

Duality of Ryu-Takayanagi surfaces inside and outside the horizon

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We study the Ryu-Takayanagi (RT) surfaces associated with timelike subregions in static spacetimes with a horizon. It is possible to find the analytical continuation of the RT surfaces that can extend into the horizon, allowing us to probe the interior of the black hole. The horizon typically divides the RT surface into two distinct parts. We demonstrate that the area of the surface inside the horizon can be reconstructed from the contributions of the surfaces outside the horizon, along with additional RT surfaces for spacelike subregions that are causally related to the timelike subregions. This result provides a concrete realization of black hole complementarity at the level of classical metric, where the spacetime in the black hole interior can be reconstructed from the degrees of freedom outside the horizon.

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Introduction. The event horizon and singularity of a black hole are crucial probes of the quantum nature of gravity. Although information within the event horizon cannot be directly transmitted to the exterior, black hole complementarity suggests that the degrees of freedom in the interior are intimately linked to those in the exterior [1,2].

In the context of Anti-de Sitter (AdS)/CFT, information within the black hole interior can be probed by observables in the dual field theories. Specifically, the bulk codimension-2 Ryu-Takayanagi (RT) surface serves as a useful holographic probe that is associated with the entanglement entropy (EE) of spacelike subregions on the boundary [3,4]. However, for RT surfaces anchored to spacelike subregions, the black hole horizon acts as a barrier [5,6], limiting the extent of the RT surface. Nevertheless, RT surfaces corresponding to subregions on the two boundaries of an eternal black hole can probe the interior of a black hole [7]. Other interesting holographic probes include correlation functions of heavy operators [8,9] and complexity [10,11].

Recently, it has been proposed that the concept of entanglement entropy can be extended to timelike subregions [12], which may be interpreted as pseudoentropy [13]. In recent years there are many works on entanglement in time from various aspects [14–23]. Timelike entanglement entropy can be well defined and computed in quantum

field theories (QFTs) through an analytical continuation of correlation functions. Although the physical meaning of entanglement in time is not yet fully understood, it is nevertheless believed that the RT formula could be applicable to timelike entanglement entropy in the context of AdS/CFT. However, a complete holographic proposal has yet to be established. Different proposals for holographic RT surfaces exist. In [12,24], the authors suggest that the RT surfaces for timelike EE are given by both spacelike RT surfaces and their timelike counterparts, while in [25], it is argued that the RT surfaces correspond to extreme surfaces in complexified geometry. In the case of AdS₃, the two proposals yield equivalent results of timelike EE, but the distinction may emerge in higher-dimensional examples. Despite the incomplete understanding of holographic timelike EE, one intriguing feature is that the RT surfaces could extend into the black hole horizon, thus probing the interior, including the singularity. This could provide a novel probe to investigate the interior of a black hole [24,26]. In [27], the authors find that timelike EE is related to spacelike EE in some examples of two-dimensional conformal field theories (CFTs), the holographic interpretation of this relation remains unclear.

The work presented in this Letter is motivated by recent studies of timelike EE and its holographic dual. We propose a method to construct RT surfaces for timelike subregions by analytically continuing their Euclidean counterparts, which correctly reproduces the expected results for timelike EE. This construction naturally leads to RT surfaces extending into a complexified geometry, consistent with the framework outlined in [25]. Importantly, the method is new and broadly applicable to general static backgrounds in arbitrary dimensions. Furthermore, we establish a relation between holographic timelike and spacelike EE

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for strip subregions in higher-dimensional backgrounds. This finding indicates that such a relation should hold in more general settings, extending beyond the specific two-dimensional CFTs case of [27].

More precisely, we will consider a static spacetime with a horizon. On the boundary, we choose a timelike strip subregion between the time interval $[0, t_0]$ (see Fig. 2 for an illustration). We would like to consider the extreme surfaces anchored to the boundary of the timelike subregion. These extreme surfaces for timelike subregions generally extend into the horizon and approach the singularity of the black hole. However, the RT surfaces for timelike subregions cannot be determined solely by the boundary conditions. In the following, we demonstrate how to construct the specific RT surfaces through analytical continuation of the Euclidean counterparts. An interesting feature is that the RT surfaces naturally extend into the interior of the horizon and the complexified geometry.

