

Potency of nilpotence

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The dynamics of $\mathcal{N} = 1$ supersymmetric gauge theories with matter in adjoint and fundamental representations and the superpotentials given by Arnold's ADE singularities has been extensively studied in the literature. It was also conjectured that supersymmetric models with W_{A_k} , $W_{D_{k+2}}$, and W_{E_7} superpotentials possess a dual description. In this paper, we revisit the analysis of the moduli space of A_k and D_{k+2} models by considering the duality along nilpotent directions on the moduli space. While our analysis provides additional evidence for the duality conjecture in W_{A_k} models, we show that the duality conjecture fails for the $W_{D_{k+2}}$ models.

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I. INTRODUCTION

One of the most striking features of $\mathcal{N} = 1$ supersymmetric gauge theories is Seiberg duality [1], which posits an infrared (IR) equivalence between seemingly different ultraviolet (UV) gauge theories. The initial examples of dualities in $SU(N)$ theories with matter in a fundamental representation were quickly expanded to more general theories and even to chains of dualities.

The duality conjectures have been tested and extended in numerous settings, particularly for theories with simple superpotentials and gauge groups. Since at least one of the two dual descriptions is typically strongly coupled in the IR, the duality is usually verified by indirect checks. First, the global symmetries of the two theories must agree. Second, the 't Hooft anomalies must match for the two theories. These are very restrictive conditions, and the successful matching of the 't Hooft anomalies is taken as strong evidence for the duality.

Further tests of the dualities involve dynamics on the moduli space or the deformation of the theory by new superpotential terms. Often, these tests are complementary—turning on moduli vacuum expectation values (VEVs) on the electric side corresponds to superpotential deformations on the magnetic side and vice versa. Typically, turning on gauge invariant operators on the electric side of duality results in gauge symmetry breaking. On the magnetic side,

this usually corresponds to superpotential deformations that may result in new Yukawa interactions, mass terms for some matter fields, and gauge symmetry breaking. For the duality to be valid, the two low energy theories must remain dual. This check is especially nontrivial along flat directions where one can show that the two low energy theories flow to the same IR fixed point.

A comprehensive study of dynamics on the moduli space has been done for some classes of models, for example, $SU(N)$ theories with matter in the fundamental representation [1]. However, due to the complexity of the moduli space, the analysis is far from complete in models with more general gauge groups, matter content, and the superpotentials.

In this paper, we will focus on $\mathcal{N} = 1$ dualities in models with chiral superfields in both fundamental and adjoint representations and nontrivial superpotentials for the adjoints. While the exhaustive study of the dynamics on the moduli space of these theories is beyond the scope of this paper, we will extend the analysis to a new class of flat directions (we will refer to such flat directions as nilpotent) and show that new stringent tests of dualities can be performed along these directions. We expect such analysis to be useful in more general models with general matter representations and nontrivial superpotentials that allow the existence of nilpotent flat directions.

The first model of $\mathcal{N} = 1$ duality with adjoint superfields was proposed by Kutasov [2–4], where the electric description was based on an $SU(N)$ gauge group with F chiral flavors in the fundamental representation, a single adjoint chiral superfield X , and a superpotential¹

$$W = \text{Tr} X^{k+1}. \quad (1)$$

¹We will ignore numerical coefficients throughout.

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The F -flatness conditions coming from the superpotential truncate the chiral ring of the theory and are an essential condition for the existence of the duality map.

Kutasov's conjecture was extended in [5] to theories with F fundamental flavors, two adjoints X and Y , and a superpotential

$$W = \text{Tr} X^{k+1} + \text{Tr} XY^2. \quad (2)$$

It was also shown [5–8] that the superpotentials in these two classes of models correspond to A_k and D_{k+2} singularities of Arnold's singularity theory [9].

