

Characterizing the merger of equatorial-eccentric-geodesic plunges in rotating black holes

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We study the gravitational waveforms generated by critical, equatorial plunging geodesics of the Kerr metric that start from an unstable-circular-orbit, which describe the test-mass limit of spin-aligned eccentric black hole mergers. The waveforms are generated employing a time-domain Teukolsky code. We span different values of the Kerr spin $-0.99 \leq a \leq 0.99$ and of the critical eccentricity e_c , for bound ($0 \leq e_c < 1$) and unbound plunges ($e_c \geq 1$). We find that, contrary to expectations, the waveform modes $h_{\ell m}$ do not always manifest a peak for high eccentricities or spins. In case of the dominant h_{22} mode, we determine the precise region of the parameter space in which its peak exists. In this region, we provide a characterization of the merger quantities of the h_{22} mode and of the higher-order modes, providing the merger structure of the equatorial-eccentric plunges of the Kerr spacetime in the test-mass limit.

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I. INTRODUCTION

The first detection of gravitational waves (GWs) [1] made by the LIGO-Virgo Collaboration [2–5] in 2015 represented a breakthrough in GW astronomy. Since this event, the LIGO-Virgo-KAGRA Collaboration [6,7] detected about a hundred signals from the coalescences of binary compact objects, like stellar-origin black holes (BHs) and neutron stars [8] and is expected to add more than ~ 200 signals by the end of the ongoing fourth observing run.

Upcoming observational runs [6] as well as the forthcoming next-generation experiments, such as the Einstein Telescope [9,10], Cosmic Explorer [11], and Laser Interferometer Space Antenna [12,13] will be characterized by higher sensitivity and by a number of detections increased by a factor $\sim 10^3$ and will be able to probe binary populations in lower frequency bands and for smaller mass ratios, which can exhibit larger eccentricities.

Eccentricity plays a crucial role in understanding BH binary (BBH) formation channels [14–17] such as dynamical interactions in dense stellar environments [18,19] and dynamical capture mechanisms [17,20–23]. Moreover, it has been shown that the effects of eccentricity in GWs cannot be ignored, since even when the eccentricities are small (e.g., $e \sim 0.1$ at 20 Hz [24]) systematic biases arise in the extraction of source properties in inference studies [24–31] and tests of general relativity [32–38]. These facts motivate the development of fast and accurate waveform models that incorporate eccentricity as an inference parameter.

A common approach in GW modeling is to employ results from numerical-relativity (NR) simulations to construct models by interpolating NR data to either fully describe the waveform [39–42] or to model specific portions of it, in particular, the merger and ringdown parts, across the parameter space. This procedure allows one to incorporate the complex dynamics of binary systems, particularly in regions where analytical methods are less effective. Other approaches blend NR and BH perturbation theory data [43–45] to develop models that can handle a wide range of mass ratios.

While the quasicircular (QC) inspiral-merger-ringdown morphology of the waveforms has been extensively studied and modeled in the last decades, a strong effort is currently

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ongoing to characterize and model the features of the gravitational waveforms emitted by eccentric BBHs. In particular, an extension from QC to eccentric merger and postmerger parts of the waveforms are of interest for models that combine analytical and numerical methods to model the entire GW signals. Among them, the models based on the effective-one-body (EOB) formalism [46,47] that stand out for their ability to combine analytical approximation methods with NR results, achieving both high accuracy and computational efficiency for the QC systems [48–60] and, recently, for eccentric systems [61–82]. These models currently incorporate eccentricity only in the inspiral part and employ QC merger-ringdown models.

In this family of models, specifically for the QC case, the ringdown portion of the waveform is modeled by fitting an ansatz to the NR ringdown and then by performing a hierarchical fit of the resulting coefficients as functions of the system’s intrinsic parameters, like the symmetric mass ratio and the spins of the BBH. To accurately model the final part of the waveform, the ringdown must be properly connected to the inspiral segment, which is computed using resummed post-Newtonian (PN) expressions of the waveform modes [83]. This attachment procedure is performed by applying continuity conditions around the merger part of the waveform in order to ensure continuity between the inspiral waveform modes and the ringdown model fitted from NR data.

For spinning BBHs with small mass ratios, the merger properties can be studied using BH perturbation theory (BHPT) results. This is done by considering waveforms generated by solving the Teukolsky equation [84] for a test mass (TM) orbiting a Kerr BH. Results from the TM limit have already been applied to inform and improve the accuracy of waveform models on the entire mass ratio regimes, especially for the family of EOB models [48,68,71,74,83,85–96]. Among these works, Refs. [68,71,74,86–88] showed the importance of modeling the merger and postmerger stages of the waveforms in the TM limit. In particular, in the QC case, Refs. [86,88] studied the merger and postmerger properties of waveforms generated with equatorial QC trajectories of a small mass orbiting a Kerr BH. These studies enriched the merger-ringdown parts of EOB models in the small mass ratio regimes.

To further improve eccentric waveform models, recent works started extending the characterization and modeling of the merger-ringdown features of eccentric systems both in the comparable and small mass ratio regimes. Among these studies, we mention Refs. [97,98], which characterized the merger properties of eccentric waveforms and the ringdown amplitudes in the comparable mass case using eccentric NR waveforms of the Rochester Institute of Technology catalog [99] and of the SXS Collaboration [39,100–110], together with numerical results from BHPT [74]. More recently, Ref. [111] investigated the impact of the relativistic anomaly on the merger features for the

comparable mass regime. Effort has also been pursued in the small mass ratio regime. In particular, Ref. [74] developed a nonspinning EOB ringdown model in the TM limit that takes into account residual eccentricity at merger. This work uses EOB trajectories to source the Teukolsky equation and generate the waveforms. A more recent study [112] investigated the merger properties of the waveforms generating the trajectories with numerical fluxes computed through the frequency-domain Teukolsky code [113].

In these EOB models, the standard procedure has been to identify the merger with the peak of the gravitational dominant mode. More specifically, the time at which the inspiral-plunge waveform is attached to the merger-ringdown one is identified with the mode peak’s time, both in the comparable and small mass ratio regimes [54,59,60] (see also Refs. [114–119]). However, it has been shown that for QC small mass ratio scenarios with high prograde spin of the central Kerr BH (e.g., $a \geq 0.95$), these peaks are not resolvable [88] or, in the near-extremal Kerr scenarios, they happen during the inspiral part of the waveform, hence many cycles before the plunge and merger part [120]. These results show that the waveform modes’ peak is not necessarily associated with the merger part of the GW’s signal in all the parameter space.

Inspired by these works, in this article, we characterize the late-stage properties of the waveforms of eccentric aligned-spin orbits of Kerr by introducing a model-independent methodology. Since we are interested in the TM limit of the final segment of these waveforms, we focus on the signal produced during the plunge phase and, thus, study waveforms associated with plunges and mergers in the TM limit. As the mass ratio vanishes, plunges become increasingly geodesic [47,121]. Recent works extended the features of the inspiral to plunge transition for small mass ratio eccentric systems [112,122]. In particular, Ref. [122] provided analytical expressions for the energy and the angular momentum of a TM orbiting a Schwarzschild BH for the eccentric transition to plunge, when the system is near the last stable orbit (LSO). Interestingly, these expressions show that the plunges approach a specific class of geodesics that start from an unstable circular orbit (UCO) in the infinite past. Analytical, closed-form expressions for this class of geodesic plunges have been derived in Refs. [123,124]. In this work, we use these analytical expressions to compute a set of this class of geodesic eccentric plunges for different values of the radius of the UCO from which they start and for different spins of the central Kerr BH. With these trajectories, we source a time-domain (TD) Teukolsky code [125–129] and produce the gravitational waveforms emitted by the trajectories. This allows us to analyze and characterize the properties at the merger in the parameter space of the TM eccentric aligned Kerr waveforms.

This article is structured as follows. In Sec. II, we describe the methodology of our work. In particular,

in Sec. II A, we introduce and describe the class of critical plunge geodesics we consider for generating the waveforms. Section II B describes how we generate the gravitational waveforms using the TD Teukolsky code. In Sec. II C, we study the existence of the peak of the (2,2) spherical harmonic mode of the strain h and provide an overview of the characterization of the waveforms. Section III is dedicated to the main results of our work. In particular, in Sec. III A, we provide a characterization of the quantities, commonly considered in GW models, evaluated at the peak of the amplitude of the (2,2) mode. In Sec. III B, we show similar results for the higher-order modes, focusing, in particular, on the (3,3) mode, and in Sec. III C we comment on the phenomenology of the peak of the amplitude of other two quantities related to the strain, the Bondi news, and the Weyl scalar. Finally, in Sec. IV, we discuss the main conclusions of our study. In Appendix A we further motivate restricting our attention to critical plunge geodesics, while in Appendix B, we show how the waveforms generated through these geodesics represent the TM limit of the generic eccentric plunge. Appendix C connects our work to the QC case studied in a previous study [88], while in Appendix D, we show how the critical plunge geodesics behave more circularly as their critical eccentricity increases. In Appendix E, we show the features at merger of the higher-order modes, and in Appendix F we list some of the values at merger of the quantities commonly considered in waveform modeling for the (2,2) mode.

A. Notation

In this work, we adopt geometric units $G = c = 1$ and consider a Kerr BH of mass M with dimensionless spin $a = J/M^2$. The Kerr metric is expressed in Boyer-Lindquist coordinates $\{\tilde{t}, \tilde{r}, \theta, \varphi\}$ and we consider scaled dimensionless variables $t = \tilde{t}/M$ and $r = \tilde{r}/M$. We restrict our analysis to the equatorial plane, $\theta = \pi/2$. For the ease of notation, we set the mass of the Kerr BH to $M = 1$.

II. METHODOLOGY

This section describes the methodology of our work. We first provide a review of the plunging geodesics that we consider to generate the waveforms and we describe how these waveforms are computed at future null infinity by sourcing the time-domain Teukolsky code with the plunging geodesics. We also describe how we characterize the merger part of the different waveforms.

A. Critical plunge geodesics

In the equatorial plane, the Kerr metric [130] expressed in dimensionless Boyer-Lindquist coordinates takes the form

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\varphi\varphi}d\varphi^2 + g_{t\varphi}dtd\varphi, \quad (1)$$

with

$$g_{tt} = -1 + \frac{2}{r}, \quad (2a)$$

$$g_{rr} = \frac{r^2}{\Delta}, \quad (2b)$$

$$g_{\varphi\varphi} = r^2 + a^2 + \frac{2a^2}{r}, \quad (2c)$$

$$g_{t\varphi} = -\frac{4a}{r}, \quad (2d)$$

and

$$\Delta = r^2 - 2r + a^2. \quad (3)$$

The evolution of equatorial geodesics in Kerr spacetime is commonly described by leveraging the existence of three constants of motion: the particle's mass μ , the orbital energy $\mathcal{E}$, and the orbital angular momentum $\mathcal{L}$. When considering these constants, the equatorial geodesics in Kerr spacetime satisfy the following equations [131]:

$$\left(\frac{dr}{d\lambda}\right)^2 = (\mathcal{E}(r^2 + a^2) - a\mathcal{L})^2 - \Delta(r^2 + (a\mathcal{E} - \mathcal{L})^2), \quad (4a)$$

$$\frac{dt}{d\lambda} = \frac{(r^2 + a^2)}{\Delta}(\mathcal{E}(r^2 + a^2)a\mathcal{L}) + a(\mathcal{L} - a\mathcal{E}), \quad (4b)$$

and

$$\frac{d\varphi}{d\lambda} = \frac{a}{\Delta}(\mathcal{E}(r^2 + a^2) - a\mathcal{L}) + \mathcal{L} - a\mathcal{E}, \quad (4c)$$

where λ is the Mino time variable, which is related to proper time τ through

$$\frac{d\tau}{d\lambda} = r^2. \quad (5)$$

The right-hand side of Eq. (4a) is a fourth-order polynomial in the variable r and can always be written in the form [132]

$$\left(\frac{dr}{d\lambda}\right)^2 = (1 - \mathcal{E}^2)(r_1 - r)(r_2 - r)(r_3 - r)r = R(r), \quad (6)$$

where r_1 , r_2 , and r_3 are roots of $R(r)$. In any situation where these roots of $R(r)$ are real and satisfy

$$\frac{1}{r_1} < \frac{1}{r_2} < \frac{1}{r_3} < \frac{1}{r_+}, \quad (7)$$

where $r_+ = 1 + \sqrt{1 - a^2}$ is the outer event horizon of the Kerr metric, then, there will be a geodesic solution satisfying

$$r_2 < r(\lambda) < r_1. \quad (8)$$

In these situations, the roots r_1 and r_2 are, respectively, named “apocenter” and “pericenter” and the geodesic solution can be described with the Keplerian parametrization in terms of the eccentricity e , the semilatus rectum p , and the relativistic anomaly ξ [133,134],

$$e = \frac{r_1 - r_2}{r_1 + r_2}, \quad p = \frac{2r_1 r_2}{r_1 + r_2}, \quad \cos \xi = \frac{p - r}{er}. \quad (9)$$

Note that when $r_1 > 0$ these geodesics describe bound orbits with $0 \leq e < 1$, and when $r_1 < 0$ they describe scattering orbits with $e > 1$.

In this work, we are interested in the study of the plunge-merger properties of GWs. As we will show in Sec. II B, we generate these waveforms by sourcing and numerically solving the Teukolsky equation in the time domain with plunge geodesics of the Kerr metric in the equatorial-eccentric case. While plunge geodesics [i.e., geodesics crossing the future (outer) event horizon at $r = r_+$] exist for any value of $\mathcal{E}$ and $\mathcal{L}$, in this work, we will be interested in a special class of orbits in which all four roots of $R(r)$ are real and satisfy

$$\frac{1}{r_1} < \frac{1}{r_2} = \frac{1}{r_3} < \frac{1}{r_+}. \quad (10)$$

When this is true, the radial equation allows three distinct type of solutions: (i) an UCO at $r = r_2 = r_3 = r_c$; (ii) so-called “homoclinic” orbits [135,136] approaching $r_c = r_2 = r_3$ from the outside in the infinite future and/or past; and (iii) critical plunge geodesics approaching $r_c = r_2 = r_3$ from the inside in the infinite past and crossing the future outer event horizon at $r = r_+$.¹ These solutions are of interest because they describe the TM limit of the transition from (bound) inspiral motion to plunging motion. Consequently, the locus of points in the configuration space of geodesics satisfying Eq. (10) is variably known as the separatrix (between bound and plunge orbits) or the LSO. In this work, we are particularly interested in the critical plunge geodesics since they describe the final part of the trajectory of an extreme mass ratio inspiral [47,121,122]. In Appendix A, we phenomenologically illustrate this.

The repeated root $r = r_2 = r_3$ in $R(r)$ guarantees that the equations for critical plunge geodesics can be solved explicitly in closed form in terms of elementary functions [123,124]. The critical plunge geodesics in a Kerr space-time with fixed spin a form a one parameter family. A possible choice is to parametrize them with the radius r_c of the associated UCO (henceforth, critical radius), which can take any value between the light ring radius [137]

$$r_{\text{LR}} = 2 + \cos\left(\frac{2}{3}\arcsin a\right) - \sqrt{3}\sin\left(\frac{2}{3}\arcsin a\right) \quad (11)$$

and the radius of the inner most stable circular orbit (ISCO) [137],

$$r_{\text{isco}} = 3 + Z_2 - a\sqrt{\frac{(3 - Z_1)(3 + Z_1 + Z_2)}{a^2}}, \quad (12a)$$

$$Z_1 = 1 + \sqrt[3]{1 - a^2}(\sqrt[3]{1 + a} + \sqrt[3]{1 - a}), \quad (12b)$$

$$Z_2 = \sqrt{3a^2 + Z_1^2}. \quad (12c)$$

That is,

$$r_{\text{LR}} < r_c \leq r_{\text{isco}}. \quad (13)$$

Keeping in mind that these plunge geodesics have the same constants of motion as the corresponding UCO [137], the energy and angular momentum of critical plunge geodesics can be expressed in terms of the parameter r_c ,

$$\mathcal{E}_c = \frac{(r_c - 2)\sqrt{r_c} + a}{\sqrt{(r_c - 3)r_c^2 + 2ar_c^{3/2}}}, \quad (14a)$$

$$\mathcal{L}_c = \frac{r_c^2 - 2a\sqrt{r_c} + a^2}{\sqrt{(r_c - 3)r_c^2 + 2ar_c^{3/2}}}. \quad (14b)$$

Instead of using r_c to parametrize the critical plunge geodesics, it can be more physically insightful to parametrize them using the eccentricity of the corresponding homoclinic orbit outside the UCO [135]. If we consider the definition of the eccentricity parameter e in Eq. (9) and we consider the critical limit in Eq. (10), for which $r_2 = r_3 = r_c$, we can define the critical eccentricity

$$e_c = \frac{r_1 - r_c}{r_1 + r_c}, \quad (15)$$

which corresponds to the eccentricity of the homoclinic orbit outside of the UCO.² Using the expressions for $\mathcal{E}_c$ and $\mathcal{L}_c$ in Eqs. (14a) and (14b), we can solve for the roots of Eq. (4a) to find

$$r_1 = \frac{2r_c(\sqrt{r_c} - a)^2}{(r_c - 4)r_c + 4a\sqrt{r_c} - a^2} \quad (16)$$

¹Technically, there also exists the time reverse of a critical plunge: a geodesic crossing the past event horizon and approaching $r_2 = r_3 = r_c$ in the infinite future.