Specifically, we uncover a duality for the RT surfaces: the area of the RT surface inside the horizon (red line in Fig. 1) is related to the area outside the horizon (black line in Fig. 1), as well as to the RT surfaces for spacelike regions on the Cauchy surface $t = 0$ in the regions $[-t_0, t_0]$. This relation is closely tied to causality. As noted above, the RT surfaces associated with spacelike subregions lie outside the horizon, suggesting that the relation can be interpreted as a duality between the RT surfaces inside and outside the horizon. This result is remarkable. In principle, one can construct infinitely many surfaces anchored on the timelike boundary subregion, some of which may cross the horizon and even reach the singularity. Generally, one would not expect the portion inside the black hole to be determined by data outside. However, we demonstrate that certain extremal surfaces do precisely this.

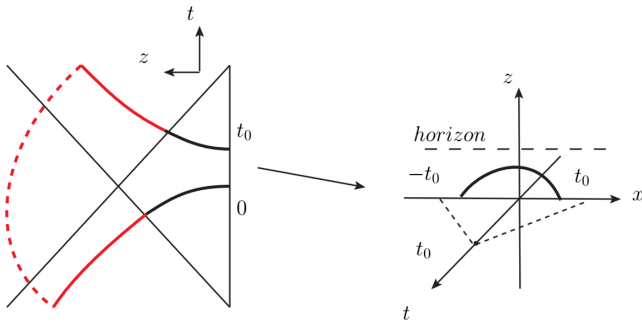


FIG. 1. Black hole horizon divides the RT surface for timelike subregions into two parts. The coordinates z , x , and t represent the holographic, spatial, and temporal directions, respectively. The red solid and dashed lines denote the portion of the RT surface inside the horizon, while the black solid line connecting the horizon to the boundary represents the RT surface outside the horizon. The duality relation states that the area of the RT surface inside the horizon equals the area outside the horizon, plus the contribution from RT surfaces for spacelike subregions on the Cauchy surface $t = 0$ (as shown in the right panel).

Moreover, the relation between timelike and spacelike EE can be applicable to black hole physics, with an interpretation of the duality between the geometric quantities inside and outside the horizon. The RT surfaces inside the horizon encode information about the black hole's interior geometry. Our duality relation asserts that this information can be reconstructed using the RT surfaces outside the horizon. This implies that an observer outside the horizon can learn the geometry inside the horizon by measuring the area of the RT surfaces outside the horizon. In the context of black hole complementarity, the information inside the black hole can be reconstructed from the exterior degrees of freedom, which means that the information in the interior of a black hole is redundant [28,29]. The duality relation in this letter can also be interpreted as a manifestation of black hole complementarity at the level of classical spacetime.

General setup for the Ryu-Takayanagi surface. Consider the asymptotically AdS_{d+1} metric

$$ds^2 = \frac{1}{z^2} \left(-f(z)dt^2 + \frac{dz^2}{g(z)} + dx^2 + d\vec{y}^2 \right), \quad (1)$$

where we set the radius of AdS $L = 1$ and $d\vec{y}^2 = \sum_{i=1}^{d-2} dy_i^2$. The AdS boundary is at $z = 0$. We assume that there exists a Killing horizon $z = z_h$ with $f(z_h) = 0$ for the Killing vector ∂_t . The spacetime is divided into regions inside and outside the horizon, as shown in Fig. 1. Generally, let us consider a timelike subregion A_L to be a strip between $(t, x, \vec{y})$ and $(t', x', \vec{y}')$, denoted by $s(t, x; t', x')$, see Fig. 2.

