

Metamorphosis of fractional instantons on a twisted $\mathbb{T}^4$ with a double-trace deformation: A numerical study

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(Received 30 June 2026; accepted 3 September 2026; published 23 September 2026)

We use numerical minimization of the lattice action of trace-deformed Yang-Mills theory on $\mathbb{T}^4$ with twisted boundary conditions to find the classical minimum action configurations of fractional topological charge. We vary the twists and ratios of torus periods to interpolate between different $\mathbb{R}^{4-k} \times \mathbb{T}^k$ geometries. This allows us to see how the corresponding minimum action saddle point configurations—monopole-instantons ($k = 1$), center vortices ($k = 2$), and fractional instantons ($k = 3, 4$)—morph into each other. We also study how the transition between them depends on the presence of a deformation potential. In particular, we argue that the recent analytic picture of chains of monopole-instantons collimating their flux into center-vortex sheets, while technically relying on the deformation potential, also holds in pure Yang-Mills theory, for tori whose shape causes the Abelianization due to the deformation to align with the one due to the twists. Our results also indicate that with nonzero deformation potential, some transitions between different minimal-action fractional charge configurations may be discontinuous and involve level crossing.

DOI: [10.1103/xnjb-v184](https://doi.org/10.1103/xnjb-v184)

I. INTRODUCTION

A. Motivation and some background

Understanding the nonperturbative dynamics of four-dimensional non-Abelian gauge theories from first principles is a challenging problem lacking a complete solution. While many approaches exist, none of them can account for all interesting aspects of the physics. This paper is devoted to one particular direction of research; the study of the weak-coupling semiclassical dynamics of $SU(N)$ gauge theories on compact spaces of the form $\mathbb{R}^{4-k} \times \mathbb{T}^k$. Here $\mathbb{T}^k$ denotes a k dimensional torus with $k = 1, 2, 3, 4$. The subscript $*$ indicates the inclusion of at least one of 't Hooft twisted boundary conditions [1,2] or a double-trace deformation [3]. We use Λ to denote the strong coupling scale of the gauge theory. We take Euclidean signature in all $\mathbb{R}^{4-k} \times \mathbb{T}^k$ cases, except on a few occasions where the use of Minkowski signature is explicitly stated.

If the size of the torus $L_{\mathbb{T}^k}$ is small,¹ such that $L_{\mathbb{T}^k} N \Lambda \ll 1$, one can show that the $\mathbb{R}^{4-k} \times \mathbb{T}^k$ theory Abelianizes; the gauge group is Higgsed, by the boundary conditions or the deformation, $SU(N) \rightarrow U(1)^{N-1}$ or $SU(N) \rightarrow \mathbb{Z}_N$, at a high scale $\frac{1}{L_{\mathbb{T}^k} N} \gg \Lambda$, ensuring that the theory is weakly coupled. Let us be slightly more precise; for $k = 1, 2$, the long-distance $\mathbb{R}^3$ or $\mathbb{R}^2$ theory is weakly coupled because of the Abelianization. For $k = 3, 4$, the theory on $\mathbb{R} \times \mathbb{T}^3$ or $\mathbb{T}^4$ is weakly coupled at small $L_{\mathbb{T}^k}$ even without the nontrivial 't Hooft twists or deformation—this is the “femtouniverse” of [4,5]. However, now the small- $L_{\mathbb{T}^k}$ and large- $L_{\mathbb{T}^k}$ theories are believed to not be continuously connected in the sense discussed below. The reason for our cautious statement is that the continuous classical vacuum degeneracy in compactifications without 't Hooft twists or deformations complicates² the semiclassical study of the ground state properties of the femtouniverse—a task that has not yet been completed, to the best of our knowledge. For this reason, we only study $\mathbb{R}^{4-k} \times \mathbb{T}^k$ theories with 't Hooft twists and/or deformation.

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¹For simplicity, the discussion here only refers to the overall size of the $\mathbb{T}^k$, denoted by $L_{\mathbb{T}^k}$, tacitly assuming that all periods of the torus are of the same order. Later, we relax this assumption, considering different ratios of periods of $\mathbb{T}^k$.

²For recent work on this problem in theories with supersymmetry, see Ref. [6].

The weak coupling at small $L_{\mathbb{T}^k}$ permits the use of theoretically controlled semiclassical methods to study ground state properties, θ -angle dependence, symmetry realization, spectra, etc.; various aspects are reviewed in [7–9]. Remarkably, it is found that the properties of the ground state of the pure gauge theory, obtained analytically at small $L_{\mathbb{T}^k}$, evolve continuously into the ones seen in the strongly-coupled $L_{\mathbb{T}^k} \rightarrow \infty$, or $\mathbb{R}^4$, limit. Demonstrating this continuity, known as “adiabatic continuity,” requires, at the current state of the art, the use of numerical simulations to leave the weakly-coupled small- $L_{\mathbb{T}^k}$ regime; for certain quantities in supersymmetric theories, one can appeal to nonrenormalization theorems (see Ref. [10]) to argue for small to large volume continuity. Here, we shall not review further details of the semiclassical dynamics nor discuss different ideas to connect the semiclassical and the strongly coupled limits; for this, we recommend the reviews cited above.

The purpose of this paper is to study the relation between the nontrivial saddles in the path integral on $\mathbb{R}^{4-k} \times \mathbb{T}_*^k$ for different k . These are finite action instantons contributing to the nonperturbative dynamics. We begin by listing the kind of finite action configurations that give the leading semiclassical contribution to the nonperturbative dynamics. We use the antisymmetric twist tensor “ $n_{\mu\nu}$,” defined (mod N) in [1], to denote the imposition of nontrivial twisted boundary conditions in the $\mathbb{T}^k$ (without, at this point, being explicit about the particular choice made). Likewise, we use “def.” to denote the addition of a double-trace deformation [3]. The upshot is that, for the different values of k , the pattern of gauge symmetry breaking, the kinds of minimal action saddles, their “quantum numbers” (e.g., topological charge Q), as well their localization properties, are found to be as follows:

$$\begin{aligned} \mathbb{R}^3 \times \mathbb{S}_{\text{def.}}^1 : SU(N) \rightarrow U(1)^{N-1}, \quad N \text{ kinds of monopole-instantons}, \quad Q = \frac{1}{N}, \\ \text{localized in } \mathbb{R}^3 \text{ and wrapped around the } \mathbb{S}^1, \\ \text{with magnetic charges under } U(1)^{N-1} \text{ labeled by } (\alpha_0, \alpha_1, \dots, \alpha_{N-1}), \end{aligned} \quad (1)$$

$$\begin{aligned} \mathbb{R}^2 \times \mathbb{T}_{n_{\mu\nu}}^2 : SU(N) \rightarrow \mathbb{Z}_N, \quad \text{center vortices}, \quad Q = \frac{1}{N}, \\ \text{localized in } \mathbb{R}^2 \text{ and wrapped around } \mathbb{T}^2, \end{aligned} \quad (2)$$

$$\begin{aligned} \mathbb{R} \times \mathbb{T}_{n_{\mu\nu}}^3 : SU(N) \rightarrow \mathbb{Z}_N, \quad \text{fractional instantons}, \quad Q = \frac{1}{N}, \\ \text{localized in } \mathbb{R} \text{ and wrapped around } \mathbb{T}^3, \end{aligned} \quad (3)$$

$$\begin{aligned} \mathbb{T}_{n_{\mu\nu}}^4 : SU(N) \rightarrow \mathbb{Z}_N, \quad \text{fractional instantons}, \quad Q = \frac{1}{N}, \\ \text{localized or extended in some or all directions, depending on } n_{\mu\nu} \text{ and ratios of } \mathbb{T}^4 \text{ periods.} \end{aligned} \quad (4)$$

In each case, we have assumed that the choice of $n_{\mu\nu}$ twists is such that the minimal $Q = \frac{1}{N}$; this choice shall be made more explicit in the body of the paper. Let us now discuss various aspects of (1)–(4):

- (1) For all values of k , the minimum action semiclassical objects have topological charge $1/N$. For $k=1$, this has been known since the classification of finite action configurations on $\mathbb{R}^3 \times \mathbb{S}^1$ of [11], which implies that $Q = 1/N$ at the center-symmetric point [3]. For $k=2, 3$, the existence of $Q = 1/N$ fractional instantons on $\mathbb{R} \times \mathbb{T}_{n_{\mu\nu}}^3$ and $\mathbb{R}^2 \times \mathbb{T}_{n_{\mu\nu}}^2$ has been realized already

in [12–15].³ For $k=4$, ’t Hooft argued [1,2] that $Q = 1/N$ for some choices of twists, later studied in great detail in [16]. These instantons are exactly self-dual (or, for $k=1$ with no supersymmetry, approximately self-dual). Thus, their semiclassical contribution to the path integral

³See Appendix B for a classification argument of finite action configurations on $\mathbb{R}^2 \times \mathbb{T}_{n_{12}}^2$ (and, by a simple extension, to $\mathbb{R} \times \mathbb{T}_{n_{12}}^3$). While the discussion of this semi-infinite volume limit classification seems absent in the literature, it appears to be known to many and we include it for completeness.

- scales as $\exp(-\frac{8\pi^2}{g^2 N})$, where g^2 is the gauge theory coupling at a scale of order $L_{\mathbb{T}^k}$. As indicated, these finite action configurations are localized in the noncompact $\mathbb{R}^{4-k}$ and have size of order $L_{\mathbb{T}^k}$.
- (2) Notice that for $k = 2, 3, 4$, there is a single, up to moduli, minimal action configuration with $Q = 1/N$. For $k = 1$, on the other hand, there are N different objects of fractional topological charge, distinguished by their $U(1)^{N-1}$ “magnetic” charges. These are labelled by the simple roots, $\alpha_1, \dots, \alpha_{N-1}$, of $SU(N)$, along with the affine root $\alpha_0 = -(\alpha_1 + \alpha_2 + \dots + \alpha_{N-1})$. These monopole-instantons are the N “constituents” of the Belavin-Polyakov-Schwarz-Tyupkin (BPST) $Q = 1$ instanton [17,18].
 - (3) In each of the cases with $k < 4$, a dilute gas of the relevant $Q = 1/N$ instantons disorders large Wilson loops in the noncompact $\mathbb{R}^3$ [case (1)] or $\mathbb{R}^2$ [case (2)], leading to area law and confinement. For $k = 1$, see Ref. [3]. For $k = 2$, confinement by center vortices is shown in [19]. For $k = 3$, Refs. [12,13] showed that tunnelling due to fractional instantons causes the correlator of two winding Wilson loops, separated in $\mathbb{R}$, to obey the area law (this is recently reviewed in Sec. 6 of [20]). In all cases, the string tensions in the pure gauge theory, computed in the leading semiclassical approximation [for case (1) see Ref. [21]] with exponential-only accuracy, are proportional to $\exp(-\frac{8\pi^2}{g^2 N})$.
 - (4) The $\mathbb{R}^3 \times \mathbb{S}_{\text{def}}^1$ and $\mathbb{R}^2 \times \mathbb{T}_{n_{\mu\nu}}^2$ cases, (1) and (2), are examples of semiclassically tractable mechanisms of confinement due to monopole-instantons and center vortices, respectively. These mechanisms have been extensively discussed in the lattice community, in the strong coupling regime of a large $\mathbb{T}^4$. There, monopoles and center vortices are identified after appropriate gauge fixing of lattice configurations belonging to an ensemble generated via Monte Carlo techniques. It is observed that removing the configurations containing center vortices/monopoles from the ensemble destroys the area law. There are many important details; for these and references, we recommend Greensite’s monograph [22].⁴
 - (5) We will not dwell much on the $k = 4$ case in this introduction. This is because the study of $Q = 1/N$ configurations on $\mathbb{T}_{n_{\mu\nu}}^4$ is really the subject of most of the rest of the paper. Here, we only mention that fractional instantons on $\mathbb{T}_{n_{\mu\nu}}^4$ have been used to calculate the gaugino condensate in super-Yang-Mills theory [10,24,25], showing complete

agreement (owing to the supersymmetric nonrenormalization theorems) with the $\mathbb{R}^4$ results reviewed in [26].

- (6) Our final remark is that there are few known analytic solutions with $Q = 1/N$ in the geometries (1)–(4). In fact, there are only two kinds; the N Bogomolnyi-Prasad-Sommerfeld (BPS) monopole instantons on $\mathbb{R}^3 \times \mathbb{S}^1$ [17] and ’t Hooft’s constant field strength fractional instantons on $\mathbb{T}^4$ [27]. Most of what is known about the saddle points mentioned in (1)–(4) has been learned by numerically minimizing lattice actions with ’t Hooft twists, largely due to the long-term efforts of the Madrid group [9]. In its use of numerical tools, this paper is not an exception.

After a further look at the various semiclassical objects appearing in (1)–(4) it should perhaps not come as a surprise that they can be related to each other. This is easiest to contemplate for the $k = 2, 3, 4$ cases, where the only “quantum number” is the topological charge $Q = 1/N$. Thus, at least naïvely (however, see Sec. VI), one imagines compactifying a center vortex, a 2D sheet wrapped on $\mathbb{T}^2$, as in (2), on an $\mathbb{S}^1$ orthogonal to the sheet, obtaining (3); this can then be further compactified to obtain (1). Conversely, one could begin with a $\mathbb{T}^4$ with appropriately chosen $n_{\mu\nu}$ twists (giving the minimal $Q = 1/N$) and then take limits of its periods such that it approximates any one of the cases with $k = 1, 2, 3$. This may be especially clear if one wants to obtain the $k = 2, 3$ cases, where the same twist in the $\mathbb{T}^2$ or $\mathbb{T}^3$ appearing in (2) or (3) can be imposed on the original $\mathbb{T}^4$ —so that one obtains the desired $\mathbb{R}^2 \times \mathbb{T}_{n_{\mu\nu}}^2$ or $\mathbb{R} \times \mathbb{T}_{n_{\mu\nu}}^3$ in the limit.⁵

Obtaining $\mathbb{R}^3 \times \mathbb{S}_{\text{def}}^1$ from (2)–(4), however, is bound to be a little trickier. First of all, as mentioned above, now there are N monopole-instantons with different $U(1)^{N-1}$ magnetic charges, as per (1), instead of a single $Q = 1/N$ fractional instanton; we call this the “multiplicity problem.” Second, if there is no deformation on the $\mathbb{T}^4$ one starts with, there is no way to strictly obtain $\mathbb{R}^3 \times \mathbb{S}_{\text{def}}^1$, a setup where Abelianization only occurs because of the deformation.⁶ These issues were tackled by the authors of [29,30], who proposed replacing $\mathbb{R}^3 \times \mathbb{S}_{\text{def}}^1$ by $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}_{\text{def}}^1$ and realized that adding a ’t Hooft twist in the thus-obtained two-torus $\mathbb{S}^1 \times \mathbb{S}_{\text{def}}^1$ solves the “multiplicity problem.” This is because an $\mathbb{S}^1$ -period translation in the presence of a

⁴For more recent work discussing newer developments on the use of effective models of ensembles of unoriented center vortices, see Ref. [23], also containing an updated list of references.

⁵A twist involving a $\mathbb{T}^4$ direction taken to infinity will not be relevant and serves only to select the desired boundary condition, one of the many possible, at $\mathbb{R}^2$ or $\mathbb{R}$ infinity; see Appendix B.

⁶As explained in Sec. IV, at finite $\mathbb{T}^4$ periods, Abelianization around a localized fractional instanton can occur without deformation potential, due to the twists alone, provided the torus periods satisfy particular relations. In some cases, this Abelianization is aligned with the one due to the deformation (however, this does not occur in the strict $\mathbb{R}^3 \times \mathbb{S}^1$ limit; see Ref. [28]).

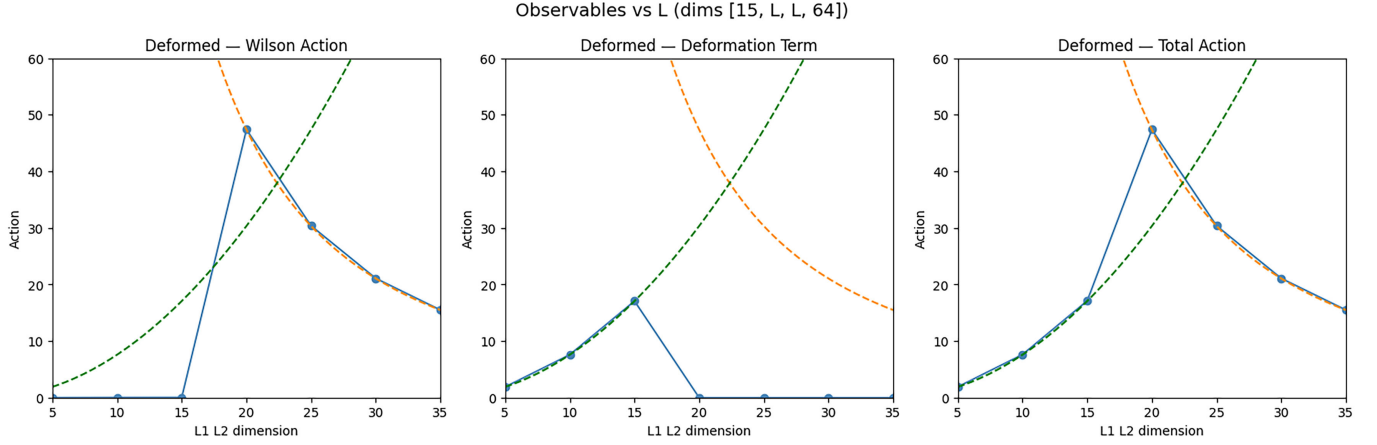


FIG. 1. The Wilson action, the deformation action, and the total action, computed for the numerically determined minimal action configuration for dYM on a $(15, L, L, 64)$ lattice, for different $L = 5, \dots, 35$. On each plot, we show the two theoretical continuum values of the actions for the flux and no-flux states of Eq. (18) as function of $L = L_1 = L_2$; the curve rising towards the origin is the action of the flux vacuum, S_{flux}/A , which is energetically favored at large L , while the curve rising to the right shows the action of the no-flux vacuum, $S_{\text{no-flux}}/A$, favored at small L . The total action plot shows that there is a transition from the no-flux vacuum, at $L < L_c$, to the flux vacuum, at $L > L_c$ at $L_c = 1.5L_0 \sim 20$. The agreement of the action determined numerically with the theoretical curves is very good, see also Table I for the numerical values.

't Hooft twist is a center symmetry transformation in the orthogonal $\mathbb{S}_{\text{def}}^1$ direction. This center symmetry cyclically permutes the N different monopole-instantons [31], thus causing them to alternate on the covering space of the $\mathbb{S}^1$, in effect creating a single object involving a monopole-instanton and all its N images under the $\mathbb{S}^1$ translation. References [29,30] then used analytic tools familiar from $\mathbb{R}^3 \times \mathbb{S}^1$ studies of deformed Yang-Mills theory [3] and classical electrodynamics to show that this single object acts like a center vortex; it localizes in the noncompact $\mathbb{R}^2$, is wrapped around $\mathbb{S}^1 \times \mathbb{S}_{\text{def}}^1$, and disorders the Wilson loops surrounding it. The resulting picture is shown, for $N = 2$, on our Fig. 2, referring the reader to Sec. VA for details.

The monopole-instanton/center-vortex continuity is an interesting new development, showing, in a theoretically controlled semiclassical framework, that these two confinement mechanisms are related. We note, however, that the relation between center vortices and monopole-instantons in the strong coupling regime has appeared earlier in the lattice literature—see Ref. [22], in particular Chap. 8 there, where pictures of gauge-fixed lattice configurations appear, identical to our Fig. 2.

However, it is satisfying to have a setup where a controllable analytic argument is available; the semiclassical limit on small compact spaces gives a clear sense as to why these configurations dominate the path integral—in contrast with the phenomenological models of the strong coupling regime. Furthermore, weak coupling semiclassics on $\mathbb{R}^{4-k} \times \mathbb{T}_*^k$ allows one to treat properties and theories which are either very challenging or simply intractable at the current level of development of lattice techniques. These include the multibranched structure and associated

θ -dependence of the vacuum⁷ and the study of theories with massless fermions in various representations, including chiral gauge theories and general supersymmetric theories, as in e.g., Refs. [34–38]. As a concrete application of the recent developments, the picture of [29,30] was used to explain some puzzles regarding confinement in supersymmetric theories [39,40]. A more speculative remark [30] is that viewing the monopole-instantons as BPST instanton constituents—recalling Comment 2 after Eqs. (1)–(4)—suggests that the continuity is a hint that both monopole-instantons and center vortices may lurk inside the gas of four dimensional instantons, in a way waiting to be made more concrete; see Refs. [41,42] for related developments.