However, already in the original paper [5] it was realized that the duality conjecture in D_{k+2} poses a puzzle. For any value of k , the theory and the dual have the same symmetries, and the global anomalies match across the theories. However, while the F -term equations lead to truncation of the chiral ring in models with odd k , the chiral ring is not truncated in models with even k . Since the truncation of the chiral ring is a necessary condition for the existence of duality, this puts in question the validity of the duality conjecture of [5], at least when k is even. While one may have more confidence in the duality for odd k , there is still a tension, which arises from the fact that one can flow between even and odd k theories either by deforming them with tree-level superpotential terms or by simply moving along the moduli space.

One possible resolution of this puzzle could be that the chiral ring of the even- k model is truncated by non-perturbative dynamics. Further research led to better understanding of the dynamics of ADE models [6], however, no dynamical effects capable of truncating the chiral ring were found.

In this work, we will take a different approach toward addressing the puzzle. In particular, we will carefully investigate a class of directions on the moduli space of the A_k and D_{k+2} models along which the VEVs of X are nilpotent,² $X^k = 0$. In the case of A_k models our work will extend the results of [10], while in the case of D_{k+2} models our analysis will be completely new. Our results will further strengthen the evidence for duality in A_k models but the tests of duality will fail in D_{k+2} models even when k is odd.

The outline of the paper is as follows. In Sec. II, we review the conjectured dualities involving adjoints. In Sec. III we discuss the tests of duality in A_k and D_{k+2} models. After reviewing the existing duality tests in Sec. III A we discuss the dynamics along nilpotent flat directions in Sec. III. We then test the duality of A_k models in Sec. III C, finding additional evidence in support of duality conjecture. Finally, in Sec. III D, we show that similar analysis of D_{k+2} models does not yield consistent results. We close with a discussion.

²When studying D_{k+2} models, we will limit our attention to $Y = 0$ directions for simplicity.

TABLE I. Matter content and global charges of the A_k electric theory.

Field	$SU(N)$	$SU(F)$	$SU(F)$	$U(1)_B$	$U(1)_R$
Q	$\square$	$\square$	1	1	$1 - \frac{2}{k+1} \frac{N}{F}$
$\bar{Q}$	$\bar{\square}$	1	$\bar{\square}$	-1	$1 - \frac{2}{k+1} \frac{N}{F}$
X	Adj	1	1	0	$\frac{2}{k+1}$

II. REVIEW OF A_k AND D_{k+2} DUALITIES

A. A_k dualities

The electric description of A_k models [2] is given in terms of an $SU(N)$ gauge group with an adjoint (Adj) superfield X and F flavors in the fundamental representation. The matter content and symmetries of the model are given by Table I. The $U(1)_R$ charges were chosen so that the following superpotential is allowed by symmetries:

$$W_{el} = \text{Tr}(X^{k+1}) + \lambda \text{Tr} X, \quad (3)$$

where we have included a Lagrange multiplier term $\lambda \text{Tr} X$ in the superpotential to enforce tracelessness of the adjoint. The equation of motion ensures that X is nilpotent up to an identity operator

$$X^k = \lambda \mathbb{I}. \quad (4)$$

We will be interested in the moduli space directions consistent with $\lambda = 0$ and therefore we will drop the Lagrange multiplier terms in the following.

As was argued in [4], this equation truncates the chiral ring of the theory and deforms the quantum moduli space. The low energy physics can then be described in terms of “dressed” chiral meson and baryon operators that belong to the truncated chiral ring. In particular, the meson operators in the chiral ring are of the form

$$M_m = \bar{Q} X^{m-1} Q, \quad m = 1, 2, \dots, k. \quad (5)$$

The duality maps dressed mesons to the elementary mesons in the magnetic theory.³ The magnetic theory is described in terms of an $SU(kF - N)$ gauge theory with matter content provided in Table II and the superpotential

$$W_{\text{mag}} = \text{Tr}(\tilde{X}^{k+1}) + \sum_{m=1}^k M_m \tilde{q} \tilde{X}^{k-m} q. \quad (6)$$

³The dressed baryons are mapped to dressed baryons of the magnetic description, but we will not write them down explicitly since we will not turn them on in our discussion.

TABLE II. Matter content and global charges of the A_k magnetic theory.

