

Hidden heavy flavor tetraquarks in the Born-Oppenheimer approximation

Bowen Kang[✉], Xi Xia[✉], and Tao Guo^{✉*}*College of Physics, Chengdu University of Technology, Chengdu 610059, China*

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The Born-Oppenheimer approximation is one of the very successful tools for solving the hydrogen atom problem. The experimental discovery of hidden heavy flavor tetraquarks $Q\bar{Q}q\bar{q}$ ($Q = c, b$ and $q = u, d, s$) provides great possibilities for the hydrogen-bond-like structure of the quantum chromodynamics version. In this work, considering that the colors of $Q\bar{Q}$ and $q\bar{q}$ are both 8, the tetraquark $Q\bar{Q}q\bar{q}$ system is formed by color coupling $8 \otimes 8 \rightarrow 1$. In order to study the mass splitting caused by the color-spin hyperfine interaction, the color-spin basis vectors of the S -wave tetraquark states are appropriately constructed. Then, we use the Born-Oppenheimer approximation to calculate the mass spectra of the S -wave hidden heavy flavor tetraquark states. The results show that some of the hidden heavy flavor exotic hadrons discovered experimentally can be well explained as this type of hydrogen-bond-like tetraquark structure. In addition, some candidates for tetraquark bound states are predicted and may be compact tetraquark states.

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I. INTRODUCTION

Quantum chromodynamics (QCD) provides a rich framework for understanding hadrons, extending beyond the conventional mesons (one quark, one antiquark) and baryons (three quarks) to encompass exotic configurations such as multiquark states, hybrids, and glueballs [1–3]. Among the multiquark states, the study of tetraquark states (composed of two quarks and two antiquarks) have received significant attention [4–6]. Notably, certain hidden heavy flavor tetraquarks $Q\bar{Q}q\bar{q}$ ($Q = c, b$ and $q = u, d, s$) can be interpreted as analogs of the hydrogen bonds within the QCD version. This allows the interaction between heavy-quark-antiquark pairs and light quarks to be treated using the Born-Oppenheimer (BO) approximation [7–13]. Based on the fact that heavy quarks are much heavier than light quarks, we can decouple the heavy-quark dynamics from the light-quark dynamics. This strategy makes it possible to systematically study the mass spectra and structures of the hidden heavy flavor tetraquark states.

In 2003, the Belle Collaboration reported the famous narrow charmoniumlike state $X(3872)$ in the $\pi^+\pi^-J/\psi$ mass spectrum of the exclusive decay process $B^\pm \rightarrow K^\pm \pi^+ \pi^- J/\psi$ [14]. The discovery of the $X(3872)$ confirmed the existence of exotic hadron states and opened up a new direction for hadron physics research. On the theoretical

side, there are various explanations for the nature of the $X(3872)$, such as the tetraquark state [15–18], molecular state [19–22], P -wave charmonium [23–26], and threshold effects [27–29]. Subsequently, more hidden heavy flavor exotic hadrons were discovered, many of which were considered to be candidates for tetraquark states and attracted much attention from particle physicists, such as $X(4140)$ [30–32], $Y(3940)$ [33–35], and $Z_b(10610)$ and $Z_b(10650)$ [36–38]; see reviews [39–42] for details. So far, although the understanding of these exotic hadrons has not yet reached consensus and is still under intense scientific debate, they deserve further investigation as candidates for tetraquark states.

Recently, there has been increasing experimental evidence proving the existence of possible hidden heavy flavor tetraquark states. In 2020, the first candidate for a charged hidden-charm tetraquark state with strangeness $Z_{cs}(3985)^-$ was discovered in the processes of $e^+e^- \rightarrow K^+ D_s^- D^{*0}$ and $K^+ D_s^{*-} D^0$ by the BESIII Collaboration [43]. Its nonzero charge characteristic enables it to be clearly distinguished from a hybrid or charmonium. Theoretically, $Z_{cs}(3985)^-$ is generally interpreted as tetraquark state $c\bar{s}\bar{c}u$ with a quantum number assignment $J^P = 1^+$ [44–48]. Later in 2021, the LHCb Collaboration reported two hidden-charm exotic structures in the decay $B^+ \rightarrow J/\psi \phi K^+$ referred to as $Z_{cs}(4000)^+$ and $Z_{cs}(4220)^+$ [49]. Their quark composition is thought to be $cu\bar{c}\bar{s}$, and their spin-parity quantum numbers are determined to be $J^P = 1^+$ and 1^+ (or 1^-), respectively [49]. However, theoretical studies suggest that the $Z_{cs}(4000)^+$ and $Z_{cs}(4220)^+$ may be candidates for charmonia [50,51], tetraquarks [52–54], or molecules [55,56]. It is worth noting that the BESIII Collaboration recently observed a new $Z_{cs}(4123)^-$ in the processes of

*Contact author: guot17@tsinghua.org.cn

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$e^+e^- \rightarrow K^+D_s^{*-}D^{*0} + c.c.^*$, and its quantum number assignment may be $J^P = 1^+$ [57]. The discovery of the $Z_{cs}(4123)^-$ may add a new member to the family of charged hidden-charm tetraquarks with strangeness. In 2023, A near-threshold peaking structure $X(3960)$ was observed in the $D_s^+D_s^-$ invariant mass spectrum by the LHCb Collaboration, and the quantum number assignment $J^{PC} = 0^{++}$ was favored [58]. Research shows that the nature of $X(3960)$ may also be a tetraquark $c\bar{s}\bar{c}\bar{s}$ [59–61]. Obviously, these exotic hadrons all have the common components $Q\bar{Q}q\bar{q}$, and figuring out their nature will help us better understand QCD.

In this work, we apply the BO approximation to calculate the mass spectra of tetraquark $Q\bar{Q}q\bar{q}$. The heavy $Q\bar{Q}$ forms a color octet, which combines with a light $q\bar{q}$ octet to produce a color-singlet tetraquark; that is, the color coupling is represented as $8 \otimes 8 \rightarrow 1$. Given the same amount of momentum, the heavy quarks move much more slowly than the light quarks. The light-quark (or antiquark) interactions can be treated as perturbations to the dominant heavy-quark potential, allowing the Schrödinger equation for the heavy-quark separation to be solved numerically. The calculation of the diquark-antidiquark paradigm was extended to the tetraquark system. Using $X(3872)$ as input, other possible tetraquark states were studied.