²In Ref. [135], the quantity that we define as e_c corresponds to e^{hc} . In our work, we prefer to use the nomenclature e_c to connect with the fact that we consider the class of critical plunge geodesics rather than homoclinic orbits.

and get

$$e_c = \frac{r_c(6 - r_c) - 8a\sqrt{r_c} + 3a^2}{r_c(r_c - 2) + a^2}. \quad (17)$$

For critical radii r_c between the radius of the innermost bound circular orbit (IBCO) [137]

$$r_{\text{ibco}} = 2 - a + 2\sqrt{1 - a} \quad (18)$$

and r_{isco} , the homoclinic orbit outside the UCO is a bound orbit and e_c varies between 0 (at r_{isco}) and 1 (at r_{ibco}). For $r_{\text{LR}} < r < r_{\text{ibco}}$ it is a scattering orbit with $1 < e_c < 3$. In this work, we also consider configurations for which $e_c > 1$, as we will show in Sec. III. In the upper panel of Fig. 1, we show two examples of critical plunge geodesics on the equatorial plane of a Kerr BH with $a = 0$. The two geodesics have $e_c = 0.4$ (blue curve) and $e_c = 0.7$ (red curve). The bottom panel of Fig. 1 shows the radial potentials $V(r)$ of the geodesics. The radial potential $V(r)$ is defined by inverting Eq. (4a) for the energy $\mathcal{E}$ at the turning points, i.e., when $dr/d\lambda = 0$. Hence, it is $V(r) = \mathcal{E}(r, dr/d\lambda = 0, \mathcal{L}, a)$. In the bottom panel, we also show the energies of the geodesics (horizontal lines) that are the same as the respective energies of the UCOs. From this panel, it is possible to notice that the critical radius r_c decreases as e_c increases, as we will make explicit in Fig. 3 for a particular spin configuration.

To understand the behavior of the critical plunge geodesics near the critical radius r_c , we can combine Eqs. (4a) and (4b) with the expressions for $\mathcal{E}$ and $\mathcal{L}$ and expand around r_c to find

$$\left(\frac{d(r_c - r)}{d\lambda}\right)^2 = \Lambda_c^2(r_c - r)^2 + \mathcal{O}(r_c - r)^3, \quad (19)$$

with

$$\Lambda_c = \frac{\sqrt{r_c(6 - r_c) - 4a\sqrt{r_c} + 3a^2}}{r_c(r_c^{3/2} + a)}. \quad (20)$$

For a plunging solution, we should have $r < r_c$ and $d(r_c - r)/d\lambda > 0$; imposing this, we obtain the approximate solution

$$r_c - r \propto \exp(\Lambda_c t), \quad (21)$$

i.e., the solution asymptotically approaches r_c as the coordinate time $t \rightarrow -\infty$.

For any geodesic, the instantaneous angular velocity is given by

$$\Omega = \frac{d\varphi}{dt} = \frac{b(r - 2) + 2a}{r^3 + a^2(r + 2) - 2ab}, \quad (22)$$

where $b = \mathcal{L}/\mathcal{E}$ is the generalized impact parameter, which for critical plunge geodesics is given by

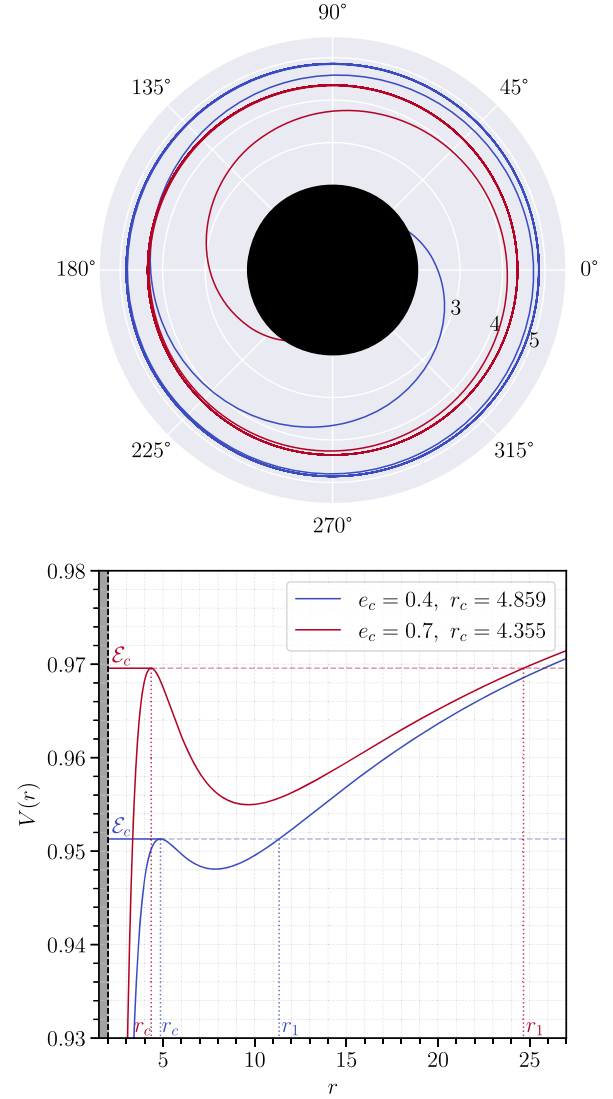


FIG. 1. Upper: we show two critical plunge geodesics for a Kerr BH with spin $a = 0$. This class of geodesics is characterized by the value of the critical eccentricity e_c , as described in Eq. (15). In the figure we show two geodesics with $e_c = \{0.4, 0.7\}$. Bottom: the radial potentials $V(r)$ of the two geodesics and their energies (horizontal lines), which correspond to the energies $\mathcal{E}_c = \{0.95130, 0.96957\}$ of their respective UCOs. We also mark the positions of the critical radii $r_c = \{4.859, 4.355\}$ and of the outer turning point $r_1 = \{11.3, 24.7\}$ with vertical dotted lines. The vertical gray area is the area inside the BH horizon.

$$b_c = \frac{r_c^2 - 2a\sqrt{r_c} + a^2}{(r_c - 2)\sqrt{r_c} + a}. \quad (23)$$

In the infinite past Ω asymptotes to the angular velocity of the UCO,

$$\Omega_c = \frac{1}{r_c^{3/2} + a}. \quad (24)$$

The angular velocity Ω will take its maximal value at

$$r_{\Omega_{\text{peak}}} = \left(1 - \frac{a}{b_c}\right) \left(1 + 2 \cos \psi\right), \quad \text{with} \quad (25)$$

$$\psi = \frac{1}{3} \arccos \left[1 - \frac{ab_c^2}{2(b_c - a)} \right]. \quad (26)$$

One can show that this is always smaller than the light ring radius r_{LR} and therefore always smaller than r_c ; critical plunge geodesics always achieve a maximal orbital frequency along the way down.

B. Waveforms computation

The class of plunge geodesics we consider in this article can be interpreted as the plunges of a small mass μ into a Kerr BH of mass M in the limit $\mu/M \rightarrow 0$. Hence, we can use BH perturbation theory as the framework to produce the gravitational waveforms sourced by critical plunge geodesics, numerically. We compute the waveforms by solving the Teukolsky master equation [84], which in Boyer-Lindquist coordinates reads

$$\begin{aligned} & - \left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] \partial_{tt} \Psi - \frac{4ar}{\Delta} \partial_{t\varphi} \Psi \\ & - 2s \left[r - \frac{(r^2 - a^2)}{\Delta} + ia \cos \theta \right] \partial_t \Psi + \Delta^{-s} \partial_r (\Delta^{s+1} \partial_r \Psi) \\ & + \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta \Psi) + \left[\frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right] \partial_{\varphi\varphi} \Psi \\ & + 2s \left[\frac{a(r-1)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta} \right] \partial_\varphi \Psi - (s^2 \cot^2 \theta - s) \Psi \\ & = -4\pi(r^2 + a^2 \cos^2 \theta) T. \end{aligned} \quad (27)$$

This equation describes the evolution of scalar, vector, and tensor perturbations of a Kerr BH. The parameter s is the “spin weight” of the field. In particular, when $s = \pm 2$ the equation describes radiative degrees of freedom, and for $s = -2$ it is $\Psi = (r - ia \cos \theta)^4 \psi_4$, where ψ_4 is the Weyl curvature scalar that describes outgoing GWs.

A system composed of a TM orbiting a Kerr BH is interpreted as a perturbed Kerr metric, and within this interpretation the source term T on the right-hand side of Eq. (27) describes a TM moving in the Kerr spacetime. The source term T of a TM orbiting a Kerr BH is constructed from Dirac- δ functions of the variables r and θ , as well as first and second derivatives of the δ functions in these variables. These terms are sourced at the location of the TM, hence the source T depends on the trajectory that the TM follows in the Kerr spacetime. Full details on construction of the TD source term can be found in Refs. [126,127]. In this analysis, the trajectories used to source the term T are the critical plunge geodesics introduced in Sec. II A.

To solve Eq. (27), we make use of the finite difference TD code developed in Refs. [125–129], which employs hyperboloidal time slicing to extract the GW modes at future null infinity. At future null infinity, the Weyl scalar ψ_4 and the waveform strain $h = h_+ - ih_\times$ are related by the expression

$$\psi_4 = \frac{1}{2} \frac{\partial^2 h}{\partial t^2} = \frac{1}{2} \left(\frac{\partial^2 h_+}{\partial t^2} - i \frac{\partial^2 h_\times}{\partial t^2} \right), \quad (28)$$

and following standard practices, the waveform strain is decomposed in -2 spin-weighted spherical harmonics

$$h = \sum_{\ell, m} {}_{-2}Y_{\ell m}(\theta, \varphi) h_{\ell m}, \quad (29)$$

where $h_{\ell m}$ are the (spherical harmonic) gravitational-waveform modes. More details on the numerical accuracy of the TD Teukolsky code can be found in Refs. [86,125–129].

C. Waveform characterization

In Fig. 2, we show the amplitudes of the (2,2) waveform mode generated by the TD Teukolsky code sourced by critical plunge geodesics with critical eccentricity in the interval $e_c = [0, 0.9]$. The upper panels show the mode’s amplitude, while the bottom panels show their first time derivatives. We show two different spin configurations of the central BH: in the left column, we consider the Schwarzschild case $a = 0$, while in the right column, we consider the spinning case with $a = 0.7$. We align the waveforms at the time of the peak of the orbital frequency $t_{\Omega_{\text{peak}}}$.

The amplitude of the h_{22} mode shows a clear dependence on e_c . Specifically, we observe that the value of the peak of the amplitude $|h_{22}^{\text{peak}}|$ increases monotonically, while, as we will also show in Fig. 6, the time t_{22}^{peak} of the peak amplitude decreases monotonically with e_c . This behavior was previously identified in Ref. [74] for the Schwarzschild case, and we observe the same trend here in the more general spinning scenario.

A further feature we identify is the dependence of the prominence of the peak of the amplitude with respect to e_c . In the QC case ($e_c = 0$), the prominence of the h_{22} amplitude peak depends on the spin a of the Kerr BH, as shown in Ref. [88]. In this work, we find that it also depends on e_c . In fact, already in the nonspinning case, it is possible to notice that as e_c increases, the peak of the amplitude is less and less prominent as depicted in the upper left panel of Fig. 2. When considering the prograde spinning case we observe that, up to a certain threshold value of the critical eccentricity, $e_c = e_c^{\text{thr}}$, it is possible to identify a peak of the amplitude; however, for values of $e_c \geq e_c^{\text{thr}}$, there is no discernable peak anymore.

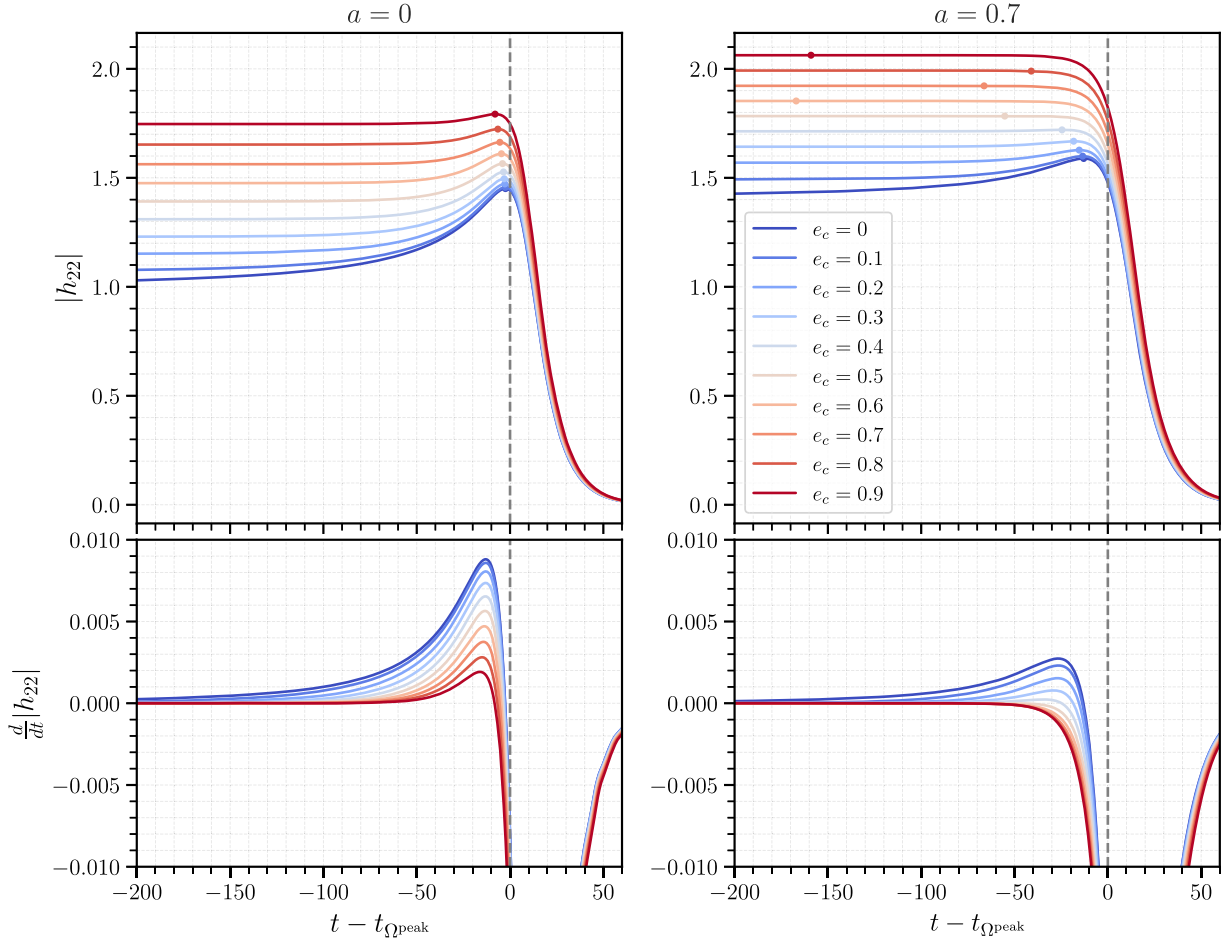


FIG. 2. In this figure we show the gravitational-waveform amplitudes of the h_{22} mode generated by equatorial critical plunge geodesics of a Kerr BH. The left column specializes to the nonspinning case ($a = 0$), while the right column to the case with spin $a = 0.7$. The upper panels show the amplitude of the mode while the bottom panels show its first time derivative. The different colors represent different critical eccentricities e_c , ranging from $e_c = 0$ to $e_c = 0.9$. The peak amplitude $|h_{22}^{\text{peak}}|$ and the time of peak amplitude t_{22}^{peak} exhibit a dependence on e_c . We find that in the case $a = 0.7$ the peak disappears starting from a certain threshold eccentricity $e_c^{\text{thr}} \approx 0.5$.

This transition—from regimes where it is possible to identify a peak of the amplitude to regimes where it is not possible anymore—is made more explicit in the bottom panels of Fig. 2, where we show the time derivative of the amplitude of the h_{22} mode. The quantity $d|h_{22}|/dt$ vanishes in the limit $t \rightarrow -\infty$ because the critical plunge geodesics asymptotically start back in time t from the UCO, as expressed in Eq. (21). When the peak of the amplitude is present, $d|h_{22}|/dt \geq 0$ back in the past and it increases until it reaches a maximum. From that moment, it starts to decrease, and it vanishes at the time of the peak t_{22}^{peak} . As e_c increases, the value of the maximum of $d|h_{22}|/dt$ decreases, eventually becoming equal to zero when $e_c = e_c^{\text{thr}}$. From this particular configuration, it is impossible to identify the peak of the amplitude of the h_{22} mode as it is no longer present.

Characterizing the peak amplitude of the h_{22} gravitational-waveform mode in the parameter space allows for precise modeling of the final stages of the waveform.

Specifically, quantities such as the time at which the amplitude peaks t_{22}^{peak} , the peak of the amplitude $|h_{22}^{\text{peak}}|$, its first $d|h_{22}|/dt|_{\text{peak}}$ and second time derivatives $d^2|h_{22}|/dt^2|_{\text{peak}}$ at the peak time, the mode frequency at the peak $\omega_{22}^{\text{peak}}$, and the frequency's time derivative at the peak $\dot{\omega}_{22}^{\text{peak}}$ are key inputs to accurately model the waveforms near the merger and to model the merger ringdown [54,58–60,86,88]. This characterization relies on the existence of a peak of the mode, however, as we summarized above, there are parts of the parameter space where no peak exists. As a first step, it is necessary to determine the part of the parameter space in which the amplitude of the h_{22} mode exhibits a peak, hence we are interested in studying e_c^{thr} for each spin a of the Kerr BH.

