stressing that positivity of $\mathcal{O}(A)$ plays no role in the argument leading to Eq. (4), and also does not force one to make the relevant assumption used there concerning the support of the observable. The alternative possibility just discussed is equally valid for a positive observable.

The discussion above can be made rigorous in the language of density-of-states functions, rewriting the expectation values $\langle \mathcal{O}(A)^l \rangle$ as

$$\langle \mathcal{O}(A)^l \rangle = \int dx x^l p_O(x, m), \quad (5)$$

where $p_O(x, m) \equiv \langle \delta(x - \mathcal{O}(A)) \rangle$, with $\int dx p_O(x, m) = 1$, is the density-of-states function (which is generally a positive distribution) associated with $\mathcal{O}(A)$. For positive $\mathcal{O}(A)$ the support of $p_O(x, m)$ is restricted to $x \geq 0$. In this language, by claiming that Eq. (1) implies Eqs. (2) and (3) [with $P(m, A)$ having the desired properties], the authors of Ref. [1] are probably implicitly assuming that the density-of-states function is of the form

$$p_O(x, m) = [1 - \alpha(m)]\delta(x) + \alpha(m)\tilde{p}(x, m), \quad (6)$$

where $\tilde{p}(x, m)$ is an ordinary positive function supported on $x \geq 0$, normalized for convenience as a probability distribution, $\int dx \tilde{p}(x, m) = 1$, and with moments $\mu_l(m) \equiv \int dx x^l \tilde{p}(x, m)$ bounded in the chiral limit; and where $0 \leq \alpha(m) \leq 1$ with $\alpha(m) = O(m^{k_0})$. In this case $\langle \mathcal{O}(A)^l \rangle / \alpha(m) \rightarrow \mu_l(0)$ as $m \rightarrow 0$, so $\langle \mathcal{O}(A)^l \rangle = O(\alpha(m)) = O(m^{k_0})$, $\forall l \geq 1$. Note, however, that finding a positive integer l_0 for which $\langle \mathcal{O}(A)^{l_0} \rangle = O(m^{k_0})$ is not sufficient to show that $\alpha(m) = O(m^{k_0})$, as it is possible that $\mu_{l_0}(0) = 0$ and so $\langle \mathcal{O}(A)^{l_0} \rangle$ vanishes in the chiral limit faster than $\alpha(m)$, and so of those $\langle \mathcal{O}(A)^l \rangle$ for which $\mu_l(0) \neq 0$. This is already a counterexample to the general statement.

If $p_O(x, m)$ is instead a Gaussian restricted to $x \geq 0$,

$$p_O(x, m) = \mathcal{N}(m)^{-1} e^{-\frac{(x-x_0(m))^2}{2\sigma(m)^2}} \theta(x), \quad (7)$$

with $\mathcal{N}(m)$ a suitable normalization factor, and if $x_0(m), \sigma(m) > 0$ for $m \neq 0$ but vanish in the chiral limit, $x_0(0) = \sigma(0) = 0$, one finds

$$\langle \mathcal{O}(A)^l \rangle_{m \rightarrow 0} \sim \begin{cases} C_\zeta^{(l)} x_0(m)^l, & \text{if } \zeta \neq \infty, \\ \tilde{C}^{(l)} \sigma(m)^l, & \text{if } \zeta = \infty, \end{cases} \quad (8)$$

where $\zeta \equiv \lim_{m \rightarrow 0} \frac{\sigma(m)}{x_0(m)}$, and $C_\zeta^{(l)}$ and $\tilde{C}^{(l)}$ are nonzero constants. Since $\langle \mathcal{O}(A)^l \rangle / \langle \mathcal{O}(A)^{l_0} \rangle \propto x_0(m)^{l-l_0}$ or $\propto \sigma(m)^{l-l_0}$, for $l < l_0$ it blows up as $m \rightarrow 0$, so if $\langle \mathcal{O}(A)^{l_0} \rangle$ vanishes like m^{k_0} in the chiral limit, then $\langle \mathcal{O}(A)^l \rangle$ for $0 < l < l_0$ vanishes more slowly and cannot be $O(m^{k_0})$. This example again contradicts the claim of Ref. [1].

An even simpler possibility is of course $p_O(x, m) = \delta(x - x_0(m))$, with $x_0(m) \geq 0$ vanishing in the chiral limit. In this case $\langle \mathcal{O}(A)^l \rangle = x_0(m)^l$, and one reaches the same conclusions as with the Gaussian functional form. Since the density-of-states function is a Dirac delta if $\mathcal{O}(A)$ is a self-averaging observable, this counterexample is certainly of physical relevance.