In conclusion of this overview, we hope to have conveyed the idea that the study of how the different non-perturbative saddles appearing in (1)–(4) morph into each other is of interest, as it sheds light on the relation between the confinement mechanisms operating in the various geometries, and perhaps, ultimately, in the $\mathbb{R}^4$ limit. We now continue to discuss in more detail the scope and results of this paper.

B. Overview and summary of results

In this paper, we further study the relation—or “metamorphosis”—between the various minimal action instantons in the geometries (1)–(4). We focus on $SU(2)$ Yang-Mills theory with a double-trace deformation potential, which we abbreviate as dYM, the deformed Yang-Mills theory of Ünsal and Yaffe [3]. We use numerical

⁷Required by consistency with various generalized anomalies [32] (these are described in the old-fashioned language close to the one used here in [33]).

minimization of the lattice action of dYM on $\mathbb{T}^4$, subject to twisted boundary conditions. To motivate the use of numerics, we note that the analytic tools used in [29,30] to study the monopole-instanton/center-vortex continuity apply in the limit where $L_{\mathbb{S}^1} \gg L_{\mathbb{S}^1}^{\text{def.}}$ (see Sec. V A). One of our goals is to use numerical methods to study the construction away from this limit. These also allow us to explore the core of the solution and the shape of the center vortex obtained from the monopole-instanton chain of Fig. 2. In addition, we are also able to study the transition between the other configurations in some detail, e.g., between (2) and (3), as alluded to in Sec. I A.

The continuity between different fractionally charged semiclassical configurations on $\mathbb{T}^4$ was previously studied

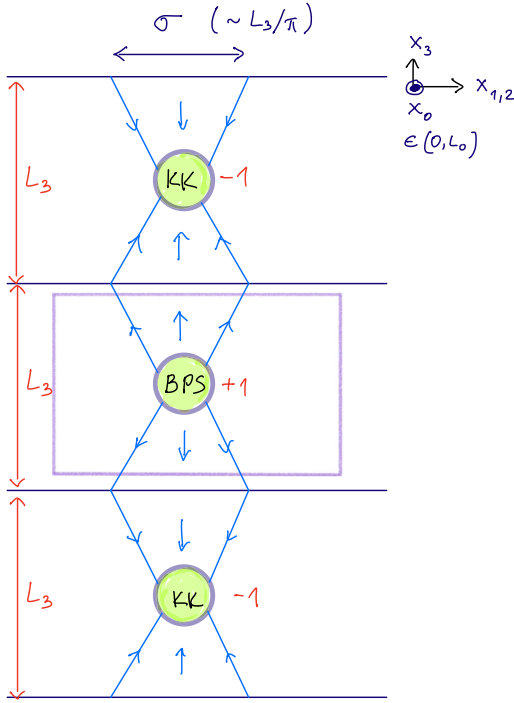


FIG. 2. A sketch of the lining up of BPS (charge α_1) and its image Kaluza-Klein (KK) (charge $\alpha_0 = -\alpha_1$, $N = 2$) monopole-instantons along the compact x_3 direction and the collimation of their flux into a center vortex sheet. There is cylindrical symmetry in the 12 plane and the small x_0 direction perpendicular to the page is not shown (there is little x_0 dependence in the actual field configuration). Blue arrows indicate the long-range Abelian “magnetic” field. The flux of the charge +1 BPS monopole-instanton is absorbed by its charge -1 KK image upon a period translation, etc. The infinite chain of images collimates the flux into a finite radius $\sigma \sim L_3/\pi$ in the 12 plane. The configuration behaves like a center vortex sheet, wrapped around the x_3 and x_0 directions, and localized in x_1, x_2 . For the actual configurations, compare the fundamental domain of the 13 plane (indicated by a square surrounding the BPS monopole-instanton) with the field distribution on Fig. 8(a). That the radial profile of the collimated flux in the 12-plane is Gaussian is shown on Fig. 8(b). The center vortex maximal radius σ is determined in Fig. 9(a) and the disordering of Wilson loops is shown on Fig. 9(c).

using cooling (see Refs. [43–45]) to find the lattice action minima [28], albeit in pure gauge theory, without the addition of a deformation potential. An important technical point is that the addition of the nonlocal deformation potential requires the use of a different method to find the minima of the action. We reduce the action using a gradient flow method, similar to the Hamiltonian evolution phase of hybrid Monte Carlo, as described in Appendix A. dYM has been the subject of Monte Carlo simulations, e.g., Refs. [46–50], but to the best of our knowledge, ours is the first study devoted to finding classical minimal action instanton configurations in dYM.

Section II: Definitions. The paper begins by presenting the lattice action of $SU(2)$ dYM. To describe our results, we now briefly go over the main features of the setup. The double trace deformation is added in the x_0 direction of period L_0 . The scaling of the deformation term, with coefficient equal to $\frac{c}{L_0^3}$ [see Eq. (6)] is motivated by the continuum one-loop expression familiar from $\mathbb{R}^3 \times \mathbb{S}^1$ studies [3,51]. In our simulations, we consider two choices of the dimensionless coefficient c of the double-trace deformation: $c = 1$ and $c = 0$. The latter choice is equivalent to pure Yang-Mills, which has been previously studied (e.g., Ref. [28]), and here is used to generate minimum action configurations to compare with $c = 1$ dYM vacua. The other directions are labelled $x_{1,2,3}$, of respective lengths $L_{1,2,3}$. We always impose a nontrivial twist in the 12 plane, $n_{12} = 1$. For the purpose of finding $|Q| = 1/2$ instantons, we turn on another nontrivial twist, $n_{03} = 1$. Finally, in all our simulations we take $L_1 = L_2$ and vary L_1 and L_3 , keeping L_0 small and fixed. This setup allows us to interpolate between the geometries (1)–(4), with the deformation in x_0 present or not.

Section III: Level crossing in dYM. We begin by studying the ground state of dYM on a spatial $\mathbb{T}^3_{(L_0, L_1, L_2)|n_{12}=1}$, with x_3 considered as time and with the only nonzero ’t Hooft twist $n_{12} = 1$. This single-twist setup explores $Q = 0$ minimum action configurations on $\mathbb{T}^4$, i.e., classical ground states. Earlier, two local minima of the energy of dYM $\mathbb{T}^3_{(L_0, L_1, L_2)|n_{12}}$ were proposed [52] as the candidate ground states: the “flux” and “no-flux” states (see Ref. [53] for earlier discussion of the flux state) Their relevant properties are described in Sec. III. As the names suggest, there is physical magnetic flux in one of these states [the “flux” one, where $SU(2) \rightarrow U(1)$] and none in the other [the “no-flux” one, where $SU(2) \rightarrow \mathbb{Z}_2$].

The main result of Sec. III is our Fig. 1. It presents numerical evidence that either the “flux” or “no-flux” state is the global energy minimum in dYM, for any shape of $\mathbb{T}^3_{(L_0, L_1, L_2)|n_{12}=1}$, with a level crossing occurring at a critical value $(L_1/L_0)_c \simeq 1.5$. For L_1/L_0 larger than the critical value, the “flux” vacuum is the minimum energy state, while for smaller values of the ratio, it is the “no-flux” vacuum. This

level crossing has implications on the transition between center vortices on $\mathbb{R}^2 \times \mathbb{T}^2$, as per (2), and fractional instantons on $\mathbb{R} \times \mathbb{T}^3$, (3), in dYM, the subject of Sec. VI.

Section IV: Reminder on pure YM theory. Here, we recall some known properties of minimum action configurations in pure YM theory on $\mathbb{T}^4$ with twists, with either a single twist ($Q = 0$) or with two twists ($|Q| = 1/2$). The most important point is that, depending on the ratio between torus periods, Abelianization around a $|Q| = 1/2$ instanton localized in some directions can also occur in pure YM theory, as noted in [54,28]. This implies that there are cases where the Abelianization due to the deformation potential in dYM and due to the twist in YM align. We shall see later in the paper that, whenever this happens, there is little qualitative difference between $|Q| = 1/2$ instantons in YM and dYM, with the deformation having the main effect of slightly raising the action above the BPS limit of $4\pi^2$.

Section V: Center vortex/monopole-instanton continuity. Here, we study the transition between fractional instantons on different $\mathbb{T}^4$ approximating $\mathbb{R}^3_{(x_1, x_2, x_3)} \times \mathbb{S}^1_{(x_0)}$ and $\mathbb{R}^2_{(x_1, x_2)} \times \mathbb{T}^2_{(x_0, x_3)}$. As usual the nonzero twists are n_{12} and n_{03} . To study this transition, in our numerical minimization, we keep L_0 small, taking $L_1 (= L_2)$ large—but by necessity finite, as opposed to the $\mathbb{R}^2$ of [29,30]—and vary L_3 from large to small, interpolating between lattices approximating $\mathbb{R}^3 \times \mathbb{S}^1$ and $\mathbb{R}^2 \times \mathbb{T}^2$, our Eqs. (1) and (2). Using numerics, we are able to relax the condition $L_3 \gg L_0$, needed to justify the analytical tools of [29,30], and thus show that the continuity persists all the way to small $L_3 \gtrsim L_0$. The dYM vacuum surrounding the localized $|Q| = 1/2$ configurations of Sec. V, for sufficiently large L_3 , is always the “flux” $SU(2) \rightarrow U(1)$ vacuum, as we always have large $L_1 > 1.5L_0$, hence the level crossing to the “no-flux” vacuum plays no role. However, for small $L_3 \sim L_0$, the

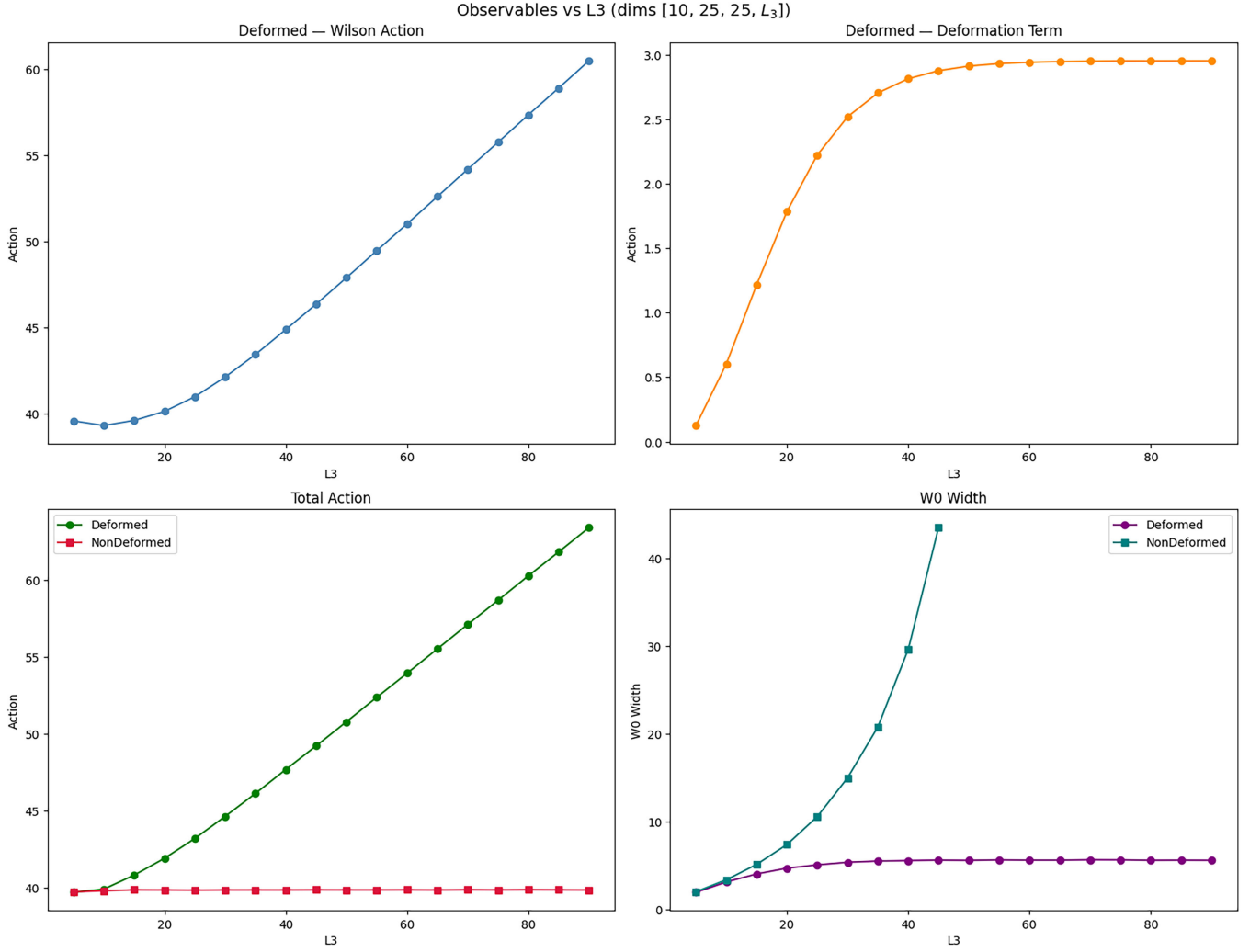


FIG. 3. Properties of the $|Q| = \frac{1}{2}$ fractional instantons for a lattice of size $(10, 25, 25, L_3)$, for $5 < L_3 < 90$. Top row: separate plots of the Wilson action and the deformation action for dYM. Bottom row: the total action and the width of the instanton, determined by $\text{tr } W_0$, for dYM and YM. For a discussion, see the text. For pure YM, at large L_3 , the 12-plane width of the Higgs field (as per Fig. 5) becomes larger than L_1 , which really signifies failure to Abelianize, as seen on the rhs plots of Fig. 7.

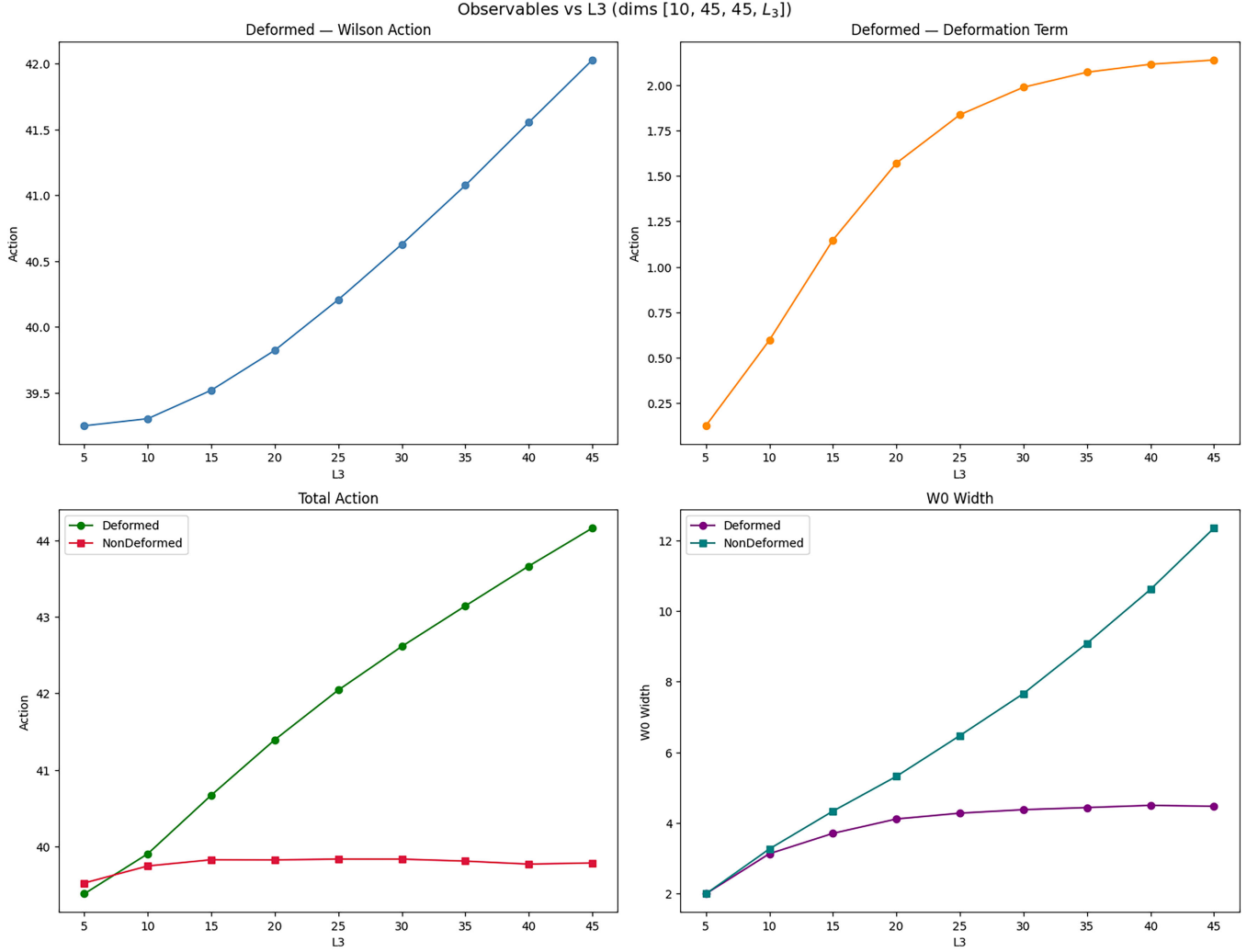


FIG. 4. Similar to Fig. 3 but for $L_1 = 45$; properties of the $|Q| = \frac{1}{2}$ fractional instantons for a lattice of size $(10, 45, 45, L_3)$, for $5 < L_3 < 45$. Top line: separate plots of the Wilson action and the deformation action for dYM. Bottom line: the total action and the width of the instanton, determined by $\text{tr } W_0$, for dYM and YM.

vacuum surrounding the center vortex in the 12 plane breaks $SU(2) \rightarrow \mathbb{Z}_2$ instead.⁸

We now summarize the results of the numerical study of Sec. V, presented on Figs. 3–10:

- (1) For the small values of $L_3 \gtrsim L_0$, the Abelianizations in YM and dYM align (as per the discussion of Sec. IV) and there is no qualitative difference between the corresponding $|Q| = 1/2$ center-vortex configurations. The small- L_3 similarity between YM and dYM minimal action configurations is clearly seen on Figs. 3, 4, 7, and 8(a), the Gaussian fit on Fig. 9 of the center-vortex profile (discussed below), as well as on Fig. 10. The difference

⁸This is yet another “no-flux” vacuum, but this time on a spatial $\mathbb{T}^3_{(L_0, L_1, L_3)}|_{n_0=1}$, with L_2 considered as time (thus relevant for large L_2). There are two zero-energy classical vacua, which are the global minima of both YM and dYM (with the deformation still in the x_0 direction) for any ratio of periods $L_{0,1,3}$.

between YM and dYM appears at larger values of L_3 —where the analysis of [29,30] is operative—here dYM Abelianizes, but the YM configurations are delocalized. This is seen upon comparing the two columns of Fig. 7, as well as the two rows of Fig. 10.