To determine the value of eccentricity $e_c^{\text{thr}}(a)$ from which the peak of the amplitude of the h_{22} mode vanishes, and its dependence on a , we employ the following method:

- (1) For every spin configuration, we first extract the quantity t_{22}^{peak} that corresponds to the time of the peak of the amplitude of the h_{22} mode.³
- (2) At this time t_{22}^{peak} , we extract the value of the radial coordinate $r_{22}^{\text{peak}} = r(t_{22}^{\text{peak}})$ from the trajectory that sources the Teukolsky code employed to generate the waveform.
- (3) Then, we compare r_{22}^{peak} with the value of the critical radius r_c introduced in Sec. II A. This quantity decreases monotonically with e_c , as depicted in Eq. (17) and as shown in the upper panel of Fig. 3, while we find that r_{22}^{peak} increases monotonically up to a certain value of eccentricity for which the value of r_{22}^{peak} reaches the value of the critical radius r_c , i.e., $r_{22}^{\text{peak}} = r_c$. When this happens, the amplitude of the h_{22} mode reaches its peak at the UCO. We identify the value of eccentricity for which $r_{22}^{\text{peak}} = r_c$ as the threshold critical eccentricity e_c^{thr} introduced above. This threshold value separates the regimes where the peak of the h_{22} mode exists to the regimes where it does not.
- (4) To compute the value of e_c^{thr} , we perform a fit of r_{22}^{peak} with respect to e_c in the regime for which it is increasing monotonic. By denoting the fit as $r_{\text{fit}}(e_c)$, we determine e_c^{thr} by computing the intersection of the fit with the curve of the critical radius

$$r_{\text{fit}}(e_c^{\text{thr}}) = r_c(e_c^{\text{thr}}). \quad (30)$$

The top panel of Fig. 3 summarizes this procedure: the red curve corresponds to the fit r_{fit} , while the black curve corresponds to r_c . The red dashed line is the extrapolation of r_{fit} performed in order to find the intersection with the r_c curve and to compute the threshold eccentricity as mentioned in Eq. (30). The dashed gray line of Fig. 3 corresponds to the threshold eccentricity e_c^{thr} .

To assess the robustness of the threshold critical eccentricity extraction method (henceforth referred to as method 1), we extract e_c^{thr} using an alternative approach (henceforth referred to as method 2), as shown in the bottom panel of Fig. 3. In this different method, we identify, for each value of a , the eccentric configurations of the trajectories for which a distinct peak in the amplitude of h_{22} exists. This is achieved by focusing on the t_{22}^{peak} values that exhibit a coherent pattern (i.e., a monotonic decreasing trend with respect to e_c), avoiding the values that show a scattered pattern due to the challenges the peak finder

³A priori, we identify t_{22}^{peak} for every waveform, even when the peak of the mode amplitude is not present and the peak finder algorithm fails, as it is shown in the upper right panel of Fig. 3 for $e_c > 0.4$. In these regimes, the algorithm cannot find the peak, and the nominal t_{22}^{peak} that is found does not exhibit the same regular pattern t_{22}^{peak} has when the peak exists.

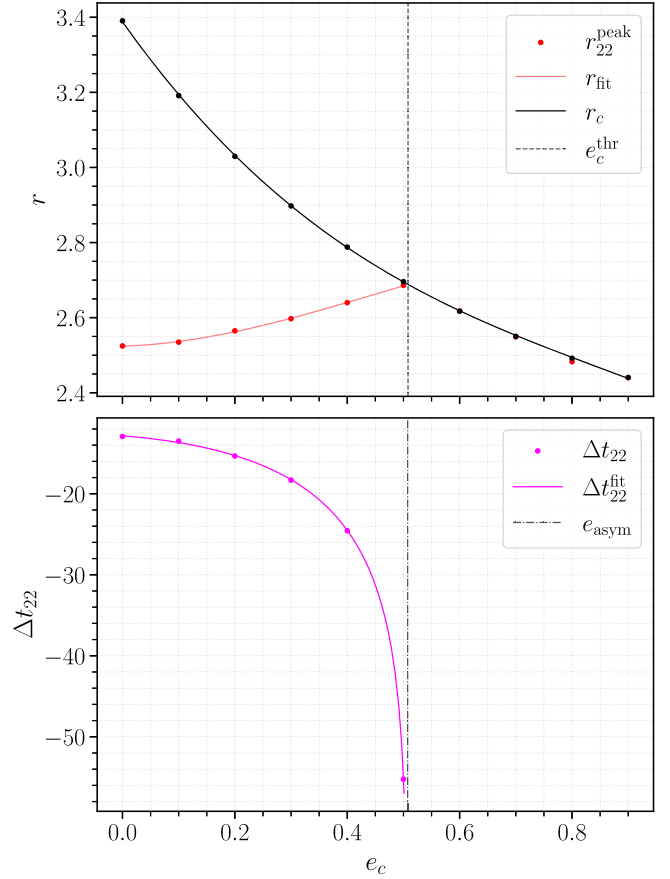


FIG. 3. This figure summarizes the two different methodologies employed to extract e_c^{thr} and e_{asym} , which are the threshold eccentricities from which the peak of the amplitude of the h_{22} mode disappears. In the figure we consider configurations with spin $a = 0.7$. Upper: shows method 1, mentioned in Sec. II C; the red curve is the fit of r_{22}^{peak} as function of e_c , while the black curve is r_c , the radius of the UCO. The threshold eccentricity e_c^{thr} corresponds to the eccentricity for which the two curves intersect. We highlight the value of e_c^{thr} with a dashed vertical black line. Bottom: shows method 2; the magenta dots represent the quantity Δt_{22} , which exhibits a logarithmic divergence at e_{asym} . After performing a fit of Δt_{22} (magenta curve), we extract the value of the asymptote e_{asym} (dot-dashed vertical black line) and we compare it with e_c^{thr} .

algorithm faces when no peak is present. This scattered pattern is explicit in the upper right panel of Fig. 2 for eccentricities $e_c \geq 0.5$. We then introduce the quantity Δt_{22} to reference the t_{22}^{peak} values to a reference time,

$$\Delta t_{22} = t_{22}^{\text{peak}} - t_{\Omega^{\text{peak}}}. \quad (31)$$

Here, $t_{\Omega^{\text{peak}}}$ corresponds to the time at which the orbital frequency Ω of the plunge geodesic, defined in Eq. (22), peaks. What we phenomenologically observe is that the Δt_{22} quantity exhibits a divergentlike behavior at a certain value of e_c .

TABLE I. Table of values e_c^{thr} and e_{asym} for different spins a of the central BH. We also compute the absolute difference δ between the two quantities.

a	e_c^{thr}	e_{asym}	δ
-0.99	1.760	1.756	3.2×10^{-3}
-0.90	1.724	1.732	7.85×10^{-3}
-0.70	1.648	1.650	1.60×10^{-3}
-0.40	1.506	1.507	1.04×10^{-3}
-0.20	1.396	1.393	3.17×10^{-3}
0.00	1.2590	1.2587	0.3×10^{-3}
0.20	1.1020	1.1010	1.0×10^{-3}
0.40	0.9072	0.9051	2.1×10^{-3}
0.70	0.5082	0.5075	0.7×10^{-3}
0.90	0.1133	0.1115	1.8×10^{-3}
0.92	0.0645	0.0619	2.7×10^{-3}
0.94	0.0201	0.0211	1.0×10^{-3}

This divergent trend can be understood and interpreted by recognizing that when $r_{22}^{\text{peak}} = r_c$, corresponding to $e_c = e_c^{\text{thr}}$ as introduced earlier, the amplitude of the h_{22} mode reaches its maximum at the UCO. However, the condition $r = r_c$ represents an asymptotic configuration for the class of critical plunge geodesics. As discussed in Sec. II A and made explicit in Eq. (21), these geodesics asymptotically approach the UCO as $t \rightarrow -\infty$, (i.e., $r \rightarrow r_c$ for $t \rightarrow -\infty$). Therefore, when the amplitude peaks at the UCO, the corresponding time is $t_{22}^{\text{peak}} = t_c \rightarrow -\infty$.

To find the value of the asymptote, we perform a fit using the functional form

$$\Delta t_{22}^{\text{fit}}(e_c) = c_0 + c_1 e_c + c_2 e_c^2 + \log(e_{\text{asym}} - e_c), \quad (32)$$

where $\{c_0, c_1, c_2, e_{\text{asym}}\}$ are the free parameters of the fit and e_{asym} is the value of the eccentricity for which Δt_{22} exhibits the asymptote. By repeating this procedure for every spin configuration of the central BH, we can extract a set of eccentricity values for the asymptotes that depend on the spin, $e_{\text{asym}} = e_{\text{asym}}(a)$. In Table I we show the values of e_c^{thr} , computed with method 1, and of e_{asym} , computed with method 2, for different values of a . The absolute differences between these two quantities is $\approx 10^{-3}$ and provides a measure of the error in the extrapolation of e_c^{thr} .

III. RESULTS

A. $\ell = 2$ $m = 2$ mode characterization

1. Amplitude peak characterization

By repeating the procedure of method 1 for different values of a in the interval $-0.99 \leq a \leq 0.99$, we extract the threshold values of the critical eccentricity e_c^{thr} from which the h_{22} mode does not exhibit a peak in its amplitude. In

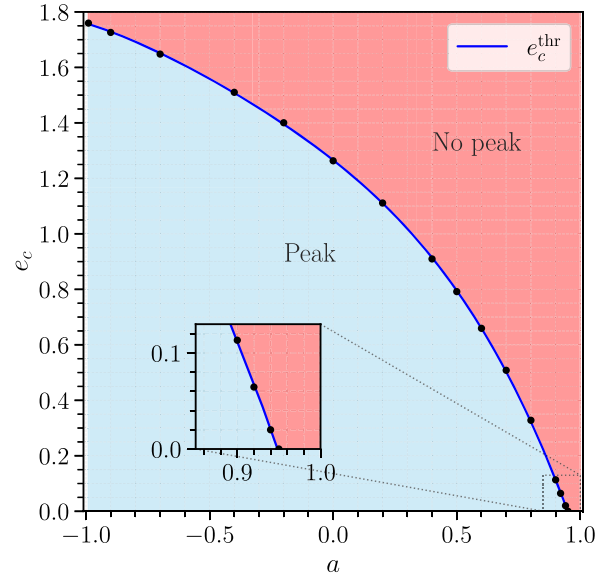


FIG. 4. Characterization over the parameter space (a, e_c) of the peak existence of the h_{22} mode amplitude. The blue area is the part of parameter space for which the peak exists, while the red area is the part for which the peak does not exist. The black dots are the values of $e_c^{\text{thr}}(a)$ computed with method 1 as explained in Sec. II C. The blue curve is a quadratic polynomial fit of e_c^{thr} as function of a . Inset: enlarged area of $e_c^{\text{thr}}(a)$ for values of the spin $0.85 \leq a \leq 1$ and critical eccentricity $0 \leq e_c \leq 1.2$. We find that for $a \geq 0.95$, it is impossible to find the peak of the amplitude of the h_{22} mode even when $e_c = 0$.

doing so, we consider values of the critical eccentricity $0 \leq e_c < 3$, i.e., we also consider critical plunge geodesics with value $e_c > 1$. As we described in Sec. II A, these configurations are associated with homoclinic orbits outside the UCO that are scattering orbits with the same energy $\mathcal{E}_c$ [see Eq. (14a)] of the critical plunge geodesics.

In Fig. 4, we characterize the regions of the (a, e_c) parameter space for which the $|h_{22}^{\text{peak}}|$ exist. The blue region corresponds to the part of parameter space where $|h_{22}^{\text{peak}}|$ exists. In this region, it is possible to identify and model all the quantities evaluated at the peak and use them as input for merger-ringdown models and respective attachment procedures to the inspiral part of the waveform [54,59,60,86]. The black dots are the values of the threshold eccentricities $e_c^{\text{thr}}(a)$ for different values of a derived with method 1 introduced in Sec. II C. The blue curve is a quadratic polynomial fit of e_c^{thr} as a function of a . This curve separates the blue region from the part of the parameter space for which it is impossible to identify a peak in the amplitude of the h_{22} mode. In Fig. 4, we highlight this portion of the parameter space with red color.

In the inset of Fig. 4, we show that, for the particular value of the spin $a = 0.95$, the peak of the amplitude of the h_{22} mode disappears in the QC configuration, $e_c = 0$. For $a \geq 0.95$ values, the h_{22} mode never peaks on the (a, e_c)

parameter space. This is similar to the result reported in Ref. [88] for equatorial QC trajectories of a Kerr BH. However, the results do not precisely match, since in that work for $a = 0.95$ the authors identify a clear peak with $\Delta t_{22} = -103$, while we do not find any peak. We believe this discrepancy comes from the fact that we are using different trajectories. In particular, the work in Ref. [88] evolved the QC trajectories with numerical fluxes up to the light ring of the Kerr BH for a TM with mass ratio $\mu/M = 0.001$, while the critical geodesics studied in the current work represent the $\mu/M = 0$ limit. These differences might be the cause of the discrepancy we observe. As we will show in Sec. III A 2, the discrepancy in the quantity Δt_{22} appears for spin values $a \geq 0.85$.

As we show in Fig. 3, the value of the radial coordinate r_{22}^{peak} for which the amplitude of the h_{22} mode peaks is a monotonic increasing function of e_c . At the same time, the critical radius r_c is a monotonic decreasing function of e_c . When introducing method 1 in Sec. II C, we mentioned that, when $r_{22}^{\text{peak}}(e_c^{\text{thr}}) = r_c(e_c^{\text{thr}})$, it happens that $|h_{22}|$ peaks at the critical radius r_c of the UCO and we identify a threshold critical eccentricity e_c^{thr} . For $e_c \geq e_c^{\text{thr}}$, $|h_{22}|$ does not peak anymore. This provides an operational and intuitive explanation for why the peak disappears. However, it shifts the question to why the peak of the waveform occurs at a relatively large radius compared to the peak of the orbital frequency. To better understand the behavior of the peak of the amplitude of the h_{22} mode (an $\ell = m$ modes), we consider the Newtonian order of the h_{22} and h_{33} modes

$$h_{22}^N \propto \dot{r}^2 + r\ddot{r} - 2r^2\Omega^2 - i(4\dot{r}r\Omega + \dot{\Omega}r^2), \quad (33a)$$

$$h_{33}^N \propto 6\dot{r}^3 - 54i\dot{r}^2r\Omega + 9\dot{r}r(2\ddot{r} - 3i\dot{\Omega}r - 9r\Omega^2) + 3r^2(\ddot{r} - i(\ddot{\Omega}r + 9\ddot{r}\Omega - 9i\dot{\Omega}r\Omega - 9r\Omega^3)). \quad (33b)$$

These expressions are derived by computing the coordinate-time derivatives of the Newtonian mass and current multipoles. Specifically, if we introduce the notation $h_{\ell m}^N = h_{\ell m}^{(N,\epsilon)}$, where ϵ is the parity of $\ell + m$, we have $h_{\ell m}^{(N,0)} \propto e^{im\varphi} I_{\ell m}^{(\ell)}$ and $h_{\ell m}^{(N,1)} \propto e^{im\varphi} S_{\ell m}^{(\ell)}$, with (ℓ) being the ℓ th time derivative and $I_{\ell m} = r^\ell e^{-im\varphi}$ and $S_{\ell m} = r^{\ell+1}\Omega e^{-im\varphi}$ the Newtonian mass and current multipoles (see, e.g., Refs. [66,138,139]). These expressions show how the Newtonian modes depend on the trajectories variables r, Ω , and their (Boyer-Lindquist) time derivatives. Using the geodesic equations (4) and Eqs. (14a) and (14b), we can express the $h_{\ell m}$ modes at Newtonian order on the critical plunges as function of r (plus a and e_c).

In the top panel of Fig. 5, we show the amplitude of the Newtonian mode h_{22}^N of Eq. (33a) as a function of the radial coordinate r . We consider critical plunge geodesics with spin $a = 0.7$ and critical eccentricities $e_c \leq 1$. In the figure,

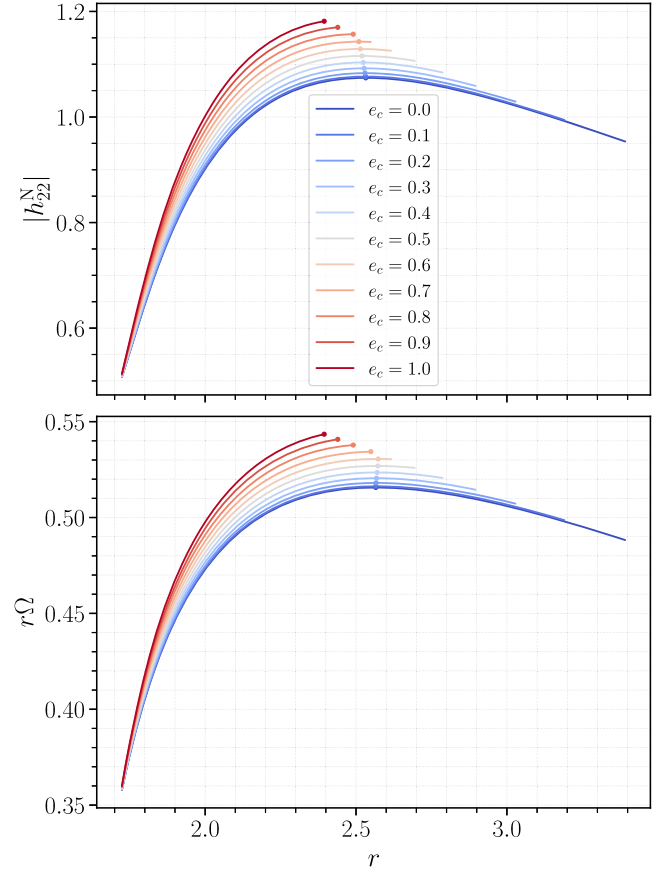


FIG. 5. Top: we show the amplitude of the h_{22}^N mode defined in Eq. (33a), which corresponds to the Newtonian order of the h_{22} mode. We compute the quantity $|h_{22}^N|$ on critical plunge geodesics with spin $a = 0.7$ and values of critical eccentricity $e_c \leq 1$, and we plot it as a function of the radial coordinate r . Bottom: the quantity $r\Omega$ of the considered critical plunge geodesics as a function of r . The different curves start at different radii because the critical plunge geodesics asymptotically start at the critical radius r_c , which is a function of the quantities a and e_c , as mentioned in Sec. II A.

the curves representing $|h_{22}^N|$ start at different radii: this is because the critical plunge geodesics asymptotically start from the critical radius r_c , as depicted in Eq. (21). We find that as e_c increases, $|h_{22}^N|$ peaks closer to the critical radius r_c (i.e., the value of r where the curves start). This is in agreement with what we observe for the numerical waveform, specifically with the fact that the curves of the radial coordinate r_{22}^{peak} for which the amplitude of the h_{22} mode peaks and the curve of the critical radius r_c intersect for a specific value of critical eccentricity e_c^{thr} , as mentioned above and summarized in the top panel of Fig. 3.