Field	$SU(kF - N)$	$SU(F)$	$SU(F)$	$U(1)_B$	$U(1)_R$
q	$\square$	$\overline{\square}$	1	$\frac{N}{kF-N}$	$1 - \frac{2}{k+1} \frac{kF-N}{F}$
$\bar{q}$	$\overline{\square}$	1	$\square$	$-\frac{N}{kF-N}$	$1 - \frac{2}{k+1} \frac{kF-N}{F}$
$\tilde{X}$	Adj	1	1	0	$\frac{2}{k+1}$
M_m	1	$\square$	$\overline{\square}$	0	$2 - \frac{4}{k+1} \frac{N}{F} + \frac{2}{k+1} (m-1)$

TABLE III. Matter content and global charges of the D_{k+2} electric theory.

Field	$SU(N)$	$SU(F)$	$SU(F)$	$U(1)_B$	$U(1)_R$
Q	$\square$	$\square$	1	1	$1 - \frac{N}{F(k+1)}$
$\bar{Q}$	$\overline{\square}$	1	$\overline{\square}$	-1	$1 - \frac{N}{F(k+1)}$
X	Adj	1	1	0	$\frac{2}{k+1}$
Y	Adj	1	1	0	$\frac{k}{k+1}$

B. D_{k+2} dualities

The D_{k+2} model [5] can be obtained by extending the A_k model with a second adjoint superfield Y and a new superpotential term. The electric description is based on an $SU(N)$ gauge group with matter content given by Table III and the superpotential⁴

$$W = \text{Tr}(X^{k+1}) + \text{Tr}(XY^2). \quad (7)$$

The equations of motion are given by

$$(k+1)X^k + Y^2 = 0, \quad (8)$$

$$XY + YX = 0.$$

Clearly, in this model X is only nilpotent if Y is. On the other hand, we can see from (8) that Y satisfies the equation

$$Y^3 = (-1)^k Y^3, \quad (9)$$

implying that it is nilpotent for odd k while remaining unconstrained for even k . Thus, the chiral ring is only truncated for odd values of k . It has been conjectured in the literature that for even k the chiral ring is truncated by nonperturbative dynamics, however, no such dynamical effects have been found so far [6].

Assuming the truncation of the chiral ring, the mesonic directions of the quantum moduli space of the electric theory are parametrized by

⁴As mentioned earlier, we have dropped Lagrange multiplier terms for brevity.

$$M_{m,l} = \bar{Q} X^{m-1} Y^{l-1} Q, \quad m = 1, 2, \dots, k; \quad l = 1, 2, 3. \quad (10)$$

As usual, the composite mesons map into the elementary mesons of the magnetic description. The dual theory has gauge group $SU(3kF - N)$ and matter content is given by Table IV. The superpotential of the magnetic dual is given by

$$W = \text{Tr} \tilde{X}^{k+1} + \text{Tr} \tilde{X} \tilde{Y}^2 + \sum_{m=1}^k \sum_{l=1}^3 M_{m,l} \tilde{q} \tilde{X}^{k-m} \tilde{Y}^{3-l} q. \quad (11)$$

III. NEW TESTS OF DUALITY CONJECTURES

A. Existing tests of the dualities

The duality conjectures in A_k and D_{k+2} theories have been tested in a variety of ways. The global symmetries and anomalies of the electric and magnetic description agree. The 't Hooft anomalies of the two theories have been shown to match (notably, for the D_{k+2} theories, this agreement occurs for both odd and even k). Certain deformations of the superpotential, e.g., by mass terms, have also been considered, and the duality conjecture continues to hold.