The procedure to construct the RT surface for A_L is as follows. The Euclidean section of the solution (1) can be obtained by $t \rightarrow -i\tau$. The Euclidean solution is defined for $z \geq z_h$. On the boundary, we will consider the subsystem A_E of the strip between $(\tau, x, \vec{y})$ and $(\tau', x', \vec{y}')$, denoted by $s_E(\tau, x; \tau', x')$. First, we solve for the RT surface associated with the subsystem A_E . By symmetry, we can parametrize the RT surface as $\tau = \tau(z)$ and $x = x(z)$. There exists two conserved constant for the RT surface, which satisfy

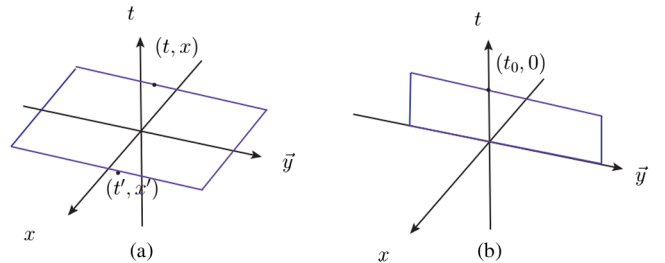


FIG. 2. (a) General timelike strip. The strip extends along the $\vec{y}$ direction, with the coordinates on the $t-x$ plane (t, x) and (t', x') where $\vec{y} = 0$. The separation between these two points is assumed timelike. (b) A special case that the strip lies along $x = 0$.

$$\begin{aligned}\tau'(z)^2 &= \frac{p_\tau^2}{f(z)g(z)[f(z)L^2z^{2(1-d)} - (f(z)p_x^2 + p_\tau^2)]}, \\ x'(z)^2 &= \frac{f(z)p_x^2}{g(z)[f(z)L^2z^{2(1-d)} - (f(z)p_x^2 + p_\tau^2)]}.\end{aligned}\quad (2)$$

In the Euclidean solution $f(z) \geq 0$, there exists a turning point of the RT surface $z = z_\tau \leq z_h$ such that $\tau'(z_\tau), x'(z_\tau) \rightarrow \infty$. This leads to the condition

$$f(z_\tau)L^2z_\tau^{2(1-d)} - (f(z_\tau)p_x^2 + p_\tau^2) = 0. \quad (3)$$

By using the conditions $\int_0^{z_\tau} dz \tau'(z) = \frac{\tau - \tau'}{2}$ and $\int_0^{z_\tau} = \frac{x - x'}{2}$, one could obtain the RT surface in the Euclidean section.

On the other hand if we consider a strip A_L in the Lorentzian metric (1), the RT surface with parametrization $t = t(z)$ and $x = x(z)$ would satisfy a relation similar to (2),

$$\begin{aligned}t'(z)^2 &= \frac{p_t^2}{f(z)g(z)[f(z)L^2z^{2(1-d)} - (f(z)p_x^2 - p_t^2)]}, \\ x'(z)^2 &= \frac{f(z)p_x^2}{g(z)[f(z)L^2z^{2(1-d)} - (f(z)p_x^2 - p_t^2)]}.\end{aligned}\quad (4)$$

If the separation between (t, x) and (t', x') is timelike, there may not exist a real turning point z_t for the RT surface. In general, these types of RT surface may pass through the horizon at $z = z_h$ and possibly extend to the singularity inside the horizon. However, unlike in the Euclidean case, there is no way to fix the values of p_t and p_x in the Lorentzian geometry.

A specific choice is the analytical continuation of the RT surface $\tau(z)$ and $x(z)$ in the Euclidean metric. By comparing the Eq. (2) with the Wick rotation $\tau \rightarrow it$ and (4), the conserved constants are fixed by

$$\begin{aligned}p_t^2 &= -p_\tau^2|_{\tau \rightarrow it, \tau' \rightarrow it'}, \\ p_x^2 &= p_x^2|_{\tau \rightarrow it, \tau' \rightarrow it'}.\end{aligned}\quad (5)$$

The turning point would also have a natural analytical result

$$z_t := z_\tau|_{\tau \rightarrow it, \tau' \rightarrow it'}. \quad (6)$$

With these conditions one could solve the RT surface $t(z)$ and $x(z)$. We will present some examples later. The procedure outlined above provides a method for constructing specific RT surfaces for timelike subregions. As mentioned, these RT surfaces extend into the horizon, offering a way to probe the interior of the black hole. After the analytical continuation (5), there are no turning points in the Lorentzian geometry. This suggests that the RT surface should be extended into the complexified geometry, consistent with the framework proposed in [25].