This paper is organized as follows: In Sec. II, we give a brief introduction of the tetraquark Hamiltonian based on the BO approximation. We obtain the BO potential using the perturbation method and construct the basis vector wave functions of the tetraquark system $Q\bar{Q}q\bar{q}$. In Sec. III, we determine the model parameters, such as the effective masses of quarks and the color coupling strengths. The mass spectra and decay modes of various hidden heavy tetraquark states are calculated and discussed in this section. We summarize in Sec. IV. The natural units $c = \hbar = 1$ are used throughout.

II. THE BORN-OPPENHEIMER THEORETICAL MODEL

A. Hamiltonian in the Born-Oppenheimer approximation

For the hidden heavy flavor tetraquark system, it is theoretically difficult to achieve a direct analytical solution. We can use the BO approximation in nonrelativistic quantum mechanics to handle the degrees of freedom of heavy and light quarks separately. The Hamiltonian of tetraquark system reads

$$\mathcal{H} = \sum_{i=1}^4 \left(\frac{p_i^2}{2m_i} \right) + V(\mathbf{x}_{A,B}) + V_I(\mathbf{x}_{A,B}, \mathbf{x}_{1,2}) + V_{\text{conf}}, \quad (1)$$

where $\mathbf{x}_{A,B}$ and $\mathbf{x}_{1,2}$ represent the coordinates of heavy quarks and light quarks, respectively, and m_i represents the

effective mass of the i th constituent quark. $\sum_{i=1}^4 \left(\frac{p_i^2}{2m_i} \right)$ is defined as the kinetic energy of the quarks. $V(\mathbf{x}_{A,B})$ describes the interaction between the heavy quarks. $V_I(\mathbf{x}_{A,B}, \mathbf{x}_{1,2})$ describes the general interactions involving light-heavy and light-light quarks. V_{conf} is defined as the color confinement part of the tetraquark states.

If we focus only on the S -wave tetraquark states, the spatial degrees of freedom can be encoded into the model parameters. Considering the energy level splitting caused by the color-spin interaction, the Hamiltonian for the S -wave tetraquark states can be simplified as [10,15]

$$H = \sum_{i=1}^4 m_i + E + H_{ss}, \quad (2)$$

with

$$H_{ss} = \sum_{i < j} 2\kappa_{ij}(s_i \cdot s_j), \quad (3)$$

where E is the energy eigenvalue of the tetraquark system obtained based on the BO approximation. The coefficient κ_{ij} is inversely proportional to the product of the masses of the i th and j th constituent quarks, and directly proportional to the color coupling strength of the corresponding quark pair. The Hamiltonian H_{ss} is the color-spin hyperfine interaction.

B. Born-Oppenheimer potential

In the BO approximation, with the positions of the heavy quarks assumed to be fixed, we first consider the Schrödinger equation for the light quarks [10]

$$\left[\left(\sum_{\text{light}} \frac{p_i^2}{2m} \right) + V_I(\mathbf{x}_{A,B}, \mathbf{x}_{1,2}) \right] f = \mathcal{E}(\mathbf{x}_{A,B})f, \quad (4)$$

where $f = f(\mathbf{x}_{A,B}, \mathbf{x}_{1,2})$ is the eigenfunction for the light quarks. $\mathcal{E}(\mathbf{x}_{A,B})$ is the eigenvalue dependent on the positions of the heavy quarks. The total wave function of the tetraquark system can be written as

$$\Psi = \psi(\mathbf{x}_{A,B})f(\mathbf{x}_{A,B}, \mathbf{x}_{1,2}), \quad (5)$$

where ψ describes the wave function of heavy quarks. We let the Hamiltonian Eq. (1) act on the wave function Eq. (5). Since the wave function f of light quarks is very sensitive to changes in the position of heavy quarks, $\nabla_i f$ cannot be ignored. But the wave function ψ of heavy quarks changes slowly with respect to the positions of the heavy quarks, and $\nabla_i \psi$ can be approximated as zero. For the kinetic term, it can be approximated as

$$\nabla_i^2 \Psi \approx (f \nabla_i^2 \psi + \psi \nabla_i^2 f). \quad (6)$$

Now we can rewrite the Schrödinger equation of heavy quarks as

$$\left(\sum_{\text{heavy}} \frac{p_i^2}{2M} + V_{\text{BO}}(\mathbf{x}_{A,B}) \right) \psi = E\psi, \quad (7)$$

with

$$V_{\text{BO}}(\mathbf{x}_{A,B}) = V(\mathbf{x}_{A,B}) + \mathcal{E}(\mathbf{x}_{A,B}) + V_{\text{conf}}. \quad (8)$$

Next, we analyze the specific form of the V_{BO} potential. In a hadron system, the color coupling between any quark and quark (or antiquark) can be expressed as

$$V(r) = \lambda_{q_1 q_2}(\mathbf{R}) \frac{\alpha_s}{r}, \quad (9)$$

with the Casimir operator

$$\lambda_{q_1 q_2}(\mathbf{R}) = \frac{1}{2} [C(\mathbf{R}) - C(q_1) - C(q_2)], \quad (10)$$

where r is the relative distance between the two quarks, and α_s is the QCD running coupling constant. The notation $\mathbf{R}$ is the color representation of any pair of particles, and q_1 and q_2 represent the irreducible representations of the quark. In the SU(3) group representation, we can get $C(3) = 4/3$, $C(6) = 10/3$, $C(1) = 0$, $C(8) = 3$, $C(\bar{q}) = C(q)$, and $C(\bar{\mathbf{R}}) = C(\mathbf{R})$.

For hidden heavy flavor tetraquarks, the color couplings of the two heavy quarks can be either 1 or 8. In the first case, however, the interaction between $Q\bar{Q}$ and $q\bar{q}$ pairs occurs through the exchange of color singlets. This is similar to the nucleon-nucleon interaction, which manifests itself as the configuration of hadronic molecular states. Therefore, we exclude the scenario where $Q\bar{Q}$ couples in the color state 1. In our calculation, we only consider the coupling of quark pairs, both $Q\bar{Q}$ and $q\bar{q}$, with color state 8.