- (2) To study the collimation of the flux of the monopole-instanton chain into a center vortex, on Fig. 8(b) we show how the dYM monopole-instanton flux profile collimates into a localized center vortex, upon increasing $L_1 L_2$. On Figs. 9(a) and 9(b), for both dYM and YM on the same size lattice, we fit the center-vortex profile to a Gaussian and estimate its width, which agrees qualitatively with the exponential falloff estimate, $\exp(-\frac{r\pi}{L_3})$, seen for $L_3 \gg L_0$ and in the $\mathbb{R}^2_{(x_1, x_2)}$ case (with r the radial coordinate in $\mathbb{R}^2$) in [29,30]. We observe that the geometry where the fit was done is one where there is little qualitative

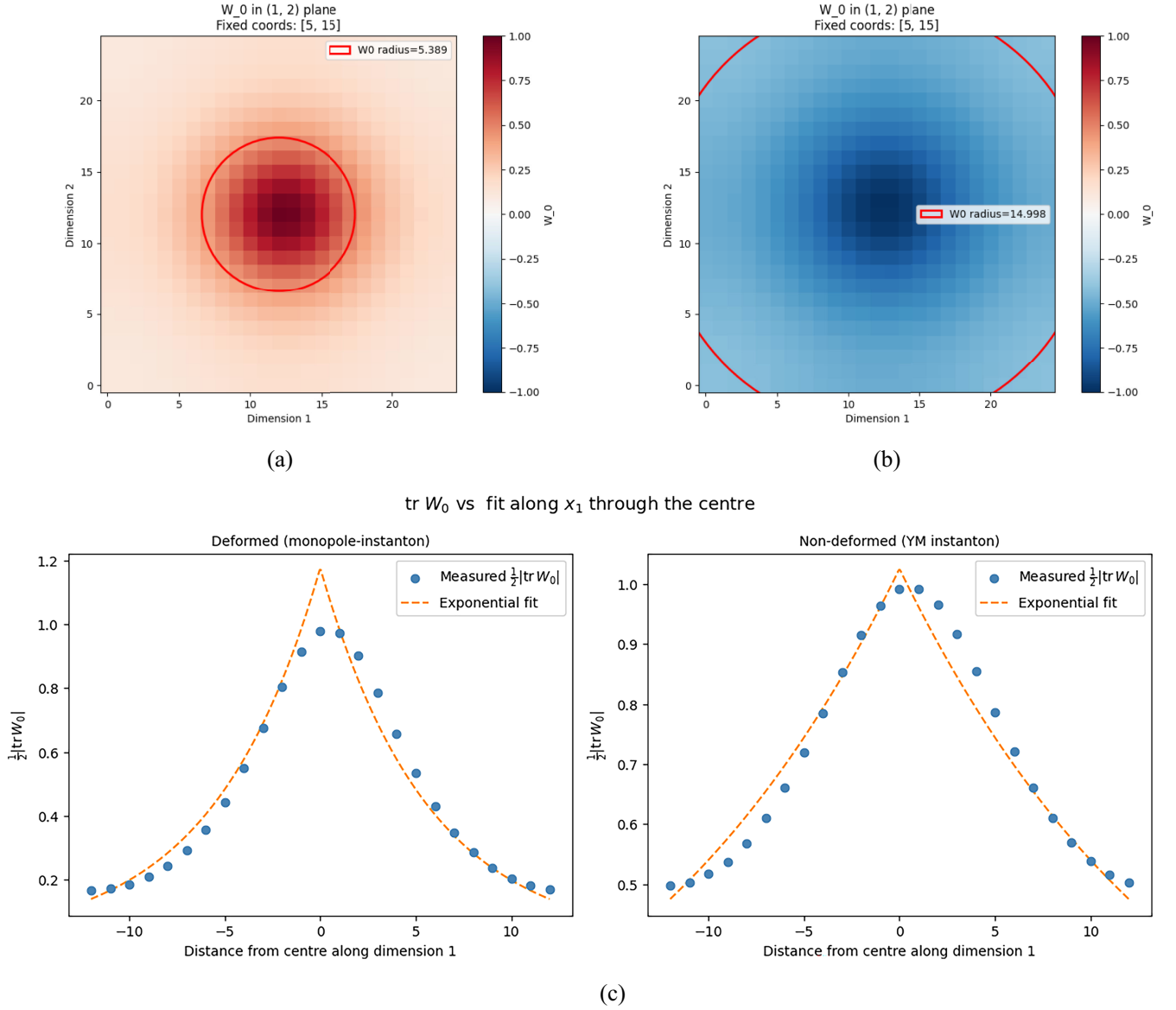


FIG. 5. Defining the localization radius of the “Higgs field”; $\text{tr} W_0(x_1, x_2)$ is fitted to the radially symmetric function $f(|x|) = ae^{-b|x|}$. The “localization radius” is then defined as the value of r where $\int_{|x|<r} d^2x f(|x|) = \frac{1}{4} \int_0^{L_1} dx_1 \int_0^{L_2} dx_2 f(|x|)$. The large radius for the YM background shown on the top right figure is because the fitting function does not represent well the actual minimum action configuration. Admittedly the fit to a simple exponential is simplistic, but sufficient for our purposes (see also Fig. 6). (a) $\text{tr} W_0$ localization radius for a dYM monopole instanton on a (10, 25, 25, 30) lattice. (b) $\text{tr} W_0$ localization radius for a YM $|Q| = \frac{1}{2}$ instanton on a (10, 25, 25, 30) lattice. (c) $\frac{1}{2}|\text{tr} W_0|$, for the lattice configurations shown above, as a function of the distance from the maximum along x_1 (of total length $L = 25$), with the other coordinates kept at the point of maximum action. On the left we show the dYM configuration vs the fit function. On the right, the YM profile decreases much more slowly, leading to an unrealistically—see Fig. 6 for further discussion—larger radius determined by the fit.

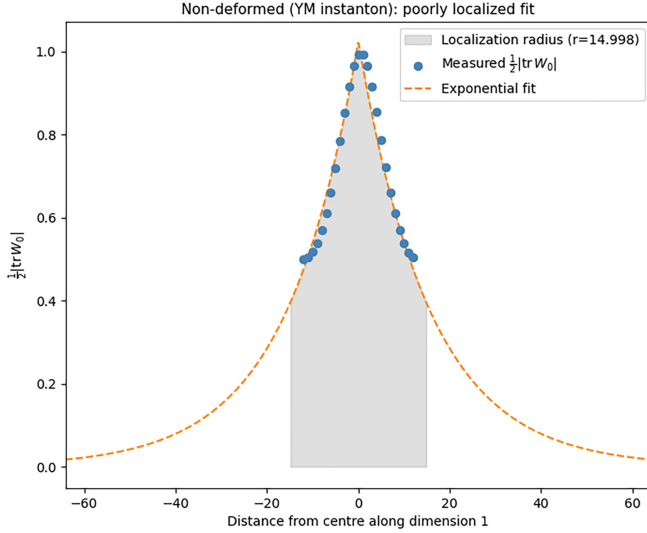


FIG. 6. Expanded version of the plot on the rhs of Fig. 5(c): $\frac{1}{2}|\text{tr } W_0|$ as a function of the distance (in x_1) to the maximum action point, for the YM configuration on a (10, 25, 25, 10) lattice. The shaded area shows the localization radius determined by procedure of the caption of Fig. 5, using the area under the simple exponential function. The obtained value $r \approx 15$, unrealistic for $L_1 = 25$, is due to the poor fit of the function to the actual configuration; it is evident, both on Fig. 5(c) and here, that the measured configuration “flattens out” before the simple exponential function.

difference between the YM and dYM configurations, owing to the alignment of Abelianizations.

- (3) On Figs. 9(c) and 9(d), we numerically demonstrate the disordering of the Wilson loop surrounding the center vortex for the small- L_3 lattice whose flux is shown on Fig. 9(a). This is similar to the analysis in pure YM from [28]. Finally, Fig. 10 shows that gauge invariants other than the action density (i.e., various Wilson loops) also evolve smoothly from large L_3 to small L_3 , both in YM and dYM.

The main lesson is that the $L_3 \gg L_0$ continuity between monopole-instantons and center vortices in dYM persists to small $L_3 \gtrsim L_0$, with all qualitative features similar between the two limits. For the smaller L_3 we studied, the same flux collimation effect is also seen in YM theory without deformation, due to the alignment of Abelianization due to twist and deformation, as reviewed in Sec. IV.⁹ However, pure YM fails to Abelianize at larger values of L_3 [see Eq. (22) and the discussion in Sec. IV that follows] and the $|Q| = 1/2$ background becomes delocalized. The alignment of Abelianizations in YM

⁹As explained in Sec. IV, in pure YM with twists, whenever (22) holds, there is a two-stage Abelianization, $SU(2) \rightarrow U(1)$ at a scale π/L_0 , and a subsequent (for $L_3 \gg L_0$) breaking $U(1) \rightarrow \mathbb{Z}_2$ at a lower scale π/L_3 . See Ref. [30] for a discussion of the two-stage Higgsing within a monopole-instanton gas effective theory framework.

and dYM makes it clear, however, that the mechanism of a monopole-instanton chain becoming a center vortex, while technically relying on the deformation potential [29,30], is more general and also applies in pure YM theory, provided the shape of the $\mathbb{T}^4$ is appropriately tuned.

Section VI: From monopole-instantons to fractional instantons on $\mathbb{R} \times \mathbb{T}^3$. Here we take a different path, studying the transition between fractional instantons on tori approximating $\mathbb{R}^3_{(x_1, x_2, x_3)} \times \mathbb{S}^1_{(x_0)}$ and $\mathbb{R}_{(x_3)} \times \mathbb{T}^3_{(x_0, x_1, x_2)}$. As before, the nonzero twists are n_{12} and n_{34} . We keep L_0 fixed and small, while L_3 is fixed and large, and vary L_1 ($= L_2$) from large $L_1 \sim L_3$, to small $L_1 \sim L_0$, interpolating between the two desired geometries, (1) and (3). As opposed to the previous study, here the dYM vacuum surrounding a localized instanton exhibits level crossing from the “flux,” at large $L_1 > 1.5L_0$, to the “no-flux” vacuum at small $L_1 < 1.5L_0$. Since, as per our results of Sec. III, this is a dYM level-crossing transition, we expect that the nature of $|Q| = 1/2$ minimum action solutions will also change discontinuously. Indeed, this is what our results on Figs. 11–14 appear to indicate.

Here, we only give a brief description of the results. For $L_1 < 1.5L_0$, we find the $\mathbb{R} \times \mathbb{T}^3_{n_{12}}$ fractional instantons studied long ago in [12,13] (seen in the low- L_1 region of Fig. 11, as discussed there). These fractional instantons, as per our Eq. (3), are well localized in L_3 . As is also well known, they disorder the $\text{tr } W_0$ Wilson loop, which approaches values ± 2 away from the solution. Both these features are seen on Fig. 12.

For $L_1 > 1.5L_0$, on the other hand, we find the flux vacuum monopole-instanton chain configurations described earlier (such a configuration at the transition point is shown on Fig. 13).

The new feature specific¹⁰ to dYM is the transition region near $L_1 \approx 1.5L_0$. Here, both the Wilson and deformation action of the minimum action configurations have a peak (as a function of L_1), seen on Fig. 11. For the critical value of L_1 , our action minimization algorithm finds two kinds of configurations, of roughly the same total action. These are either monopole-instantons in the flux vacuum, localized in x_3 and shown on Fig. 13, or the delocalized (in x_3) configurations shown on Fig. 14 and described there in detail. This apparent degeneracy is consistent with a level crossing transition, however, we do not have a detailed understanding of the delocalized configurations; their study requires a more fine-grained resolution of the transition region, which is beyond our scope here.

¹⁰We recall that along the same path in pure YM theory, for the lattices we study, the transition proceeds, instead, via the maximally delocalized constant-field strength instanton [27], which has minimal action when $L_1 L_2 = L_0 L_3$, studied in [28].

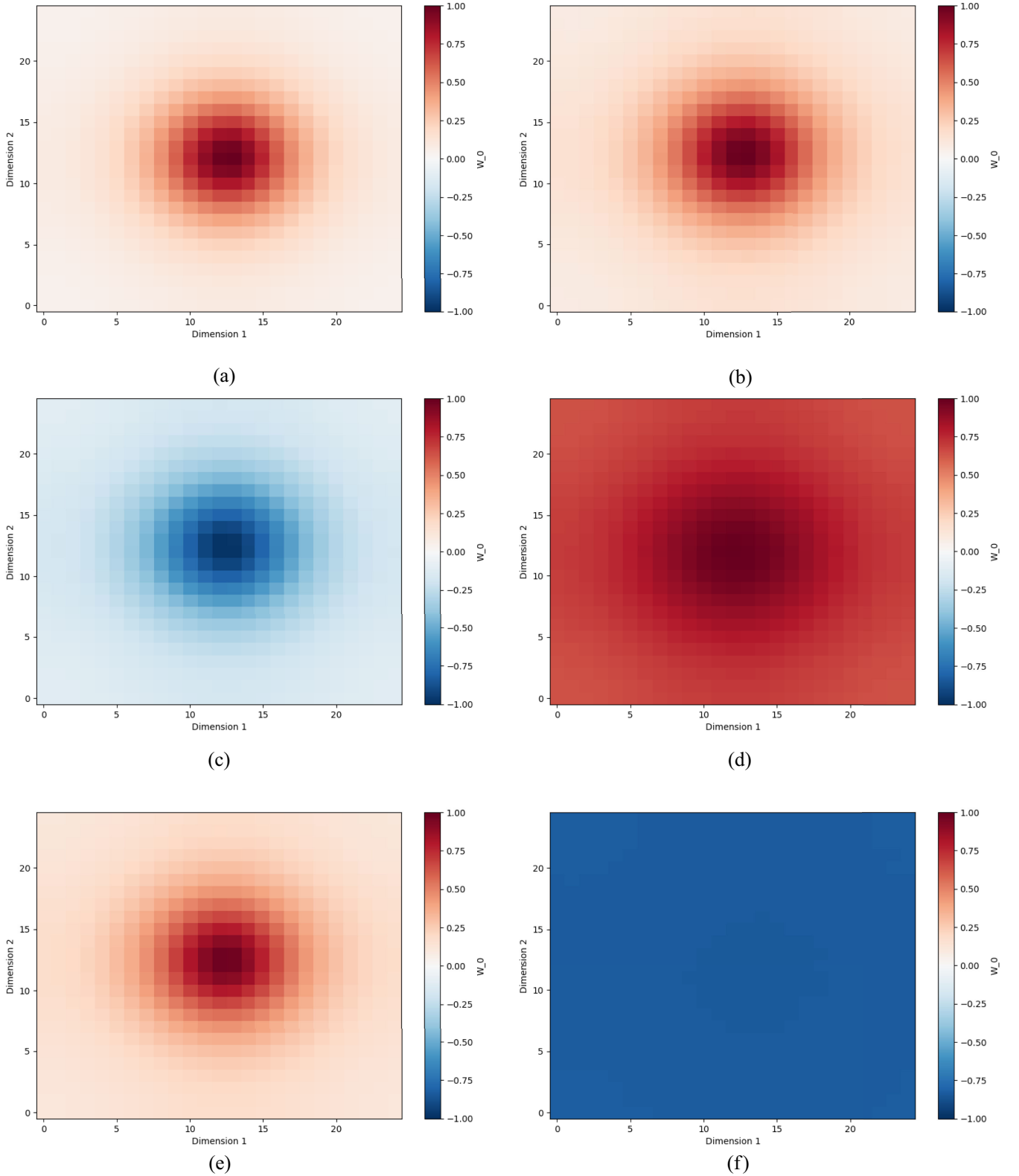


FIG. 7. Instanton core size in dYM (left column) and YM (right column) for three different lattice sizes. Top line: $L_0 L_3 \ll L_1 L_2$ —Abelianization directions due to deformation in dYM and twist in YM are aligned. Middle line: $L_0 L_3 \lesssim L_1 L_2$ dYM Abelianizes due to deformation, no Abelianization in YM. Bottom line: $L_0 L_3 \gtrsim L_1 L_2$ dYM Abelianizes, no Abelianization in YM (almost constant solution). (a) (10, 25, 25, 15): dYM, $\text{tr } W_0$ in 12 plane. (b) (10, 25, 25, 15): YM, $\text{tr } W_0$ in 12 plane. (c) (10, 25, 25, 40): dYM, $\text{tr } W_0$ in 12 plane. (d) (10, 25, 25, 40): YM, $\text{tr } W_0$ in 12 plane. (e) (10, 25, 25, 70): dYM, $\text{tr } W_0$ in 12 plane. (f) (10, 25, 25, 70): YM, $\text{tr } W_0$ in 12 plane.

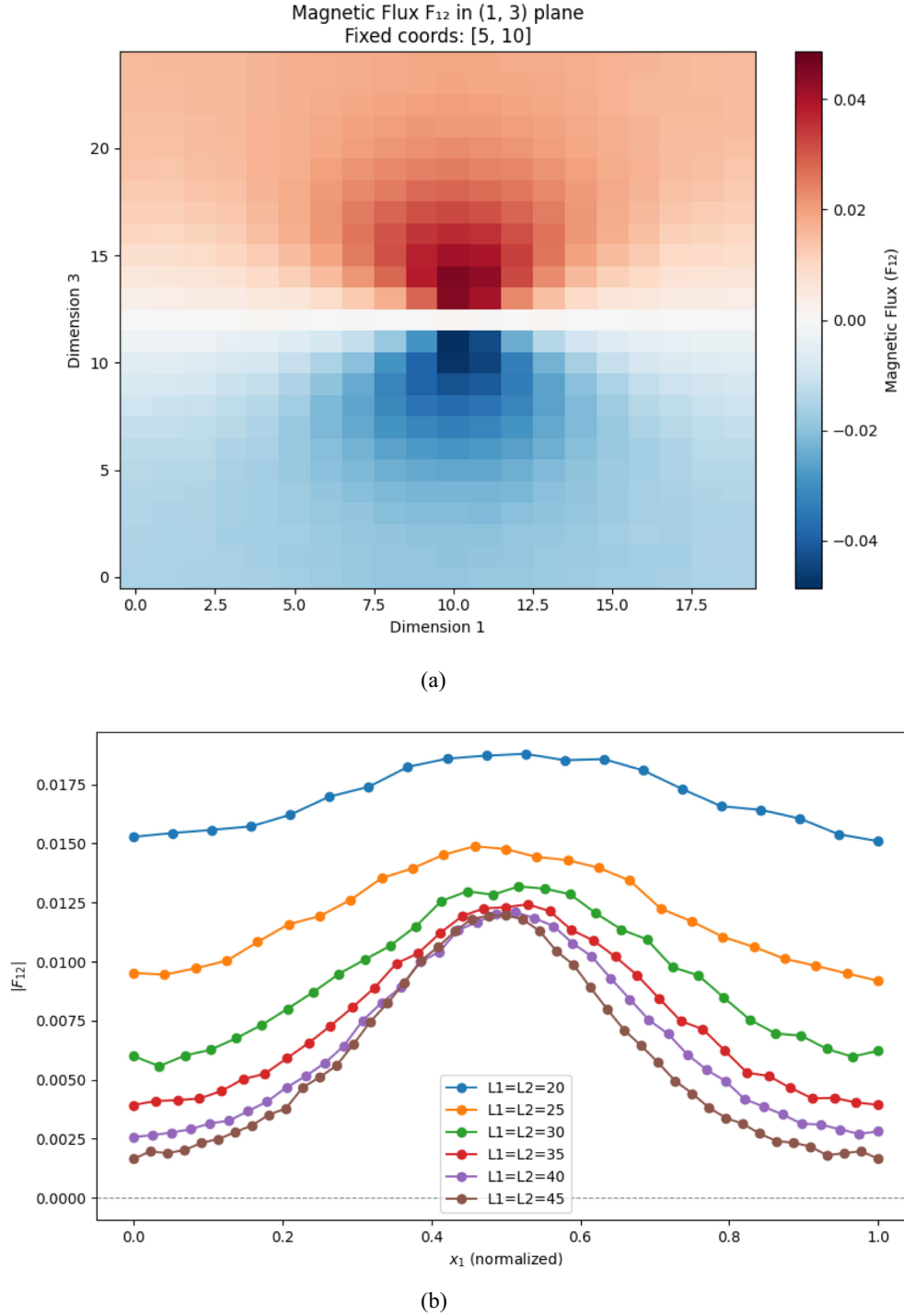


FIG. 8. The chain of monopole-instantons (recall Fig. 2) and their center-symmetry/translation images collimate the flux into a center vortex; see also Fig. 9. (a) $F_{12}^{U(1)}$, or the x_3 -component (on the plot, upward pointing) of the “magnetic” field of a dYM monopole-instanton as a function of x_1 and x_3 on a $(10, 20, 20, 25)$ lattice at $x_0 = 5$, $x_2 = 10$. There is cylindrical symmetry in the 12 plane. The BPS monopole-instanton flux is absorbed by its oppositely charged image upon L_3 translations. The picture is almost identical to the one in pure YM theory with the same twists, due to the aligned abelianizations in YM and dYM for this lattice size [see Fig. 25 of [28] for a YM monopole-instanton on a $(6, 18, 18, 24)$ lattice]. (b) Collimation of the magnetic flux $F_{12}^{U(1)}$ of a monopole-instanton in dYM into a center vortex on a $(10, L_1, L_1, 25)$ lattice. The monopole-instanton is at $x_3 = L_3/2$, $x_1 = x_2 = L_1/2$ and is extended in x_0 . The value of $F_{12}^{U(1)}$ at $x_3 = 0$, $x_2 = L_1/2$, halfway between the BPS and its KK image, is plotted as a function of x_1/L_1 for $L_1 = 20, \dots, 45$. The infinite- L_1 limit calculation shows the collimation of the flux into a tube, a center-vortex of an estimated radius $\sim L_3/\pi \sim 8$. This value is consistent with our plots for $L_1 = 40, 45$; for smaller L_1 , there are additional images in the $x_{1,2}$ direction. See the Gaussian fit on Fig. 9 for the $L_1 = 45$ curve.