A different set of tests considers the dynamics along certain directions on the moduli space. One such example is the analysis of [10]. These authors considered the dynamics of an A_2 model along the Higgs branch parametrized by the meson $M_1 \sim Q\bar{Q}$. Choosing the rank of the meson matrix to be $F_M < N - 1$, one finds the electric theory gauge group is broken to $SU(N - F_M)$. The adjoint X decomposes into an adjoint X' , F_M fundamental flavors Q' , $\bar{Q}'$ and gauge singlets. Since F_M fundamental flavors are eaten by the super-Higgs mechanism, the number of light fundamentals is still F . On the other hand, the global symmetry is broken to $SU(F - F_M)$ due to a superpotential term

$$\Delta W_{el} = \sum_f^{F_M} \bar{Q}'^f X' Q'_f, \quad (12)$$

which arises from the X^3 superpotential of the UV theory.

A naive duality transformation suggests that on the Higgs branch the magnetic dual is described in terms of an $SU(2F - (N - F))$ theory, i.e., it the gauge group of this dual is larger than the $SU(2F - N)$ magnetic group in the UV. The resolution of this apparent puzzle arises from

TABLE IV. Matter content and global charges of the D_{k+2} magnetic theory.

Field	$SU(3kF - N)$	$SU(F)$	$SU(F)$	$U(1)_B$	$U(1)_R$
q	$\square$	$\overline{\square}$	1	$\frac{N}{3kF-N}$	$1 - \frac{3kF-N}{F(k+1)}$
$\bar{q}$	$\overline{\square}$	1	$\square$	$-\frac{N}{3kF-N}$	$1 - \frac{3kF-N}{F(k+1)}$
$\tilde{X}$	Adj	1	1	0	$\frac{2}{k+1}$
$\tilde{Y}$	Adj	1	1	0	$\frac{k}{k+1}$
$M_{m,l}$	1	$\square$	$\overline{\square}$	0	$2 - \frac{2}{k+1}\frac{N}{F} + \frac{2(m-1)+k(l-1)}{k+1}$

the observation that the superpotential deformation (12) leads to tadpoles in the superpotential of the magnetic theory, which, in turn, break the $SU(2N - (F - N))$ group down to the expected $SU(2F - N)$.

A more general analysis of ADE models was performed in [6,7]. While the authors found that the duality conjectures worked in most cases, the duality conjecture failed along some flat directions in even- k D_{k+2} models. The authors of [6] performed an extensive analysis of the dynamics but did not find nonperturbative effects that would lift the dangerous flat directions and restore the duality conjecture. Going beyond D_{k+2} models, it was shown in [6] that similar puzzles arise in E_7 superconformal field theories where different renormalization group (RG) flow trajectories resulted in a different number of higher-dimensional representations in the IR.

B. Dynamics of nilpotent flat directions

Despite a large body of work, the dynamics along the flat directions of ADE models has not been studied exhaustively. While a complete analysis of the moduli space is beyond the scope of our work, we will here consider a large new class of flat directions, the nilpotent flat directions along which $X \neq 0$ while $X^k = 0$. These flat directions are parametrized by mesons of the form $M_m \sim \bar{Q}X^{m-1}Q$. While some of these flat directions were considered in [10] in the case of A_k models, they have not been studied in D_{k+2} models.

When we move along a nilpotent direction, we can analyze the RG flow into the IR in two different ways. In the first approach, we will start from the magnetic dual theory in the UV. The moduli space of this theory consists of the Higgs and Coulomb branches, parametrized by dressed mesons and baryons of the magnetic theory, as well as a mesonic branch parametrized by gauge singlet mesons M_m . Moving along along our chosen nilpotent flat direction on the electric side corresponds to turning on the VEVs of these mesons and moving onto “the meson branch of the magnetic theory.” The magnetic gauge group remains unbroken along the meson branch but the magnetic superpotential is modified.

In the second approach, we start from the electric theory and move along the nilpotent flat direction, thus partially breaking the electric gauge group. Integrating out the heavy

degrees of freedom, we find an effective description in terms of an A_k or D_{k+2} theory with a smaller gauge group and a deformed superpotential. We will refer to this theory as the “electric theory on the Higgs branch.” At this point we can perform the duality transformation, thus constructing a “magnetic dual of the Higgs branch theory.” Somewhat surprisingly, the gauge group of this magnetic dual is larger than the magnetic gauge group found in the first approach. This is not necessarily a failure of the duality. Indeed, we shall see that the superpotential deformation of the Higgs branch electric theory implies a spontaneous symmetry breaking in its magnetic dual, leading to a possible agreement with the analysis employed in our first approach. If the duality works, the RG flow may be graphically represented as in Fig. 1.