In order to derive $\mathcal{E}$, we need to solve Schrödinger Eq. (4). Given that the impact of heavy quarks on light quarks is non-negligible, the hidden heavy flavor tetraquark state can be considered a hydrogenlike molecular system of the QCD version. Then, we can extend the calculation methods for hydrogen molecules to study the tetraquark $Q\bar{Q}q\bar{q}$. Specifically, a light quark is paired exclusively with one heavy quark (or antiquark) forming two symmetrical orbits. To analyze the interaction between the orbit $\bar{Q}q$ or $Q\bar{q}$, we introduce the Cornell potential of the nonrelativistic quark model,

$$V_{\bar{Q}q}(r) = -\frac{4}{3} \frac{\alpha_s}{r} + \sigma r + V_0, \quad (11)$$

where σ is the string tensor strength of QCD and is proportional to $|\lambda|$. The parameter $V_0 \simeq -0.315$ GeV can

be obtained by fitting the quarkonium. For the orbit Qq or $\bar{Q}\bar{q}$, we modify the interaction as follows:

$$V_{Qq}(r) = -\frac{1}{3} \frac{\alpha_s}{r} + \frac{1}{4} \sigma r + V_0. \quad (12)$$

The radial variation wave function is given by

$$R(r) = \frac{A^{3/2}}{\sqrt{4\pi}} e^{-Ar}, \quad (13)$$

where A is the variational parameter. Substituting the trial wave function and interaction into the Hamiltonian, we can obtain the ground state energy ($\langle H \rangle_{\text{min}}$) and the corresponding wave function (ζ or ξ) of a heavy-light-quark pair by using a variational calculation.

We consider a nonperturbative process for the orbits of two noninteracting heavy-light-quark pairs. Because of the symmetry of the tetraquark system, the wave function of light quarks can be expressed as

$$f = \zeta(\boldsymbol{\rho})\xi(\boldsymbol{\eta}) = R(|\mathbf{x}_1 - \mathbf{x}_A|)R(|\mathbf{x}_2 - \mathbf{x}_B|), \quad (14)$$

and the ground state energy corresponding to the wave function reads

$$E_{Qq \leftrightarrow \bar{Q}\bar{q}} = 2\langle H \rangle_{\text{min}}. \quad (15)$$

For the orbit formed by the light-quark-antiquark pair, we regard it as the perturbation term. Then, the perturbation Hamiltonian is

$$H_p = -\frac{7}{6} \alpha_s \left(\frac{1}{|\mathbf{x}_1 - \mathbf{x}_B|} + \frac{1}{|\mathbf{x}_2 - \mathbf{x}_A|} \right) + \frac{1}{6} \alpha_s \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}. \quad (16)$$

For simplicity, the perturbative energy is written as

$$\Delta E = \langle f | H_p | f \rangle. \quad (17)$$

By calculation, ΔE is written as

$$\Delta E = -\frac{7}{6} \alpha_s 2I_1(r_{AB}) + \frac{1}{6} \alpha_s I_2(r_{AB}), \quad (18)$$

where $r_{AB} = |\mathbf{x}_B - \mathbf{x}_A|$. The functions I_1 and I_2 are

$$I_1(r_{AB}) = \int d^3\rho |\zeta(\boldsymbol{\rho})|^2 \frac{1}{|\boldsymbol{\rho} - \mathbf{r}_{AB}|}, \quad (19)$$

$$I_2(r_{AB}) = \int d^3\rho d^3\eta |\zeta(\boldsymbol{\rho})|^2 |\xi(\boldsymbol{\eta})|^2 \frac{1}{|\boldsymbol{\rho} - \boldsymbol{\eta}|}. \quad (20)$$

Now, we can obtain the final energy as

$$\mathcal{E}(\mathbf{x}_{A,B}) = E_{Qq \leftrightarrow \bar{Q}\bar{q}} + \Delta E. \quad (21)$$

For the hidden heavy flavor tetraquark system, the overall color coupling needs to ensure a color singlet. We need to include a linear string tension to simulate the confinement effect [62,63]. Considering the tetraquark configuration $|(Q\bar{Q})_8(q\bar{q})_8\rangle_1$, the linearly increasing confinement term reads [10]

$$V_{\text{conf}} = \sigma \times (r - R_0) \times \theta(r - R_0). \quad (22)$$

R_0 is the relative distance between quarks within a hadron. In general, R_0 can be treated as an adjustable free parameter and can be obtained by fitting the hadron spectrum. Therefore, the Born-Oppenheimer potential can be expressed as

$$V_{\text{BO}}(r_{AB}) = \frac{1}{6}\alpha_s \frac{1}{r_{AB}} + E_{Qq \leftrightarrow \bar{Q}\bar{q}} + \Delta E + \sigma \times (r_{AB} - R_0) \times \theta(r_{AB} - R_0). \quad (23)$$

Substituting Eq. (23) into Eq. (7), we can obtain the binding energy E of the hidden heavy flavor tetraquark state under the BO approximation.

C. Wave function

In order to study the energy level splitting caused by color-spin hyperfine interaction, we can construct the basis vector wave functions of the tetraquark state. In color space, the tetraquark state is divided into a meson-meson configuration $|(Q\bar{Q})(q\bar{q})\rangle$ [or $|(Q\bar{q})(q\bar{Q})\rangle$] and a diquark-antidiquark configuration $|(Qq)(\bar{Q}\bar{q})\rangle$, and these two configurations can be connected by Fierz transformation. In this calculation, we focus only on the diquark-antidiquark configuration. Obviously, the total spin of all possible S -wave tetraquark states is 0, 1, and 2. For the scalar tetraquark states with quantum number $J^P = 0^+$, there are two basis vectors

$$\begin{aligned} |0^{+(+)}\rangle_1 &= |0_{Qq}, 0_{\bar{Q}\bar{q}}; J = 0\rangle, \\ |0^{+(+)}\rangle_2 &= |1_{Qq}, 1_{\bar{Q}\bar{q}}; J = 0\rangle. \end{aligned} \quad (24)$$

The symbol $(+/-)$ represents charge conjugation when a tetraquark state is in flavor symmetry, and the positive and negative signs in quantum numbers have the same meaning below, unless otherwise stated.