By studying the expressions in Eq. (33a), we can connect the peak disappearance of the mode with the features of the dynamical quantities of the critical plunge geodesics. In particular, we find that the quantity $r\Omega$, which appears in Eqs. (33a) and (33b), also manifests a disappearing peak as

e_c increases, as we show in the bottom panel of Fig. 5. The disappearance of the peak of $r\Omega$ is straightforwardly understood. As noted in Sec. II A, Ω is a function of a , b , and r that is universal for all equatorial Kerr geodesics. Consequently, location of its peak does not depend on whether the geodesic is critical or not, depending (weakly) only on the value of b_c . By extension, the same is true for $r\Omega$, but with a peak at a larger value of r , allowing the possibility that a critical plunge starts at a radius r_c smaller than the position of the peak of $r\Omega$.

Through an explicit computation, we find that the quantity $r\Omega$ dominates the different contributions in the Newtonian mode expressions h_{22}^N and h_{33}^N in Eqs. (33a) and (33b). In particular, we note that as e_c increases the radial evolution $\dot{r}$ becomes smaller over the entire plunge (see Appendix D for more detail). Consequently, the dominance of $r\Omega$ tends to increase with increasing critical eccentricity. Thus, the behavior of the peak of $r\Omega$ also explains the disappearance of the peak of the h_{22} mode.

2. Mode features at the peak

In GW modeling, the time of the mode amplitude peak $t_{\ell m}^{\text{peak}}$ is a feature inferred from NR waveforms and employed as a calibration parameter to correctly attach ringdown models to the end of the inspiral-plunge waveform. In EOB models, for example, the values of $t_{\ell m}^{\text{peak}}$ are expressed in terms of time intervals $\Delta t_{\ell m}$ with respect to a reference time associated with particular configurations of the dynamics of the binary, like the peak of the orbital frequency Ω or the ISCO crossing [54,59,60,86].

In our work, we adopt the same scheme and we refer $t_{\ell m}^{\text{peak}}$ with respect to $t_{\Omega^{\text{peak}}}$, i.e., we consider

$$\Delta t_{\ell m} = t_{\ell m}^{\text{peak}} - t_{\Omega^{\text{peak}}}, \quad (34)$$

which is a generalization of Eq. (31) in the case of the h_{22} mode. In Fig. 6, we show the quantity Δt_{22} as a function of the eccentricity e_c . The different colors represent different spin a configurations of the Kerr BH. In the figure, we focus on the section of parameter space for which the peak of the h_{22} mode exists, as depicted in Fig. 4.

The time interval Δt_{22} exhibits a monotonic decreasing trend with respect to the eccentricity for all the spins. Moreover, for every fixed e_c configuration, it also shows a monotonic decreasing behavior with respect to the spin. As already pointed out in Sec. II C, at a certain value of e_c (which depends on the spin), the Δt_{22} quantity exhibits a divergent trend because the peak has to move to the infinite past before disappearing. We highlight the presence of these divergences in Fig. 6 by plotting the asymptote values e_{asym} introduced in Sec. II C with vertical dashed lines; the colors of these lines correspond to the color of the spin configuration for which the asymptotes occur at the value $e_{\text{asym}}(a)$. In the Δt_{22} study, we also compared with previous

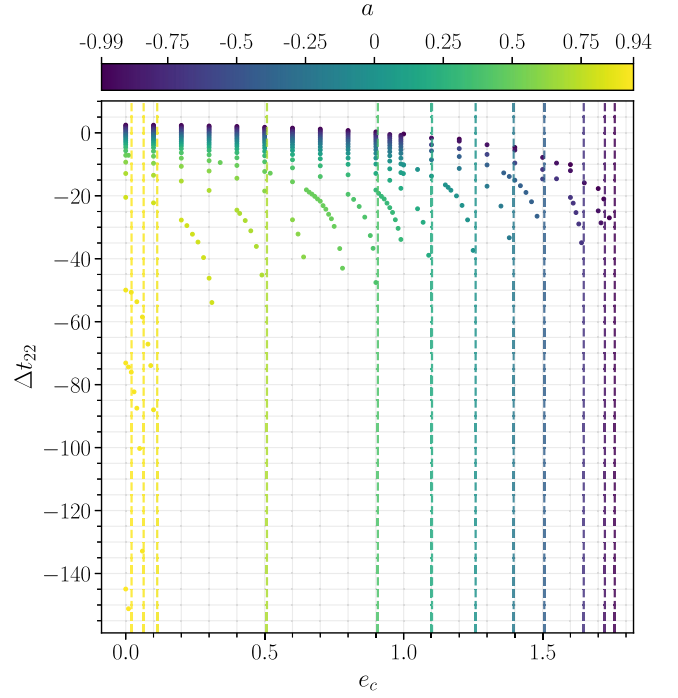


FIG. 6. This figure shows the dependence of the quantity Δt_{22} as a function of e_c . The color scale represents the spin a of the Kerr BH. In the figure, we consider $-0.99 \leq a < 0.95$. The vertical dashed lines correspond to the asymptote values e_{asym} in Table I; the colors of the lines correspond to the color of the spin configuration for which the asymptotes occur. For spin values $a \geq 0.95$, we can no longer find a peak even when $e_c = 0$, as shown in Fig. 4. For these spins, the quantity Δt_{22} is not defined and not shown in the figure.

results in Ref. [88] for the QC case. We find a good agreement of Δt_{22} for the $e_c = 0$ case with the values shown in Fig. 13 of Ref. [88] up to spin values $a \sim 0.85$. However, for $a = 0.9$ and $a = 0.95$, we find a discrepancy between the values of Δt_{22} we computed from the waveforms produced in our work and the values of Δt_{22} in Ref. [88]. As we mentioned in Sec. III A 1, we believe this discrepancy is because the trajectories from Ref. [88] are considered at finite mass ratio. These differences are particularly evident when considering Δt_{22} for high spins. In Appendix C, we provide a direct comparison of the Δt_{22} quantity extracted considering waveforms produced with critical plunge geodesics in the QC case (i.e., with $e_c = 0$) with the same quantity extracted in Ref. [88].

In Fig. 7, we show the quantities employed in typical attachment procedures and in the ringdown fits of the GW models based on the EOB formalism. These quantities correspond to the amplitude $|h_{\ell m}^{\text{peak}}|$, second-order time derivative $d^2|h_{\ell m}|/dt^2|_{\text{peak}}$, frequency $\omega_{\ell m}^{\text{peak}}$, and first-order time derivative $\dot{\omega}_{\ell m}^{\text{peak}}$ of the $h_{\ell m}$ modes evaluated at the peak [54,59,60,86]. We specialize our analysis to the dominant h_{22} mode, and we parametrize the quantities with respect to the e_c of the plunging geodesic. We restrict our analysis to

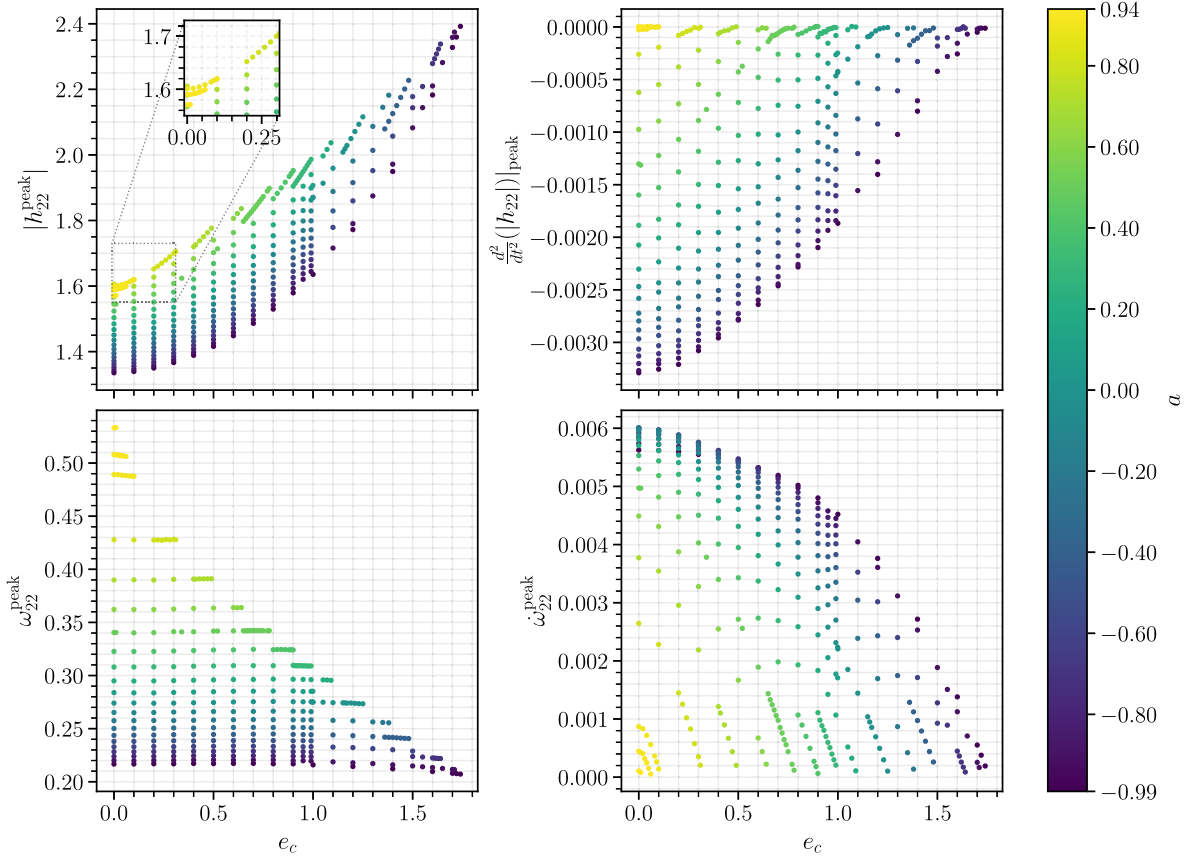


FIG. 7. In this figure we show the dependence of the amplitude $|h_{22}^{\text{peak}}|$, second-order time derivative of the amplitude $\frac{d^2|h_{22}|}{dt^2}|_{\text{peak}}$, frequency $\omega_{22}^{\text{peak}}$, and first-order time derivative of the frequency $\dot{\omega}_{22}^{\text{peak}}$ of the h_{22} mode evaluated at the peak of its amplitude as a function of the critical eccentricity e_c . We span different values of the spin $-0.99 \leq a \leq 0.99$ of the central Kerr BH, represented by different colors. We consider configurations for which the peak of the h_{22} exists, as illustrated in the blue area of the parameter space in Fig. 4. For spin values $a \geq 0.95$, we can no longer find a peak even when $e_c = 0$, as shown in Fig. 4. For these spins, it is not possible to define quantities at the peak, and they are not shown in the figure.

the section of parameter space for which the peak of the h_{22} mode exists.

In the top left panel of Fig. 7, we show that the peak of the amplitude increases with respect to e_c and a , showing an overall monotonic pattern. The exception to this trend is given by low values of critical eccentricity, $e_c \leq 0.08$, for the highest prograde spin scenario (i.e., $a = [0.9, 0.92, 0.94]$). In these cases, the peak of the amplitudes is smaller than the $e_c = 0$ case with spin $a = 0.8$, as shown in the inset of the top left panel of Fig. 7. We mention that this result was already found in Ref. [88] and is compatible with their value within the numerical error. Moreover, we find that for $a \geq 0.9$ the values of the amplitude of the h_{22} at the peak for fixed values of e_c is not an increasing monotonic function of a , e.g., the peak amplitude for $a = 0.94$ at $e_c = 0$ is smaller than the peak amplitude for $a = 0.92$ at the same critical eccentricity, as shown in the inset of the top left panel of Fig. 7. To quantify how much the eccentricity changes the value of the amplitude at the peak with respect to the QC case, we mention the fact that, for $e_c \leq 0.5$, we find that

the fractional differences of the amplitude at the peak of the eccentric h_{22} mode (i.e., with $e_c \neq 0$) compared to the amplitudes at the peak of the QC modes (i.e., with $e_c = 0$) is $\lesssim 10\%$.

The second-order time derivative of the amplitude at the peak $\frac{d^2|h_{22}|}{dt^2}|_{\text{peak}}$ also manifests a monotonic trend with respect to both the spin a and the critical eccentricity e_c . We highlight that, when e_c approaches e_c^{thr} , it happens that $\frac{d^2|h_{22}|}{dt^2}|_{\text{peak}}$ starts to vanish. This is connected to the fact that the mode's amplitude is more and more flattened in this regime. Regarding the mode frequency ω_{22} , we find that it has an almost flat trend with respect to eccentricity, for all the spin values of the central BH, showing an overall independence from e_c . However, we find a very mild dependence on e_c for values of $e_c \geq 1.1$ for the high retrograde spin configurations. Finally, for the time derivative of the mode frequency $\dot{\omega}_{22}$ we find that this quantity is always positive and that it is characterized by an overall monotonic trend, going to zero as e_c approaches e_c^{thr} . However, we also notice that, for small values of the

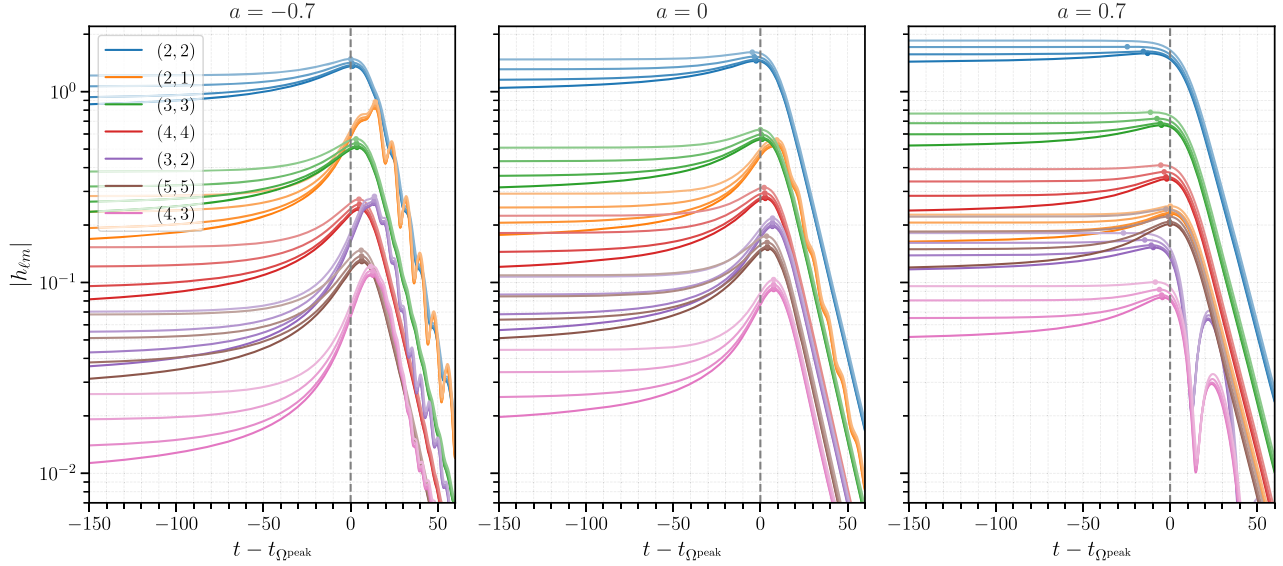


FIG. 8. Amplitude of the modes h_{22} (in blue), h_{21} (in orange), h_{33} (in green), h_{32} (in purple), h_{44} (in red), h_{43} (in pink), and h_{55} (in brown). The first panel on the left shows the case with spin $a = -0.7$, the second panel shows the case with $a = 0.0$, and the third panel shows the case with $a = 0.7$. The different shades of the colors represent different values of e_c with $e_c = [0.0, 0.2, 0.4, 0.6]$. Darker shades of the colors are linked to lower values of eccentricity. We align the waveform at the time of the peak of the orbital frequency $t_{\Omega\text{peak}}$.

eccentricity, $\dot{\omega}_{22}$ is not monotonic for some negative values of the spin. This result was already found for the QC case (i.e., $e_c = 0$) in Fig. 15 of Ref. [88], and we find this holds also for the eccentric scenarios. As we show in Fig. 16 in Appendix E, this small dependence for high values of eccentricity is also present for the time derivative of the frequency $\dot{\omega}_{\ell m}$ of the $\ell = m$ higher-order modes. In Appendix F, we provide the values for a subset of the data produced and shown in Fig. 7.