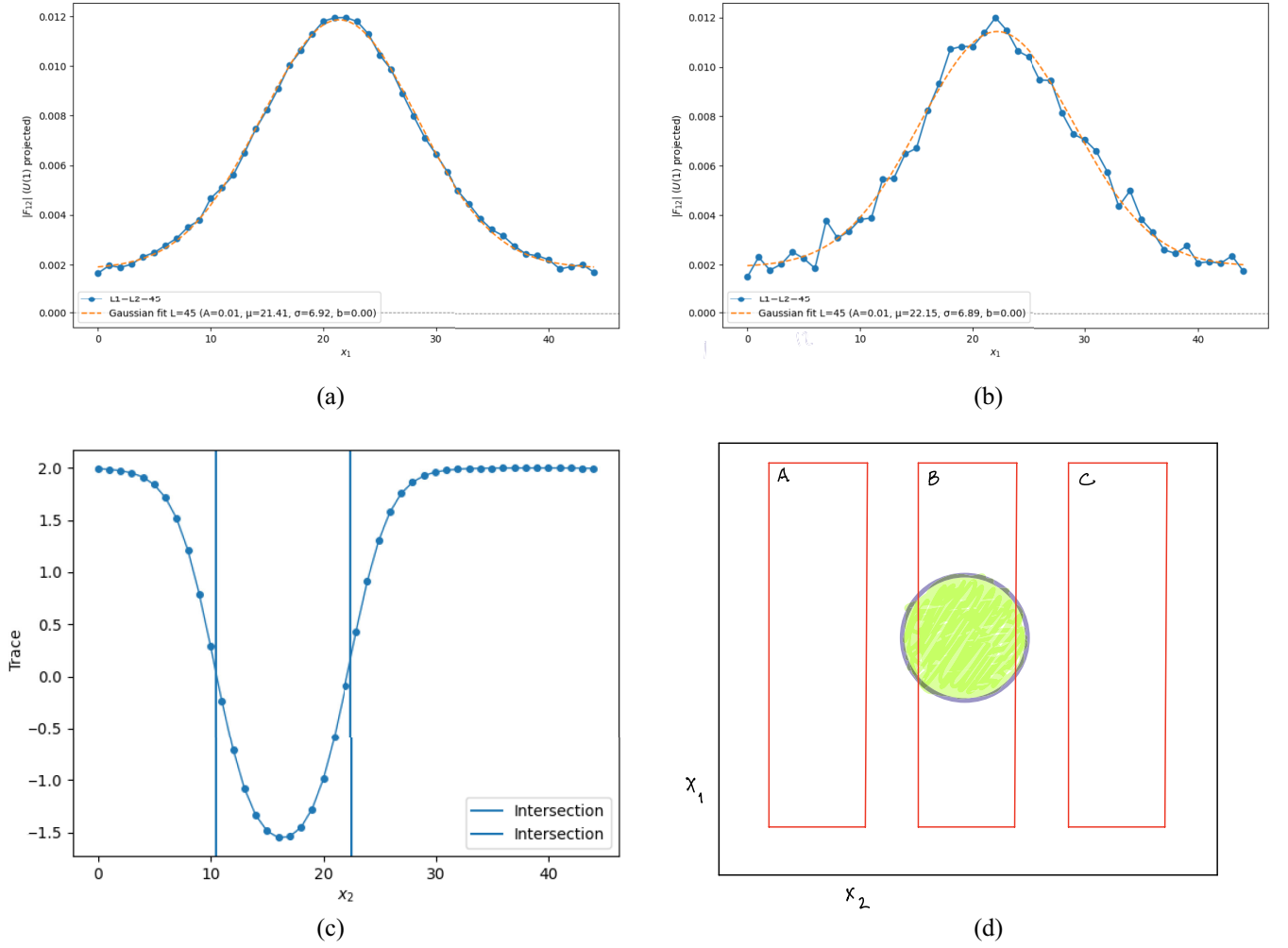


FIG. 9. The profile of the field of the center vortex and the disordering of the Wilson loop. Top row [(a) and (b)]: a Gaussian fit of the magnetic field, midway between the monopole-instanton and its image upon L_3 translation, for the $L_1 = 45$ curve of Fig. 8(b). $F_{12}^{U(1)}$, at $x_3 = 0, x_2 = L_2/2$, is fitted to $Ae^{-\frac{1}{2}(\frac{x_1-\mu}{\sigma})^2} + b$. The best fit parameters shown are essentially the same in YM and dYM, due to the alignment of Abelianizations, as per (22). Bottom row [(c) and (d)]: The left (c) shows the value of trace of a 43×12 Wilson loop in the 12-plane, evaluated in the “center vortex” minimum action configuration in dYM on a $(10, 45, 45, 25)$ lattice. The loop is dragged along the x_2 direction across the center vortex localized in x_1, x_2 . The Wilson loops at $x_2 < 10$ and $x_2 > 25$ do not enclose the center vortex, but the ones at $10 < x_2 < 25$ do (partially), as in the examples A, B, C on (d). There is clear indication of the sign difference between the loops that enclose the center vortex and the ones that do not. The vertical lines on (c) represent the points where the edges of the Wilson loop cross the peak of the center vortex, which is located at the midpoint of the lattice. (a) $F_{12}^{U(1)}$ in dYM on $(10, 45, 45, 25)$ lattice fitted to a Gaussian: $A = 0.01, \mu = 21.41, \sigma = 6.92, b = 0$. (b) $F_{12}^{U(1)}$ YM on $(10, 45, 45, 25)$ lattice fitted to a Gaussian: $A = 0.01, \mu = 22.15, \sigma = 6.89, b = 0$. (c) The center vortex disordering of the Wilson loop: values of $x_1 < 10, x_2 > 25$ correspond to loops A and C [see (d)] not surrounding the center vortex, while $x_2 \sim 17$ corresponds to the loop B. (d) The center vortex disordering of the Wilson loop: three positions of the 12 plane Wilson loop, A, B, C, with respect to the center vortex (schematically indicated by the circle).

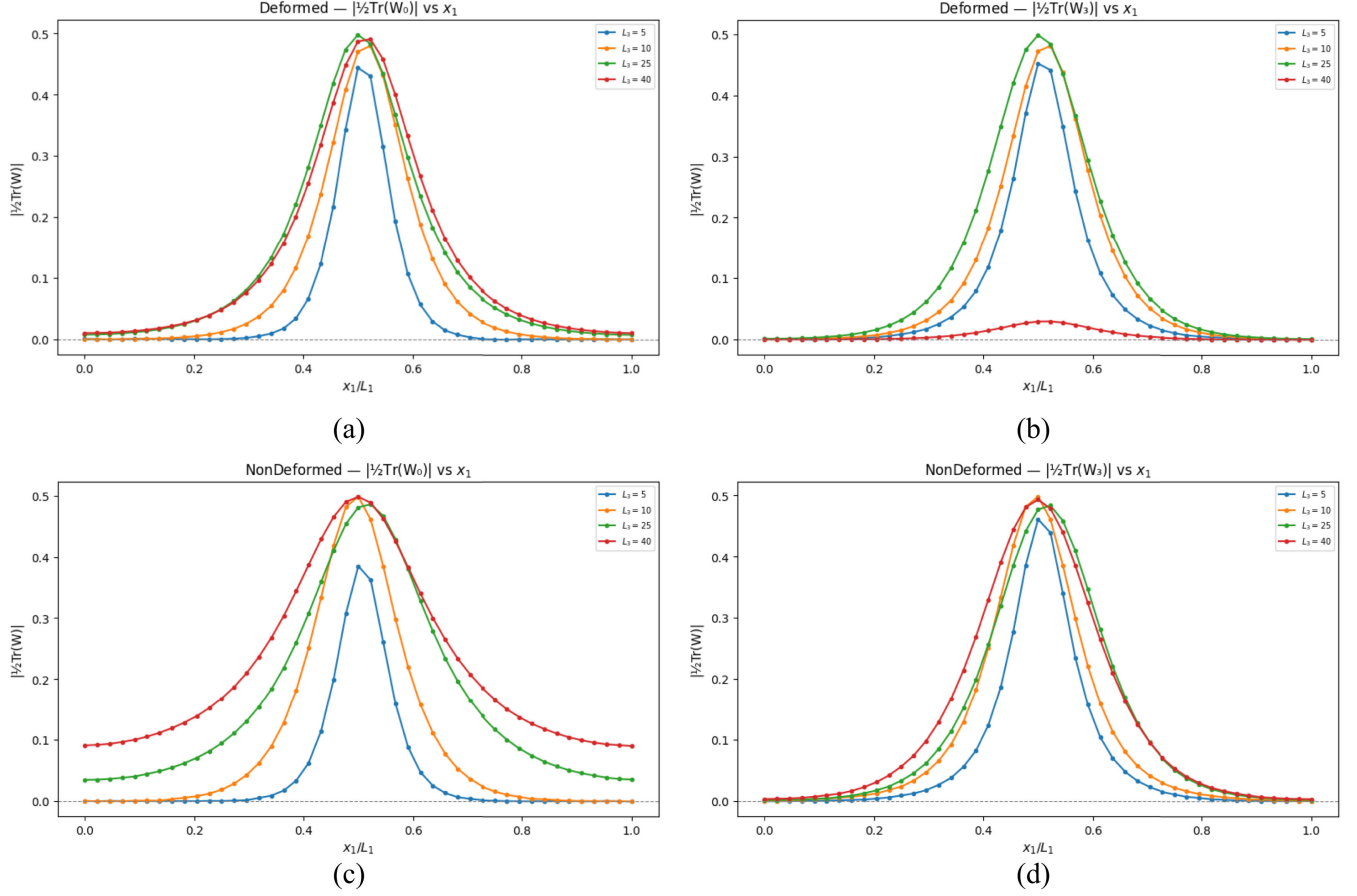


FIG. 10. Smoothly interpolating from large to small L_3 on a $(10, 45, 45, L_3)$ lattice. Recall that $L_3 \gg L_0$ is the regime where the analytic construction of [29,30] and the picture of Fig. 2 hold. On the other hand, for small $L_3 \sim L_0$, there is no qualitative difference between the YM and dYM center vortex with $L_0 L_3 \ll L_1 L_2$, due to the alignment of Abelianizations. Here we compare $\text{tr } W_0(\frac{x_1}{L_1})$ and $\text{tr } W_3(\frac{x_1}{L_1})$ (the other coordinates are taken at the point of maximal action density) for dYM (top row) and YM (bottom row) for four different values of L_3 . It is clearly seen that for small $L_3 \sim L_0$, blue curve, $\text{tr } W_0$ and $\text{tr } W_3$ in YM and dYM are qualitatively similar. On the other hand, for the largest $L_3 = 40$, the YM solution becomes more delocalized, as seen on (c), while dYM remains localized, as per the behavior of $\text{tr } W_0$ on (a), but with $\text{tr } W_3$, on (b), becoming small. (a) dYM: $\text{tr } W_0(\frac{x_1}{L_1})$ for $L_3 = 5, 15, 25, 40$. (b) dYM: $\text{tr } W_3(\frac{x_1}{L_1})$ for $L_3 = 5, 15, 25, 40$. (c) YM: $\text{tr } W_0(\frac{x_1}{L_1})$ for $L_3 = 5, 15, 25, 40$. (d) YM: $\text{tr } W_3(\frac{x_1}{L_1})$ for $L_3 = 5, 15, 25, 40$.

C. Outlook

The study of this paper further confirms that the set of minimal action configurations listed in (1)–(4) are related to each other in intricate ways, forming a rich set of interconnected saddle points related by changing the twists and ratios of periods of $\mathbb{T}^4$. In this regard, our study could be furthered by considering in more detail the “critical” region of the transition between the monopole-instanton and fractional instanton configurations of (1) and (3), studied in Sec. VI and associated with a transition between the flux and no-flux vacua of dYM. This, however,

requires a significantly more fine-grained study of the transition and the associated computer resources. Additionally, the gradient flow technique used to generate minimum action configurations could in principle be used to place upper bounds on the height of the energy barriers between vacua near the critical point. By giving configurations in one vacuum progressively stronger “kicks” or fictitious kinetic energy, until the lattice settles into the other vacuum, it could be possible to trace the path through field space the lattice takes. By computing the energy of these intermediate configurations it would be possible to place an upper bound on the height of the energy barrier.

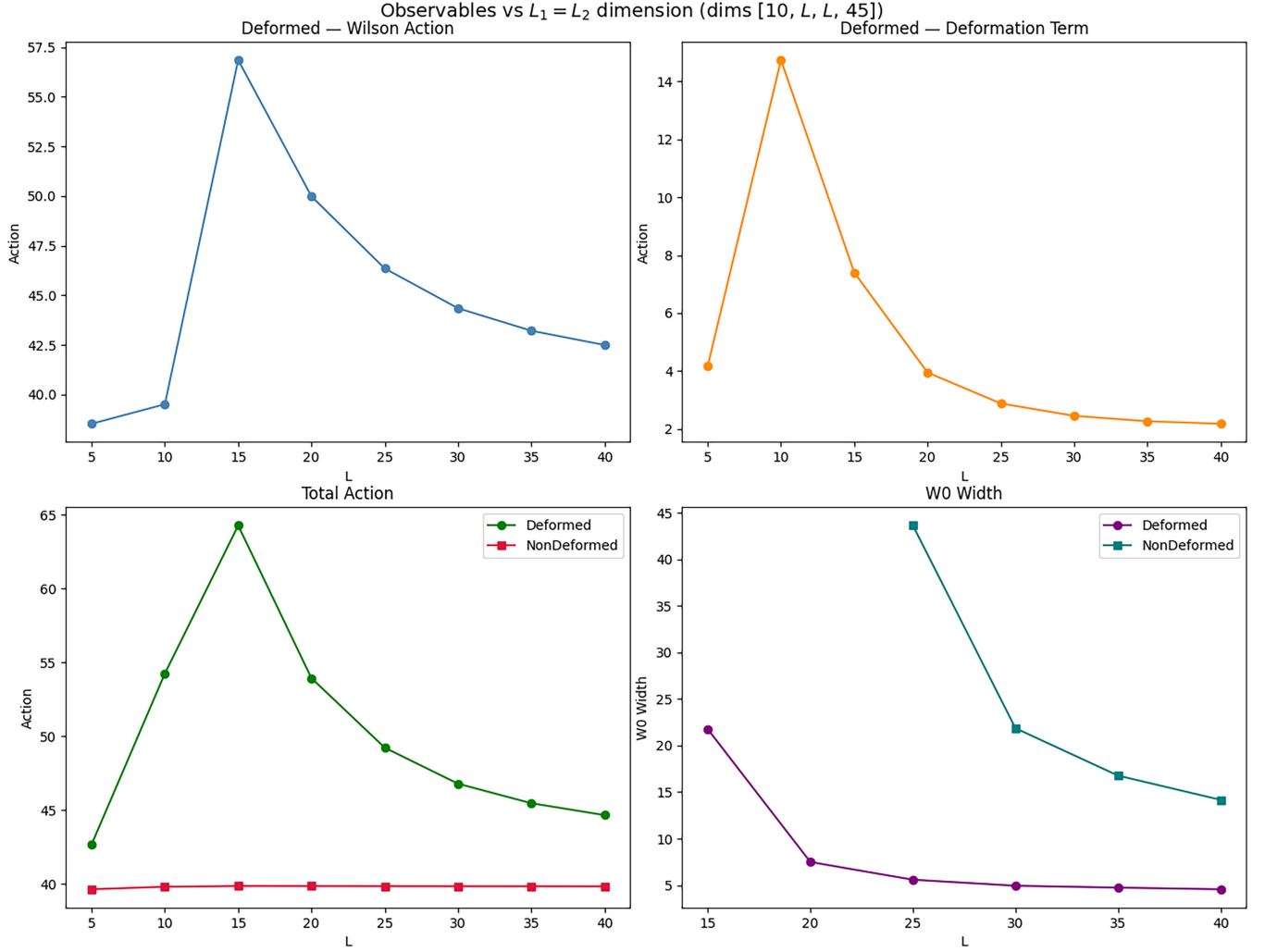


FIG. 11. Results for a $(10, L, L, 45)$ lattice interpolating between $\mathbb{T}^3 \times \mathbb{R}$ for small L and $\mathbb{S}^1 \times \mathbb{R}^3$ for large L . All quantities are shown for $5 \leq L \leq 40$. Top row: the Wilson and deformation contributions to the action of dYM. Bottom row: the total actions of YM and dYM, as well as the $\text{tr } W_0$ width in the x_1, x_2 -plane, determined as in Fig. 5. The W_0 width in the 12 plane is not shown for $L < 15$. For dYM, this is because the theory transitions to the no-flux vacuum. For YM minimum action configurations, the W_0 width becomes larger than L as $L \rightarrow 25$ from above, showing that the solutions delocalize on the $\mathbb{T}^3$ and Abelianization in YM along $\text{tr } W_0 = 0$, i.e., aligned with dYM flux vacuum, fails. On the other hand, for $L < 15$, the no-flux vacua of dYM and YM are aligned, as per (22) (see the discussion in the text).

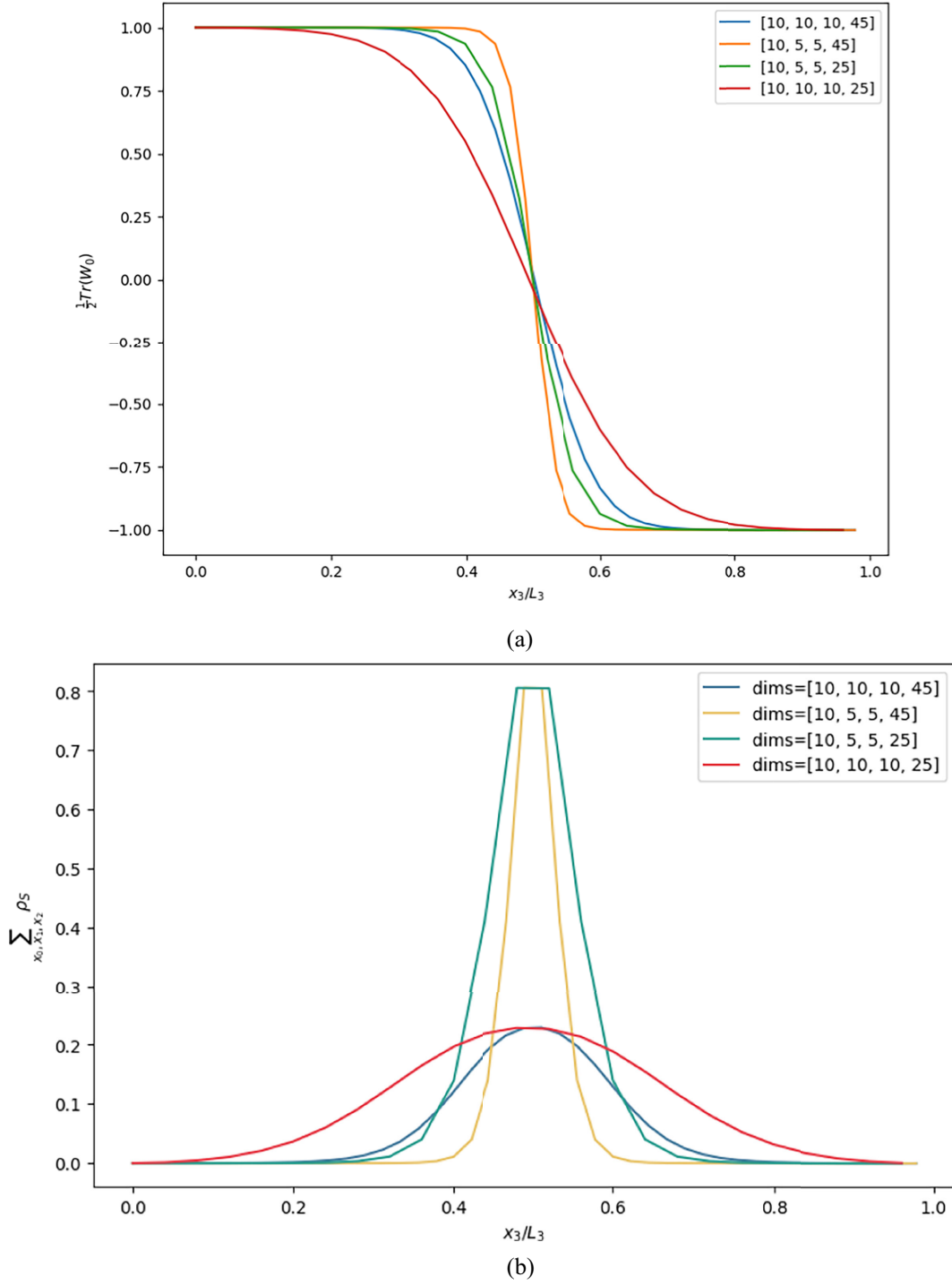


FIG. 12. (a) The disordering effect of the monopole instanton on $\mathbb{R}_{(x_3)} \times \mathbb{T}^3_{(x_0, x_1, x_2)|_{n_{12}=1}}$ on the Wilson loop winding in the x_0 direction, for $\mathbb{T}^3$ of sizes (10, 10, 10) and (10, 5, 5) for two different values of $L_3 = 25, 45$. This is the counterpart of the center-vortex disorder of Fig. 9(c). (b) The instanton is localized at $x_3/L_3 = 0.5$. The action density plots indicate that the instanton more strongly localized at smaller $L_1 L_2 = L_1^2$, with its scale set by L_1 . The small- $\mathbb{T}^3$ regime studied here is where the YM and dYM Abelianizations align and the effect of the deformation is small. (The continuous curves on the plot are obtained by interpolating between discrete values of x_3 , which are not shown.) (a) The dYM fractional instanton on $\mathbb{T}^2 \times \mathbb{R}$ disordering the Wilson loop $\text{tr } W_0$. (b) The action density of the fractional instanton in dYM integrated over $\mathbb{T}^3$.