Indeed, when studying the A_k models, we will show that the theories follow the flows indicated by this diagram, further confirming the validity of the duality conjecture. On the other hand, we will argue that in D_{k+2} models the RG flow diagram cannot close for any value of k , thus invalidating the duality conjecture in this class of models.

To be more specific, we note that the UV magnetic gauge group in the UV remains unbroken along the mesonic branch. On the other hand, we will find complementary pairs of nilpotent directions of the electric theory along which the effective description will be given in terms of D_{k+2} models with different gauge groups yet the same superpotential deformations. Since, in the magnetic description, the superpotential deformations appear as symmetry breaking tadpoles, this precludes the possibility

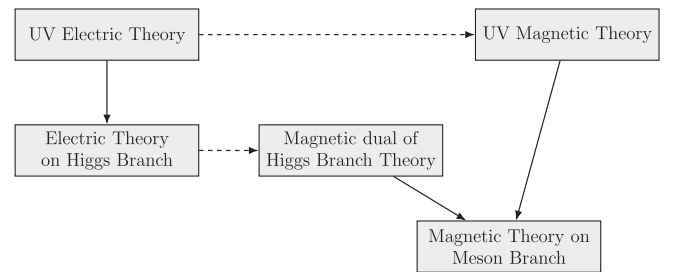


FIG. 1. Flows between electric and magnetic theories under Higgsing and meson-branch deformations. The horizontal dashed lines indicate duality maps, and the vertical or diagonal lines represent flows from the UV theories to the IR theories.

that the duals of the complementary nilpotent directions can flow to the same gauge group of the UV magnetic dual on the meson branch. It is interesting that the failure mode will be similar to the one found in [6]—this time different RG trajectories will lead to IR physics with the same superpotential but different magnetic gauge groups.

C. Duality tests for A_k theory

1. The electric theory on the Higgs branch

We consider a flat direction parametrized by a VEV of a single meson $M_n = \bar{Q}^F X^{n-1} Q_F$, $1 < n \leq k$. In terms of the elementary fields (and up to gauge and global symmetry transformations), this flat direction is given by

$$X_2^1 = X_3^2 = \dots = X_n^{n-1} = Q_F^n = \bar{Q}_1^F = v, \quad (13)$$

where the indices on the fundamental and antifundamental are color indices. All other components of the fields are 0. It is easy to see that the adjoint is nilpotent along this direction, $X^n = 0$.

Along this direction the unbroken gauge group is $SU(N - n)$ and the adjoint X decomposes into an adjoint X' , n fundamental flavors, and several gauge singlets. At the same time, n fundamental flavors are eaten by the super-Higgs mechanism. Thus, the low energy $SU(N - n)$ theory still has F light flavors, one of which is still coupled in the superpotential.

The light degrees of freedom and their charges are given by Table V, where we omitted gauge singlet fields with the R charge less than $2/3$ since they saturate the unitarity bound and become free, decoupling from the interacting physics at the IR fixed point [1,11–13]. The electric superpotential of the low energy theory takes the form

$$\text{Tr}(X^{k+1}) = \text{Tr}((X')^{k+1}) + \bar{x}(X')^{k-n}x + \dots, \quad (14)$$

where x and $\bar{x}$ represent the light linear combination of the moduli left uneaten by the super-Higgs mechanism.

We note that one can interpret this effective theory as a deformation of an A_k model with $N' = N - n$ colors and F flavors by a superpotential term involving just one

fundamental flavor. The deformation breaks the global symmetry to $SU(F - 1)$.