B. Higher-order mode characterization

1. Mode hierarchy and eccentricity

As a matter of fact, the hierarchy of the different $h_{\ell m}$ modes, which matters when all modes are summed in Eq. (29) to provide the total strain h , depends on the spin of the central BH [86,88]. In this section, we aim to study the dependence of the hierarchy with respect to e_c . In Fig. 8, we show the different modes amplitudes $|h_{\ell m}|$ for three configurations of spin: in the first panel, we consider the case with spin $a = -0.7$, in the second panel the case with $a = 0.0$, and in the third panel, we consider the case with $a = 0.7$. In each panel, we consider the modes h_{22} (in blue), h_{21} (in orange), h_{33} (in green), h_{32} (in purple), h_{44} (in red), h_{43} (in pink), and h_{55} (in brown). The different shades of the colors, for each mode, represent different values of e_c . We consider eccentricity values $e_c = [0.0, 0.2, 0.4, 0.6]$ and, in Fig. 8, darker shades of the colors correspond to lower values of eccentricity. As in Fig. 2, we align the waveforms at the time of the peak of the orbital frequency $t_{\Omega\text{peak}}$ of the critical plunge geodesics used to produce the waveforms.

From Fig. 8, we notice that, regardless of the value of a , the eccentricity has more impact on the value of the amplitude of the modes during the premerger part of the waveform (i.e., when the critical plunge geodesics evolve on the UCO) rather than on the value of the amplitude in the merger part of the waveforms (i.e., close to the peak, just before the beginning of the ringdown part). From Fig. 8, we also notice that the hierarchy of the modes mostly depends on the spin a of the Kerr BH, while the eccentricity seems not to play a significant role. In fact, by inspecting the three panels of the figure, it turns out that waveforms connected to the same eccentricity (same shade of the different colors) do not exhibit significant changes in the hierarchy (i.e., in the relative magnitude of their amplitudes). However, in Fig. 8, there are some configurations for which the independence of the hierarchy from eccentricity does not always hold. In particular, we observe that for all the three spin scenarios the h_{32} (purple) and the h_{55} (brown) modes become more and more comparable for times $t \leq 20$ when the eccentricity increases at the point that, when $e_c > 0.2$, their amplitudes are almost the same. However, we remark that this effect is not present close to merger. In the $a = 0.7$ case (third panel of Fig. 8), we observe that different eccentricities affect the hierarchy of the h_{21} (orange) and the h_{55} (brown) modes. In fact, for low eccentricities (i.e., $e_c \leq 0.2$) the h_{21} is slightly more dominant, but as e_c increases, the two modes become more and more comparable. However, in all these cases, the change in mode hierarchy from different eccentricities is smaller than the much larger impact due to the change in the spin.

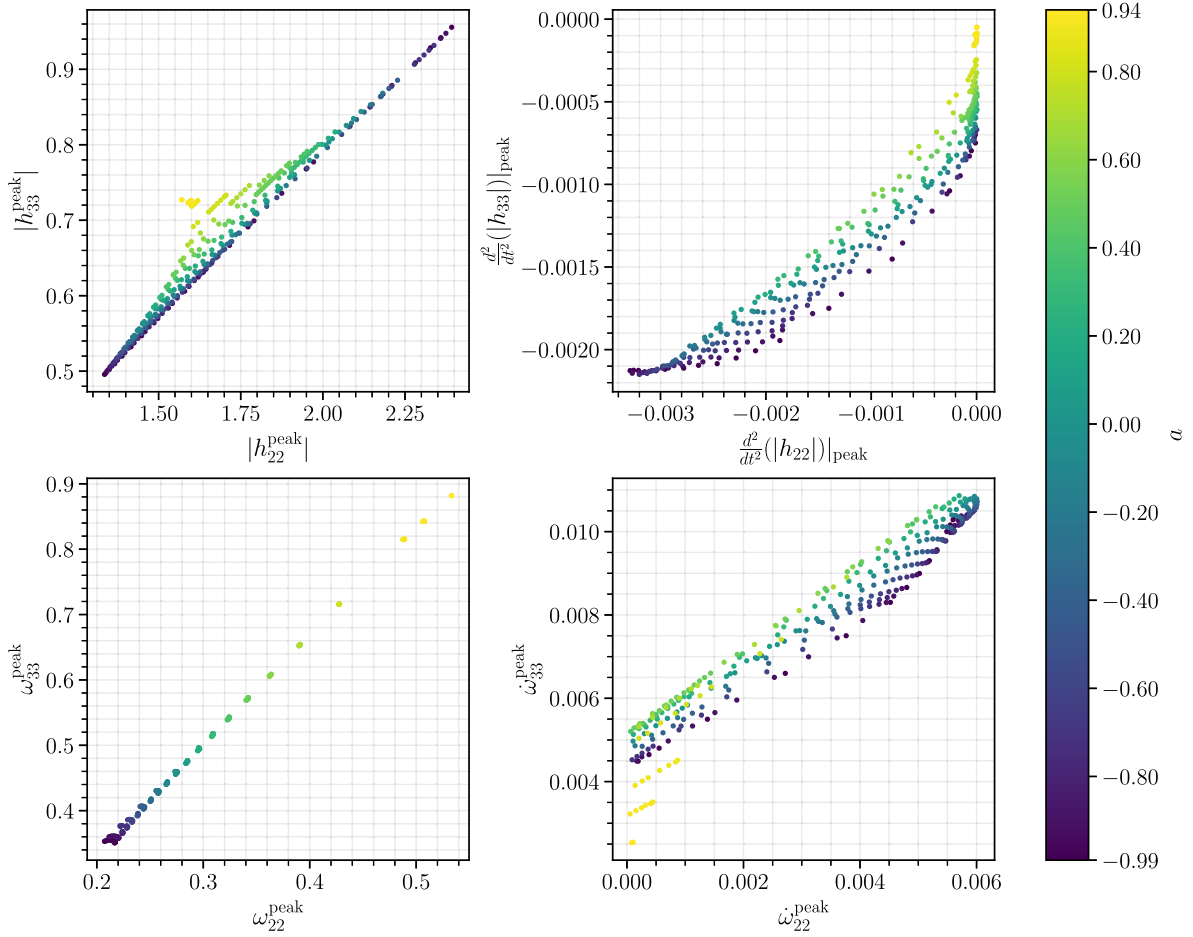


FIG. 9. In this figure we show the dependence of the amplitude $|h_{33}^{\text{peak}}|$, second-order time derivative of the amplitude $d^2|h_{33}|/dt^2|_{\text{peak}}$, frequency $\omega_{33}^{\text{peak}}$, and first-order time derivative of the frequency $\dot{\omega}_{33}^{\text{peak}}$ of the h_{33} mode evaluated at the peak of its amplitude as a function of the same corresponding quantities for the h_{22} mode. We span different values of the spin $-0.99 \leq a \leq 0.99$ of the central Kerr BH, represented by different colors. We consider configurations for which the peak of the h_{22} exists, as illustrated in the blue area of the parameter space in Fig. 4. For spin values $a \geq 0.95$, we can no longer find a peak even when $e_c = 0$, as shown in Fig. 4. For these spins, it is not possible to define quantities at the peak, and they are not shown in the figure.

2. $\ell = m$ modes

For the $\ell = m$ higher-order modes, similar to the dominant h_{22} mode, we also observe that the amplitude peak disappears from a certain threshold value of e_c , as can be noticed from the third panel of Fig. 8. However, what we notice is that this transition occurs at larger e_c compared to the h_{22} mode case. Phenomenologically, we find that when the h_{22} mode has a peak, also the amplitude of the $\ell = m$ higher-order modes we consider, i.e., h_{33} , h_{44} , and h_{55} , have peaks. By inspecting the Newtonian terms in Eqs. (33a) and (33b) on the critical plunge geodesic, we find that the threshold eccentricity for h_{33}^N is higher than that for h_{22}^N . This shows that the hierarchy of the different threshold critical eccentricities among the $\ell = m$ modes is evident already at the Newtonian level and agrees with what we find with the numerical waveforms.

Moreover, we observe that the features and patterns we identified for the h_{22} mode in Fig. 7 are similar also for the higher-order modes. In this section, we limit to show the results for the quantities related to the h_{33} mode and we refer the reader to Appendix E for the other modes.

In Fig. 9, we show $|h_{33}^{\text{peak}}|$, $d^2(|h_{33}|)/dt^2|_{\text{peak}}$, $\omega_{33}^{\text{peak}}$, and $\dot{\omega}_{33}^{\text{peak}}$ of the h_{33} mode. Different from Sec. III A, where we show these quantities for the h_{22} mode as a function of e_c , we choose to show h_{33} quantities with respect to the corresponding quantities of the h_{22} mode. In doing so, we consider the region of the parameter space for which the peak of the h_{22} exists, as depicted in Fig. 4.

In the top left panel of Fig. 9, we observe that, for each spin configuration, the $|h_{33}^{\text{peak}}|$ shows a linear dependence with respect to the $|h_{22}^{\text{peak}}|$ quantity. This suggests the value of $|h_{33}^{\text{peak}}|$ is a (spin-dependent) multiple of $|h_{22}^{\text{peak}}|$.

Concerning $d^2(|h_{33}|)/dt^2|_{\text{peak}}$ (top right panel of Fig. 9), we find a monotonic increasing trend with respect to the corresponding h_{22} quantity and with respect to the spin of the central BH. We also highlight the fact that, for configurations close to the threshold eccentricity $e_c^{\text{thr}}(a)$ (for which the peak of the amplitude of the h_{22} mode is not anymore manifest), when $d^2(|h_{22}|)/dt^2|_{\text{peak}} = 0$, the corresponding h_{33} quantity $d^2|h_{33}|/dt^2|_{\text{peak}} < 0$. This confirms what we mention at the beginning of this section, i.e., that whenever the peak of the amplitude of the h_{22} mode exists, also the peak of the amplitude of the higher-order modes exists, being its second-order time derivative of the amplitude nonvanishing.

From the bottom left panel of Fig. 9 we observe that the relationship between the frequencies $\omega_{22}^{\text{peak}}$ and $\omega_{33}^{\text{peak}}$ does not depend on e_c , except a small dependence for high retrograde spins, which is inherited from the small dependence of $\omega_{22}^{\text{peak}}$ with respect to e_c for values of eccentricity $e_c \geq 1.1$ (bottom left panel of Fig. 7), as we already commented in Sec. III A. This reflects the fact that, also for the higher-order modes, there is no dependence of the frequency on e_c , as shown for the h_{22} mode in Sec. III A.

Finally, regarding $\dot{\omega}_{33}^{\text{peak}}$ (bottom right panel of Fig. 9) we observe an increasing monotonic behavior with respect to the relative quantity of the h_{22} mode but, as for the h_{22} mode, the dependency on the spin of the central BH is not monotonic.

3. $\ell \neq m$ modes

In our work, we also consider higher-order modes with $\ell \neq m$; in particular, we consider the off-diagonal h_{21} , h_{32} , and h_{43} modes. As for the $\ell = m$ case, we first qualitatively inspect the behavior of the peak of the different modes on the parameter space. As in the $\ell = m$ case, we find that whenever the peak of the amplitude of the h_{22} mode exists also the peak of these higher-order modes is present, as can be observed in Fig. 8 for a subset of waveforms. From the figure, it is also possible to notice how the $\ell \neq m$ modes are more affected by the mode mixing, also when considering eccentric systems. Without going into more details, we only mention that for retrograde spins the mixing is mostly with the mirror fundamental quasinormal mode (see, e.g., Ref. [140]), while for prograde spins the mixing comes primarily from the lower- ℓ harmonics with the same m . Many works have studied the nature of this mixing, and since its analysis is not among the scope of this article, we refer the reader to Ref. [88] for more details, noting that the inclusion of eccentricity does not seem to have a large impact on the phenomenology relative to the QC case.

In this work, we perform the same analysis we do for the $\ell = m$ case for the considered $\ell \neq m$ modes, and we find that, in general, the h_{21} , h_{32} , and h_{43} modes have a merger structure that is different from the structure of the merger of the $\ell = m$ modes. However, we find that the

h_{21} , h_{32} , and h_{43} have a similar merger structure among them. Furthermore, we also find that the merger structure of the negative spin scenarios presents differences with respect to the merger structure of the positive spin scenarios. We link these different behaviors to the presence of the mode mixing mentioned above and we provide more details about this analysis and characterization in Appendix E.

C. Bondi news and ψ_4 peak characterization

In Sec. II C we characterized the existence of the peak of the amplitude of the h_{22} mode in the parameter space. For the sake of completeness, we aim to understand how the amplitude peaks behave for two other quantities related to the waveform modes, the Weyl scalar ψ_4 defined in Eq. (28) and the Bondi news N , defined by

$$N = \frac{1}{2} \frac{\partial h}{\partial t} = \frac{1}{2} \left(\frac{\partial h_+}{\partial t} - i \frac{\partial h_\times}{\partial t} \right). \quad (35)$$

In particular, as for the strain h , we are interested in the (2,2) spherical harmonic mode of these two quantities, N_{22} and $\psi_{4,22}$. In the second and third rows of Fig. 10, we show the amplitudes of these quantities $|N_{22}|$ and $|\psi_{4,22}|$ generated by trajectories with critical eccentricities $e_c = [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9]$ and for two spin configurations of the central BH: in the left column, we consider $a = 0$, while in the right column we consider $a = 0.9$. The first row of Fig. 10 shows the amplitude of the h_{22} mode: for $a = 0$ the values of e_c are such that the peak of the amplitude is always present. For $a = 0.9$, this is not the case, since we consider values of critical eccentricity that are greater than the threshold value e_c^{thr} we identified in our characterization performed in Sec. II C and illustrated in Fig. 4. Note that this result was already found for the QC case (i.e., $e_c = 0$) for values of spin $a > 0.95$ in Ref. [88].

When analyzing the N_{22} and $\psi_{4,22}$, we find that, similar to the h_{22} mode, their amplitude peak ceases to exist beyond certain threshold values of e_c . However, we observe a different pattern when considering the N_{22} and $\psi_{4,22}$, which we show in the second and third rows of Fig. 10, respectively. We find that the peaks' existence phenomenology of these quantities is different. In fact, by focusing on the $a = 0.9$ case, we observe that the threshold values of the critical eccentricity, from which the peak of the amplitudes disappears, for the Bondi news $e_{N_{22}}^{\text{thr}}$, and for the Weyl scalar $e_{\psi_{4,22}}^{\text{thr}}$ are different compared to the h_{22} mode, and they appear to be greater, i.e., $e_c^{\text{thr}} < e_{N_{22}}^{\text{thr}} < e_{\psi_{4,22}}^{\text{thr}}$. For example, for $a = 0.9$ and $e_c = 0.4$, the h_{22} does not show any peak. However, both the Bondi news and the Weyl scalar manifest a peak at this value of the critical eccentricity. A possible explanation of this behavior is linked to the fact that $N_{22} \sim \dot{h}_{22} \approx \omega_{22} h_{22}$ and $\psi_{4,22} \sim \ddot{h}_{22} \approx (\omega_{22})^2 h_{22}$, where ω_{22} is the frequency of the

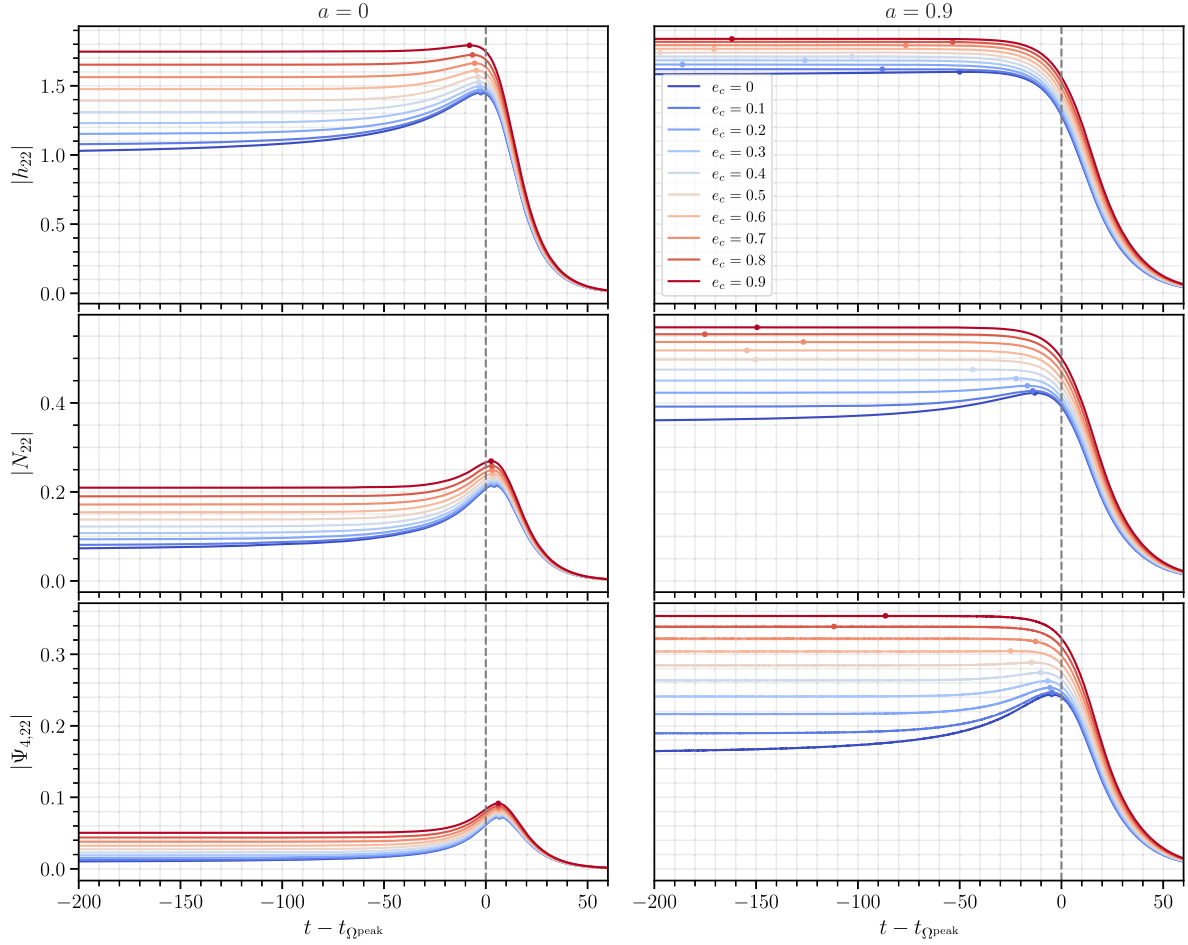


FIG. 10. We show the amplitude peak behavior of the (2,2) spin-weighted spherical harmonic mode of the strain h_{22} (first row), Bondi news N_{22} (second row), and Weyl scalar $\psi_{4,22}$ (third row). Left column: we consider spin $a = 0$ configurations; right column: spin $a = 0.9$ configurations. The different colors represent different critical eccentricities, ranging from $e_c = 0$ to $e_c = 0.9$. The peak amplitudes of all the quantities and the time of the peak amplitude exhibit a dependence on the eccentricity, as already mentioned in Fig. 2. We notice that both the amplitudes of N_{22} and of $\psi_{4,22}$ are characterized by regimes for which their peaks disappear, as for the strain h_{22} . However, the threshold eccentricities for which they disappear are different from the threshold eccentricity $e_c^{\text{thr}}(a)$ we identified for the h_{22} mode in Sec. II C.

h_{22} mode. Hence, the presence of the peak of the amplitudes of N_{22} and $\psi_{4,22}$ changes on the parameter space (a, e_c) due to the fact that $|N_{22}|$ and $|\psi_{4,22}|$ inherit the features of the factor ω_{22} .