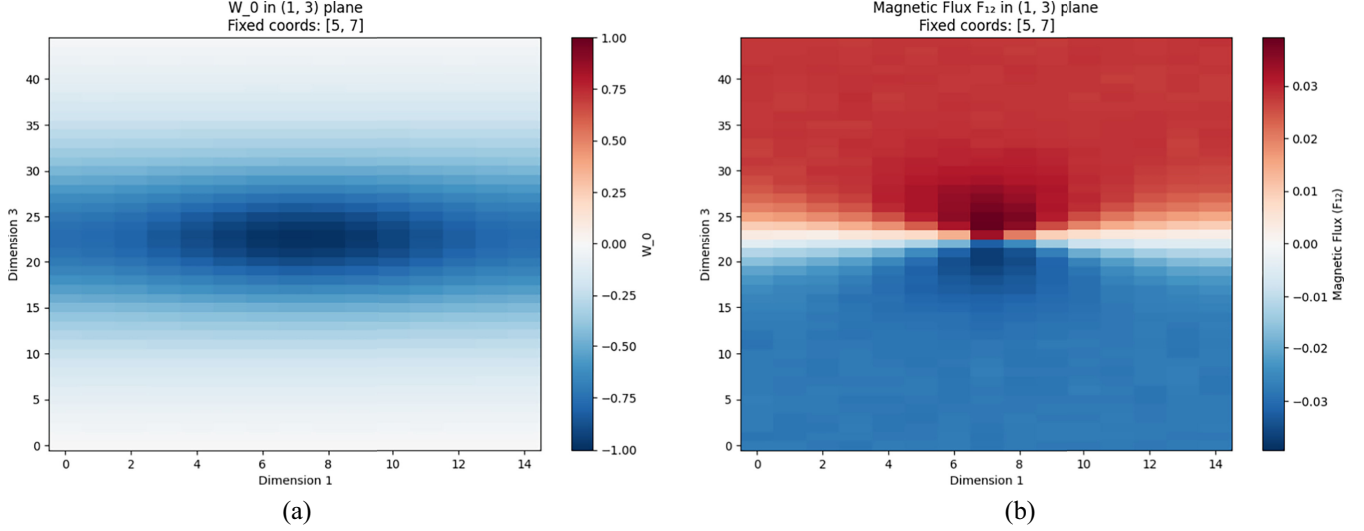


FIG. 13. A small number (2 out of 9) of minimum action dYM configurations with $|Q| = 1/2$ at $\frac{L_1}{L_0} = 1.5$, on a $(10, 15, 15, 45)$ lattice, at the transition point between the flux and no-flux vacua of dYM, are found to look like monopole instantons in the flux vacuum of dYM. $\text{tr } W_0$ shows localization in x_3 , with width of order L_1 , and approaches 0 far away from the core of the monopole instanton showing Abelianization in dYM. (a) $(10, 15, 15, 45)$: $\text{tr } W_0$ in 13 plane. (b) $(10, 15, 15, 45)$: $F_{12}^{U(1)}$ in 13 plane

A more general question—interesting from both mathematics and physics points of view—concerns the nature and moduli of classical self-dual instantons of (fractional) charge $Q = r/N$, $\forall r \in \mathbb{N}$, in $SU(N)$ YM theory on a

twisted $\mathbb{T}^4$. For general $r > 1$, the moduli space of such field configurations is only understood locally, as in [55,56], using a combination of numerical and analytic tools. On the numerical side, it would be interesting to

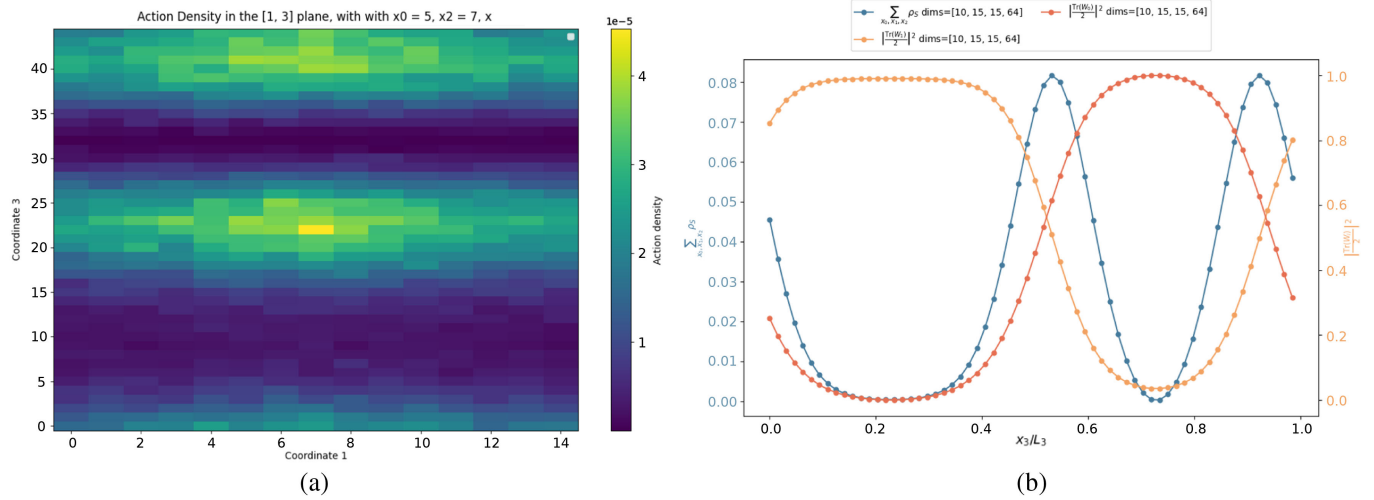


FIG. 14. The majority (7 out of 9) of the minimum action dYM configurations with $|Q| = 1/2$, on a $(10, 15, 15, 45)$ lattice, at the transition point $\frac{L_1}{L_0} = 1.5$ between the flux and no-flux vacua of dYM, are found to have the two maxima structure of their Wilson action density, shown on the (a), but a single deformation action peak, seen on the rhs. On the rhs, we compare $|\text{tr } W_0/2|^2$, $|\text{tr } W_1/2|^2$, and the Wilson action density as a function of x_3 , for a lattice with a larger L_3 at the same critical value of L_1 (we do not show $|\text{tr } W_2|^2$ as its behavior is identical to that of $|\text{tr } W_1|^2$ shown). These configurations show no localization in x_3 , with the region of nonzero Wilson and deformation action density taking roughly half the L_3 length. (This is true for $L_3 = 45$ shown as well as for $L_3 = 65$, which we do not show). These configurations have, within our precision, roughly the same (or a bit smaller) total action as the flux vacua ones on Fig. 13, but a smaller Wilson and larger deformation action, due to the $\text{tr } W_0$ behavior shown. (a) The Wilson term action density for a $(10, 15, 15, 45)$ lattice in the 13 plane. Its two-hump profile as a function of x_3 , taken at the point of maximal action in 12 plane, is also shown on the right, albeit for a $(10, 15, 15, 64)$ lattice with larger L_3 . (b) The deformation action, $|\text{tr } W_0/2|^2$ (the curve with a single maximum, with scale given on the rhs), along with $|\text{tr } W_1/2|^2$ and the Wilson term action density, displaying a two maxima structure, as a function of x_3 .

study whether one could make further progress, first for $N = 2$, for any $r > 1$, by generalizing the methods of [43,44].

II. THE LATTICE SETUP OF DEFORMED YANG-MILLS THEORY

We study Wilson's lattice gauge theory for an $SU(2)$ gauge group, on a rectangular $\mathbb{T}^4$ lattice of periods L_μ , $\mu = 0, 1, 2, 3$. The theory obtained by the addition of a double-trace deformation [3] in the L_0 direction with coordinate $x_0 \in \{1, 2, \dots, L_0\}$ converts it into dYM. It is well-known that the double-trace deformation can be due to massive adjoint fermions [3,51,57] but this interpretation is not relevant for us, as we are focused on studying classical minima.

We also add a 't Hooft flux background defined by six (mod 2) integers $n_{\mu\nu} = -n_{\nu\mu}$. The latter will be chosen to be only nontrivial in the 03 and 12 planes,

$$n_{03} = 1 \quad \text{or} \quad 0, \quad n_{12} = 1. \quad (5)$$

As indicated, we sometimes turn off the 03-plane flux by choosing $n_{03} = 0$, notably in Sec. III. Further, in all our studies, we take $L_1 = L_2$ and vary L_1 along with L_3 , usually keeping L_0 the smallest dimension. Thus, upon comparing different dimensions, we shall often refer to ratios of L_0, L_3 to L_1 only. We begin with the lattice action of dYM,

$$\begin{aligned} S_{\text{total}} &= A(S_{\text{Wilson}} + S_{\text{def}}) \\ &= A \left(\sum_x \sum_{\mu\nu} \text{tr}(\mathbf{1} - B_{\mu\nu}(x) \square_{\mu\nu}(x)) \right. \\ &\quad \left. + \frac{c}{L_0^3} \sum_{\vec{x}} |\text{tr} W_0(\vec{x})|^2 \right). \end{aligned} \quad (6)$$

Here, $\square_{\mu\nu}(x) = U_\mu(x) U_\nu(x + \hat{e}_\mu) U_\mu^\dagger(x + \hat{e}_\nu) U_\nu^\dagger(x)$ is the $\mu\nu$ -plane plaquette, located at the lattice point x , with $\hat{e}_\mu$ a unit vector in the x_μ direction and $U_\mu(x)$ —the $SU(2)$ -valued link variables, periodic upon a shift of x_μ by L_μ . x_0

denotes the coordinate in the direction with the double-trace deformation, and sometimes we use $\vec{x}$ to label coordinates in the $x_{1,2,3}$ directions. $B_{\mu\nu}$ is the two-form (i.e., plaquette-based) topological background for the $\mathbb{Z}_2^{(1)}$ 1-form symmetry, or a 't Hooft twist. Explicitly, the fundamental representation Wilson loop W_0 winding in the x_0 direction, starting from $x_0 = 1$, and the two-form background are¹¹

$$W_0(\vec{x}) = \prod_{j=1}^{j=L_0} U_0(j, \vec{x}), \quad \text{for } \vec{x} \in \mathbb{Z}^3, \quad (7)$$

$$B_{\mu\nu}(x) = \begin{cases} (-1)^{n_{\mu\nu}}, & x_\mu = L_\mu, x_\nu = L_\nu \\ 1, & \text{else} \end{cases}. \quad (8)$$

The 't Hooft fluxes are inserted in a corner of the relevant 2-plane; this choice is inessential as the backgrounds are topological. Whenever both twists in (5) are nontrivial, the two-form background consists of two intersecting non-dynamical center vortices. When $n_{03} = 0$, there is only a single one; a 2-plane extending in x_3, x_4 and located at $x_1 = L_1, x_2 = L_2$. The double-trace deformation term in (6) is the one proportional to $|\text{tr} W_0|^2$. The L_0^{-3} scaling of the coefficient of the deformation term is motivated by the usual one-loop expression obtained on $\mathbb{R}^3 \times \mathbb{S}^1$ in the continuum. Our studies are for the particular value $c = 1$, but we also take the pure YM limit $c = 0$ (whose relevant properties are reviewed in Sec. IV). Throughout, as we further discuss in Sec. III, the x_0 direction of extent L_0 is associated with the small spatial circle familiar from continuum studies on $\mathbb{R}^3 \times \mathbb{S}^1$.

Our purpose is to study the minima of the classical action, the one in the brackets in (6), upon varying the twists and the ratio of sides of the torus. Thus, the overall coefficient A will be adjusted at will during the simulation and, after computing the action of a minimum action configuration, it is divided out, see Appendix A. When referring to our results, we always list the lattice size in the order (L_0, L_1, L_2, L_3) , taking $L_1 = L_2$. We study minima of the action (6) with two choices of twists (5). The first choice only involves a single nontrivial twist, n_{12} ,

$$\overbrace{(L_0, L_1, L_2, L_3)}^{n_{03}=0, n_{12}=1} \Rightarrow \text{minimum action configurations with } Q = 0, \text{ or classical vacua on } \mathbb{T}_{(L_0, L_1, L_2)|_{n_{12}=1}}^3, \quad (9)$$

and is studied in Sec. III. The second case is when both twists are nontrivial [1,2,16],

¹¹Winding Wilson loops W_μ in the x_μ directions are defined similar to (7), with $0 \rightarrow \mu$.

$$\underbrace{(L_0, L_1, L_2, L_3)}_{\substack{n_{03}=1 \\ n_{12}=1}} \Rightarrow \text{minimum action configurations with } |Q| = \frac{1}{2}, \quad \text{or fractional instantons.} \quad (10)$$

For $|Q| = 1/2$, for $c = 0$, the minimal action solutions obey the BPS bound, $\frac{1}{A} S_{\text{Wilson}} = \frac{1}{A} S_{\text{BPS}} = 4\pi^2$.

III. DYM VACUA ON $\mathbb{T}^3_{(L_0, L_1, L_2)|_{n_{12}=1}}$: FROM ‘FLUX’ TO ‘NO-FLUX’ UPON CHANGING THE $\mathbb{T}^3$ SHAPE

In this section, we study the ground states of dYM on $\mathbb{T}^3_{(L_0, L_1, L_2)|_{n_{12}=1}}$, upon changing the ratio L_1/L_0 (recall that we always take $L_1 = L_2$). Our lattice minimization of the action (6) shows that one of the two continuum states discussed in [52], known to be local minima, is always a global minimum of the energy, with the transition between them being a level crossing. These two states also played important role in the study of [30].

We begin with the classical continuum limit of the dYM lattice action (6),

$$\begin{aligned} \frac{S_{\text{total}}}{A} \Big|_{\text{cont.}; n_{03}, n_{12}} &= \sum_{\mu, \nu} \int_{\mathbb{T}^4} d^4x \frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu} \\ &+ \frac{c}{L_0^3} \int_{\mathbb{T}^3} d^3\vec{x} |\text{tr} W_0(\vec{x})|^2, \end{aligned} \quad (11)$$

where L_0 now denotes the physical size of the x_0 dimension. As indicated in (11), the continuum fields are subject to twisted boundary conditions determined by the twists (5), as in [1,2]. The action (11) is the well-known action of continuum dYM theory [3].

We now consider L_3 as the time direction and minimize the energy on the spatial $\mathbb{T}^3$ of size L_0, L_1, L_2 , with a nontrivial twist $n_{12} = 1$. We take $n_{03} = 0$, consistent with the interpretation that we are looking for energy minima on a spatial $\mathbb{T}^3$ with a single twist. The continuum energy functional, following from the Minkowski version of (11), see Ref. [52], in the $A_3 = 0$ gauge,¹² upon setting the momentum variables (electric fields, or derivatives w.r.t. x_3 time) to zero and taking generators obeying $\text{tr} t^a t^b = \delta^{ab}/2$, is

$$\begin{aligned} \frac{E}{A} &= \int_0^{L_1} dx_1 \int_0^{L_2} dx_2 \\ &\times \int_0^{L_0} dx_0 \left(\frac{1}{2} F_{12}^a F_{12}^a + \frac{1}{2} F_{0i}^a F_{0i}^a + \frac{c}{L_0^4} |\text{tr} W_0(\vec{x})|^2 \right), \end{aligned} \quad (12)$$

¹²We hope that the choice of x_3 to label the time direction will not be too confusing.

with a sum over $a = 1, 2, 3$ and $i = 1, 2$ implied. The gauge fields obey twisted boundary conditions in the 12 plane and are periodic in the third spatial direction of length L_0 .

There are two competing classical states in the $\mathbb{T}^3$ theory with a deformation and a ’t Hooft twist $n_{12} = 1$, discussed in great detail in [52]. We refer the reader to that reference for details of the choice of gauge for the transition functions and for the gauge-field backgrounds associated with these states. We call these the “flux” and “no-flux” states.

The essential properties of the flux state are

$$\begin{aligned} \text{“flux”}: F_{12} &= \frac{2\pi}{L_1 L_2} \frac{\sigma^3}{2}, \quad F_{01} = F_{02} = 0, \\ \text{tr} W_0 &= 0, \quad \text{tr}(W_0 F_{12}) = \pm i \frac{2\pi}{L_1 L_2}, \\ \text{tr} W_1 &= 2 \cos \frac{\pi x_2}{L_2}, \quad \text{tr} W_2 = 2 \cos \frac{\pi x_1}{L_1}, \end{aligned} \quad (13)$$

where we ignored the arbitrary origin of x_1, x_2 in the Wilson lines above and chose a particular gauge [52] to describe F_{12} . Notice that both W_1 and W_2 undergo a center symmetry transformation upon a period shift of x_2 or x_1 , respectively. Thus, W_1 is antiperiodic under upon a x_2 translation and vice versa. The antiperiodicity of W_1 with respect to a period translation in x_2 , etc., is a general property due to the nontrivial ’t Hooft twist n_{12} .

For the “no-flux” vacuum, we have instead,

$$\begin{aligned} \text{“no-flux”}: F_{12} &= F_{01} = F_{02} = 0, \quad \text{tr} W_0 = \pm 2, \\ \text{tr} W_1 &= \text{tr} W_2 = \text{tr} W_1 W_2 = 0. \end{aligned} \quad (14)$$

Both the flux and no-flux classical ground states are twofold degenerate, the degenerate states related by the “broken” 1-form $\mathbb{Z}_2$ center symmetry in the x_0 direction, which acts as $W_0 \rightarrow -W_0$. From (12), we immediately obtain for the energies of the states (13) and (14),

$$\begin{aligned} \frac{E_{\text{flux}}}{A} &= 2\pi^2 \frac{L_0}{L_1 L_2}, \\ \frac{E_{\text{no-flux}}}{A} &= 4c \frac{L_1 L_2}{L_0^3} \Rightarrow \frac{E_{\text{flux}}}{E_{\text{no-flux}}} = \frac{\pi^2}{2c} \left(\frac{L_0}{\sqrt{L_1 L_2}} \right)^4. \end{aligned} \quad (15)$$

Thus, the classical flux state has lower energy at small $L_0/\sqrt{L_1 L_2}$, while the no-flux has lower energy for large $L_0/\sqrt{L_1 L_2}$. The two energies are of the same order when $\sqrt{L_1 L_2}/L_0 = (\pi^2/2)^{1/4}$, for $c = 1$, thus we expect a

transition between these two ground states at a critical value of the ratio,

$$\left. \frac{L_1}{L_0} \right|_{\text{crit.}} = \left(\frac{\pi^2}{2} \right)^{\frac{1}{4}} \simeq 1.49, \quad (16)$$

where we took $L_1 = L_2$. We can summarize the “phase structure” implied by (16) as

$$\begin{aligned} & \text{“} \frac{L_1}{L_0} > 1.5 \Leftrightarrow \text{flux vacuum”} \\ \text{and } & \text{“} \frac{L_1}{L_0} < 1.5 \Leftrightarrow \text{no-flux vacuum”}. \end{aligned} \quad (17)$$

Numerically, we found that (16) and (17) correctly locate the transition between the flux and no-flux vacua on the lattices we study: a minimization of our lattice action (6) with $c = 1$, a trivial $n_{03} = 0$, and $n_{12} = 1$, exhibits this transition precisely near this value (see Fig. 1 below).

To connect the above analysis to the lattice minimization of the Euclidean action (6), we note that the two vacua (13) and (14) are associated with the following actions, found by simply multiplying the energies (15) by the extent of the time direction L_3 :

$$\begin{aligned} \frac{S_{\text{flux}}}{A} &= 2\pi^2 \frac{L_3 L_0}{L_1 L_2} \Big|_{L_1=L_2} = 4 \frac{L_3}{L_0} \frac{\pi^2}{2} \left(\frac{L_0}{L_1} \right)^2, \\ \frac{S_{\text{no-flux}}}{A} &= 4 \frac{L_3 L_1 L_2}{L_0^3} \Big|_{L_1=L_2} = 4 \frac{L_3}{L_0} \left(\frac{L_1}{L_0} \right)^2. \end{aligned} \quad (18)$$

For use below, we also give the lattice definition of the gauge invariant $U(1)$ flux, given in the continuum in the last term of the first line in (13),¹³

$$F_{\mu\nu}^{U(1)} = \frac{\text{tr}(\square_{\mu\nu} W_0) - \text{tr} W_0}{\sqrt{1 - (\frac{1}{2} \text{tr} W_0)^2}}. \quad (19)$$

For small L_0 , as implied by (17), the integral of $F_{12}^{U(1)}$ over the 12 plane is indeed equal to $\pm i2\pi$ in the flux vacua (13).