More generally, if there is some $x_0(m)$, with $x_0(0) = 0$, for which the function $\pi_O(x, m) \equiv p_O(x/x_0(m), m)$ has a well-defined chiral limit $\pi_O(x, 0)$ with finite moments, one finds

$$\langle \mathcal{O}(A)^l \rangle = x_0(m)^l \int_0^\infty dx x^l \pi_O(x, 0) + o(x_0(m)^l), \quad (9)$$

and one reaches the same conclusions as above.

These counterexamples show that without additional assumptions one cannot deduce Eq. (3) as a necessary consequence of Eq. (1), and more generally one cannot conclude that $\langle \mathcal{O}(A)^{l_0} \rangle = O(m^{k_0})$ for some l_0, k_0 always implies $\langle \mathcal{O}(A)^l \rangle = O(m^{k_0})$ for any integer $l \geq 1$ for positive observables that depend only on gauge fields. Of course, one may assume that the support of the relevant

observables is such that Eqs. (2) and (3) can be derived [see the discussion around Eq. (4)], but there seems to be no compelling reason to do so, and the applicability of such an assumption in realistic cases is called into question by the simplicity of the counterexamples.

In Ref. [1] the authors make the additional m^2 -analyticity assumption mentioned above, namely that in the symmetric phase the expectation value in the infinite-volume limit (if it exists at $m \neq 0$) of an m -independent functional of the gauge fields, $\mathcal{O}(A)$, is an analytic function of m^2 near $m = 0$. It is not clear whether this assumption can be used to derive Eq. (3), and if so, how, and if it was used in Ref. [1] to this end. For nonlocal functionals (such as those used in Ref. [1]) such an assumption cannot be directly justified using the properties of a local quantum field theory in the symmetric phase (see Ref. [11]), and should be regarded as an independent assumption.

In any case, under such an unrestricted m^2 -analyticity assumption the counterexamples given above do not apply. In order for $\langle \mathcal{O}(A)^l \rangle$ with l integer to be analytic in m^2 near zero, $x_0(m)$ must be proportional to m^{2k} for some integer k , in all cases except for the Gaussian density-of-states function if $\zeta = \infty$, in which case this requirement applies to $\sigma(m)$. [Conversely, requiring that $x_0(m)$, and for the Gaussian density-of-states function also $\sigma(m)$, be analytic functions of m^2 guarantees m^2 analyticity of $\langle \mathcal{O}(A)^l \rangle$ for any integer l .] It is then clear that $\langle \mathcal{O}(A)^\alpha \rangle$ will not be m^2 analytic near $m = 0$ for suitably chosen noninteger powers α . The possibility to derive Eq. (3) from Eq. (1) under this additional assumption remains therefore open.

However, an unrestricted m^2 -analyticity assumption has very peculiar consequences that lead one to question its applicability. Let $\mathcal{L}(A)$ be any m -independent functional that depends only on the lattice gauge fields $A(y)$ for y in some finite region of the lattice, kept fixed in the thermodynamic limit. Let $\mathcal{L}(A_x)$ be the translated functionals obtained by replacing $A(y)$ with $A_x(y) = A(y + x)$ in the argument of $\mathcal{L}(A)$, and construct the “intensive” gauge observable

$$\mathcal{O}_{\mathcal{L}}(A) \equiv \frac{1}{V} \sum_x \mathcal{L}(A_x), \quad (10)$$

where the sum runs over the lattice sites and V is the lattice volume. Since interactions are short range this observable is self-averaging, and in the thermodynamic limit the corresponding density-of-states function reads as

$$p_{\mathcal{O}_{\mathcal{L}}}(x, m) = \delta(x - \ell_{\mathcal{L}}(m)), \quad \ell_{\mathcal{L}}(m) \equiv \langle \mathcal{O}_{\mathcal{L}}(A) \rangle. \quad (11)$$