IV. CONCLUSIONS

In this work, we characterized the properties at merger of the GW modes for equatorial-eccentric configurations of the Kerr metric in the TM limit. This is done by considering a particular class of critical plunge geodesics parametrized by the spin a of the central Kerr BH and by the critical eccentricity e_c introduced in Sec. II A. We employed a TD Teukolsky code to produce the GW signals generated by this class of plunge geodesics.

By inspecting different configurations of the spin a and different e_c , we found that the amplitude of the h_{22} mode

exhibits a peak in a particular part of the parameter space, while this peak disappears in a disjoint part of the parameter space. In Sec. II, we introduced a methodology that allows one to characterize the existence of the peak of the amplitude of the h_{22} mode and we identified the region of the parameter space for which its peak exists, as we show in Fig. 4. This behavior is qualitatively understood by studying the analytic expression for h_{22}^{N} , the Newtonian order part of the h_{22} , which, in turn, is dominated by the quantity $r\Omega$. The location of the peak of $r\Omega$ is at a fixed value of r , and the disappearance of the peak for larger values of e_c is explained by the critical plunge geodesics starting at a radius r_c that is smaller than this value.

We restricted the merger characterization of the waveform modes to the scenarios for which the peak of the amplitude of the h_{22} mode exists. On the part of parameter

space (a, e_c) for which the peak of the h_{22} mode exists, we described the quantities usually employed in the modeling of the merger and postmerger parts of gravitational waveforms. These quantities correspond to the amplitude $|h_{\ell m}^{\text{peak}}|$, second-order time derivative of the amplitude $d^2|h_{\ell m}|/dt^2|_{\text{peak}}$, frequency $\omega_{\ell m}^{\text{peak}}$, and first-order time derivative $\dot{\omega}_{\ell m}^{\text{peak}}$ of the $h_{\ell m}$ waveform modes evaluated at the peak of the amplitude of the modes [54,59,60,86]. In our work, we also inspected the higher-order h_{33} , h_{44} , h_{55} , h_{21} , h_{32} , and h_{43} modes. For the $\ell = m$ modes, we found that the peak of the amplitude also disappears at a certain value of e_c . However, this threshold value is larger than the value we found for the h_{22} mode. Consequently, when the peak of the h_{22} mode exists, the peaks of the other $\ell = m$ modes also exist. For these higher-order modes, we found that the quantities at the peak exhibit a similar dependence on e_c and a as for the case of the h_{22} mode, as shown in Fig. 16, and in Fig. 9 we showed a different way of parametrizing the quantities at the peak. This different parametrization describes the quantities $|h_{\ell m}^{\text{peak}}|$, $d^2|h_{\ell m}|/dt^2|_{\text{peak}}$, $\omega_{\ell m}^{\text{peak}}$, and $\dot{\omega}_{\ell m}^{\text{peak}}$ of the $\ell = m > 2$ modes as function of the same quantities of the h_{22} mode. In Fig. 9, we specialized to the case $\ell = m = 3$.

Finally, we also explored what happens to the peak of the amplitude of the (2,2) mode of the Bondi news N and of the Weyl scalar ψ_4 . We observed that also for these quantities the peak of the amplitude of the dominant spin-weighted spherical harmonic mode disappears at a certain value of e_c . This threshold value is different from the threshold value we found and characterized for the h_{22} mode and appears to follow the general rule that $e_{\text{thr}} < e_{\text{thr}}^{N_{22}} < e_{\text{thr}}^{\psi_{4,22}}$. Although, we have not exhaustively probed the parameter space to confirm this.

The methodology we employed in this work unveils the properties of the merger and postmerger parts of the gravitational waveforms in the TM limit of the eccentric equatorial Kerr scenario. This work can be extended in multiple directions: to study the merger properties of the waveforms in the more general case of non-equatorial-eccentric critical plunge geodesics of Kerr and to study the impact of eccentricity in the ringdown part of the waveforms. Furthermore, it can be possible to extend current phenomenological EOB QC ringdown models in the TM limit [60,86,88] to take into account eccentricity and spin.

We believe all these directions will provide important insights about the nature of the merger and postmerger of the waveforms emitted by a TM plunging and merging in a Kerr BH and they will be essential in informing the current [63,76] and future gravitational-waveform models with TM limit information.

This work makes use of the Black Hole Perturbation Toolkit [141].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: WHY CRITICAL PLUNGE GEODESICS?

The class of critical plunge geodesics, for which analytical closed-form expressions can be found in Refs. [123,124] and that we reviewed in Sec. II A, describes the final part of the trajectories of a TM in Kerr spacetime. In this limit, when the symmetric mass ratio $\nu = \mu/M \rightarrow 0$, the dynamics of the plunge becomes geodesic [47,121], and it converges to a critical plunge geodesic, see [122] for an analytical treatment in the Schwarzschild case.

To have better insight into this, it is instructive to inspect the phenomenology of the eccentric plunge transition of a small mass μ orbiting the equatorial plane of the Kerr spacetime and then consider the TM limit. During the inspiral phase, the bound system evolves adiabatically and the orbital motion of the small mass is represented as a sequence of eccentric geodesics, which at any given instant can be parametrized by the eccentricity e , the semilatus rectum p , and the relativistic anomaly ξ expressed in Eq. (9).

In the eccentric case, as the system approaches the LSO, the plunge dynamics that follow exhibit richer phenomenology compared to the QC case, which, by contrast, follows a universal behavior [47,121]. Notably, the features of the eccentric plunge depend sensitively on the value of the relativistic anomaly ξ_{LSO} at the LSO [111,112].

Combining the expressions in Eq. (9) of e and p evaluated at the LSO (i.e., with $r_2 = r_3 = r_c$, as summarized in Sec. II A) in the definition of the relativistic anomaly ξ , it reads

$$\cos \xi_{\text{LSO}} = \frac{2r_1 r_c - r_{\text{LSO}}(r_1 + r_c)}{r_{\text{LSO}}(r_1 - r_c)}, \quad (\text{A1})$$

where r_c is the critical radius introduced in Sec. II A, r_{LSO} corresponds to the value of the radial coordinate r when the TM is at the LSO, and where r_1 is the apocenter radial coordinate at the LSO. We remark that Eq. (A1) typically has two solutions between 0 and 2π . We solve this degeneracy by considering the time derivative of the radial coordinate $\dot{r}$ at the LSO.

Depending on the value of ξ_{LSO} , it is possible to identify three possible scenarios that characterize the eccentric plunge transition:

- (i) when $\xi_{\text{LSO}} = 0^-$,^{4,5} Eqs. (A1) and (4a) imply $r_{\text{LSO}} = r_c$ and $(dr/d\lambda)_{\text{LSO}} = 0$, the TM orbits the UCO and then plunges into the BH;
- (ii) when $\xi_{\text{LSO}} = 0^+$, the TM orbits the UCO but then performs a last radial cycle before plunging; and
- (iii) when $0 < \xi_{\text{LSO}} < 2\pi$, the LSO occurs when $r_c < r_{\text{LSO}} < r_1$ and the TM completes a last fraction of radial cycle before plunging.

To better understand how the plunges of these scenarios converge to the critical plunge geodesic in the TM limit, it is instructive to consider the difference between the energy of the system E and the energy $\mathcal{E}_{\text{UCO}}$ of the UCO with the same angular momentum L ,

$$\Delta E_{\text{UCO}} = E - \mathcal{E}_{\text{UCO}}(L). \quad (\text{A2})$$

As long as E_{UCO} is negative, the centrifugal barrier keeps the particle in an eccentric orbit outside the UCO. However, when $E_{\text{UCO}} > 0$ the particle is free to pass across the radius of the UCO and start its plunge. We remark on the fact that, while the system evolves, both E and $\mathcal{E}_{\text{UCO}}$ decrease because of the gravitational radiation (in particular, $\mathcal{E}_{\text{UCO}}$ evolves through the evolution in time of L), however, they generally evolve with different rates.

We numerically study the behavior of the quantity ΔE_{UCO} by considering trajectories generated by evolving the Hamiltonian system described in Eqs. (5a)–(5d) of Ref. [94]. The system incorporates the dissipative effects due to GW emission through a radiation-reaction (RR) force that includes resummed PN eccentric corrections

⁴We are considering $\xi_{\text{LSO}} \bmod 2\pi$.

⁵The critical case $\xi_{\text{LSO}} = 0$ produces the same critical phenomenon for critical scattering as studied (in Schwarzschild) in Ref. [142], but reaching the UCO from a bound orbit rather than a scattering orbit; the TM continues orbiting the UCO, which radius r_c grows under the loss of energy, until it reaches a transition regime when it reaches the ISCO.

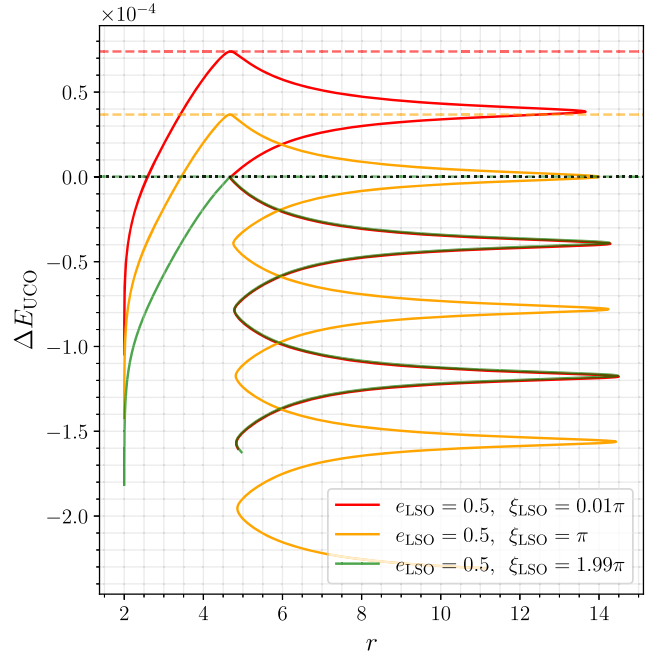


FIG. 11. Evolution of the quantity ΔE_{UCO} as a function of the radial coordinate r for three trajectories characterized by $\nu = 10^{-3}$, $e_{\text{LSO}} = 0.5$, and three distinct values of $\xi_{\text{LSO}} = [0.01\pi, \pi, 1.99\pi]$ corresponding, respectively, to the scenarios (ii) (red curve), (iii) (orange curve), (i) (green curve) mentioned in Appendix A. The quantity ΔE_{UCO} exhibits an increasing monotonic trend until it reaches a maximum (after the LSO crossing) and starts to decrease monotonically during the plunge. The horizontal dashed lines highlight the maximum value $\Delta E_{\text{UCO}}^{\text{onset}}$ which defines the onset of the plunge.

[61,64,94]. In Fig. 11, we show the evolution of the quantity ΔE_{UCO} with respect to the radial coordinate r for three trajectories with $\nu = 10^{-3}$, $e_{\text{LSO}} = 0.5$, and $\xi_{\text{LSO}} = [0.01\pi, \pi, 1.99\pi]$ orbiting Schwarzschild. These three values of ξ_{LSO} yield, respectively, the scenarios (ii) (red curve), (iii) (orange curve), and (i) (green curve) described above. Figure 11 shows that during the inspiral $\Delta E_{\text{UCO}} < 0$ and monotonically increases. When the system reaches the LSO configuration, by definition

$$E = E_{\text{LSO}} = \mathcal{E}_{\text{UCO}} = \mathcal{E}_c, \quad (\text{A3})$$

and $\Delta E_{\text{UCO}} = 0$. Note that the three curves in Fig. 11 reach this configuration having different values of the radial coordinate corresponding to the different radial phase ξ_{LSO} values.

After the LSO, the three scenarios evolve differently before plunging, but for all of them the plunge begins when $r \approx r_c$. In this configuration, ΔE_{UCO} reaches its maximum value and it starts to decrease monotonically until the end of the plunge. We define the moment for which ΔE_{UCO} is in its maximum as the “onset” of the plunge. In Fig. 11, we highlight this configuration with three horizontal dashed

lines of the same color of the related ΔE_{UCO} curves. Since the onset of the plunge happens after the LSO crossing and when $r \approx r_c$ we have $\Delta E_{\text{UCO}}^{\text{onset}} \geq 0$. As we motivate in the following, if $\Delta E_{\text{UCO}}^{\text{onset}} \rightarrow 0$ in the TM limit, then the different plunges of scenarios (i)–(iii) converge to the critical plunge geodesic. In fact, as $\nu \rightarrow 0$, the plunge becomes geodesic [47,121] and the value of the energy of the plunge geodesic freezes at the value it has at its onset, i.e., $E^{\text{plunge}} = E^{\text{onset}}$. Hence, by showing that $\Delta E_{\text{UCO}}^{\text{onset}} \rightarrow 0$ in the TM limit, then, from Eq. (A3) we get $E^{\text{plunge}} = E^{\text{onset}} \rightarrow \mathcal{E}_c$.

To study the behavior of $\Delta E_{\text{UCO}}^{\text{onset}}$ when $\nu \rightarrow 0$, it is insightful to consider the total time derivative of ΔE_{UCO} ,

$$\begin{aligned} \Delta \dot{E}_{\text{UCO}} &= \dot{E} - \dot{\mathcal{E}}_{\text{UCO}} \\ &= \dot{E} - \frac{d\mathcal{E}_{\text{UCO}}}{dL} \dot{L} \\ &= \dot{E} - \Omega_{\text{UCO}} \dot{L}, \end{aligned} \quad (\text{A4})$$

where Ω_{UCO} corresponds to the orbital frequency of the UCO. In the last passage of Eq. (A4) we employ the fact that $L(t) = \mathcal{L}_{\text{UCO}}(t)$ and that $d\mathcal{E}_{\text{UCO}}/d\mathcal{L}_{\text{UCO}} = \Omega_{\text{UCO}}$ [137]. Since both $\dot{E}$ and $\dot{L}$ scale with ν , also $\Delta \dot{E}_{\text{UCO}} \sim \nu$. Accordingly, we introduce the notation $\Delta \dot{E}_{\text{UCO}} = \nu \mathcal{F}_{\Delta E_{\text{UCO}}}$. Furthermore, we highlight the fact that the quantity $\mathcal{F}_{\Delta E_{\text{UCO}}}$ vanishes at the UCO, since on circular orbits $\dot{E} = \Omega \dot{L}$ [137].