For the axial vector tetraquark states with quantum number $J^P = 1^+$, there are three basis vectors

$$\begin{aligned} |A\rangle &= |0_{Qq}, 1_{\bar{Q}\bar{q}}; J = 1\rangle, \\ |B\rangle &= |1_{Qq}, 0_{\bar{Q}\bar{q}}; J = 1\rangle, \\ |C\rangle &= |1_{Qq}, 1_{\bar{Q}\bar{q}}; J = 1\rangle. \end{aligned} \quad (25)$$

In charge conjugation, considering the symmetric and antisymmetric combinations of states $|A\rangle$ and $|B\rangle$, we obtain a C -odd and a C -even state, and the charge conjugation of $|C\rangle$ is odd. Therefore, for the hidden heavy flavor tetraquark states with $J^P = 1^+$, the basis vectors with definite charge conjugation can be rewritten as

$$\begin{aligned} |1^{+(+)}\rangle &= \frac{1}{\sqrt{2}}(|A\rangle + |B\rangle), \\ |1^{+(-)}\rangle_1 &= \frac{1}{\sqrt{2}}(|A\rangle - |B\rangle), \\ |1^{+(-)}\rangle_2 &= |C\rangle. \end{aligned} \quad (26)$$

For the $J^P = 2^+$ state, the basis vector is

$$|2^{+(+)}\rangle = |1_{Qq}, 1_{\bar{Q}\bar{q}}; J = 2\rangle. \quad (27)$$

Now we have constructed the basis vector wave functions of the hidden heavy flavor tetraquark system based on the diquark-antidiquark configuration. Substituting these basis vectors into Eq. (2), we can obtain the corresponding Hamiltonian matrices. Then, by diagonalizing the Hamiltonian matrices, we can obtain the mass spectra and corresponding wave functions of the tetraquark states. Finally, considering the characteristics of the meson-meson configuration, we analyze the possible decay channels of these calculated tetraquark states.

III. RESULTS AND DISCUSSION

A. Model parameters

In the specific numerical calculation, we first need to determine the model parameters. By fitting the experimental observations and lattice calculations of heavy flavor conventional hadrons, we extract the effective masses of the constituent quarks: $m_n = 308$ MeV, $m_s = 484$ MeV, $m_c = 1667$ MeV, and $m_b = 5005$ MeV. Here, n represents u and d quarks. Without loss of generality, we take α_s and σ based on the charmonium and bottomonium mesons: $\alpha_s(2m_c) = 0.30$, $\alpha_s(2m_b) = 0.21$, and $\sigma = 0.15$ GeV². Other strong coupling constants α_s come from Ref. [64]. In addition, other model parameters required for our calculation are listed in the Tables I and II.

So far, since the $s\bar{s}$ meson with quantum number $J^P = 0^-$ and the $b\bar{c}$ meson with quantum number $J^P = 1^-$ have not been observed experimentally, the parameters $\kappa_{s\bar{s}}$ and $\kappa_{b\bar{c}}$ can be extracted by using a method similar to that used in Ref. [15]. For a hidden heavy flavor tetraquark system, according to the Fierz transformation, we can get the parameter $\kappa_{q\bar{q}} = \frac{1}{4}(\kappa_{q\bar{q}})_0$. In addition, it is worth emphasizing that these parameters have a certain symmetry, namely, $\kappa_{ij} = \kappa_{ji}$.

To solve the mass spectra of the hidden heavy flavor tetraquark states, we choose $X(3872)$ as input to determine

TABLE I. Parameter κ (in MeV) for quark-antiquark systems. These parameters are derived from fitting the S -wave mesons.

	$n\bar{n}$	$s\bar{n}$	$s\bar{s}$	$c\bar{n}$	$c\bar{s}$	$c\bar{c}$	$b\bar{n}$	$b\bar{s}$	$b\bar{c}$	$b\bar{b}$
$(\kappa_{ij})_0$	315	195	121	70	72	59	23	24	20	30

TABLE II. Parameter κ (in MeV) for quark-quark (or antiquark-antiquark) systems. These parameters are derived from fitting the S -wave baryons.

	nn	sn	cn	cs	bn	bs
$(\kappa_{ij})_3$	103	64	22	25	6.6	7.5

R_0 . Assuming that $X(3872)$ is a tetraquark state $c\bar{c}n\bar{n}$ with quantum number $J^{PC} = 1^{++}$, we can substitute the basis vector $|1^{++}\rangle$ into the Hamiltonian Eq. (2) to obtain the corresponding mass spectrum

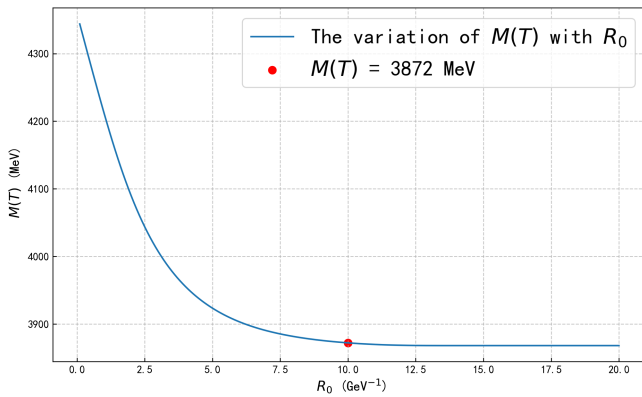
$$M(T) = 2(m_c + m_q) + E + \langle H_{ss} \rangle, \quad (28)$$

where T represents the tetraquark state $c\bar{c}n\bar{n}$. E is a function of R_0 . $\langle H_{ss} \rangle$ is the expectation value of the color-spin hyperfine interaction and can be expressed as

$$\langle H_{ss} \rangle = \frac{1}{4} \left[\frac{1}{2} (\kappa_{n\bar{n}})_0 + \frac{1}{2} (\kappa_{c\bar{c}})_0 - (\kappa_{c\bar{n}})_0 \right] - (\kappa_{cn})_3. \quad (29)$$

Substituting the parameters in Table I into Eq. (29), we have $\langle H_{ss} \rangle = 7.25$ MeV.