Here it is assumed that $\ell_{\mathcal{L}}(m)$ does not vanish, e.g., due to symmetry reasons, at $m \neq 0$; note that $\ell_{\mathcal{L}}(m)$ must be analytic in m^2 by the m^2 -analyticity assumption. An obvious example of an observable of this type is the usual gauge action density. More general examples are the

average over lattice translations, in the sense of Eq. (10), of multilocal gauge observables, e.g., products of Wilson loops of fixed but arbitrary shapes and relative positions, which involve gauge fields in a fixed finite lattice region, although disconnected. Now, for the observables $\tilde{\mathcal{O}}_{\mathcal{L}}^{(\alpha)}(A) \equiv |\mathcal{O}_{\mathcal{L}}(A) - \ell_{\mathcal{L}}(0)|^\alpha$, which are again m -independent functionals of gauge fields only, one finds

$$\begin{aligned} \langle \tilde{\mathcal{O}}_{\mathcal{L}}^{(\alpha)}(A) \rangle &= \int dx |x - \ell_{\mathcal{L}}(0)|^\alpha p_{\mathcal{O}_{\mathcal{L}}}(x, m) \\ &= |\ell_{\mathcal{L}}(m) - \ell_{\mathcal{L}}(0)|^\alpha, \end{aligned} \quad (12)$$

that cannot be analytic in m^2 near $m = 0$ for every $\alpha > 0$ unless it vanishes identically for sufficiently small m , i.e., unless $\ell_{\mathcal{L}}(m) = \ell_{\mathcal{L}}(0)$ if $|m| < m_{\mathcal{L}}$, for some possibly observable-dependent $m_{\mathcal{L}} > 0$. [The m^2 -analyticity assumption then does not apply for $\alpha < 0$, since in this case the expectation value, Eq. (12), diverges.] Using translation invariance, this would imply that $\langle \mathcal{L}(A) \rangle = \ell_{\mathcal{L}}(m)$ becomes m independent for $|m| < m_{\mathcal{L}}$, and so is not m^2 analytic at some finite nonzero $m = \pm m_{\mathcal{L}}$ (unless it were m independent for all m , which is very unlikely; for sure, there exists some $\mathcal{L}$ whose expectation value is not m independent for all m).

Under the unrestricted m^2 -analyticity assumption, in the simplest case where $\inf_{\mathcal{L}} m_{\mathcal{L}} \neq 0$, there would then be a phase at small m where all $\langle \mathcal{L}(A) \rangle$ are m independent; otherwise there would have to be infinitely many phase-transition lines, with these expectation values becoming m independent gradually. This would apply in particular to all the n -point correlation functions of local functionals of the gauge fields (in the usual sense of involving products of gauge link variables in a fixed connected region of the lattice). The existence of such a phase seems highly unlikely, even as a lattice artifact, given that it is at small m that the backreaction of the fermionic determinant on the gauge-field dynamics is the most effective.

The fact that the unrestricted m^2 -analyticity assumption implies the existence of such a peculiar phase at small m calls, if not for its rejection, at least for some suspicion on its applicability. In Ref. [1] the m^2 -analyticity assumption, although formulated without restrictions, is used explicitly only for certain nonlocal observables and their integer powers, and so the argument above would not necessarily invalidate their results (although one would have to properly justify why m^2 analyticity applies to the relevant observables). Clearly, if one restricts the m^2 -analyticity assumption in this way then the counterexamples provided above are perfectly valid. (One can of course make the counterexamples inapplicable to the observables of interest by requiring m^2 analyticity for the expectation value of their noninteger powers, but such an *ad hoc* assumption would have to be thoroughly justified.)

In Sec. III G of Ref. [1] [see after their Eq. (92)], the implication that if $\langle \mathcal{O}(A)^{l_0} \rangle = O(m^{k_0})$ for some positive integer l_0 then $\langle \mathcal{O}(A)^l \rangle = O(m^{k_0})$ for any integer $l \geq 1$ (in particular for $1 \leq l < l_0$) is crucially used, together with technical assumptions on the spectral density on a fixed configuration (including in practice a restricted form of the m^2 -analyticity assumption), to prove that in the symmetric phase χ_t and $\rho(0; m)$ vanish faster than any power of m in the chiral limit, and therefore identically below some small but nonzero m due to analyticity in m^2 near $m = 0$. However, the counterexamples discussed above show that the status of this implication is problematic, and these conclusions of Ref. [1] should be critically reconsidered. These counterexamples could be invalidated by making stronger assumptions on the m^2 analyticity of expectation values in the symmetric phase, but these assumptions would have to be properly justified. In particular, assuming

m^2 analyticity for the expectation value of any m -independent gauge-field functional (when this exists at $m \neq 0$) would lead to a very peculiar phase at small but nonzero m , casting serious doubts on the applicability of this assumption.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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