In general, the value of $\Delta E_{\text{UCO}}^{\text{onset}} \geq 0$ and it is bounded by the value of $\Delta E_{\text{UCO}}^{\text{onset},ii}$ of scenario (ii), which represents the configuration for which $\Delta E_{\text{UCO}}^{\text{onset}}$ reaches its maximum among all the possible scenarios, as depicted in Fig. 11,

$$\begin{aligned} \Delta E_{\text{UCO}}^{\text{onset}} \leq \Delta E_{\text{UCO}}^{\text{onset},ii} &= \nu \int_{r_c}^{r_{\text{la}}} \mathcal{F}_{\Delta E_{\text{UCO}}} \frac{dt}{dr} dr \\ &+ \nu \int_{r_{\text{la}}}^{r_{\text{on}}} \mathcal{F}_{\Delta E_{\text{UCO}}} \frac{dt}{dr} dr, \end{aligned} \quad (\text{A5})$$

where r_c is the critical radius introduced in Sec. II A, r_{la} corresponds to the radial position of the last apocenter passage of scenario (ii), and r_{on} is the value of the radial coordinate of the TM when it is at the onset of the plunge. While the second integral (computed between r_{la} and r_{on}) is always finite, this is not as obvious for the first integral, since at $r = r_c$ the quantity dt/dr diverges. However, as mentioned above, at the UCO $\mathcal{F}_{\Delta E_{\text{UCO}}}$ vanishes, and we find empirically that this happens quick enough to render the integral finite and conjecture that this is true, in general, and remains bounded in the $\nu \rightarrow 0$ limit. Equation (A5) shows that $\Delta E_{\text{UCO}}^{\text{onset},ii} \sim \nu$ and since this quantity bounds $\Delta E_{\text{UCO}}^{\text{onset}}$, it holds

$$\Delta E_{\text{UCO}}^{\text{onset}} \rightarrow 0, \quad \text{when } \nu \rightarrow 0. \quad (\text{A6})$$

As introduced above, when considering the TM limit, the plunge becomes geodesic, and its energy corresponds to the

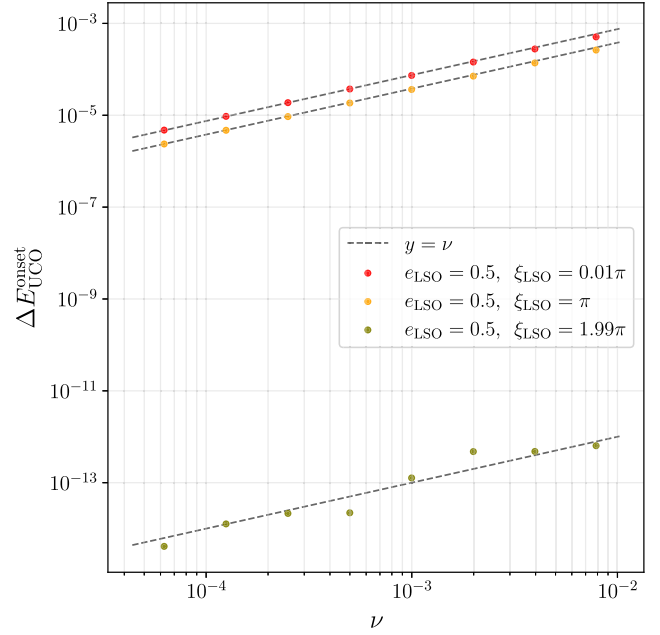


FIG. 12. Scaling of $\Delta E_{\text{UCO}}^{\text{onset}}$ with respect to the symmetric mass ratio ν . The data are shown for fixed eccentricity $e_{\text{LSO}} = 0.5$ and three configurations of the radial phase at the LSO, $\xi_{\text{LSO}} = [0.01\pi, \pi, 1.99\pi]$, respectively, represented with red, orange, and green dots. These configurations correspond, respectively, to the scenarios (ii), (iii), (i) mentioned in Appendix A. We observe a scaling $\Delta E_{\text{UCO}}^{\text{onset}} \sim \nu$ for all the different ξ_{LSO} .

value $E^{\text{plunge}} = E^{\text{onset}}$ the system has at the onset of the plunge. Since in the TM limit $\Delta E_{\text{UCO}}^{\text{onset}} \rightarrow 0$, it follows from Eq. (A3) that the plunge becomes geodesic with energy $E^{\text{plunge}} \rightarrow \mathcal{E}_c$, converging to a critical plunge geodesic with energy $\mathcal{E}_c$.

We numerically test the limit in Eq. (A6) and study the scaling of $\Delta E_{\text{UCO}}^{\text{onset}}$ by considering a set of eccentric trajectories like in Fig. 11 but with different symmetric mass ratios. The trajectories we consider have residual eccentricity at the LSO $e_{\text{LSO}} = 0.5$, different symmetric mass ratios $\nu = [0.06, 0.13, 0.25, 0.50, 1.00, 2.00, 4.00, 8.00] \times 10^{-3}$, and we examine the same three families of trajectories characterized by values of the relativistic anomaly at LSO, $\xi_{\text{LSO}} = [0.01\pi, \pi, 1.99\pi]$ that correspond, respectively, to scenarios (ii), (iii), and (i) as in Fig. 11.

For each of these trajectories, we compute the quantity $\Delta E_{\text{UCO}}^{\text{onset}}$ at the onset of the plunge, when $r \approx r_c$. In Fig. 12, we show the scaling of $\Delta E_{\text{UCO}}^{\text{onset}}$ with respect to the symmetric mass ratio ν for the three families of trajectories we considered. Red dots are the values connected to scenario (ii), orange dots to scenario (ii), and green dots to scenario (i). Figure 12 shows that when $\nu \rightarrow 0$ the excess $\Delta E_{\text{UCO}}^{\text{onset}}$ scales as a power of ν which does not depend on the value of the radial phase at the LSO, since the slope of the lines is the same for every value of ξ_{LSO} . This scaling corroborates what we described above,

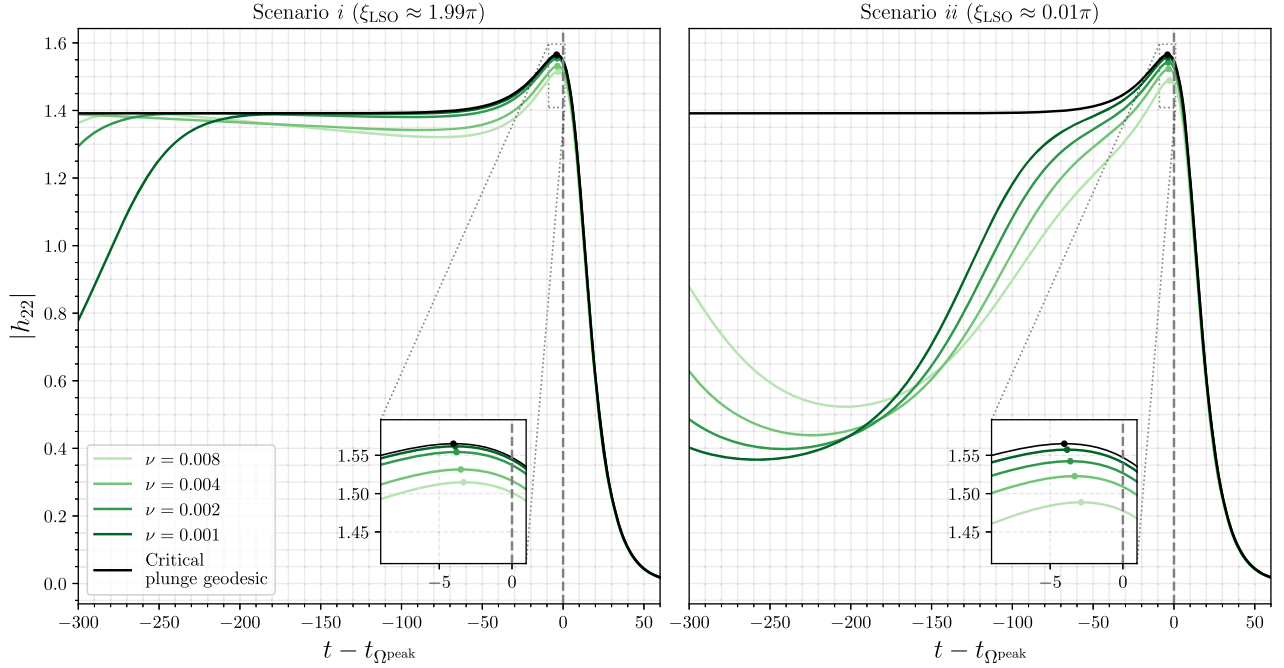


FIG. 13. Comparison of the amplitude of the h_{22} mode generated considering different trajectories with different mass ratios and same eccentricity $e_{\text{LSO}} = 0.5$ at the LSO. The different green shades represent the $|h_{22}|$ generated considering a subset of the trajectories evolved with the RR force of Ref. [94] introduced in Appendix A. We consider the subset with different symmetric mass ratios $\nu = [0.008, 0.004, 0.002, 0.001]$ and $\xi_{\text{LSO}} = [1.99\pi, 0.01\pi]$. The black curve corresponds to the amplitude of the h_{22} mode of the critical plunge geodesic having the same properties of the RR evolved trajectories at the LSO, i.e., having $e_c = e_{\text{LSO}} = 0.5$. Left: we show the case with $\xi_{\text{LSO}} = 1.99\pi$, corresponding to scenario (i) mentioned in Appendix A. Right: we show the case with $\xi_{\text{LSO}} = 0.01\pi$ corresponding to scenario (ii). With this test, we phenomenologically corroborate the fact that as the symmetric mass ratio ν decreases, the merger properties of the mode converge to the merger properties of the mode generated employing critical plunge geodesics.

that in the TM limit, the plunge converges to the class of critical plunge geodesics discussed in Sec. II A. Figure 12 also strongly suggests that $\Delta E_{\text{UCO}}^{\text{onset}}$ scales linearly with ν , i.e., $\Delta E_{\text{UCO}}^{\text{onset}} \sim \nu$.

APPENDIX B: TEST-PARTICLE LIMIT CONVERGENCE OF ECCENTRIC WAVEFORM MERGER PROPERTIES

In order to illustrate and corroborate the convergence of the merger properties of the waveforms to the merger properties of the critical plunge geodesic waveforms, we consider a subset of the trajectories mentioned in Appendix A, which are evolved through the use of the RR force of Ref. [94] and we generate the waveforms produced by these trajectories. In particular, we consider the subset of trajectories with symmetric mass ratio $\nu = [0.008, 0.004, 0.002, 0.001]$ of the two families having relativistic anomaly at LSO $\xi_{\text{LSO}} = [0.01\pi, 1.99\pi]$, which correspond respectively to scenarios (ii) and (i) introduced in Appendix A. In Fig. 13 we represent in different shades of green the amplitude of the h_{22} mode, generated considering the different symmetric mass ratios. These waveforms have the same features at LSO, i.e., the same values

of eccentricity at LSO $e_{\text{LSO}} = 0.5$. The black curve represents the h_{22} mode generated considering a critical plunge geodesic with $e_c = e_{\text{LSO}} = 0.5$, i.e., with critical eccentricity equal to the same value of eccentricity at LSO of the trajectories evolved with the radiation-reaction force of Ref. [94]. The left panel of Fig. 13 illustrates the configurations in scenario (i), i.e., with $\xi_{\text{LSO}} = 1.99\pi$, while the right panel considers scenario (ii) with $\xi_{\text{LSO}} = 0.01\pi$. From Fig. 13, we observe that, as the symmetric mass ratio ν decreases (curves with darker shades of green), the merger properties of the waveforms generated through the RR force evolved trajectories approach the merger properties of the waveform generated employing critical plunge geodesic, as highlighted by the inset plots in the figure.

From the figure, it is also possible to observe the impact of the value at LSO of the relativistic anomaly ξ_{LSO} on the merger features of the waveforms. In particular, we mention the effect of the evolution of the small mass on the UCO, when $\xi_{\text{LSO}} = 1.99\pi$. From the left panel of Fig. 13, we observe that the coordinate-time interval Δt_{UCO} the small mass spends on the UCO ranges in between $\Delta t_{\text{UCO}} \sim 200$ when $\nu = 0.001$ and $\Delta t_{\text{UCO}} \sim 300$ for the higher mass ratio case with $\nu = 0.008$. For the cases characterized by higher

values of ν , the amplitude shows a clear decreasing trend. This trend is due to the fact that, when the small mass evolution is stuck on the UCO, its orbital frequency Ω decreases and the radial coordinate r [which corresponds to the radius $r_{\text{UCO}}(t)$ of the UCO] increases with time, because of the evolution in time of the UCO configuration due to the radiated angular momentum in the emission of GWs. The decreasing trend of Ω influences the waveform properties and, in particular, it manifests with a decreasing behavior of the amplitude of the modes (and of their frequencies), as can be observed in the left panel of Fig. 13 for the cases with higher values of ν (curves with lighter shades of green). The reason why this decreasing behavior is more manifest for higher values of ν is connected to the fact that the amount of radiated angular momentum scales with ν ; consequently, these effects are more evident with higher values of ν .

Finally, another feature that we mention here, which is explored deeper in Refs. [111,112], is the fact that the merger properties of the waveforms depend on the value of ξ_{LSO} . By comparing the inset plots in the two panels of Fig. 13 it is possible to notice, for example, how the values of the peak of the amplitudes of the h_{22} mode differ for waveforms that have the same LSO configuration but not the same ξ_{LSO} . However, in the TM limit, as $\nu \rightarrow 0$, these differences vanish and the waveform's merger converges to the critical plunge geodesic merger, corroborating the fact that in this limit the eccentric aligned geodesic plunges converge to this particular class of geodesics.

APPENDIX C: COMPARISON OF Δt_{22} FOR $e_c = 0$ WITH PREVIOUS WORKS

In order to assess the consistency of our computation of the quantity Δt_{22} defined in Eq. (31) with previous results in the literature, we perform a direct comparison with the values of Δt_{22} obtained in Ref. [88], hereafter referred to as Taracchini *et al.* In that work, the authors generated waveforms for the Kerr equatorial QC case, using the same methodology employed in the present analysis, but considering trajectories evolved through numerical fluxes computed via a frequency-domain Teukolsky code [51,87]. The specific scenario considered in Taracchini *et al.* corresponds to a system with $\nu = 0.001$.

In Fig. 14, we show the comparison between the Δt_{22} values extracted in Taracchini *et al.* and the values extracted in our work considering critical plunge geodesics in the QC case, i.e., with $e_c = 0$. In Fig. 14, the black curve represents Δt_{22} computed in Taracchini *et al.*, while the dashed orange curve represents Δt_{22} extracted considering the geodesics. We observe that, for spins up to $a \sim 0.85$, the values of Δt_{22} extracted considering waveforms produced with the critical plunge geodesics are in agreement with the corresponding values computed by Taracchini *et al.*

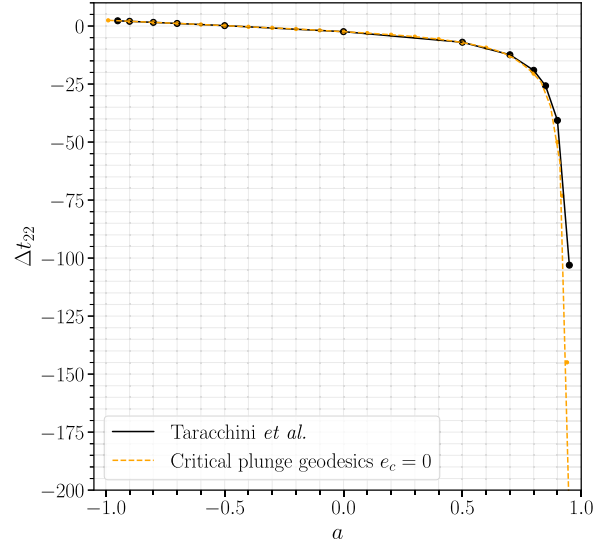


FIG. 14. In this figure we show the values of quantity Δt_{22} defined in Eq. (31). In orange, we show Δt_{22} values extracted from the h_{22} mode generated using critical plunge geodesics with $e_c = 0$. In black, we plot the same quantity as computed in Taracchini *et al.* [88].

However, for higher spin values $a > 0.85$, a discrepancy begins to emerge between the two works. We interpret this deviation as a consequence of the fact that the trajectories of the two works are different. Specifically, while Taracchini *et al.* consider trajectories at finite $\nu = 0.001$, our computation strictly corresponds to the $\nu = 0$ limit. For lower spin values ($a \lesssim 0.85$), the influence of the difference in the symmetric mass ratio on Δt_{22} remains negligible, resulting in a good agreement. As the spin increases, however, these corrections become increasingly significant, since Δt_{22} diverges for geodesics while remaining finite for finite values of ν .

APPENDIX D: MORE ECCENTRIC PLUNGES BEHAVE MORE CIRCULAR

We study the behavior of the time derivative of the radial coordinate, i.e., the quantity dr/dt of the critical plunge geodesics as a function of r . In Fig. 15, we show dr/dt for spin values $a = -0.7$ (first column), $a = 0$ (second column), and $a = 0.7$ (third column). For every spin case, we consider critical eccentricities $e_c \leq 1$. As in Ref. [88] for the QC case ($e_c = 0$), we find that the radial velocity dr/dt reaches maximum values that decrease with a , meaning that for large spins the plunge becomes more circular. Interestingly, we find that, when keeping the spin at a fixed value, the maximum values of dr/dt decrease with e_c . We remark that this result can be found in Ref. [74] for the case $a = 0$. We extended the result to the more general Kerr case. Hence, paradoxically, by increasing the critical eccentricity the plunge becomes more circular.

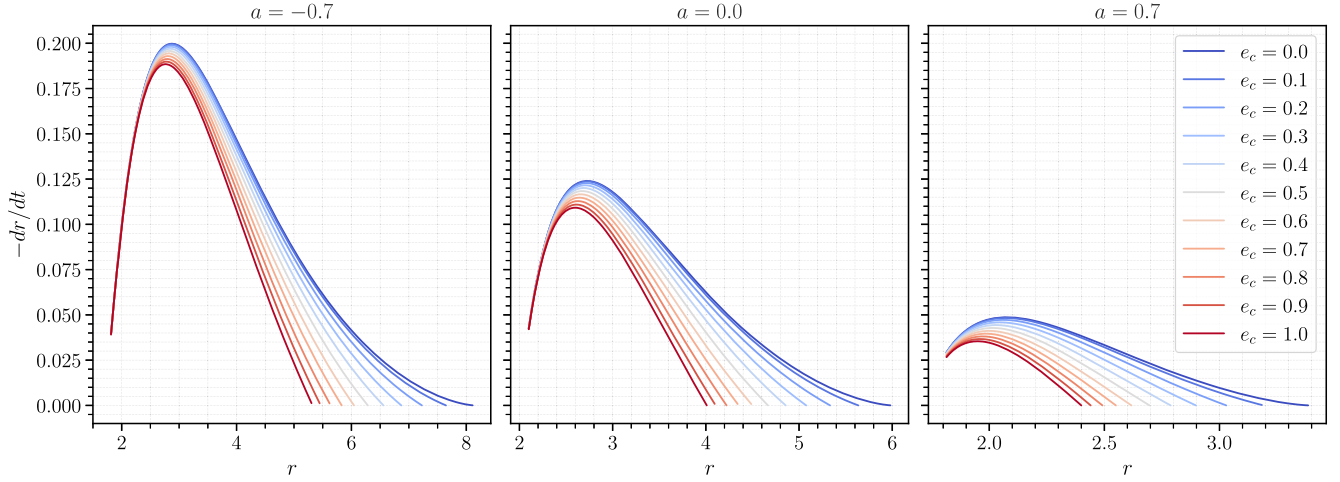


FIG. 15. In this figure we show the time derivative of the radial coordinate dr/dt of the critical plunge geodesics as function of r . We consider spin values $a = -0.7$ (first column), $a = 0$ (second column), and $a = 0.7$ (third column). For every spin case, we consider critical eccentricities $e_c \leq 1$.