Now we plot the mass $M(T)$ as R_0 changes; see Fig. 1. It can be found that the mass $M(T)$ decreases as R_0 increases. It is noteworthy that when $R_0 \geq 10 \text{ GeV}^{-1}$, the mass of the tetraquark state $c\bar{c}n\bar{n}$ tends to be constant. This indicates that the confinement between $c\bar{c}$ disappears, and the tetraquark state decays into two open-charm mesons.

FIG. 1. For the tetraquark state $c\bar{c}n\bar{n}$ with quantum number $J^{PC} = 1^{++}$, the mass spectrum of the ground state decreases with increasing parameter R_0 .TABLE III. Mass spectra of tetraquark $c\bar{c}n\bar{n}$ at varying R_0 .

$R_0 \text{ (GeV}^{-1}\text{)}$	Mass (MeV)			
	$M(0^{++})$	$M(1^{+-})$	$M(1^{++})$	$M(2^{++})$
$R_0 = 6$	3792, 3898	3785, 3913	3903	3982
$R_0 = 8$	3770, 3876	3763, 3891	3881	3960
$R_0 = 10$	3761, 3867	3754, 3882	3872	3951
$R_0 = 12$	3757, 3863	3750, 3878	3868	3947

Furthermore, we calculated the masses of the tetraquark state $c\bar{c}n\bar{n}$ at different R_0 ; see Table III. Among them, when $R_0 = 10 \text{ GeV}^{-1}$, the mass of the tetraquark state $c\bar{c}n\bar{n}$ with quantum number 1^{++} is 3872 MeV. In this case, the total energy eigenvalue E is -85.1 MeV.

B. Hidden-charm tetraquark states with $c\bar{c}n\bar{n}$, $c\bar{c}n\bar{s}$, and $c\bar{c}s\bar{s}$ configurations

In this section, we systematically investigate the mass spectra of the S -wave hidden-charm tetraquark states by properly considering the BO approximation and the color-spin hyperfine interaction. The quantum number assignments of these tetraquark states are $J^{P(C)} = 0^{++}, 1^{+(\pm)},$ and 2^{++} . The model parameters and specific calculation scheme have been explained in the previous section. We take $R_0 = 10 \text{ GeV}^{-1}$, and the mass spectra of the tetraquark states $c\bar{c}n\bar{n}$, $c\bar{c}n\bar{s}$, and $c\bar{c}s\bar{s}$ are plotted in Fig. 2, respectively. For ease of comparison, the corresponding meson-meson thresholds are also plotted in these subfigures. Additionally, we have also calculated the mass spectra of the tetraquarks $c\bar{c}n\bar{s}$ and $c\bar{c}s\bar{s}$ at varying R_0 values, as shown in Tables IV and V.

1. The $c\bar{c}n\bar{n}$ configuration

For the tetraquark state $c\bar{c}n\bar{n}$ with different quantum numbers, the mass spectra and the corresponding meson-meson thresholds are exhibited in Fig. 2(a). In this tetraquark system, if the two light quarks are of the same flavor, the isospin of $c\bar{c}n\bar{n}$ is $I = 0$, and it is electrically neutral. If one of the two light quarks is a u quark and the other is a d quark, the isospin of $c\bar{c}n\bar{n}$ is $I = 1$, and it is a charged particle. At the same time, the possible final state decay channels of these calculated tetraquark states are analyzed according to the quantum numbers and meson thresholds.

There are two S -wave hidden-charm tetraquark states with $J^{P(C)} = 0^{++}$, whose masses are 3761.1 and 3867.2 MeV, respectively. The state with the lowest mass, 3761.1 MeV, lies above the $D\bar{D}$ and $\eta_c\pi$ thresholds. Considering the conservation of quantum numbers ($J, P,$ and C), this state can decay naturally into mesons $D\bar{D}$ or $\eta_c\pi$. It shows that this state may be observed in the invariant mass spectrum of mesons $D\bar{D}$ or $\eta_c\pi$. In addition, in this article, it should be emphasized that we only analyze the S -wave decay process

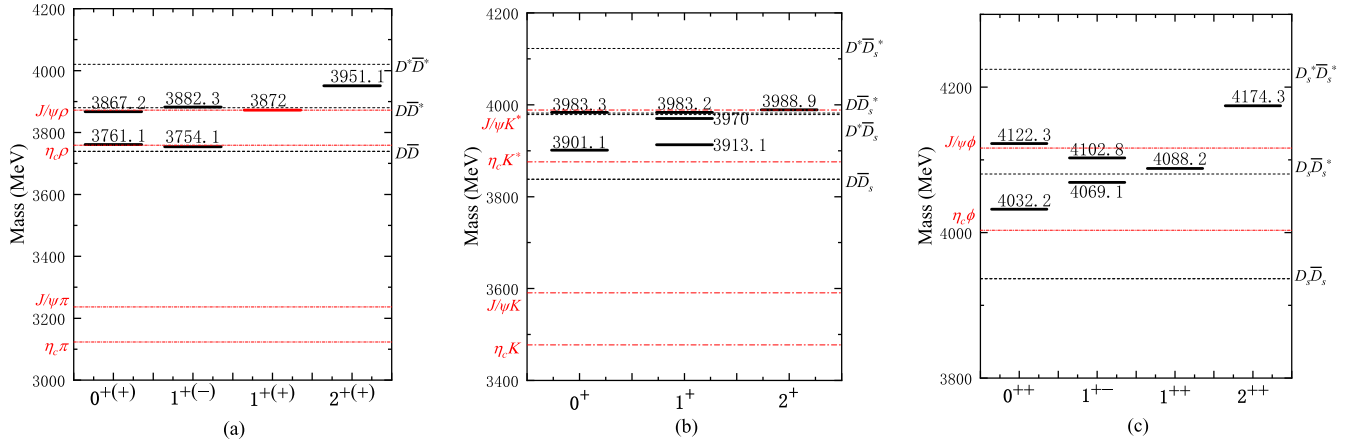


FIG. 2. The mass spectra of the S -wave hidden-charm tetraquark states: (a) $c\bar{c}n\bar{n}$, (b) $c\bar{c}n\bar{s}$, and (c) $c\bar{c}s\bar{s}$. The black dashed lines and red dash-dotted lines represent the corresponding meson-meson thresholds.

of the tetraquark state and ignore the higher-order P -wave and D -wave decay modes. The reason is that the higher-order decay channels are generally small and are not the main decay modes. The state 3867.2 MeV is also above the $\eta_c\pi$ and $D\bar{D}$ thresholds but slightly below the $J/\psi\rho$ threshold. Therefore, this state is allowed to decay into mesons $\eta_c\pi$, $D\bar{D}$, or $J/\psi\rho$. It can be considered a candidate for $X(3915)$.