The area of the RT surface is given by

$$\begin{aligned}\mathcal{A}(t, x; t', x') &= 2 \int_C dz \mathcal{L}, \\ \mathcal{L} &= \frac{R^{d-2}}{z^{2(d-1)}} \frac{\sqrt{f(z)}}{\sqrt{g(z)[f(z)L^2z^{2(1-d)} - (f(z)p_x^2 - p_t^2)]}},\end{aligned}\quad (7)$$

where R is the IR cutoff for the y_i directions, C is a path on the complex z plane connecting the boundary $z = \delta$ to the turning point z_t , which is generally a branch point of the integrand in (7). To evaluate the integral (7), one must specify the path C . However, our method does not directly determine this choice. In principle, we expect the RT surface for the strip to be unique, corresponding to the path C_R on the complex- z plane. Since (7) is a contour integral, the result remains unchanged for different paths as long as the regions enclosed by them contain no singularities. Therefore, one may equivalently choose the path C shown in Fig. 3, in which case the associated bulk surface extends across the black hole horizon and reaches the singularity.

We can thus define and evaluate the surface areas on both sides of the horizon,

$$\begin{aligned}\mathcal{A}_{\text{out}} &= 2 \int_\delta^{z_h} dz \mathcal{L}, \\ \mathcal{A}_{\text{in}} &= 2 \int_{z_h}^{z_t} dz \mathcal{L}.\end{aligned}\quad (8)$$

By definition we have $\mathcal{A}(t, x; t', x') = \mathcal{A}_{\text{in}} + \mathcal{A}_{\text{out}}$.

In the case of higher-dimensional black holes, analytical results can only be obtained via perturbative methods. Alternatively, numerical methods can be used to determine turning points. Let us focus on the strip $s(0, 0; t_0, 0)$.

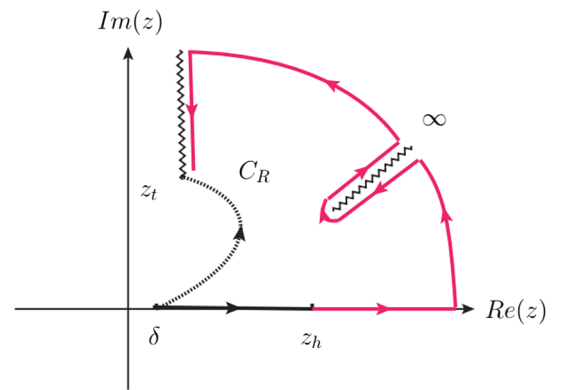


FIG. 3. Evaluation of the integral (7) requires specifying a path C in the complex z plane that connects $z = \delta$ to $z = z_t$. We assume that the path C_R (dotted line) corresponds to the correct RT surface. For our purpose, however, we choose a path that first runs from $z = \delta$ to $z = z_h$ (solid black line), and then continues from z_h to z_t (solid red line).

For this setup, we solve Eq. (4) with $p_x = 0$. The key issue is to determine the turning point z_t , which is generally complex valued. By symmetry, we expect the boundary conditions for Eq. (4) to be $t(0) = 0$ and $t(z_t) = 0$. Consequently, we can numerically solve the differential equation in a manner similar to the spacelike case, except that both z_t and $\mathcal{A}$ are complex valued. In the following, we present numerical results for the higher-dimensional black hole examples.