APPENDIX E: HIGHER-ORDER MODES MERGER FEATURES

In this appendix, we assess what was mentioned in Sec. III A regarding the fact that the higher-order $\ell = m$ modes are characterized by a merger that has the same structure and similar trend of the h_{22} mode case. Specifically, we investigate the dependence of the waveform quantities at merger, like the mode amplitude $|h_{\ell m}^{\text{peak}}|$, the second-order time derivative of the amplitude $d^2|h_{\ell m}|/dt^2|_{\text{peak}}$, the mode frequency $\omega_{\ell m}^{\text{peak}}$, and its time derivative $\dot{\omega}_{\ell m}^{\text{peak}}$ with respect to the critical eccentricity e_c , for the range of spin values considered in our analysis.

In the first two rows of Fig. 16, we plot these quantities for the h_{33} and h_{44} modes as function of e_c . As in Fig. 7, we restrict to the region of the parameter space for which the peak of the amplitude of the h_{22} mode exists, as depicted in the blue region of Fig. 4. By comparing the results in Fig. 16 with the results in Fig. 7 of Sec. III, we confirm that the higher-order $\ell = m$ modes merger structure is similar to the structure of the h_{22} mode merger. More precisely, the values of the plotted quantities $|h_{\ell m}^{\text{peak}}|$, $d^2|h_{\ell m}|/dt^2|_{\text{peak}}$, $\omega_{\ell m}^{\text{peak}}$, and $\dot{\omega}_{\ell m}^{\text{peak}}$ have a trend similar to the same quantities of the h_{22} mode. In particular, we find that the peak of the amplitude $|h_{\ell m}^{\text{peak}}|$ exhibits a monotonic increase with e_c and spin a . The second-order time derivative of the amplitude $d^2|h_{\ell m}|/dt^2|_{\text{peak}}$ increases as e_c approaches the threshold eccentricity e_c^{thr} , indicating a flattening of the modes amplitude at merger when e_c increases. The mode frequency $\omega_{\ell m}^{\text{peak}}$ remains largely insensitive to variations in e_c , showing a nearly flat trend across all spin configurations. The time derivative of the mode frequency $\dot{\omega}_{\ell m}^{\text{peak}}$ generally follows a monotonic decreasing pattern with respect to e_c . As in the h_{22} mode case, we observe deviations from

monotonicity concerning the spin dependence, and we find that these deviations exhibit the same structure for all the $\ell = m$ modes, as can be noticed in the first two panels of the third column of Fig. 16. The fact that for fixed eccentricity $\dot{\omega}_{\ell m}^{\text{peak}}$ is not monotonic (especially for lower values of e_c) for all the $\ell = m$ modes strongly suggests this behavior is a physical feature of the waveform modes close to merger.

The third row of Fig. 16 shows the same quantities mentioned above evaluated at merger for the h_{21} mode. From the figure, it is possible to notice how the merger structure presents differences with respect to the cases where $\ell = m$. In fact, for the amplitude of the mode at the peak $|h_{\ell m}^{\text{peak}}|$ we observe a decreasing monotonic trend with respect to the spin for all the values of e_c , contrary to the increasing monotonic trend observed in the $\ell = m$ cases. Concerning the dependence on the eccentricity, we still observe that $|h_{\ell m}^{\text{peak}}|$ increases when e_c increases. We also notice a different structure regarding the quantity $d^2|h_{\ell m}|/dt^2|_{\text{peak}}$, which has significant smaller values for the retrograde spin cases compared to the prograde ones. This is connected to the fact that in the retrograde spin configurations the mixing of the mode with its mirror quasinormal mode with complex frequency $-\omega_{\ell m}^*$ is significant close to merger. This mixing is due to the frame dragging which characterizes the dynamics of retrograde spin trajectories in the regimes of the Kerr ergosphere. As a matter of fact, the resulting inversion of the azimuthal motion of the trajectories due to the dragging significantly excites the mirror quasinormal modes. For this reason, in the retrograde scenarios the mixing with the mirror modes generates a peak of the h_{21} mode which always happens at times $t > t_{\Omega^{\text{peak}}}$, as can be observed in the first panel of Fig. 8. This mode mixing also impacts the structure of the frequency $\omega_{\ell m}^{\text{peak}}$ of the mode since it has a different pattern

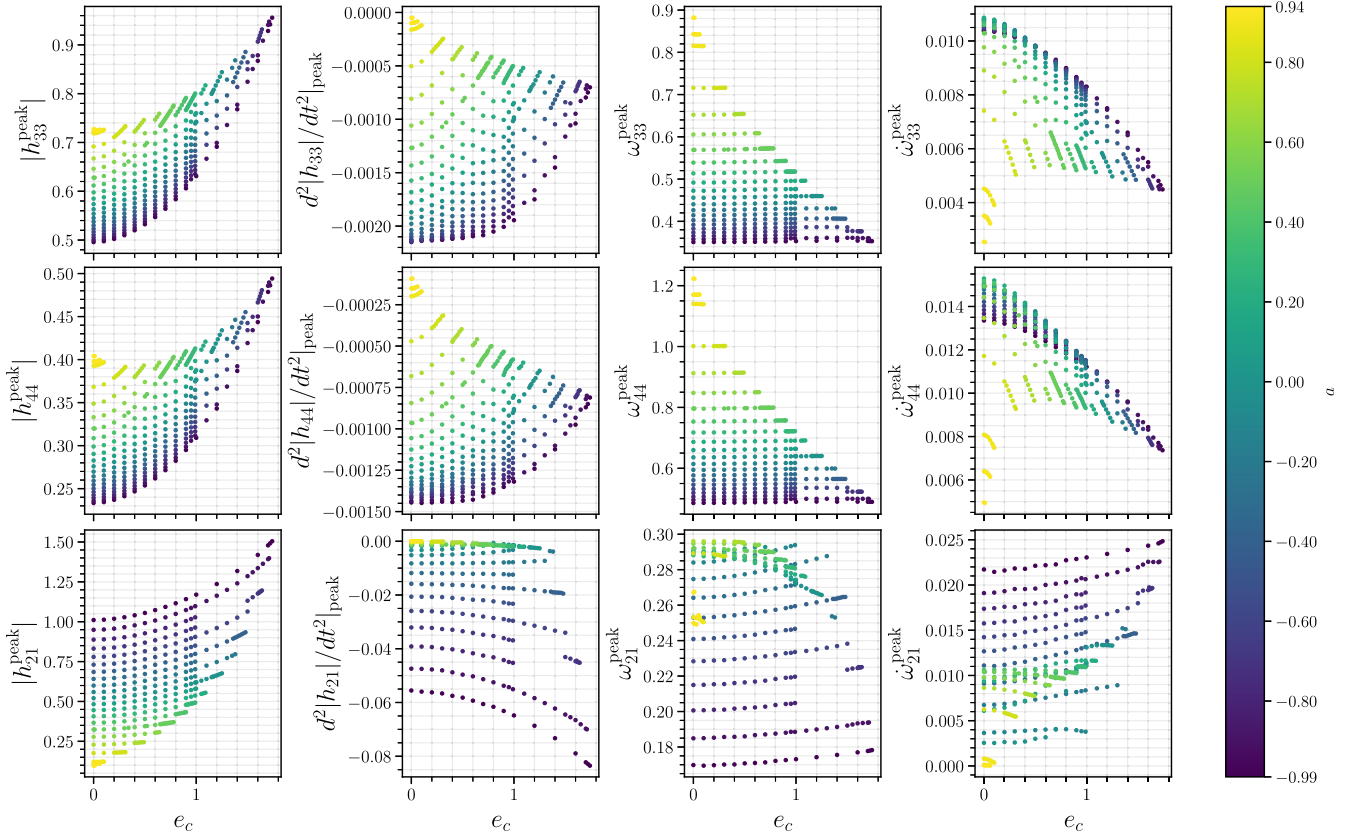


FIG. 16. In this figure we show the dependence of the amplitude $|h_{\ell m}^{\text{peak}}|$, second-order time derivative of the amplitude $d^2|h_{\ell m}|/dt^2|_{\text{peak}}$, frequency $\omega_{\ell m}^{\text{peak}}$, and first-order time derivative of the frequency $d\omega_{\ell m}^{\text{peak}}/dt$ of the h_{33} , h_{44} , and h_{21} modes evaluated at the peak of their respective amplitudes as function of the critical eccentricity e_c . We span different values of the spin $-0.99 \leq a \leq 0.99$ of the central Kerr BH, represented by different colors. We consider the section of parameter space for which the amplitude of the h_{22} mode has a peak, i.e., the blue area of Fig. 4. For spin values $a \geq 0.95$, we can no longer find a peak of the amplitude of the h_{22} mode even when $e_c = 0$, as shown in Fig. 4. These configurations are not shown in the figure.

from the frequencies of the $\ell = m$ cases and this holds also for the quantity $\dot{\omega}_{\ell m}^{\text{peak}}$. Furthermore, examining both the quantities $\omega_{\ell m}^{\text{peak}}$ and $\dot{\omega}_{\ell m}^{\text{peak}}$ in Fig. 16 it is possible to notice how the dependence on eccentricity of the merger of the h_{21} mode is different between the retrograde and prograde spin scenarios. In fact, $\omega_{\ell m}^{\text{peak}}$ and $\dot{\omega}_{\ell m}^{\text{peak}}$ increase monotonically with respect to e_c for the retrograde spin cases, while they decrease with e_c for the prograde spin scenarios. As we mentioned above, we believe this different structure between retrograde and prograde orbits, and with respect to the $\ell = m$ cases, is due to the mode mixing that affects the h_{21} mode merger and which makes its merger structure less trivial than the $\ell = m$ case. However, we found (without showing in Fig. 16) that the h_{32} and h_{43} modes have a similar merger structure as the h_{21} mode.

APPENDIX F: $\ell = 2$ $m = 2$ MERGER VALUES

In this section, we present a subset of the data for the quantities of the h_{22} mode studied in Sec. III A, a part of which is shown in Fig. 7. The subset is selected considering values of the critical eccentricity $e_c \leq 0.9$ and values of the spin $a = [-0.9, -0.7, -0.5, -0.3, -0.1, 0.0, 0.1, 0.3, 0.5, 0.7, 0.9]$. In Table II we show the values of the peak of the amplitude $|h_{22}^{\text{peak}}|$ of the h_{22} mode in the section of parameter space we consider. In Tables III–V we show, respectively, the values for the quantities $d^2|h_{22}|/dt^2|_{\text{peak}}$, $\omega_{22}^{\text{peak}}$, and $\dot{\omega}_{22}^{\text{peak}}$. In the tables, the entries containing the three dots represent regions of the parameter space where the peak of the amplitude of the h_{22} mode is not present.

TABLE II. A subset of the values of the peak of the amplitude $|h_{22}^{\text{peak}}|$ of the h_{22} mode.

e	a										
	−0.9	−0.7	−0.5	−0.3	−0.1	0.0	0.1	0.3	0.5	0.7	0.9
0.0	1.343	1.361	1.383	1.407	1.435	1.451	1.467	1.504	1.545	1.589	1.601
0.1	1.347	1.366	1.387	1.412	1.441	1.456	1.473	1.511	1.553	1.599	1.620
0.2	1.358	1.377	1.400	1.426	1.455	1.472	1.490	1.529	1.575	1.627	...
0.3	1.375	1.395	1.419	1.447	1.478	1.496	1.515	1.557	1.608	1.668	...
0.4	1.397	1.419	1.445	1.474	1.508	1.527	1.547	1.594	1.650	1.720	...
0.5	1.425	1.449	1.476	1.508	1.545	1.565	1.588	1.639	1.702	1.783	...
0.6	1.459	1.484	1.514	1.548	1.588	1.610	1.635	1.692	1.763	...	...
0.7	1.497	1.525	1.558	1.595	1.638	1.663	1.690	1.753	1.834	...	...
0.8	1.542	1.572	1.608	1.648	1.696	1.723	1.753	1.824	...	...	...
0.9	1.593	1.626	1.665	1.709	1.762	1.792	1.826	1.905	...	...	...

TABLE III. A subset of the values of the second-order time derivative of the amplitude $d^2|h_{22}|/dt^2|_{\text{peak}}$ of the h_{22} mode evaluated at its amplitude peak.

e	a										
	−0.9	−0.7	−0.5	−0.3	−0.1	0.0	0.1	0.3	0.5	0.7	0.9
0.0	−0.00327	−0.00313	−0.00297	−0.00280	−0.00258	−0.00247	−0.00230	−0.00189	−0.00130	−0.00062	−0.00003
0.1	−0.00320	−0.00311	−0.00293	−0.00278	−0.00252	−0.00243	−0.00223	−0.00183	−0.00122	−0.00055	−0.00000
0.2	−0.00315	−0.00303	−0.00289	−0.00266	−0.00243	−0.00229	−0.00210	−0.00169	−0.00107	−0.00038	...
0.3	−0.00304	−0.00292	−0.00278	−0.00256	−0.00230	−0.00213	−0.00196	−0.00150	−0.00089	−0.00022	...
0.4	−0.00292	−0.00281	−0.00259	−0.00237	−0.00211	−0.00195	−0.00176	−0.00129	−0.00064	−0.00006	...
0.5	−0.00279	−0.00263	−0.00246	−0.00222	−0.00191	−0.00174	−0.00154	−0.00103	−0.00043	−0.00000	...
0.6	−0.00260	−0.00248	−0.00226	−0.00202	−0.00168	−0.00152	−0.00130	−0.00077	−0.00021	...	...
0.7	−0.00244	−0.00229	−0.00205	−0.00179	−0.00143	−0.00126	−0.00103	−0.00051	−0.00006	...	...
0.8	−0.00226	−0.00207	−0.00184	−0.00153	−0.00118	−0.00100	−0.00075	−0.00027	...	...	...
0.9	−0.00203	−0.00183	−0.00158	−0.00128	−0.00091	−0.00071	−0.00049	−0.00010	...	...	...

TABLE IV. A subset of the values of the frequency $\omega_{22}^{\text{peak}}$ of the h_{22} mode evaluated at its amplitude peak.

e	a										
	−0.9	−0.7	−0.5	−0.3	−0.1	0.0	0.1	0.3	0.5	0.7	0.9
0.0	0.220	0.228	0.238	0.250	0.265	0.274	0.284	0.308	0.341	0.390	0.489
0.1	0.220	0.228	0.238	0.250	0.265	0.274	0.284	0.308	0.340	0.390	0.487
0.2	0.220	0.228	0.238	0.250	0.265	0.274	0.284	0.308	0.340	0.390	...
0.4	0.221	0.229	0.238	0.251	0.266	0.274	0.284	0.309	0.341	0.391	...
0.5	0.221	0.229	0.239	0.251	0.266	0.275	0.285	0.309	0.342	0.391	...
0.6	0.220	0.229	0.239	0.251	0.266	0.275	0.285	0.309	0.342	...	...
0.7	0.221	0.229	0.239	0.251	0.266	0.275	0.285	0.309	0.342	...	...
0.8	0.221	0.229	0.239	0.251	0.266	0.276	0.285	0.309	...	...	...
0.9	0.220	0.228	0.239	0.251	0.266	0.275	0.285	0.310	...	...	...

TABLE V. A subset of the values of the first-order time derivative of the frequency $\dot{\omega}_{22}^{\text{peak}}$ of the h_{22} mode evaluated at its amplitude peak.

e	a										
	−0.9	−0.7	−0.5	−0.3	−0.1	0.0	0.1	0.3	0.5	0.7	0.9
0.0	0.00574	0.00591	0.00599	0.00601	0.00596	0.00588	0.00580	0.00553	0.00497	0.00377	0.00087
0.1	0.00573	0.00589	0.00596	0.00597	0.00590	0.00582	0.00572	0.00544	0.00482	0.00354	0.00014
0.2	0.00569	0.00583	0.00589	0.00586	0.00577	0.00565	0.00554	0.00521	0.00448	0.00296	...
0.3	0.00563	0.00574	0.00576	0.00571	0.00557	0.00543	0.00530	0.00487	0.00403	0.00219	...
0.4	0.00555	0.00562	0.00559	0.00549	0.00531	0.00515	0.00498	0.00445	0.00340	0.00121	...
0.5	0.00545	0.00546	0.00540	0.00525	0.00502	0.00481	0.00461	0.00395	0.00271	0.00009	...
0.6	0.00532	0.00529	0.00516	0.00497	0.00464	0.00444	0.00418	0.00337	0.00189	...	...
0.7	0.00517	0.00507	0.00489	0.00463	0.00424	0.00399	0.00366	0.00269	0.00098	...	...
0.8	0.00498	0.00480	0.00457	0.00424	0.00378	0.00349	0.00309	0.00193	...	...	...
0.9	0.00472	0.00449	0.00419	0.00381	0.00326	0.00290	0.00245	0.00111	...	...	...

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