There are three S -wave hidden-charm states with $J^P = 1^+$. Their masses are 3754.1, 3872.0, and 3882.3 MeV, respectively. This type of tetraquark state can be classified into two categories based on their charge conjugation C -parity. For quantum number $J^{PC} = 1^{+-}$, the lowest state 3754.1 MeV is slightly below the $\eta_c\rho$ threshold. It can be considered a compact tetraquark state. Normally, this state can decay into mesons $\eta_c\pi^+\pi^-$ via an off-shell process for meson ρ . In addition, this state can also decay into mesons $J/\psi\pi$.

Similarly, the state with a mass of 3882.3 MeV is close to the thresholds $D\bar{D}^*$ and $J/\psi\rho$, suggesting that it can easily decay into mesons $D\bar{D}^*$ or $J/\psi\rho$. Of course, it can also decay into mesons $\eta_c\rho$ or $J/\psi\pi$. For the state with quantum number $J^{PC} = 1^{++}$, its mass is very close to the thresholds $J/\psi\rho$ and $D\bar{D}^*$. Therefore, this state tends to decay into mesons $J/\psi\rho$ and $D\bar{D}^*$, and $J/\psi\rho$ can decay naturally into mesons $J/\psi\pi^+\pi^-$, which is consistent with previous studies [65]. Obviously, it can be concluded that the $|X(3872)\rangle = (c\bar{c}u\bar{u} - c\bar{c}d\bar{d})/\sqrt{2}$ can be considered to be an electrically neutral tetraquark state.

There is an S -wave hidden-charm tetraquark state with $J^{P(C)} = 2^{++}$. Its mass is 3951.1 MeV, which is above the threshold $J/\psi\rho$. Thus, it is possible for the state to decay into mesons $J/\psi\rho$. Again, this state is slightly below the threshold $D^*\bar{D}^*$, so it may also be found in the $D^*\bar{D}^*$ invariant mass spectrum.

TABLE IV. Mass spectra of tetraquark $c\bar{c}n\bar{s}$ at varying R_0 .

R_0 (GeV $^{-1}$)	Mass (MeV)		
	$M(0^+)$	$M(1^+)$	$M(2^+)$
$R_0 = 6$	3929, 4011	3942, 3998, 4011	4017
$R_0 = 8$	3909, 3991	3922, 3978, 3991	3997
$R_0 = 10$	3901, 3983	3913, 3970, 3983	3989
$R_0 = 12$	3898, 3980	3910, 3967, 3979	3986

TABLE V. Mass spectra of tetraquark $c\bar{c}s\bar{s}$ at varying R_0 .

R_0 (GeV $^{-1}$)	Mass (MeV)			
	$M(0^{++})$	$M(1^{+-})$	$M(1^{++})$	$M(2^{++})$
$R_0 = 6$	4060, 4150	4096, 4130	4115	4201
$R_0 = 8$	4041, 4130	4077, 4111	4096	4182
$R_0 = 10$	4032, 4122	4069, 4103	4088	4174
$R_0 = 12$	4030, 4120	4066, 4101	4086	4172

2. The $c\bar{c}n\bar{s}$ configuration

The mass spectra of tetraquark $c\bar{c}n\bar{s}$ system are shown in Fig. 2(b). For comparison and decay channel analysis, the corresponding meson-meson thresholds are also presented in this figure. For the tetraquark state $c\bar{c}n\bar{s}$, the C -parity is undetermined, so the quantum numbers of the S -wave states are $J^P = 0^+, 1^+, \text{ and } 2^+$. In addition, we take $R_0 = 6, 8, 10$, and 12 GeV $^{-1}$; the calculated mass spectra of the S -wave tetraquark state $c\bar{c}q\bar{s}$ with varying quantum numbers are listed in Table IV.

There are two S -wave tetraquark states with $J^P = 0^+$. Their masses are 3901.1 and 3983.3 MeV, respectively. The lighter state at 3901.1 MeV is above the thresholds $\eta_c K$ and $D\bar{D}_s$, and can easily decay into mesons $\eta_c K$ and $D\bar{D}_s$. However, due to the large mass gap, the decay widths may be large, making it difficult to observe experimentally in the decay $\eta_c K$ and $D\bar{D}_s$ channels. The state at 3983.3 MeV can not only decay naturally into the mesons $\eta_c K$ and $D\bar{D}_s$, but

also decay into the mesons $J/\psi K^*$. Obviously, this state is close to the threshold $J/\psi K^*$, so it is easier to be found in the $J/\psi K^*$ invariant mass spectrum.

For $J^P = 1^+$, we find three S -wave tetraquark states with masses of 3913.1, 3970, and 3983.2 MeV, respectively. The lightest state 3913.1 MeV is above the thresholds $J/\psi K$ and $\eta_c K^*$, and its decay channels are likely to include $J/\psi K$ or $\eta_c K^*$. Similarly, the state at 3970 MeV can also decay naturally into the mesons $J/\psi K$ or $\eta_c K^*$. Of course, this state is very close to the $J/\psi K^*$ and $D\bar{D}_s^*$ thresholds, and it is very likely to decay into the corresponding mesons. The state at 3983.2 MeV is significantly above the thresholds $J/\psi K$ and $\eta_c K^*$, allowing it to decay naturally into the mesons $J/\psi K$ and $\eta_c K^*$. At the same time, this state is located near the thresholds $J/\psi K^*$, $D^*\bar{D}_s$, and $D\bar{D}_s^*$. There is a high probability that this state will be found in the invariant mass spectrum of $J/\psi K^*$, $D^*\bar{D}_s$, or $D\bar{D}_s^*$. Obviously, these three hidden-charm tetraquark partner states with strangeness can be candidates for exotic hadrons $Z_{cs}(3985)^-$, $Z_{cs}(4000)^+$, or $Z_{cs}(4123)^-$.