2. The magnetic theory on the meson branch

The magnetic dual of the A_k theory along the nilpotent direction of the moduli space can be constructed in two ways. First, we note that the $(M_n)_F^F$ direction on the Higgs branch of the electric theory corresponds to a gauge singlet direction of the magnetic theory (we will refer to it as a mesonic direction). Below the scale of the meson VEV, the physics is still described in terms of the $SU(kF - N)$ magnetic gauge group but now with the superpotential

$$W_{\text{mag}} = \text{Tr}(\tilde{X}^{k+1}) + \sum_m M_m \bar{q} \tilde{X}^{k-m} q + \bar{q}^F \tilde{X}^{k-n} q_F, \quad (15)$$

although one should remember that the Yukawa coupling in the last term is given by a VEV of a massless field.

3. The dual of the electric theory on the Higgs branch

We can now perform the duality transformation on the electric theory on the meson branch. Following [10] we find that this magnetic theory has an $SU(kF - N + n)$ gauge group, matter content given by Table VI, and the superpotential

$$W'_{\text{mag}} = \text{Tr}(\tilde{X}^{k+1}) + \sum_m M'_m \bar{q} \tilde{X}^{k-m} q + (M'_{k-n})_F^F. \quad (16)$$

To fully analyze the vacuum structure of the deformed magnetic dual, one must observe a presence of a tadpole for the M'_{k-n} meson in the superpotential. This tadpole drives the VEVs of the composite $\bar{q} \tilde{X}^n q$ meson to nonzero value

$$0 \neq \bar{q} \tilde{X}^{k-(k-n)} q = \bar{q} \tilde{X}^n q. \quad (17)$$

At the minimum of the superpotential, the unbroken gauge group is $SU(kF - N)$ in an agreement with the gauge group of the magnetic theory on the meson branch. Moreover, integrating out the heavy fields, one arrives at the superpotential (15).

We therefore find that the duality conjectures work along these directions in the moduli space of the A_k theory.

D. Duality tests for D_{k+2} theory

We will now proceed in the same manner for the D_{k+2} theory. However, to analyze the D_{k+2} theory we will consider two related flat directions, parametrized by a pair of dressed mesons $M_{\tilde{n},1} = \bar{Q}^F X^{\tilde{n}-1} Q_F$, where the first meson has $\tilde{n} = n$ and the second meson has $\tilde{n} = k - n$. It is clear that the gauge symmetry breaking pattern along these flat directions can be inferred from the analysis of the A_k models. Thus, we can build on the results obtained in Sec. III C 1.

TABLE V. Matter content and global charges of the A_k IR electric theory.

Field	$SU(N - n)$	$SU(F - 1)$	$SU(F - 1)$	$U(1)_B$	$U(1)_R$
Q'	$\square$	$\square$	1	1	$\frac{F(k+1)+n-2N-1}{(F-1)(k+1)}$
$\bar{Q}'$	$\bar{\square}$	1	$\bar{\square}$	-1	$\frac{F(k+1)+n-2N-1}{(F-1)(k+1)}$
x	$\square$	1	1	1	$\frac{n+1}{k+1}$
$\bar{x}$	$\bar{\square}$	1	1	-1	$\frac{n+1}{k+1}$
X'	Adj	1	1	0	$\frac{2}{k+1}$

TABLE VI. Matter content and global charges of the A_k dual of the electric theory on the Higgs branch.

Field	$SU(kF - N + n)$	$SU(F - 1)$	$SU(F - 1)$	$U(1)_B$	$U(1)_R$
q	$\square$	$\overline{\square}$	1	$\frac{N-n}{kF-N+n}$	$\frac{F(1-k)+2N-n}{(F-1)(k+1)}$
$\bar{q}$	$\overline{\square}$	1	$\square$	$-\frac{N-n}{kF-N+n}$	$\frac{F(1-k)+2N-n}{(F-1)(k+1)}$
q_F	$\square$	1	1	$\frac{N-n}{kF-N+n}$	$-\frac{n}{k+1}$
$\bar{q}^F$	$\overline{\square}$	1	1	$-\frac{N-n}{kF-N+n}$	$-\frac{n}{k+1}$
$\tilde{X}$	Adj	1	1	0	$\frac{2}{k+1}$
$(M'_m)^{f'}$	1	$\square$	$\overline{\square}$	0	$\frac{2(m(F-1)+Fk+n-2N-1)}{(F-1)(k+1)}$
$(M'_m)^f_F$	1	1	$\overline{\square}$	0	$\frac{F(2m+k+n+1)-2(m+N+1)}{(F-1)(k+1)}$
$(M'_m)^f_f$	1	$\square$	1	0	$\frac{F(2m+k+n+1)-2(m+N+1)}{(F-1)(k+1)}$
$(M'_m)^F_F$	1	1	1	0	$\frac{2(m+n+1)}{k+1}$
$(M'_{k-n})^F_F$	1	1	1	0	2