The duality. The RT surfaces we construct provide a holographic realization of timelike EE for the strip subregions. In the QFTs, the timelike EE can be evaluated by analytically continuing correlators involving twist operators. Our construction of RT surfaces for timelike subregions thus serves as the bulk dual of this analytic continuation. Consider the specific case $A_L = s(0, 0; t_0, 0)$, as illustrated in Fig. 2(b). The timelike EE is associated with the correlator of twist operator Σ_n , which are generally nonlocal. Specifically, $S_E \sim \langle \Sigma_n(t_0, 0) \Sigma_n(0, 0) \rangle_{\text{QFT}^n}$, where the subscript denotes n copies of the theory. By causality, the operators $\Sigma_n(t_0, 0)$ can be decomposed as operators on the Cauchy surface $t = 0$ within the region $x \in [-t_0, t_0]$, that is the strip $s(0, -t_0; 0, t_0)$. Consequently, S_E is expected to be represented by correlators on the Cauchy surface, which may also exhibit bulk geometric duality. In the following examples, we demonstrate that only $\mathcal{A}(0, x; 0, x')$ and $\partial_t \mathcal{A}(0, x; 0, x')$ are required, both of which depend solely on data outside the black hole horizon. For more general states, causality implies that the timelike EE should still depend only on the expectation values of operators localized in the spacelike strip $s_L(0, -t_0; 0, t_0)$. However, in such cases, additional contributions from other operators in the theory are also expected to appear [30]. According to the subregion/subregion duality, or equivalently the entanglement wedge reconstruction theorem [31–33] in AdS/CFT, these new contributions can still be mapped to the bulk entanglement wedge associated with $s(0, -t_0; 0, t_0)$, which lies outside the horizon. In summary, we expect that the bulk surface encoding the timelike EE can be reconstructed entirely from information outside the horizon. Thus, the duality relation remains valid for general states, although the resulting expressions may be too complicated to be practically useful.

BTZ black hole and AdS-Rindler coordinate. The metric of a Baños–Teitelboim–Zanelli (BTZ) black hole is given by

$$ds^2 = \frac{1}{z^2} \left[-f(z) dt^2 + \frac{dz^2}{f(z)} + dx^2 \right], \quad (9)$$

with $f(z) = 1 - \frac{z^2}{z_h^2}$, where $z = z_h$ is the black hole horizon. The singularity is at $z = \infty$. We consider the timelike interval $[0, t_0]$ on $x = 0$. For the strip $s_E(0, 0; \tau_0, 0)$ we can obtain $p_{\tau_0} = \frac{\sin \frac{t_0}{z_h}}{z_h(1 - \cos \frac{t_0}{z_h})}$. By using (5) we have

$$p_{t_0}^2 = \frac{1}{z_h^2} \left(\coth \frac{t_0}{2z_h} \right)^2. \quad (10)$$

The turning point is given by $z_{t_0} = iz_h \sinh \frac{t_0}{2z_h}$. The length of RT surface inside and outside the horizon is given by

$$\begin{aligned} \mathcal{A}_{\text{in}} &= 2 \log \left[\left(1 + \coth \frac{t_0}{2z_h} \right) \sinh \frac{t_0}{2z_h} \right] + i\pi, \\ \mathcal{A}_{\text{out}} &= 2 \log \left[\frac{2z_h}{\delta} \frac{1}{1 + \coth \frac{t_0}{2z_h}} \right]. \end{aligned} \quad (11)$$

See Supplemental Material [34] for calculation details. The above result yields the expected value for the timelike EE of the interval $[0, t_0]$. It can be shown that the RT surfaces inside and outside the horizon satisfy the relation

$$\begin{aligned} \mathcal{A}_{\text{in}} &= \frac{1}{2} (\mathcal{A}(0, -t_0; 0, 0) + \mathcal{A}(0, 0; 0, t_0)) \\ &+ \frac{1}{2} \int_{-t_0}^{t_0} dx \partial_t \mathcal{A}(0, x; 0, 0) - \mathcal{A}_{\text{out}}. \end{aligned} \quad (12)$$

The detailed derivation is presented in Supplemental Material [34]. For $z_h = 1$, the metric corresponds to AdS-Rindler coordinates, which cover only a portion of AdS.