Before continuing to show that the “phase transition” (17) is corroborated by our numerical minimization, let us make some comments regarding the flux and no-flux states:

- (1) The flux and no-flux states are local minima of the energy functional in dYM on $\mathbb{T}^3$ with $n_{12} = 1$. For the no-flux state in pure YM theory, this has been

¹³The terms $\sim \text{tr} W_0$ vanish in the flux vacuum (13). Now, a general $SU(2)$ group element is $W_0 = \cos \alpha + i \sin \alpha \hat{n} \cdot \vec{\sigma}$. The inclusion of $\text{tr} W_0$ in (19) is to eliminate the α dependence for $\text{tr} W_0 \neq 0$, or $\alpha \neq \pi/2$. To explain, we note that in our further study we will use the definition (19) to study the $U(1)$ field around a monopole-instanton. Near its core, $\cos \alpha \neq 0$, only approaching zero asymptotically. Naturally, near $\alpha = 0$, the definition (19) breaks down, as $|\text{tr} W_0| = 2$ and the theory is non-Abelian.

known since [58] (see Ref. [59] for a calculation of the massive spectrum). The argument trivially generalizes to dYM. For the flux state, see Ref. [60] and, for details of the massive spectrum [52];

However, we stress that there is no proof, but only heuristic arguments [52,60] (for example, continuity at $L_1 L_2 \rightarrow \infty$, where the flux state approaches the dYM ground state on $\mathbb{R}^3 \times \mathbb{S}^1$) that one of these two states represent the global minimum of the energy in dYM for any shape of $\mathbb{T}^3$. Below, see Fig. 1, by minimizing the lattice action of dYM for various shapes of $\mathbb{T}^3$, we numerically find that, indeed, either the flux or the no-flux state represents the true ground state in the appropriate regime (17). This is one of the new results of this paper.

- (2) Next, we note that in the flux vacuum the theory classically Abelianizes. To see this, we note that upon a choice of gauge, we can describe the flux vacuum (13) by taking $\langle W_0 \rangle = \pm i\sigma_3$, along with the value of F_{12}^3 given in (13). The winding Wilson loop W_0 acts as a unitary Higgs field, whose vacuum expectation value (VEV) breaks $SU(2) \rightarrow U(1)$ at a scale π/L_0 (the VEV $i\sigma_3$ only commutes with the $U(1)$ Cartan subgroup and the non-Cartan gauge bosons have mass of order $1/L_0$). Then, the last term of (13) is the usual gauge invariant definition of the flux in an unbroken $U(1)$ via the Higgs field [with lattice version (19)]. The massless and massive spectra are explicitly given in [52].

The quantum theory in the flux vacuum remains weakly coupled at $L_0 \Lambda \ll \pi$, with Λ the strong coupling scale of the theory (in this limit, the scale of the breaking is $1/L_0 \gg \Lambda$, ensuring weak coupling). Thus, the classical flux vacuum is close to the true quantum vacuum for such values of L_0 .

As we saw above, the flux state is the preferred classical ground state when $L_0/\sqrt{L_1 L_2} \ll 1$ [see Eqs. (16) and (17)] and certainly in the large $L_1 L_2$ limit (e.g., $\mathbb{R}^3 \times \mathbb{S}^1$). From the above, it remains close to the quantum ground state when, in addition, L_0 obeys $L_0 \Lambda \ll 1$.

- (3) Similarly, we could describe the no-flux vacuum (14) by taking the Higgs field VEV $\langle W_0 \rangle = \pm \mathbf{1}$, a unit matrix commuting with all generators, apparently implying that there is no gauge symmetry “breaking”. However, there are two more unitary Higgs fields, whose expectation values are $\langle \text{tr} W_1 \rangle = \langle \text{tr} W_2 \rangle = 0$. These correspond to non-commuting unitary adjoint Higgs fields W_1 and W_2 whose VEVs can be taken the shift and clock matrices, e.g., $i\sigma_1$ and $i\sigma_3$ for $SU(2)$, thus breaking $SU(2) \rightarrow \mathbb{Z}_N$.¹⁴

¹⁴As further elaborated in Sec. IV, if one takes $L_1 \neq L_2$ the breaking proceeds in two stages.

TABLE I. Measured Wilson action, deformation term action, and total action values for the $[15, L, L, 64]$ lattice, alongside the theoretical predictions $S_{\text{flux}} = 2\pi^2 \frac{L_0 L_3}{L^2}$ and $S_{\text{no-flux}} = 4L^2 \frac{L_3}{L_0^3}$. For better visualization, we have bolded the measured value of the total action and the theoretical prediction it best agrees with. The numerical data shown here are used to make the plots on Fig. 1.

L	Wilson action	Deformation term	Total action	S_{flux}	$S_{\text{no-flux}}$
5	0.01155	1.896	1.908	758	1.896
10	0.03513	7.585	7.62	189.5	7.585
15	0.04314	17.07	17.11	84.22	17.07
20	47.38	5.125×10^{-14}	47.38	47.37	30.34
25	30.37	3.706×10^{-8}	30.37	30.32	47.41
30	21.08	3.62×10^{-8}	21.08	21.06	68.27
35	15.49	2.676×10^{-7}	15.49	15.47	92.92

In the absence of a deformation (i.e., in pure YM theory), the no-flux vacuum is the classical ground state of the $\mathbb{T}_{(L_0, L_1, L_2)}^3$ theory with $n_{12} = 1$, for any L_0, L_1, L_2 [58]. This is clear from the fact that all field strengths in (14) vanish; hence, in the absence of a deformation, the classical energy is zero, the minimum possible one. Thus, it is also clear that this vacuum is expected to be the lowest energy classical state when the deformation contribution is subdominant, i.e., at $L_0/\sqrt{L_1 L_2} \gg 1$, or the more precise Eq. (17).

For general L_1, L_2 the quantum theory in the no-flux vacuum is strongly coupled. However, at small $\sqrt{L_1 L_2}$, such that $\sqrt{L_1 L_2} \Lambda \ll 1$, the gauge group, as described above, can be considered broken to $\mathbb{Z}_N$ by the 't Hooft boundary conditions. Taking $L_1 = L_2$, for such values of L_1 , all fields have mass $\sim 1/L_1 \gg \Lambda$ (see Ref. [59]). The classical theory at distances larger than the inverse mass gap is a $\mathbb{Z}_2$ topological quantum field theory (TQFT), a theory whose Hilbert space consists of the two degenerate no-flux states (14). Quantum mechanically, the degeneracy of the two states is lifted by tunnelling associated with fractional instantons. The area law of the appropriate Wilson loop, e.g., the long distance correlator of two $\text{tr } W_0$ operators on $\mathbb{T}^3 \times \mathbb{R}$, is a consequence of the lifting of this degeneracy. This can be described in different ways; in semiclassical terms, [12,13], recently also reviewed in Sec. 6 of [20], or, in modern language, as the deformation of the $\mathbb{Z}_2$ TQFT by a term leading to the restoration of the $\mathbb{Z}_2^{(1)}$ 1-form symmetry [61].

Now, on Fig. 1, we present the numerical evidence for the transition (17) between the two minima of the dYM action with $n_{12} = 1$ and $n_{03} = 0$. As seen from the numerical data, there is a transition at a critical value $L_c \sim 20$ [22.5 is the theoretical continuum value of Eq. (17)]. This provides numerical evidence that, indeed, one of the two local minima, the flux (13) and no-flux (14) one, is the dYM ground state on $\mathbb{T}_{(x_0, x_1, x_2)}^3|_{n_{12}=1}$ in the

appropriate parts of the “phase diagram”. We notice that since both vacua are local minima, the transition is “first order”; there is level crossing and the flux and the no-flux state are degenerate at the transition point. Our limited numerical study here does not locate precisely this point [at any rate, the continuum critical value (16) is irrational], nor can it measure the height of the energy barrier separating the two. The point shown at $L_1 = 20$ has vanishing deformed and non-vanishing Wilson action, i.e., belongs to the flux vacuum. The numerical values obtained for the Wilson action, deformation action, and total action are compared to the theoretical predictions (18) in Table I, showing good agreement.

The level crossing and associated transition from the flux to the no-flux state evident on the rhs plot of Fig. 1 will be important in our Sec. VI, when we study the transition from monopole instantons on $\mathbb{R}^3 \times \mathbb{S}^1$ to fractional instantons on $\mathbb{R} \times \mathbb{T}^3$.

IV. REVIEW OF MINIMUM ACTION CONFIGURATIONS OF PURE YM WITH DIFFERENT (n_{03}, n_{12}) ON $\mathbb{T}^4$ OF VARYING SHAPE

Here, we turn off the deformation and recall some pertinent features of pure YM theory on $\mathbb{T}^4$ with twists (5). The reason for this review is that the features of pure YM theory on a twisted $\mathbb{T}^4$ described here are important for the interpretation of some of our results of the following sections, notably the comparison between dYM and YM observables.

First, when only a single twist is nonzero, there are 2^2 classical zero action configurations in pure YM theory on $\mathbb{T}^4$ [58,62]. Using gauge invariant terms, these are easy to describe. We begin with $n_{12} = 1$ being the only nonzero twist, where the zero action configurations are

$$\begin{aligned}
 (n_{03} = 0, n_{12} = 1): \quad & S_{\min} = 0, \quad \text{tr } W_1 = \text{tr } W_2 = 0, \\
 & \text{tr } W_0 = \pm 2, \quad \text{tr } W_3 = \pm 2, \\
 & \text{tr } W_1 W_2 = 0,
 \end{aligned} \tag{20}$$

with all signs uncorrelated. This is simply the no-flux vacuum of dYM described in Sec. III (twice repeated, i.e.,

by taking the time direction to be either x_3 or x_0). In pure YM, the action vanishes due to the absence of deformation potential. The zero-action configurations (20) are related by the action of the classically broken $\mathbb{Z}_2$ center symmetries in the 0 and 3 directions, the ones in the plane with vanishing twist. Likewise, when only $n_{03} = 1$ is nonzero, the roles of the 12 and 03 directions are reversed, giving rise to four configurations related by the broken $\mathbb{Z}_2$ center symmetry in the 1 and 2 directions,

$$(n_{03} = 1, n_{12} = 0): S_{\min} = 0, \quad \text{tr } W_1 = \pm 2, \quad \text{tr } W_2 = \pm 2, \\ \text{tr } W_0 = \text{tr } W_3 = 0, \quad \text{tr } W_0 W_3 = 0. \quad (21)$$

Next, when both twists are nonzero, the action saturates the BPS bound for a $|Q| = 1/2$ instanton, $S_{\text{BPS}} = 4\pi^2$ [1,2,16]. The corresponding instanton, apart from the particular case¹⁵ of an (almost) tuned torus shape,

$$(n_{03} = 1, n_{12} = 1): S_{\min} = 4\pi^2, \quad \begin{array}{l} \text{asymptotics away} \\ \text{from the core} \end{array} \rightarrow \begin{cases} \text{tr } W_{1,2} = 0, |\text{tr } W_{0,3}| = 2, & \text{for } L_1 L_2 \ll L_0 L_3, \\ |\text{tr } W_{1,2}| = 2, \text{tr } W_{0,3} = 0, & \text{for } L_1 L_2 \gg L_0 L_3. \end{cases} \quad (22)$$

We now give a heuristic argument in favor of the top line above. Let us argue that $L_1 L_2 \ll L_0 L_3$ leads to the asymptotics stated.¹⁶ Begin by assuming the opposite of (22): suppose that away from the core, the solution localized in L_0, L_3 (and maybe also in one of L_1, L_2 , if one of the two is large) approaches the configuration with $|\text{tr } W_{1,2}| = 2$. As we already mentioned in our discussion after (13), the nonzero twist in the 12 plane implies that $\text{tr } W_1$ must change sign as x_2 traverses a period L_2 and $\text{tr } W_2$ must change sign as x_1 traverses a period L_1 . But since at least one of L_1 or L_2 is small, say L_1 , this change of sign implies that $\text{tr } W_2$ exhibits a large variation (from +2 to -2) over a small distance, which should increase the action cost. This action cost is avoided if the asymptotics is simply $\text{tr } W_2 = 0$, i.e., if the asymptotics is the one shown on the top line in (22). On the other hand, since L_3 is large, the fact that $\text{tr } W_0$ must change sign (again, from +2 to -2) upon traversing a large distance L_3 does not result in large action cost; likewise for $\text{tr } W_3$.

Let us now discuss two examples of the use of (22), which will be relevant further:

- (1) Consider the theory with two twists and take a $\mathbb{T}^4$ with $L_0 \ll L_{1,2,3}$, but obeying $L_0 L_3 \ll L_1 L_2$.

¹⁵In the tuned $L_1 L_2 = L_0 L_3$ case, the action density of the solution in pure YM is uniform through the entire $\mathbb{T}^4$ [27].

¹⁶The argument proceeds with the obvious changes for the opposite case $L_1 L_2 \gg L_0 L_3$.

$L_1 L_2 = L_0 L_3$, is generically found to be localized in at least some of the $\mathbb{T}^4$ directions. The details of the localization, however, depend on the specific choices of shape of the torus. We consider two explicit examples in the paragraphs after Eq. (22) below, taken from [28]. A generic feature seen is that, whenever there are directions in which the instanton is localized, the minimum action configuration approaches, as one moves away from the core of the instanton, one of the two zero action density configurations, (20) or (21). Clearly, this is necessary if the finite action solution is to persist as a localized finite-action “blob” in the limit when the size of these directions (the ones in which the solution is localized) is taken to infinity.

The rule-of-thumb observation of [28] is that the asymptotics of the minimum action $|Q| = 1/2$ configuration away from the core of the instanton depends on the shape of the torus as follows:

According to the bottom line in (22), away from the core of the solution with $|Q| = 1/2$, localized in the large $\mathbb{T}^3_{(x_1, x_2, x_3)}$, we have that $\text{tr } W_{0,3} = 0$, while $|\text{tr } W_{1,2}| = 2$. Notice that $\text{tr } W_0 = 0$ means, semi-classically, that the theory Abelianizes, at a scale $1/L_0$. This is because we can take, asymptotically, $W_0 = i\sigma_3 = \exp(i\frac{\sigma_3}{2}L_0 A_0^3)$, thus the “Higgs field” VEV is $A_0^3 = \frac{\pi}{L_0}$. Notice, however, that there is a second unitary Higgs field which also has an asymptotic VEV, since also $\text{tr } W_3 = 0$ far away from the solution. The corresponding Higgs VEV can be taken $A_3^2 = \frac{\pi}{L_3}$, thus not commuting with the first one—as the vacuum (21) breaks $SU(2)$ to $\mathbb{Z}_2$ —but its effect is small since $L_3 \gg L_0$. This hierarchy thus realizes a two-stage Higgsing in the vacuum surrounding the instanton,

$$SU(2) \xrightarrow[\pi/L_0]{} U(1) \xrightarrow[\pi/L_3]{} \mathbb{Z}_2, \quad \text{with } \pi/L_0 \gg \pi/L_3. \quad (23)$$

The $|Q| = 1/2$ solution in the $L_0 L_3 \ll L_1 L_2$, $L_0 \ll L_3$, geometry was numerically studied in [28] (and much earlier in [54]) and it was shown that it, indeed, has the properties of a monopole instanton on $\mathbb{R}^3_{(x_1, x_2, x_3)} \times \mathbb{S}^1_{(x_0)}$. It is localized in $\mathbb{R}^3$

and extended in the small $\mathbb{S}^1$. Its long-distance Abelian field is given by the projection of the $SU(2)$ field onto the direction set by the asymptotics of W_0 , as in (19). The L_0 extent fixes the size of the core of the solution, and the Abelian magnetic flux and charge are as expected. This is clearly seen on the plots of the action density, asymptotics of Wilson loops, and magnetic field/charge for this solution; see, respectively, Figs. 16–18 in [28],¹⁷ for a $\mathbb{T}^4$ with $(L_0, 48, 48, 48)$ for $L_0 = 6, 9, 12$.

- (2) Now consider the same two-twist theory, taking $L_0 L_3 \gg L_1 L_2$, but with equal $L_1 = L_2$. The resulting $|Q| = 1/2$ finite action configuration is localized in the large $\mathbb{T}^2_{(x_0, x_3)}$, with a core size determined by the small L_1 . The asymptotics of the configuration is as in the top line in (22), with $\text{tr } W_{1,2} = 0$ and $|\text{tr } W_{0,3}| = 2$, i.e., the vacuum (20). Notice that because $L_1 = L_2$, here the breaking pattern is

$$SU(2) \xrightarrow{\pi/L_1} \mathbb{Z}_2. \quad (24)$$

The instanton is a center vortex, localized in the large $L_{0,3}$. It disorders the Wilson loop, see Refs. [14,15]. In the recent [28], using a $\mathbb{T}^4$ of size $(48, 12, 12, 48)$, the localization in the 03 plane is shown on Fig. 13, the disordering effect on the Wilson loop surrounding the center vortex is demonstrated on Fig. 14, while Fig. 15 shows that the winding Wilson loops asymptotics at large $x_{0,3}$ precisely follows our Eq. (22), i.e., the solution approaches at large distances the zero action density background of our Eq. (20).

The above properties of pure YM theory with both twists nonzero imply that in either of the limits shown in (22), Abelianization occurs already in YM theory without deformation. For us, the most important lesson is that, for $L_1 L_2 \gg L_0 L_3$, Abelianization in YM occurs, with $\text{tr } W_0 = 0$ (at $L_3 \gg L_0$), exactly as in the flux phase of dYM (13). This implies (and we shall see numerical confirmation of this expectation) that in this limit, the $|Q| = 1/2$ solutions in dYM and YM are similar, as the Abelianization forced by the twist and by the deformation align. The main effect of the deformation will be seen to lift the action above the BPS limit, $S > 4\pi^2$.

V. THE ‘FLUX’ VACUUM OF DYM: MONOPOLE-INSTANTONS AND THEIR TRANSMUTATION INTO CENTER-VORTICES

We begin our study of the fractionally charged instantons in dYM by considering the flux vacuum on $\mathbb{T}^3_{(x_0, x_1, x_2)}$ with $n_{12} = 1$. As per Eq. (17), it is the state of lowest energy at

sufficiently small L_0 , $L_0 < \frac{L_1}{1.5}$. To numerically study instantons with $|Q| = 1/2$, we also turn on $n_{03} = 1$, as in Eq. (10).

A. Brief review of the analytic study of the monopole/center vortex continuity

Before we begin with our numerical study of this transition, we shall review the compelling analytical picture describing the transition of monopole-instantons in dYM on $\mathbb{R}^3_{(x_1, x_2, x_3)} \times \mathbb{S}^1_{(x_0)}$ to center vortices on $\mathbb{R}^2_{(x_1, x_2)} \times \mathbb{T}^2_{(x_0, x_3)}$, with a ‘t Hooft twist in $\mathbb{T}^2_{(x_0, x_3)}$.¹⁸ This picture was recently developed in the continuum in [29,30].

A side remark is due first, however. The picture of monopole flux collimating to create center vortices has been discussed previously in the lattice literature, see Ref. [22], in particular Chapter 8 there, where pictures like our Fig. 2, obtained after appropriate gauge fixing, appear. The novelty of the observations of [29,30] is that in the deformed theory, one can argue for the relation between confinement mechanisms using an analytical treatment within a valid semiclassical approximation. Apart from being satisfactory on its own, this semiclassical construction was later useful [39,40] in explaining some puzzles regarding confinement in supersymmetric theories.

To describe the construction [29,30], we first recall that there are two kinds of monopole instantons in $SU(2)$ dYM theory on $\mathbb{R}^3_{(x_1, x_2, x_3)} \times \mathbb{S}^1_{(x_0)}$, both of the same topological charge $Q = 1/2$, but with opposite magnetic charges ± 1 [17,18,63]; an introduction/review is in [8]. These are sometimes called BPS and KK monopole-instantons respectively (and, for convenience, we adopt these names here), and can be thought of as the two constituents of a $Q = 1$ instanton. It is also well-known that in dYM, the size of the core of monopole-instantons, an object localized in $\mathbb{R}^3_{(x_1, x_2, x_3)}$ and extended in $\mathbb{S}^1_{(x_0)}$, is set by L_0 , the size of the circle. At distances larger than L_0 in $\mathbb{R}^3$, the long distance field of a monopole instanton is Abelian and can be described, using 3D Abelian duality, in terms of a dual photon.