1. The electric theory on the Higgs branch

Along the $M_{\tilde{n},1}$ direction of the Higgs branch, the $SU(N)$ gauge group is broken to the $SU(N - \tilde{n})$ subgroup. Each of the adjoints, X and Y , decomposes into an adjoint (X' and Y' , respectively), $\tilde{n}$ new fields in the fundamental representation ($x, \bar{x}$, and $y, \bar{y}$, respectively), and several gauge singlet fields. The low energy superpotential takes the form

$$W = \text{Tr}((X')^{k+1}) + \text{Tr}(X'(Y')^2) + \bar{x}(X')^{k-\tilde{n}}x + \bar{y}_1 Y' x + \bar{x} Y' y_{\tilde{n}} + \sum_{f=1}^{\tilde{n}} \bar{y}_f X' y_f + \sum_{f=1}^{\tilde{n}-1} \bar{y}_{f+1} y_f + \dots, \quad (18)$$

where $x, \bar{x}, y_f$, and $\bar{y}_f$ represent the $2\tilde{n}$ new flavors in the fundamental representation. It is easy to see that this superpotential gives mass to $\tilde{n} - 1$ flavors, while $\tilde{n}$ flavors are eaten by the super-Higgs mechanism. After integrating out the massive fields, the superpotential becomes

$$W = \text{Tr}((X')^{k+1}) + \text{Tr}(X'(Y')^2) + \bar{x} X'^{k-\tilde{n}} x + \bar{y} Y' x + \bar{x} Y' y + \bar{y}(X')^{\tilde{n}} y, \quad (19)$$

TABLE VII. Matter content and global charges of the D_{k+2} IR electric theory.

Field	$SU(N - n)$	$SU(F - 1)$	$SU(F - 1)$	$U(1)_B$	$U(1)_R$
Q'	$\square$	$\square$	1	1	$\frac{F(k+1)-N-1+n}{(F-1)(k+1)}$
$\bar{Q}'$	$\overline{\square}$	1	$\overline{\square}$	-1	$\frac{F(k+1)-N-1+n}{(F-1)(k+1)}$
x	$\square$	1	1	1	$\frac{n+1}{k+1}$
$\bar{x}$	$\overline{\square}$	1	1	-1	$\frac{n+1}{k+1}$
y	$\square$	1	1	1	$1 - \frac{n}{k+1}$
$\bar{y}$	$\overline{\square}$	1	1	-1	$1 - \frac{n}{k+1}$
X'	Adj	1	1	0	$\frac{2}{k+1}$
Y'	Adj	1	1	0	$\frac{k}{k+1}$

where $\bar{y} = \bar{y}_1$ and $y = y_n$, while the matter content of the theory is summarized in Table VII.

2. The magnetic theory on the meson branch

Starting in the UV, the magnetic dual of the D_{k+2} model is given in terms of an $SU(3kF - N)$ gauge group discussed in Sec. II B. The nilpotent flat directions of the Higgs branch considered in the previous subsection correspond to the mesonic branch of the magnetic description. While the gauge group of the magnetic description remains unbroken, the meson VEV converts one of the superpotential terms into a tadpole for a dressed magnetic meson

$$M_{\tilde{n},1} \bar{q} \tilde{X}^{k-\tilde{n}} \tilde{Y}^2 q \rightarrow \tilde{M}_{k-n,3}. \quad (20)$$

This tadpole drives the magnetic theory to a new IR fixed point.