Higher dimensional examples. Let us firstly consider the pure AdS $_{d+1}$ ($d \geq 2$). The metric is given by (1) with $f(z) = g(z) = 1$. There exists a horizon at $z = \infty$. We consider the strip $s(0, 0; t_0, 0)$. Using the same procedure, we can evaluate the RT surface in the Euclidean metric for the subsystem $s_E(0, 0; \tau_0, 0)$. We find

$$p_{\tau}^2 = z_{\tau}^{2-2d}, \quad \text{with} \quad z_{\tau} = \frac{\tau_0 \Gamma(\frac{1}{2(d-1)})}{2\sqrt{\pi} \Gamma(\frac{d}{2(d-1)})}. \quad (13)$$

With the analytical continuation $\tau_0 \rightarrow it_0$, we obtain

$$p_t^2 = i^{2(2-d)} \left(\frac{t_0 \Gamma(\frac{1}{2(d-1)})}{2\sqrt{\pi} \Gamma(\frac{d}{2(d-1)})} \right)^{2-2d}. \quad (14)$$

The turning point can also be obtained by analytical continuation $z_t = \frac{it_0 \Gamma(\frac{1}{2(d-1)})}{2\sqrt{\pi} \Gamma(\frac{d}{2(d-1)})}$. The area of the RT surface is given by

$$\mathcal{A}(0, 0; t_0, 0) = \frac{2R^{d-1}}{(d-2)\delta^{d-2}} + 2\kappa_d \frac{R^{d-1}(-i)^d}{t_0^{d-2}}, \quad (15)$$

with $\kappa_d = \frac{\pi^{\frac{d-1}{2}} 2^{d-2}}{d-2} \left(\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})} \right)^{d-1}$. See Supplemental Material [34] for calculation details. The results are consistent with [25,35]. For the spacelike strip, the first-order temporal derivative satisfies the following relation [35],

$$x^{d-2} \partial_t \mathcal{A}(0, x; 0, 0) = 2iR^{d-1} \kappa_d (d-2) \pi \delta(x). \quad (16)$$

Just like the AdS_3 case, the RT surface approaches the horizon $z = \infty$ and extends into the complexified geometry. The areas of the RT surface inside and outside the horizon can be evaluated. Guided by Eq. (16) and inspired by the AdS_3 relation (12), we construct the following equation

$$\begin{aligned} \mathcal{A}(0, 0; t_0, 0) &= \frac{1}{2} (\mathcal{A}(0, -t_0; 0, 0) + \mathcal{A}(0, 0; t_0, 0)) \\ &+ \frac{i[(-i)^{d-2} - 1]}{t_0^{d-2}(d-2)\pi} \int_{-t_0}^{t_0} dx x^{d-2} \partial_t \mathcal{A}(0, x; 0, 0). \end{aligned} \quad (17)$$

See Supplemental Material [34] for the derivation. One can directly verify the above formula using Eqs. (15) and (16). Notably, in the limit $d \rightarrow 2$, the formula reduces to the AdS_3 case given by Eq. (12).

Now, let us consider the black hole background with $f(z) = g(z) = 1 - \frac{z^d}{z_h^d}$. We will still consider the timelike strip $s(0, 0; t_0, 0)$. In this case, obtaining an analytical expression for the RT surface area is challenging. However, for small strips satisfying $t_0 \ll z_h$, the problem can be solved perturbatively. In this section, we set $z_h = 1$ and assume $t_0 \ll 1$, keeping only the leading-order contributions.