References [29,30] used this long-distance description. The construction begins by making the x_3 direction of $\mathbb{R}^3_{(x_1, x_2, x_3)} \times \mathbb{S}^1_{(x_0)}$ compact, of size L_3 , and further imposing a ‘t Hooft twist n_{03} in the resulting $\mathbb{T}^2_{(x_0, x_3)}$. Using the Abelian long-distance description of the monopole instantons and the dual photon’s transformation under center symmetry [31], they argued that the $n_{03} = 1$ twist causes the BPS and KK monopole instantons to line up along the x_3 direction. The twist causes a BPS monopole-instanton of charge $+1$, after traversing a period L_3 , to convert into a KK monopole-instanton of charge -1 . In other words, the

¹⁷We use the labeling convention of our Eq. (10). Notice that in Ref. [28], the x_0 and x_3 labels are flipped.

¹⁸We label all coordinates according to the conventions of the present paper, Eq. (10).

image of the BPS monopole upon a translation on a period L_3 in the x_3 direction is the KK monopole, whose image after another L_3 translation is a BPS monopole instanton, etc. Thus, on the covering space of $\mathbb{S}_{(x_3)}^1$, there is an infinite chain of alternating BPS and KK monopole-instantons of charges $\dots/ +1/ -1/ +1/ \dots$, etc.¹⁹ The effect of this chain of alternating charges, computed in [29,30] using the long-distance Abelian description of the monopole-instantons, is that the magnetic flux collimates into the x_1, x_2 plane (i.e., in $\mathbb{R}_{(x_1, x_2)}^2$) and has maximum size²⁰ of order L_3/π , i.e., determined by L_3 . The use of the long-distance approximation to the monopole-instanton field requires $L_3 \gg L_0$ (the core size), i.e., the picture assumes the hierarchy of scales,

$$L_0 \ll L_3 \ll \sqrt{L_1 L_2} (\rightarrow \infty), \quad (25)$$

where we indicated that they considered infinite L_1, L_2 . In other words, the analytic discussion holds on an asymmetric $\mathbb{T}_{(x_0, x_3)}^2$. Further, ref. [29,30] showed that on $\mathbb{R}_{(x_1, x_2)}^2$, this localized “blob” of magnetic flux of size $\sim L_3/\pi$ was, in fact, a center vortex. Concretely, a 12-plane fundamental representation Wilson loop which encloses the flux gets multiplied by -1 , compared to a Wilson loop that does not enclose it.

In summary, what has been achieved is to show that center-vortex semiclassical confinement on $\mathbb{R}_{(x_1, x_2)}^2 \times \mathbb{T}_{(x_0, x_3)}^2$ with unit ’t Hooft flux and L_0, L_3 obeying (25), and the monopole-instanton confinement in dYM on $\mathbb{R}_{(x_1, x_2, x_3)}^3 \times \mathbb{S}_{(x_0)}^1$ are continuously connected, via the deformation described in the previous two paragraphs. The collimation of the flux of monopole instantons into center vortices is illustrated on Fig. 2.

B. Numerical study of the monopole/center vortex continuity

Via numerical minimization, we are able to relax the condition $L_3 \gg L_0$ and show that the continuity between monopole-instantons and center vortices persists to smaller L_3 all the way down to a symmetric $\mathbb{T}_{(x_0, x_3)}^2$, where $L_3 \sim L_0$. Numerics will also allow us to explore the core of the instanton and to compare dYM and pure YM solutions.

We begin our discussion with Figs. 3 and 4, where we plot various quantities for dYM theory with $n_{03} = n_{12} = 1$, for $L_0 = 10$, and two different values of L_1 : $L_1 = 25$ (on Fig. 3) and $L_1 = 45$ (on Fig. 4), as a function of L_3 . On the top line of both figures, we separately plot the dYM Wilson action and the action due to the deformation term. On the

bottom line in each figure, we compare the total action for dYM to the one for YM ($c = 0$) as well as the corresponding widths of the localization of the “Higgs field” $\text{tr } W_0$ in the 12 plane around the instanton (i.e., the core of the instanton). All quantities are shown as function of L_3 varying in the ranges shown.

Let us now discuss the features of the different regimes seen on Figs. 3 and 4:

- (1) First, we stress that on both figures, $\frac{L_1}{L_0} > 1.5$, thus according to Eq. (17) of Sec. III, this regime would correspond to the flux vacuum of dYM on the $\mathbb{T}_{(x_0, x_1, x_2)}^3$ —in the absence of a n_{03} twist or at large enough (strictly infinite) L_3 . Thus, for large enough L_3 , the vacuum surrounding the localized fractional instanton is the dYM flux vacuum; further evidence for this is discussed in the following point 2. Recall also that the dYM flux vacuum (13) approaches the $\mathbb{R}^3 \times \mathbb{S}^1$ dYM vacuum at infinite $L_{1,2}$. We shall see that at large L_3 , the $|Q| = 1/2$ minimum action configurations have an interpretation as the $\mathbb{R}^3 \times \mathbb{S}^1$ monopole-instantons.
- (2) Next, beginning with large L_3 , we observe that the dYM Wilson action of the instanton is linear in L_3 , while the deformation action levels off as a function of L_3 , as seen on the lhs plots on both Figs. 3 and 4. The fact that the action of an instanton grows with the volume may be unusual but should not be surprising: this is because it includes the action of the vacuum, which is nonzero for the flux vacuum of dYM, recall (18).²¹ The fact that the deformation action levels off is also expected, as $\text{tr } W_0 \neq 0$ only in the core of the instanton (and $\text{tr } W_0 = 0$ otherwise) and the core size, for large L_3 , is L_3 independent. Consistent with the picture above, the linear growth of the total action with L_3 seen on the figures is precisely that of the action (18) of the flux vacuum (13).²²
- (3) On the bottom left figure we plot the total action in dYM vs YM, for the same twists and lattice sizes. The YM action equals the BPS action, $4\pi^2 \sim 39.5$, for all values of L_3 , while the total action in dYM linearly grows with L_3 , as per the discussion above.
- (4) On the bottom right in both Figs. 3 and 4, we plot the width of the “Higgs field” $\text{tr } W_0$ in the instanton background, measured in the 12 plane (this is really

¹⁹The lining up of BPS and KK monopole-instantons due to the twist was already understood in the lattice studies of [54].

²⁰As a function of x_3 , with the maximum size reached for x_3 taken half-way between the monopole instanton and its image.

²¹The instanton interpolates between the two degenerate flux vacua [60].

²²For example, on Fig. 3, the slope, for $L_3 > 40$, is easily estimated as $S/L_3 \simeq 0.3$, while (18) gives $S/L_3 = 2\pi^2 L_0 L_1^{-2} = 0.31$. A similar estimate holds for Fig. 4 with a slope ~ 0.1 . On both figures, for large L_3 , the total action can be fit by a fixed value, roughly of order the BPS action (but we stress that a more precise study of interpolations and scaling is needed, which is beyond our scope here) plus the action of the flux vacuum (18) accounting for the linearity in L_3 .

the core size of the instanton). The width of $\text{tr } W_0$ in the 12 plane is determined by a simple fit, sufficient for our purposes (details are explained in the caption of Fig. 5). We see that the core size of the instanton levels off in dYM, consistent with the fact that the instanton size is set by the deformation potential, for large L_3 .

- (5) Regarding the pure YM core size, we note that on Fig. 3, the W_0 width in YM grows as L_3 increases and becomes bigger than L_1 around $L_3 \sim 40\text{--}50$ (here $L_1 L_2 = 625$ while $L_0 L_3 = 500$, which is close to the $\mathbb{T}^4$ shape leading to the constant- F self-dual solution [27]). The similarity of the core sizes in dYM and YM at small L_3 , and the growth of the YM core size with L_3 are also seen on Fig. 4. Notice that the core size growth beyond L_1 , seen on Fig. 3, does not appear on Fig. 4; here $L_1 L_2 = 2025$ and in order that $L_0 L_3$ exceeds $L_1 L_2$ one must go to $L_3 \sim 200$, beyond the scope of our numerics. (We also refer to Fig. 7 and the discussion that follows.)
- (6) At smaller L_3 , outside of the analytic regime (25), we notice that the core sizes (as measured by the W_0 width defined in Fig. 5) in YM and dYM appear to coincide.

Now, we recall the features of YM discussed in Sec. IV, in particular our Eq. (22), arguing that for $L_0 L_3 \ll L_1 L_2$, Abelianization due to W_0 around the localized instanton also occurs in YM theory. Precisely in this limit, the pure YM Abelianization is aligned with the one due to the deformation term. Thus, at the smallest values of L_3 the $\text{tr } W_0$ widths seen are similar in YM and dYM (this is seen even more clearly on our next Fig. 7). The smallness of the deformation action for $L_3 = 5, 10$ is due to the small width of $\text{tr } W_0$.²³ We note that the vacuum surrounding the solution in the 12 plane is now $\text{tr } W_0 = \text{tr } W_3 = 0$.²⁴ At $L_0 \sim L_3$ the limit where the two-stage breaking (23) becomes the one-stage (24) (with the obvious interchange $03 \leftrightarrow 12$ in the latter). See also Fig. 10 and its caption.

To further elaborate on the similarity and difference between dYM and pure YM, at the same twists and lattice sizes, we now move to Fig. 7. Here, we plot the Higgs field localization in the 12 plane (or the core size or the fractional instanton) in YM and dYM for three different lattice sizes. The point is to show that at $L_0 L_3 \ll L_1 L_2$, the

Abelianization in YM due to twists, discussed in Sec. IV, and in dYM due to the deformation potential are aligned in a manner consistent with Eq. (22). The fractional instanton solutions in this regime are qualitatively similar and the role of the deformation potential is only to raise the action above the BPS limit (and to slightly decrease the core size in dYM compared to YM, as a careful look at the top two plots shows). However, as one increases L_3 , going through the “transition” of $L_0 L_3 = L_1 L_2$ (where the YM fractional instanton is position independent), Abelianization in YM disappears, while it persists in dYM. The Abelianized regime in dYM on Figs. 7(c) and 7(e), the ones where $L_3 > L_0$ [recall (25)] is where the considerations of [29,30] are valid.

We continue, on Fig. 8(a), by studying the monopole-instanton’s magnetic field, $F_{12}^{U(1)}$ defined in Eq. (19), in dYM on a (10, 20, 20, 25) lattice. This is, in fact, very similar to the monopole-instantons in pure YM theory on a lattice of a similar size and with the same twists studied in [28]; as in that reference, the total magnetic flux surrounding the monopole-instanton can be seen to be 4π (in particular, the integral of the flux of $F_{12}^{U(1)}$ over each 12 plane taken half-way (in x_3) between the monopole instanton and its x_3 -translation image equals 2π , as in that reference and as suggested in [60]). As already discussed, the similarity is due to the fact that for this size YM theory also Abelianizes in a direction aligned with the deformation. The image of the BPS monopole-instanton upon L_3 translations is a KK monopole-instanton absorbing its flux, etc. The collimation of the flux in the 12 plane in a region of size estimated in [29,30] as $L_3/\pi \sim 8$ is difficult to see for this lattice size (as the result for the collimation size of is obtained in the infinite L_1, L_2 limit, while here we also have images upon translations in L_1, L_2).

Next, we study the collimation of the monopole-instanton flux into a tube stretched along x_3 and localized in the 12 plane. We first refer to Fig. 8(b), where we plot $F_{12}^{U(1)}$ halfway between the BPS and its KK image, as a function of x_1/L_1 —equivalently, due to the approximate cylindrical symmetry, this can be thought of as the dimensionless radial coordinate in the 12 plane. We see that for the larger L_1 sizes (so that images in x_1, x_2 directions can be neglected), an exponential localization occurs, within, roughly, a L_3/π radius. Further, on Figs. 9(a) and 9(b) we show that the flux in both in dYM and YM fits well to a Gaussian and determine its parameters, shown in the caption. That this tube of flux with a finite thickness is a center vortex is seen clearly by studying the disordering effect on Wilson loops surrounding the tube, shown on Figs. 9(c) and 9(d).

We note that Refs. [29,30], working in the $L_{1,2} \rightarrow \infty$ limit, used the long-distance Abelian field of a monopole-instanton to argue that the effect of the infinite ...BPS-KK-BPS-... chain of Fig. 2 is to collimate the magnetic flux into a center-vortexlike configuration. They showed that the this collimation is such that, in the midpoint between a

²³In fact, the width shown on the bottom rhs figure can be used to obtain a reasonable estimate of the numerical value of the deformation action of the top rhs figure (assuming e.g., that $|\text{tr } W_0| = 2$ inside a region of order the width and $= 0$ otherwise).

²⁴This background can, in fact, be thought as the vacuum of dYM on $\mathbb{T}^3_{(L_0, L_1, L_3)}|_{n_{03}=1}$, with L_2 taken as time (this is relevant since we are considering the large L_2 limit); now $\text{tr } W_0 = \text{tr } W_3 = 0$ and $\text{tr } W_1 = \pm 2$ are the two zero energy vacua. These are also the pure YM classical vacua in the same geometry.

monopole-instanton and its image, as $|x_{1,2}| \rightarrow \infty$, the magnetic field falls off as $e^{-\frac{|x|}{L_3}}$ (this falloff can serve as an approximate measure of the localization of the flux). The actual Gaussian shape seen on Figs. 9(a) and 9(b) and its width were not determined. Thus, their observation and determination are new results of this (finite $L_{1,2}$) study.

VI. FROM THE ‘FLUX’ TO THE ‘NO-FLUX’ VACUUM: FROM MONOPOLE-INSTANTONS ON $\mathbb{R}^3 \times \mathbb{S}^1$ TO FRACTIONAL INSTANTONS ON $\mathbb{R} \times \mathbb{T}^3$

We now study the transition of monopole-instantons in dYM on $\mathbb{R}^3 \times \mathbb{S}^1$ to fractional instantons on $\mathbb{R} \times \mathbb{T}^3$. In this case, there is no analytical picture of the transition between the corresponding semiclassical configurations as explicit as the one of Fig. 2 (studied numerically in our previous Sec. V).

We begin our numerical study of the $\mathbb{R}_{(x_3)}^3 \times \mathbb{S}_{(x_0)}^1$ to $\mathbb{R}_{(x_3)} \times \mathbb{T}_{(x_0, x_1, x_2)}^3$ transition in dYM by focusing on Fig. 11, presenting results for a $(10, L, L, 45)$ lattice.²⁵ The plots shown are similar to the ones on Figs. 3 and 4. However, while there we varied L_3 instead, keeping L_0 and $L_1 (= L_2)$ fixed, here we vary $L = L_1$, from 5 to 40, keeping fixed the small L_0 and the large L_3 . For large values of $L_1 \sim L_3$, our lattice can be thought of as $\mathbb{R}_{(x_1, x_2, x_3)}^3 \times \mathbb{S}_{(x_0)}^1$, while for $L_1 \sim L_0$ we approach $\mathbb{T}_{(x_0, x_1, x_2)}^3 \times \mathbb{R}_{(x_3)}$; this will be substantiated by the results that we present below.

The transition from the asymmetric (large $L \sim L_3$) to the symmetric (small $L \sim L_0$) $\mathbb{T}^3$ studied on Fig. 11 is related to the transition from the flux to no-flux vacua in dYM as determined by (17). Recall that the transition from the flux vacuum (large L_1/L_0) of dYM on $\mathbb{T}_{(x_0, x_1, x_2)}^3$ to the no-flux vacuum (small L_1/L_0) occurs at $L_1 \simeq 1.5L_0$, or $L_1 = 15$ for the value of L_0 chosen on the figures. A quick glance on Fig. 11 shows that both the Wilson and the total action of dYM show a maximum at values of $L_1 = L_c \sim 15$. The discontinuous nature of the flux to the no-flux transition from Sec. III (recall the level crossing from the flux to the no-flux vacuum seen on Fig. 1) suggests that the associated change of the nature of the minimal action fractional instantons upon changing the $\mathbb{T}^4$ shape is also discontinuous. We begin our discussion by first discussing the small- and large- L limits and then focusing on the transition between the two.

A. Small- L_1 , $\mathbb{T}^3 \times \mathbb{R}$ regime

Let us first focus on the small- L_1 regime on Fig. 11. The $L_1 = 5, 10$ data can be thought as approximating $\mathbb{T}_{(x_0, x_1, x_2)}^3|_{n_{12}=1} \times \mathbb{R}_{(x_3)}$. The first feature we observe is that the deformation action is very small at $L_1 = 5, 10$.

We can estimate the value of the deformation action as follows. Let us assume that the configuration around the localized solution is the no-flux vacuum of dYM with $|\text{tr } W_0| = 2$ and let us take the deformation action in (11) to be $4(L_3 - L_1) \frac{L_1^2}{L_0^2}$. In writing this expression, we replaced $|\text{tr } W_0|^2 \rightarrow 4$ in (11). We also assumed that $\text{tr } W_0$ vanishes for some segment along x_3 , whose length is of order L_1 (for now, this is just our guess for the width in x_3 of the finite action solution). Otherwise, $\text{tr } W_0$ equals ± 2 , for an extent in the x_3 direction of length $L_3 - L_1$. Remarkably, for $L_1 = 5$ we find that this expression for the deformation action equals 4, precisely matching the values shown on Fig. 11. Likewise, for $L_1 = 10$ we find for the deformation action 14, also perfectly matching the numerics.²⁶ Thus, the dYM no-flux vacuum surrounding the localized (in x_3) solution explains the value of the deformation action for both $L_1 = 5$ and $L_1 = 10$, the two points below the transition at $L_1 = 15$.

Next, we can check the assumptions used above to explain the value of the deformation action for $L_1 = 5, 10$. We do this by plotting, on Fig. 12, the action density and $\text{tr } W_0$ for the solution as a function of x_3 . These plots show that the solution localizes in a segment of length of order L_1 on the x_3 axis, as we now discuss. On Fig. 12(b), we plot the results for the dYM action density (integrated over x_0, x_1, x_2) as a function of x_3/L_3 , for two values of $L_3 = 25, 45$ and for $L_1 = 5, 10$. It is easy to infer from the plot that the action density is localized over a region of size L_1 .

Further, we note that despite the fact that we are plotting the properties of the dYM minimum action configuration, the resulting configuration is very close to the one in pure YM theory, due to the fact that for lattices such as plotted here, the Abelianizations due to the twist n_{12} in YM and in dYM align, as per (22). Concretely, as in our discussion in Sec. V, we notice that for our choice of parameters, $L_1 = 5, L_1 L_2 = 25 \ll L_0 L_3 (= 450)$, values for which the pure YM argument of Eq. (22) implies that far from the core of the solution, $\text{tr } W_0 = \pm 2$ (a similar argument albeit leading to a somewhat less strong inequality applies for $L_1 = 10$). Thus, in the small- L_1 regime of Figs. 11 and 12, the deformation is not qualitatively important, as the same structure surrounding the solution localized in x_3 , the no-flux vacuum (14), is implied by both the dYM criterion (17), and the YM one (22), due to the 12-plane twist.

Finally, we study the disordering of the $\text{tr } W_0$ Wilson loop by the $\mathbb{T}^3 \times \mathbb{R}$ fractional instanton. The top plot, Fig. 12(a), shows the variation of $\text{tr } W_0$, taken at the point of maximal action in x_1, x_2 , as a function of x_3/L_3 , on $\mathbb{T}^3 \times \mathbb{S}_{(x_3)}^1$ (the circle size is L_3 , approximating $\mathbb{R}$) for $\mathbb{T}^3$ of sizes

²⁵As always, we use the notation and choice of twists from (10).

²⁶We also studied the smaller lattice with $L_3 = 25$, where similar estimates at the smallest L_1 values work as well.

(10, 10, 10) and (10, 10, 5). We see, in particular, that $\text{tr } W_0$ jumps from +2 to -2 as one crosses the solution. This is the one-dimensional analogue (i.e., on $\mathbb{T}^3 \times \mathbb{R}$) of the disordering of the Wilson loop by a center vortex on $\mathbb{R}^2 \times \mathbb{T}^2$, observed long ago in [12,13] and argued to lead to semiclassical center symmetry restoration/confinement.

B. The large- L_1 , $\mathbb{R}^3 \times \mathbb{S}^1$ regime

We now move to the large- $L_1 \geq 20$ regime on Fig. 11. In this large- L_1 regime, the solutions are the monopole-instantons in the flux vacuum of dYM already discussed in Sec. V. That this is so is already clear from the plot of the Higgs field ($\text{tr } W_0$) width in the 12 plane shown on the bottom rhs plot on Fig. 11, which, in dYM, remains finite and smaller than L even for $L = 20$.

It is also easy to see that for the values of L_1 and L_3 that match the ones of Figs. 3 and 4 all quantities agree with Fig. 11. Finally, we note that here, $L_0 L_3 = 450$ and $L_1 L_2 = L_1^2 = (400, \dots, 1600)$ as $L_1 = (20, \dots, 40)$. Thus, the lower range of the $20 < L_1 < 40$ regime plotted, is similar to the one on Figs. 7(c) and 7(e), where dYM Abelianizes but YM does not. This is not entirely surprising since the level crossing transition at $L_1 = 1.5L_0$ from flux to no-flux vacuum is one appearing in dYM. In YM, on the other hand, the transition is through the completely delocalized constant field strength minimal action solutions at $L_0 L_3 = L_1 L_2$.