3. The dual of the electric theory on the Higgs branch

As in the case of the A_k models, the analysis is more involved if we construct a dual of the low energy $SU(N - \tilde{n})$ electric description. The magnetic theory has an $SU(N')$ gauge group with $N' = 3k(F + 1) - N + \tilde{n}$. The matter content consists of two adjoint supermultiplets, $F + 1$ flavors in the fundamental representation, and a set of elementary mesons. However, the global non-Abelian symmetry under which the fundamentals and mesons transform is only $SU(F - 1)_L \times SU(F - 1)_R$. The superpotential of the dual description is given by a sum of the usual magnetic superpotential (11) of a D_{k+2} and the deformation superpotential

$$\Delta W'_{\text{mag}} \sim (M'_{k-\tilde{n},1})_F^F + (M'_{1,2})_{F+1}^F + (M'_{1,2})_F^{F+1} + (M'_{\tilde{n},1})_{F+1}^{F+1}. \quad (21)$$

4. The failure of the duality map

Let us now compare the duals of the electric theories on the Higgs branch for the cases $\tilde{n} = n$ and $\tilde{n} = k - n$. It is easy to see that the two theories have the same unbroken global symmetry and the same superpotential (21). On the other hand, they have different gauge groups [specifically $SU(3k(F+1) - N + n)$ and $SU(3k(F+1) - N + k - n)$]. In both cases, the number of colors in this dual description is greater than the number of colors in the UV magnetic theory $N' > 3kF - N$.

Just as in the case of A_k models, the gauge group of these magnetic theories has to be broken down to $SU(3kF - N)$ if duality is to work. Indeed, the tadpoles (21) induce VEVs for four dressed magnetic mesons,

$$\begin{aligned} (M'_{k-n,1})_F^F &\rightarrow \bar{q}_F \tilde{X}^n \tilde{Y}^2 q_F \neq 0, \\ (M'_{1,2})_F^{F+1} &\rightarrow \bar{q}_F \tilde{X}^{k-1} \tilde{Y} q_{F+1} \neq 0, \\ (M'_{1,2})_{F+1}^F &\rightarrow \bar{q}_{F+1} \tilde{X}^{k-1} \tilde{Y} q_F \neq 0, \\ (M'_{n,1})_{F+1}^{F+1} &\rightarrow \bar{q}_{F+1} \tilde{X}^{k-n} \tilde{Y}^2 q_{F+1} \neq 0, \end{aligned} \quad (22)$$

partially breaking the magnetic $SU(N')$ group. However, our choice of $\tilde{n}$ ensures that these VEVs are invariant under the change $n \rightarrow k - n$. It is clear that the same VEVs cannot break both $SU(3k(F+1) - N + n)$ and $SU(3k(F+1) - N + k - n)$ down to $SU(3kF - N)$. In other words, the duality must fail for at least one of these nilpotent directions.

IV. DISCUSSION

Duality conjectures in $\mathcal{N} = 1$ supersymmetric theories have been tested in a variety of ways, including matching of symmetries and 't Hooft anomalies. We have analyzed here a new set of tests, which involve analysis of dynamics along nilpotent directions on the moduli space. The construction of the duality map along such flat directions leads to two potential magnetic duals that must be equivalent in the infrared.

We have applied these tests to $\mathcal{N} = 1$ duality with adjoint superfields and nontrivial superpotentials. We have found that models with A_k superpotential theory indeed pass this test, lending further support to the duality conjecture. On the other hand, the models with D_{k+2} theories with two adjoints fail the test, suggesting that the conjecture is invalid. While previous authors had already pointed out difficulties on the duality conjecture for even k , we emphasize that our tests show that the conjecture fails for both k odd and k even.

There are several further directions to pursue. These tests can be applied to other theories with superpotentials, as for example, theories with symmetric and antisymmetric tensors [14,15]. It would also be interesting to study a maximization in these theories. We leave these and other questions for future work.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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