Both the timelike and spacelike RT surfaces receive thermal corrections. Let $\mathcal{A}_{bh}(0, 0; 0, x)$ denote the area of the RT surface for the spacelike strip with width x on the Cauchy surface $t = 0$. For $x \ll 1$, the leading-order thermal correction is given by

$$\begin{aligned} \mathcal{A}_{bh}(0, 0; 0, x) &= \mathcal{A}(0, 0; 0, x) + \delta \mathcal{A}_{bh}(0, 0; 0, x), \\ \text{with } \delta \mathcal{A}_{bh}(0, 0; 0, x) &= \alpha_d x^2 + O(x^{2+d}), \end{aligned} \quad (18)$$

where $\delta \mathcal{A}_{bh}$ represents the thermal correction to the EE of the spacelike strip. The results for α_d are provided in Supplemental Material [34].

For the timelike strip $s(0, 0; t_0, 0)$ we can also obtain the thermal correction. The procedure is the same as before. We consider the strip $s_E(0, 0; \tau_0, 0)$ in the Euclidean spacetime. By some calculations, we obtain the turning point z_{τ_0} . The explicit results are provided in Supplemental Material [34]. After performing the analytical continuation $\tau \rightarrow it$, we obtain the turning point $z_{t_0} = z_{\tau_0}|_{\tau_0 \rightarrow it_0}$. We consider the RT surface connecting $z = \delta$ to z_{t_0} , with the path C on the complex z plane chosen similarly to the vacuum case. Given this path, we can then evaluate the area of the RT surface.

$$\mathcal{A}_{bh}(0, 0; t_0, 0) = \mathcal{A}(0, 0; t_0, 0) + \beta_d t_0^2 + O(t_0^{2+d}), \quad (19)$$

where β_d is a constant that depends only on d . The explicit results are provided in Supplemental Material [34].

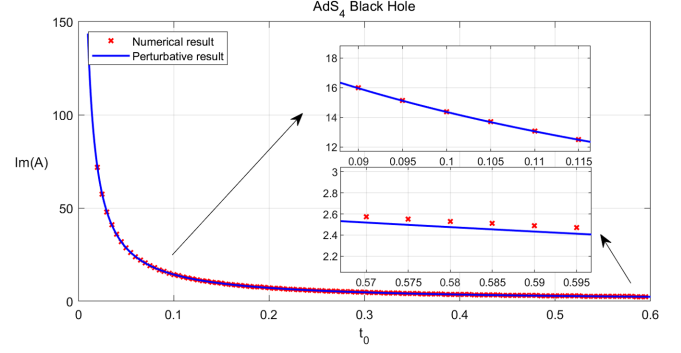


FIG. 4. Plot of the imaginary part of the RT surface area $\mathcal{A}(0, 0; t_0, 0)$ versus interval length t_0 for AdS_4 black hole. The red crosses denote numerical results, while the blue curve represents the perturbative analytical solution. We set $z_h = R = 1$. The interval length t_0 is sampled over $(0, 0.6)$ with spacing 0.005.

For higher-dimensional black holes, the area of the RT surface can also be obtained numerically. In Fig. 4 (AdS_4 black hole) and Fig. 5 (AdS_5 black hole), we compare the numerical results with those from the perturbative approach. The agreement holds at small $t_0 \ll z_h$, with discrepancies appearing at $t_0 \sim z_h$.

We now demonstrate that the relation (16) remains valid in the black hole background. Consider a strip $s(t, x; 0, 0)$ with $t \ll 1$ while ensuring $-t^2 + x^2 > 0$, i.e., a spacelike strip. The holographic EE can be computed perturbatively in this regime. Due to time-reversal symmetry, thermal corrections to the RT surface $\mathcal{A}_{bh}(t, x; 0, 0)$ cannot introduce terms linear in t ; the leading-order corrections must be at most $O(t^2)$. Consequently, we find $\partial_t \mathcal{A}_{bh}(t, x; 0, 0)|_{t \rightarrow 0} = \partial_t \mathcal{A}(t, x; 0, 0)|_{t \rightarrow 0}$, implying that thermal corrections do not affect the first-order temporal derivative of the spacelike RT surface.