For $J^P = 2^+$, there is an S -wave tetraquark state with a mass of 3988.9 MeV. Its mass spectrum lies above the $J/\psi K^*$ threshold, allowing it to decay naturally into the mesons $J/\psi K^*$. Also, this state is below the threshold $D^*\bar{D}_s^*$, allowing possible decay to the corresponding mesons via off-shell processes. Similarly, this state can also be considered a candidate for a compact tetraquark state.

3. The $c\bar{c}s\bar{s}$ configuration

The S -wave hidden-charm tetraquark $c\bar{c}s\bar{s}$ system has a well-defined C -parity. The possible quantum numbers are $J^{PC} = 0^{++}$, 1^{+-} , and 2^{++} . Given $R_0 = 10 \text{ GeV}^{-1}$, the calculated mass spectra of the tetraquark states with the

$c\bar{c}s\bar{s}$ configuration are shown in Fig. 2(c). At the same time, we also show the corresponding meson-meson thresholds. For varying R_0 , we calculated the mass spectra of the tetraquark $c\bar{c}s\bar{s}$ system under different quantum number assignments; see Table V.

For S -wave tetraquark state $c\bar{c}s\bar{s}$ with $J^{PC} = 0^{++}$, we find two states with masses of 4032.2 and 4122.3 MeV, respectively. The lowest state 4032.2 MeV is above the threshold $D_s\bar{D}_s$, so its possible decay channel is mesons $D_s\bar{D}_s$. These results are in good agreement with the experimental observations of the exotic hadron $X(3960)$ by the LHCb Collaboration. Therefore, the tetraquark state can be regarded as a good candidate for $X(3960)$. The state at 4122.3 MeV, which lies just below the threshold $D_s^*\bar{D}_s^*$, can naturally decay into mesons $D_s\bar{D}_s$ and $J/\psi\phi$.

There are two S -wave tetraquark states with $J^{PC} = 1^{+-}$ having masses of 4069.1 and 4102.8 MeV, respectively. The lighter state 4069.1 MeV is well above the $\eta_c\phi$ threshold, allowing it to naturally decay into mesons $\eta_c\phi$. The state at 4102.8 MeV, in addition to the decay mode into mesons $\eta_c\phi$, also has a possible decay mode into mesons $D_s\bar{D}_s^*$. It is worth noting that this state is slightly below the threshold $J/\psi\phi$, and a corresponding off-shell decay process may be allowed. Furthermore, there is an S -wave tetraquark state with $J^{PC} = 1^{++}$ and a mass of 4088.2 MeV. For this state, both $\eta_c\phi$ and $D_s\bar{D}_s^*$ decay modes are allowed. This result suggests that the tetraquark state may be a candidate for the exotic hadrons $Y(3940)$ or $X(4140)$.

For $J^{PC} = 2^{++}$, we observe an S -wave tetraquark state with a mass of 4174.3 MeV. Considering J^{PC} conservation, this tetraquark state can decay into the mesons $J/\psi\phi$. Of course, the channel for the tetraquark state to decay into mesons $D_s^*\bar{D}_s^*$ still cannot be prohibited.

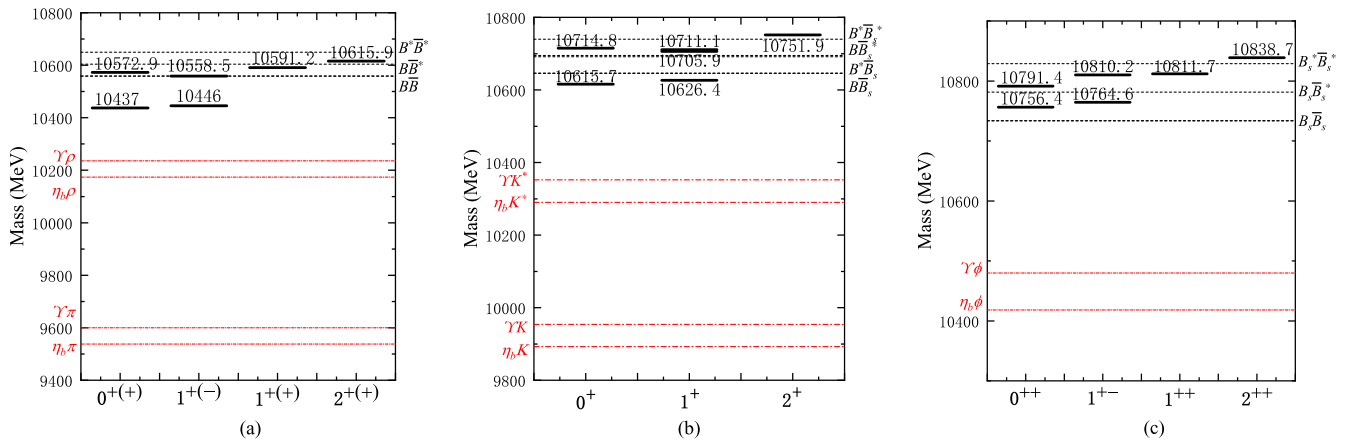


FIG. 3. The mass spectra of the S -wave hidden-bottom tetraquark states: (a) $b\bar{b}n\bar{n}$, (b) $b\bar{b}n\bar{s}$, and (c) $b\bar{b}s\bar{s}$. The black dashed lines and red dash-dotted lines represent all possible meson-meson thresholds.