C. The transition region, near $L_1 = L_c \sim 15$

We now move on to discuss the transition seen at $L_1 \sim 15$, associated, as per discussion above, with the dYM transition from the flux to the no-flux vacuum.

First we note that on Fig. 11, the action peak for the (10, L , L , 45) lattice is at $L_c = 15$ is at $L_1 L_2 = 225$ while $L_0 L_3 = 450$. In pure YM these values would be not far from the ones where the minimal action configuration is the constant- F one (which would occur at $L_1 = L_2 \simeq 21$). Thus in YM, one would expect delocalized configurations, as was found in [28] (this is clear already from the $\text{tr } W_0$ 12-plane localization plot for YM on the bottom rhs).

In dYM, the nature of the configurations appearing at $L = 15$ is as follows. In a small number (2 out of 9) of the minimum action configurations for a (10, 15, 15, 45) lattice found by our minimization algorithm, we observe configurations with properties like those of the flux vacuum monopole instantons, as shown on Fig. 13. However, the majority (7 out of 9) of minimal action configurations identified by our algorithm, are configurations of similar action, which are not localized in x_3 . These configurations show a two-peak Wilson action density structure in x_3 , see Fig. 14, but a single-peak

deformation action density, with $|\text{tr } W_0|^2/2$ shown on the rhs of the same figure.²⁷

Our final comment on the transition region is that its more detailed study requires significantly more resources: in particular, a more fine-grained study of the transition region, as well as studies of larger lattices and possibly different values of c may be warranted, a task that goes beyond the scope of this work. However, the point made clear by our results is that the small- L and large- L values correspond to the no-flux and flux vacua surrounding the localized finite action solution, with the detailed study of the transition region left for future work.

ACKNOWLEDGMENTS

This work is supported by an NSERC Discovery Grant. We also acknowledge the Digital Research Alliance of Canada for giving us access to computer resources. We thank Rajamani Narayanan for helpful discussions of numerical methods.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: IMPLEMENTING THE GRADIENT FLOW

Typically minimization of the Wilson action in $SU(2)$ pure Yang-Mills is done using cooling techniques, see [43–45] or the recent [28,55]. However, cooling techniques rely on plaquettes in the Wilson action being first order in the link variables, and therefore will not work for more general actions. Hamiltonian Monte Carlo [64] techniques are more flexible, but reproduce the statistics of the path integral rather than generating minimum action configurations. To circumvent this limitation, we used a modified version of the hybrid Monte Carlo (HMC) algorithm, where rather than giving a configuration random momentum in between evolution periods, the configuration's momentum was set to zero. If the Hamiltonian evolution periods are short enough, this ensures that each evolution period will reduce the action of the configuration, unless numerical errors have caused the algorithm to not conserve energy, in which case the configuration is rejected and the evolution is tried again with a smaller $\delta\tau$. The procedure will be summarized below.

²⁷We note that on the rhs we show the data for the (10, 15, 15, 64) lattice of larger L_3 (where we found that all minimum action configurations at $L = 15$ were of the “two-hump” variety shown). We stress, however, that the identical identical structure of all quantities shown appears for the 7 out of 9 configurations on the (10, 15, 15, 45) lattice.

The overall action used was the same as in (6), repeated here for convenience,

$$S_{\text{total}} = A(S_{\text{Wilson}} + S_{\text{def.}}) = A\left(\sum_x \sum_{\mu\nu} \text{tr}(\mathbf{1} - B_{\mu\nu}(x)\square_{\mu\nu}(x)) + \frac{c}{L_0^3} \sum_{\vec{x}} |\text{tr} W_0(\vec{x})|^2\right). \quad (\text{A1})$$

Following the procedure in [64,65] we derive equations of motion, where $x \in \mathbb{R}^4$ and τ being the fictitious time evolution parameter,

$$\begin{aligned} \dot{U}_\mu(x, \tau) &= \pi_\mu(x, \tau) U_\mu(x, \tau), \\ \dot{\pi}_\mu(x, \tau) &= A \left(\underbrace{\frac{c}{L_0^3} (\text{Tr}(W_0^\dagger) W_0 - \text{Tr}(W_0) W_0^\dagger)}_{\text{Contribution from } S_{\text{def.}}} + \underbrace{(V_\mu^\dagger U_\mu^\dagger - U_\mu V_\mu)}_{\text{Contribution from } S_{\text{Wilson}}} \right) (x, \tau). \end{aligned} \quad (\text{A2})$$

where $W_0 := W_0(x)$, i.e., the Wilson loop winding in the 0 direction starting at site x , and the staple $V_\mu(x)$ is defined by

$$V_\mu(x) = \sum_{\nu \neq \mu} (B_{\mu\nu}(x) U_\nu(x + \hat{e}_\mu) U_\mu^\dagger(x + \hat{e}_\nu) U_\nu^\dagger(x) + B_{\mu\nu}(x - \hat{e}_\nu) U_\nu^\dagger(x - \hat{e}_\nu + \hat{e}_\mu) U_\nu^\dagger(x - \hat{e}_\nu) U_\nu(x - \hat{e}_\nu)). \quad (\text{A3})$$

where $B_{\mu\nu}$ is the 2-form topological background (8), as in (9) or (10). The $\dot{\pi}_\mu(x)$ equation is derived by defining the Hamiltonian

$$\mathcal{H} = \sum_{x,\mu} \text{tr} \left(\frac{\pi_\mu \pi_\mu^\dagger}{2} \right) + S_{\text{total}}[U_\mu]. \quad (\text{A4})$$

and then enforcing $\dot{\mathcal{H}} = 0$ and $\dot{U}_\mu(x, \tau) = \pi_\mu(x, \tau) U_\mu(x, \tau)$. Then, the procedure for generating a configuration is as follows:

- (1) Generate a lattice with random link variables $U_\mu(x)$.
- (2) Assign each link variable an $\mathfrak{su}(2)$ -valued momentum $\pi_\mu(x) = \mathbf{0}$. The initial total energy is thus $\mathcal{H} = S_{\text{total}}$.
- (3) Numerically evolve the links and momenta for some length of fictitious time τ_0 according to Eq. (A2), implemented using a leapfrog time-stepping method to conserve $\mathcal{H}$.
- (4) Verify that S_{total} has decreased. If it has, accept the configuration. Otherwise, increase numerical precision (by reducing A and shrinking the time step $\delta\tau$) and repeat.
- (5) If accepted, set all $\pi_\mu(x) = \mathbf{0}$ and repeat steps 3 and 4 for the new configuration of links.
- (6) Repeat this process until action has stopped changing or a predetermined number of evolution periods have elapsed.

This procedure is the same as in standard HMC, with the exception that momenta are set to zero rather than randomized between evolution periods and the acceptance criterion is a reduction in the action, not of $\mathcal{H}$. In contrast with HMC, this acceptance step plays no statistical role

and is simply a check that the algorithm is numerically stable.

To speed up the rate of convergence, the parameter A can be adjusted, which increases the “forces” present in Eq. (A2). However, the “true” action of a configuration, the value shown in figures above, was always calculated by dividing out the parameter A . Each time the change in true action after a period of evolution dropped below 1%, the parameter A was multiplied by 10 for the next period of evolution. This continued until the program started to encounter numerical instability, at which point the multiplication stopped. Any subsequent numerical instability resulted in A being halved.

This procedure was able to easily reproduce the known numerical results on pure Yang-Mills twisted lattices found in [28]. For most lattices, if a true minimum of the action was found, the program had reduced the action to that minimum within around 300 periods of evolution for a fictitious time $\tau = 10$. Frequently a local minimum of the action would be encountered far above the true minimum. These local minima appeared frequently when reducing the action of dYM configurations, and became more common as lattice sizes increased. Pure Yang-Mills configurations very rarely encountered local minima, if at all.

Given the lack of a BPS bound for dYM, the numerical value of actions designated “true” vacua were determined by performing this minimization process on a large number of configurations, and taking the smallest action found to be the true minimum. Usually, a significant fraction of configurations clustered around this “true” minimum value, and others clustered around various local minima above the true vacuum.

APPENDIX B: THE CLASSIFICATION OF FINITE ACTION EUCLIDEAN SOLUTIONS ON $\mathbb{R}^2 \times \mathbb{T}_{n_{12}}^2$

Let us work on $\mathbb{R}_{(x_0, x_3)}^2 \times \mathbb{T}_{(x_1, x_2)}^2$ with a twist $n_{12} = 1$, in an $SU(2)$ pure YM theory.²⁸ We want to classify the finite Euclidean action configurations in this setup. We begin by taking the $A_0 = 0$ gauge, which, we recall, is also very convenient in the usual analysis of finite action instantons on $\mathbb{R} \times \mathbb{R}^3$, with $\mathbb{R}$ denoting Euclidean time. A finite action field configuration should have the property that at $\mathbb{R}^2$ -infinity, it should satisfy $F_{ij} = 0$ and $F_{0i} = 0$, where $i = 1, 2, 3$, i.e., should go to a pure gauge. The second condition, in the $A_0 = 0$ gauge, implies that $\partial_0 A_i = 0$ at infinity in $\mathbb{R}_{(x_0, x_3)}^2$. It is convenient to think of infinity in $\mathbb{R}^2$ as a square, whose four sides correspond to $x_0 \rightarrow \pm\infty$ (for any x_3) or $x_3 \rightarrow \pm\infty$ (for any x_0).

We then ask what are the minimum (zero) energy classical configurations, $F_{ij} = 0$, on $\mathbb{R}_{(x_3)} \times \mathbb{T}_{(x_1, x_2)}^2|_{n_{12}=1}$, which should be the configurations the finite action Euclidean solutions should approach as $x_0 \rightarrow \pm\infty$. To proceed, we take constant transition functions in the 12 plane, $\Gamma_1, \Gamma_2 \in SU(2)$, with the gauge field; $A \equiv A_i dx^i$ (a sum over $i = 1, 2, 3$ is implied) obeying twisted boundary conditions on $\mathbb{T}^2$,

$$A(\vec{x} + \vec{e}_1 L_1) = \Gamma_1 A(\vec{x}) \Gamma_1^\dagger, \quad (\text{B1})$$

$$A(\vec{x} + \vec{e}_2 L_2) = \Gamma_2 A(\vec{x}) \Gamma_2^\dagger, \quad (\text{B2})$$

$$\Gamma_1 \Gamma_2 = e^{i\pi} \Gamma_2 \Gamma_1 \quad (\text{B3})$$

Clearly, $F_{ij} = 0$ implies that $A \equiv A_i dx^i$ is pure gauge at $x_0 \rightarrow \pm\infty$. The solution to this problem has been known for a long time [58],

$$A^{(k)} = -iT^k dT^{-k}, \quad k = 0, 1,$$

$$\text{where } T = T(x_1, x_2, x_3) \equiv T(\vec{x}) \in SU(2). \quad (\text{B4})$$

In other words, the pure-gauge configurations which the finite action instanton should approach as $x_0 \rightarrow \pm\infty$ are either $A^{(0)} = 0$, or $A^{(1)} = -iTdT^{-1}$. The properties that $T(\vec{x})$ obeys are

$$T(\vec{x} + \vec{e}_1 L_1) = \Gamma_1 T(\vec{x}) \Gamma_1^\dagger,$$

$$T(\vec{x} + \vec{e}_2 L_2) = \Gamma_2 T(\vec{x}) \Gamma_2^\dagger,$$

$$T(x_1, x_2, x_3 \rightarrow \infty) = e^{i\pi} T(x_1, x_2, x_3 \rightarrow -\infty). \quad (\text{B5})$$

²⁸For brevity we take $SU(2)$ gauge group, noting that the argument is easily generalized to any $SU(N)$ with a general twist n_{12} , see the comment at the end. Also, it generalizes to dYM, with the deformation either in the x_3 direction (which then would have to be taken compact, replacing $\mathbb{T}^2$ with $\mathbb{T}^3$) or in one of the directions of $\mathbb{T}^2$, e.g., the x_1 .

An explicit expression of a representative²⁹ of T can be written using the function $g(x_1, x_2)$ and $f(x_3/L_1)$ obeying the following properties:

$$T(x_1, x_2, x_3) = g(x_1, x_2) e^{-i\pi\sigma^3 f(x_3/L_1)} g^{-1}(x_1, x_2),$$

where $g(x_1 + L_1, x_2) = \Gamma_1 g(x_1, x_2)$,

$$g(x_1, x_2 + L_2) = \Gamma_2 g(x_1, x_2) e^{-i\pi\sigma^3 \frac{x_1}{L_1}},$$

$$f(y \rightarrow +\infty) = 1,$$

$$f(y \rightarrow -\infty) = 0. \quad (\text{B6})$$

An explicit expression for the function $g(x_1, x_2)$ can be found in [52],³⁰ where $\Gamma_1 = i\sigma^1$ and $\Gamma_2 = i\sigma^3$, but all we need to verify that $T(\vec{x})$ of (B6) obeys (B5) is its existence and properties. The function $f(y)$ can be any smooth function with the given limits, for example $f(y) = \frac{1}{2}(1 + \tanh y)$ and we note that the scale L_1 appearing in $f(x_3/L_1)$ on the first line in (B6) can be replaced by any fixed scale.

Further, we note that as $x_3 \rightarrow \pm\infty$, the two configurations $A^{(0)}$ and $A^{(1)}$ from (B4) (the two limits of the finite action solution at $x_0 \rightarrow \pm\infty$) approach zero exponentially fast because $e^{i\pi\sigma^3 f}$ approaches ± 1 times the unit matrix at spatial infinity, thus T becomes trivial and A vanishes as $|x_3| \rightarrow \infty$. In each case, as $A = 0$ at $|x_3| \rightarrow \infty$, $\partial_0 A_i = F_{0i} = 0$ is obeyed there, as required. The configurations at $x_0 = \infty$ and $x_0 = -\infty$ could both be the same or differ, i.e., correspond to the same or different value of k from (B4). Thus, a finite action Euclidean configuration on $\mathbb{R}^2 \times \mathbb{T}^2$ with $n_{12} = 1$ can have one of four possible boundary conditions at the $\mathbb{R}^2$ infinity. In the case where the gauge field approaches the same limit at $x_0 \rightarrow \pm\infty$, i.e., either both have $k = 0$ (or $k = 1$), the topological charge is integer [owing to the fact that the large gauge transformations $\mathbb{S}^3 \rightarrow SU(2)$, not included in (B4), can have different integer winding in the two limits, recall footnote 29].

Consider then the gauge field approaching different limits, for definiteness, $k = 1$ at $x_0 \rightarrow \infty$

²⁹This is because T itself is defined up to large gauge transformations with integer winding number, which can be thought of as maps from $\mathbb{S}^3$ to $SU(2)$, see Ref. [33] for a recent discussion. In other words, as written, the two configurations (B4) only capture the fractional part of the topological charge. The integer action classification is the same as on $\mathbb{R}^4$.

³⁰The meaning of $g(x_1, x_2)$ is that it is the function [itself a rather complicated map from the covering space of $\mathbb{T}^2$ to $SU(2)$] that maps the constant to the Abelian transition functions. This will not be important in our discussion here. We also note that if L_3 is taken finite, then in T we replace $f(x_3/L_1)$ by x_3/L_3 ; T then has the meaning as the center symmetry generator in the x_3 direction. The two configurations $A^{(0)}$ and $A^{(1)}$ from (B4) are distinguished by the value of the winding Wilson loop in x_3 $\text{tr } W_3$ it equals 2 for $k = 0$ and -2 for $k = 1$. We note that with the definition (B6) this distinction remains true in the infinite L_3 limit, where $\text{tr } W_3$ now includes an x_3 integral over the real axis.

(i.e., $A|_{x_0 \rightarrow \infty} = -iTdT^{-1}$) and $k = 0$ at $x_0 \rightarrow -\infty$ (i.e., $A|_{x_0 \rightarrow -\infty} = 0$). The (fractional part of the) topological charge of such a configuration is then calculated, upon integrations by parts, in terms of its asymptotics, following the steps outlined below:

$$Q|_{k=0}^{k=1} = \frac{1}{8\pi^2} \int_{\mathbb{R}^2_{(x_0, x_3)} \times \mathbb{T}^2_{(x_1, x_2)}} \text{tr}(TdT^{-1})^3, \quad (\text{B7})$$

$$\begin{aligned} & \frac{1}{24\pi^2} \int_{\mathbb{R}_{(x_3)} \times \mathbb{T}^2_{(x_1, x_2)}} \text{tr}(TdT^{-1})^3 \\ &= \frac{i}{4\pi} \int_{\mathbb{R}_{(x_3)} \times \mathbb{T}^2_{(x_1, x_2)}} dx_1 dx_2 dx_3 \partial_3 f \text{tr}[\sigma_3(X_\alpha(1 - \cos^2 \pi f) - \sigma^3 X_\alpha \sigma^3 \sin^2 \pi f) \epsilon^{\alpha\beta} X_\beta] \\ &= \int_{-\infty}^{\infty} dx_3 (1 - \cos 2\pi f) \partial_3 f \int_0^1 dx_1 \int_0^1 dx_2 \partial_2 \tilde{f}^2(x_2) = (f|_{x_3 \rightarrow \infty} - f|_{x_3 \rightarrow -\infty})(\tilde{f}^2(1) - \tilde{f}^2(0)) = \frac{1}{2}. \end{aligned} \quad (\text{B8})$$

Obtaining the last line requires using the properties of $g(x_1, x_2)$, along with behavior of $f(x_3)$ from (B6), as well as the already mentioned definitions of X_α . We rescaled all coordinates appropriately so that $L_1 = L_2 = 1$. The function $\tilde{f}(x_2)$ on the last line is the “bump” function entering the definition of $g(x_1, x_2)$, see Appendix A in [52]; most importantly, it has the property that $\tilde{f}^2(1) - \tilde{f}^2(0) = \frac{1}{2}$. The entire calculation follows the one given in Sec. 3.1 of [52] for the $\mathbb{R} \times \mathbb{T}^3$ case, the only difference being that the function $f(x_3)$ is taken to be the one appropriate for $\mathbb{R}^2 \times \mathbb{T}^2$. Combining (B7) and (B8), we obtain that the fractional part of the topological charge with different asymptotics of the gauge field at $x_0 \rightarrow \pm\infty$ is $Q|_{k=0}^{k=1} = 1/2$. The generalization to $\mathbb{R} \times \mathbb{T}^3$ with a twist n_{12} in $\mathbb{T}^3$ follows essentially the same steps.

The moral of the discussion here is that one can argue for the fractionality of the topological charge already in the semi-infinite limit $\mathbb{R}^k \times \mathbb{T}^{4-k}_{n_{12}}$, $k = 1, 2$. The argument is familiar from $\mathbb{R}^4$ and is based on the conditions on the gauge field configurations at infinity ensuring finite Euclidean action (made convenient by choosing $A_0 = 0$) and on the understanding of the kinds of locally pure-gauge configurations that are allowed in the presence of ’t Hooft twists. This had been understood already by the authors of [12–15] who observed that the fractional instantons persist as localized configurations of finite action when the

The equality above follows from (B4) upon integration by parts in x_0 , similar to [1,2] (also given in detail in [33,52]; we stress that the use of constant transition functions is important).

To further calculate the winding number [which appears on the rhs in (B7)] of T , considered a map $\mathbb{R} \times \mathbb{T}^2 \rightarrow SU(2)$ obeying the boundary conditions (B5), we use the expression for T from (B6) and introduce the shorthand notation $X_\alpha \equiv g^{-1} \partial_\alpha g$, $\alpha = 1, 2$ (explicit expressions for X_α are given in Appendix A in [52]),

appropriate semi-infinite volume limit is taken. As already noted, we are not aware of a more formal argument, along the lines given here, in the published literature,³¹ which is why we included it for completeness.

The argument generalizes to dYM as well, in both the flux and no-flux vacua. We shall not give any details, which can be filled in by the reader, but will only mention that if one considers the no-flux vacuum of dYM, the argument is essentially the same as the one already given. On the other hand, in the flux vacuum, the fractional topological charge corresponds to tunnelling events between the two different flux vacua, a fact that has been recognized already in [60].

Our final comment concerns the generalization to $SU(N)$.³² Here, one can similarly show that the topological charges are $Q = \frac{p}{N'} \pmod{1}$ with $p = 0, \dots, N' - 1$ and $N' = N/\text{gcd}(n_{12}, N)$. The explicit detailed expression for the map T , which we used in (B8) to calculate the winding number (B7), is not really needed in order to obtain the fractional part of Q , see for example [16] or the Appendixes of [33].

³¹However, A. González-Arroyo has told us that he is familiar with the argument given here.

³²The only advantage for considering $SU(2)$ is that all quantities entering Eqs. (B6)–(B8) are explicitly worked out in [52].

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