Notably, the corrections to the spacelike and timelike strips differ, i.e., $\alpha_d \neq \beta_d$. However, we find that they obey the universal relation

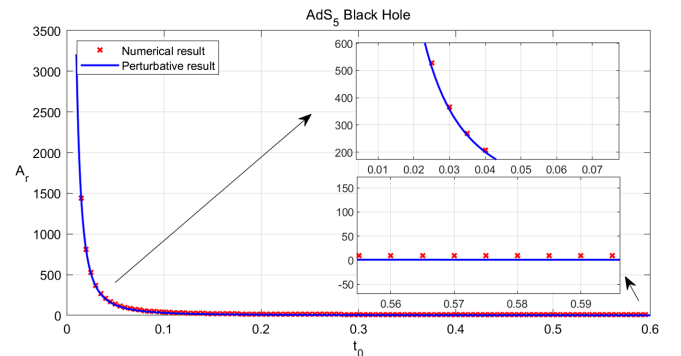


FIG. 5. Plot of the regularized RT surface area $\mathcal{A}_r = \mathcal{A}_{bh}(0, 0; t_0, 0) - \frac{2}{3\delta^3}$ versus interval length t_0 for AdS_5 black hole. The red crosses denote numerical results, while the blue curve represents the perturbative calculation. Here, we set $z_h = R = 1$. The interval length t_0 is sampled over $(0, 0.6)$ with spacing 0.005.

$$\frac{\beta_d}{\alpha_d} = -\frac{d-3}{d-1}. \quad (20)$$

See Supplemental Material [34] for more details. Thus, the relation (17) can be modified as

$$\begin{aligned} \mathcal{A}_{bh}(0, 0; t_0, 0) &= \frac{1}{2} (\mathcal{A}_{bh}(0, -t_0; 0, 0) + \mathcal{A}_{bh}(0, 0; 0, t_0)) \\ &\quad - \frac{d-2}{d-1} (\delta \mathcal{A}_{bh}(0, -t_0; 0, 0) + \delta \mathcal{A}_{bh}(0, 0; 0, t_0)) \\ &\quad + \frac{i[(-i)^{d-2} - 1]}{t_0^{d-2}(d-2)\pi} \int_{-t_0}^{t_0} dx x^{d-2} \partial_t \mathcal{A}_{bh}(0, x; 0, 0). \end{aligned} \quad (21)$$

The derivation is presented in Supplemental Material [34]. The right-hand side of the above equation only includes information outside the black hole horizon. For $d = 2$ there are no thermal corrections on the right-hand side, which is consistent with formula (12) for the BTZ black hole. We can divide the RT surface for the timelike strip into the inside and outside parts of the horizon. Thus, the RT surface inside the horizon can be constructed using quantities outside the horizon.

Discussion. In this letter, we demonstrate how to construct the RT surface for the timelike strip via analytical continuation. We also uncover a duality between the RT surfaces inside and outside the black hole horizon, which can be interpreted as a realization of the concept of black

hole complementarity: the information inside the horizon can be reconstructed from the degrees of freedom outside the horizon.

In our construction, it seems necessary to consider the RT surface in the complexified geometry. The role of the complexified geometry in understanding the interior of black holes remains unclear. If matter is included in the bulk, we would generally have $f \neq g$. It has been shown in [9] that the metric near the singularity would resemble a more general Kasner universe [36,37]. It would be interesting to investigate whether it is possible to construct the duality relation in this case and explore the properties of the singularity through this duality. Additionally, it would be interesting to investigate whether a similar duality persists in a black hole with Hawking radiation [38–41].

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Data availability. The data are not publicly available. The data are available from the authors upon reasonable request.

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- [34] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/xndj-9ftm>, Section I presents examples in AdS₃, including pure AdS, BTZ black hole, and AdSRindler cases. Section II gives the detailed calculations of RyuTakayanagi surfaces for higher-dimensional strips in the vacuum state, highlighting the difference between even and odd dimensions. Section III extends the analysis to higher-dimensional black holes, providing perturbative results for both timelike and spacelike strips, with explicit examples for $d = 3, 4, 5, 6$.
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