C. Hidden-bottom tetraquark states with $b\bar{b}n\bar{n}$, $b\bar{b}n\bar{s}$, and $b\bar{b}s\bar{s}$ configurations

The hidden-bottom tetraquark states can be realized by converting the charm quarks of the hidden-charm tetraquark states into bottom quarks, and these hidden-bottom tetraquark states can be classified into three configurations: $b\bar{b}n\bar{n}$, $b\bar{b}n\bar{s}$, and $b\bar{b}s\bar{s}$. The possible quantum number assignments for these S -wave tetraquark states are the same as for the hidden-charm tetraquark states, i.e., $J^{P(C)} = 0^{+(+)}$, $1^{+(\pm)}$, and $2^{+(+)}$. Similarly, given $R_0 = 10 \text{ GeV}^{-1}$, we calculated the mass spectra of these states and plotted them in Fig. 3. In addition, considering the influence of different R_0 on the mass spectra of hidden-bottom tetraquark states, we take $R_0 = 6, 8, 10$, and 12 GeV^{-1} , respectively, and the corresponding calculated mass spectra listed in Tables VI–VIII. The analytical approach for these states is similar to that used for the hidden-charm tetraquark states. Therefore, we will not discuss each state in detail.

In this subsection, we focus on the two exotic hadrons observed experimentally: $Z_b(10610)$ and $Z_b(10650)$. They are found in the decay channels $\pi^\pm \Upsilon(nS)$ and $\pi^\pm h_b(mP)$, where $n = 1, 2, 3$ and $m = 1, 2$. This suggests that the component configuration of these two exotic hadrons may be a tetraquark state $b\bar{b}n\bar{n}$, and their quantum numbers are likely $J^P = 1^+$, but there is no information regarding their C -parity. In our calculations, we prefer to interpret these two exotic states as the tetraquark states $b\bar{b}d\bar{u}$ or $b\bar{b}u\bar{d}$ with quantum number assignment $I(J^{PC}) = 1(1^{+-})$, as shown in Fig. 3(a). However, the calculated masses of the tetraquark states are smaller than the corresponding experimental values for the two exotic hadrons. One obvious reason is that we chose a larger parameter R_0 ; see Table VI.

TABLE VI. Mass spectra of tetraquark $b\bar{b}n\bar{n}$ at varying R_0 .

$R_0 \text{ (GeV}^{-1}\text{)}$	Mass (MeV)			
	$M(0^{++})$	$M(1^{+-})$	$M(1^{++})$	$M(2^{++})$
$R_0 = 6$	10583, 10447	10456, 10598	10601	10625
$R_0 = 8$	10575, 10439	10448, 10591	10593	10618
$R_0 = 10$	10573, 10437	10446, 10559	10591	10616
$R_0 = 12$	10572, 10436	10445, 10588	10591	10615

TABLE VII. Mass spectra of tetraquark $b\bar{b}n\bar{s}$ at varying R_0 .

$R_0 \text{ (GeV}^{-1}\text{)}$	Mass (MeV)		
	$M(0^+)$	$M(1^+)$	$M(2^+)$
$R_0 = 6$	10637, 10736	10648, 10727, 10732	10773
$R_0 = 8$	10622, 10721	10633, 10712, 10717	10758
$R_0 = 10$	10616, 10715	10626, 10706, 10711	10752
$R_0 = 12$	10613, 10712	10624, 10704, 10709	10750

TABLE VIII. Mass spectra of tetraquark $b\bar{b}s\bar{s}$ at varying R_0 .

$R_0 \text{ (GeV}^{-1}\text{)}$	Mass (MeV)			
	$M(0^{++})$	$M(1^{+-})$	$M(1^{++})$	$M(2^{++})$
$R_0 = 6$	10799, 10764	10772, 10818	10819	10846
$R_0 = 8$	10793, 10758	10766, 10812	10813	10840
$R_0 = 10$	10791, 10756	10765, 10810	10812	10839
$R_0 = 12$	10791, 10756	10764, 10810	10811	10838

A deeper reason is that the two quarks inside a hadron form a diquark pair Qq , which manifests as an attractive interaction. Compared with the hidden-charm tetraquark system, the parameter R_0 corresponding to the hidden-bottom tetraquark system should be smaller.

IV. SUMMARY

In this paper, we investigate the hidden heavy flavor tetraquark states $Q\bar{Q}q\bar{q}$ composed of two heavy quarks and two light quarks. A remarkable feature of these tetraquark states is that they conform to a hydrogen-bond-like structure in the quantum chromodynamics version, which can be treated using the BO approximation. Subsequently, we systematically calculated the mass spectra of the S -wave tetraquark $Q\bar{Q}q\bar{q}$ system with various quantum numbers ($J^P = 0^{+(+)}$, $1^{+(\pm)}$, and $2^{+(+)}$) and analyzed their decay channels.

Based on the fact that heavy quarks are much heavier than light quarks, we assume that the wave functions of heavy and light quarks in the tetraquark state $Q\bar{Q}q\bar{q}$ can be treated separately. We consider the interaction between light quarks as a perturbation and introduce the BO potential. In specific calculations, the BO potential can be divided into three parts: the QCD Coulomb interaction, the eigenenergy of the light-quark systems, and a linear term introduced to account for the color confinement of the tetraquark state. Analogous to the treatment of H_2 in QED, we solve the two-body Schrödinger equation to obtain the zeroth-order approximation of the energy occupied by light quarks within the tetraquark state, and thus obtain the eigenvalues of the light-quark Hamiltonian. For better comparison, we use $X(3872)$ as input to determine $R_0 = 10 \text{ GeV}^{-1}$, and study other hidden heavy flavor tetraquark states.

Our calculation results indicate that the exotic hadrons $X(4140)$, $Y(3940)$, $Z_b(10610)$, and $Z_b(10650)$ remain strong candidates for tetraquark states. Additionally, we propose that $Z_{cs}(3985)^-$, $Z_{cs}(4000)^+$, and $Z_{cs}(4123)^-$ may be partner states of the tetraquark state $cu\bar{c}\bar{s}/cs\bar{c}\bar{u}$ with quantum number $J^P = 1^+$. The $X(3960)$ may be a tetraquark state $cs\bar{c}\bar{s}$ with quantum number $J^{PC} = 0^{++}$. In addition, we predict several other possible hidden heavy flavor tetraquark states and look forward to further experimental and theoretical research. Finally, it is worth emphasizing that although the difference in quark masses

may have some impact on the mass spectra of tetraquark states, this hydrogen-bond-like structure in quantum chromodynamics should not be overlooked.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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