Ten-dimensional localization

Christopher Couzens,[*] Alice Lüscher,[†] and James Sparks[‡]

Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, United Kingdom

(Received 28 August 2026; accepted 8 October 2026; published 28 October 2026)

We construct equivariantly closed polyforms for the fluxes and the action of (massive) type IIA and type IIB supergravity directly in ten dimensions. We illustrate applications of our formalism to two distinct classes of solutions in massive type IIA. Our analysis straightforwardly reproduces and vastly generalizes all known results for on shell actions in these classes, with no need for a consistent truncation. Where known, we match the corresponding field theory partition functions at leading order, and our results extend easily to other solutions of type II supergravity.

DOI: 10.1103/xk94-kksm

I. INTRODUCTION

There are two main classes of solutions of interest when considering supersymmetric backgrounds relevant for holography and AdS/CFT. The first class studies supersymmetric anti–de Sitter (AdS) solutions embedded in ten- and 11-dimensional supergravity, i.e., backgrounds of the form $\mathrm{AdS}_p \times X_{D-p}$. Via the AdS/CFT correspondence these are dual to $(p-1)$-dimensional CFTs on a Minkowski background. The details of the CFT, for example the symmetries and field content, are encoded in the choice of compact internal space X_{D-p} and one can learn about the supersymmetric protected observables such as anomalies, free energy and conformal dimensions of BPS operators by studying this geometry. There is a rich history of classifying the most general supersymmetry-preserving AdS solutions using G-structure techniques [1–3], and more recently generalized complex geometry [4,5].

The second class of solutions involves replacing AdS with a space M_p which is asymptotically locally AdS. One now fibers X_{D-p} over M_p so that the spacetime takes the form $M_p \ltimes X_{D-p}$, where X_{D-p} may also be warped. This is then interpreted as placing the CFT on the nontrivial background ∂M_p and tells us about deformations of the CFT. This allows us, for example, to probe how the observables of the CFT depend on the background, and we can compute protected observables such as indices and twisted partition functions. Typically one works in a p-dimensional gauged supergravity theory which is a consistent truncation of the D-dimensional theory.

While consistent truncations provide an invaluable bridge between higher-dimensional string theory and lower-dimensional gauged supergravity, they are only available for special internal manifolds and restricted field content. Indeed, for some of the examples we will present, such consistent truncations are not currently known. Furthermore, many questions can only be answered when studying the higher-dimensional parent theories. For example, studying probe branes in a black hole background requires analyzing the embedding of the branes in the full parent theory, and not merely within the consistent truncation of which the black hole is a solution [6–10]. Moreover, questions such as studying resolutions of punctured Riemann surfaces require a higher-dimensional embedding (or coupling the gauged supergravity to non-Abelian vector multiplets with no known uplifts [11,12]). For these, and many other reasons, a higher-dimensional description, involving both the external and internal spacetime, is thus desirable. This, however, is no easy task, as one needs to solve Einstein's equations in ten and eleven dimensions. Even with supersymmetry one still typically has nonlinear partial differential conditions to solve.

Equivariant localization in supergravity, first developed in [13], is a new method allowing for the computation of supergravity observables without the need to solve the equations of motion explicitly. This has been applied to a variety of setups, either in gauged supergravity or for the internal spaces of classes of AdS solutions [9,13–38]. To apply the method one requires the existence of a symmetry, with the Berline-Vergne–Atiyah-Bott (BV–AB) theorem stating that the integral receives contributions only from the fixed points of the symmetry. Typically, supersymmetric solutions possess such a symmetry, dual to the R-symmetry,

[*]Contact author: christopher.couzens@maths.ox.ac.uk
[†]Contact author: alice.luscher@maths.ox.ac.uk
[‡]Contact author: james.sparks@maths.ox.ac.uk

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

with the vector field generating the symmetry constructed from spinor bilinears. One then constructs equivariantly closed polyforms with respect to this symmetry with the top-form the observable of interest.

We will apply this formalism directly to (massive) type IIA and type IIB supergravity. This allows us to simultaneously address both cases outlined above. For supersymmetric solutions preserving a minimal amount of supersymmetry, the ten-dimensional manifold generically has a Killing vector [5]. We use a subset of the necessary and sufficient conditions for such a supersymmetric solution to exist [5,39] to compute equivariantly closed polyforms for the Page fluxes and the on shell action. This allows us to appropriately compute the quantization of fluxes for any supersymmetric supergravity background and to then compute the on shell action. We find that there is a very universal structure for these polyforms. Our results are applicable to whatever topologies and fluxes one wishes to consider. We will apply this formalism to geometries with asymptotically AdS spaces; however, it is applicable more broadly, including to asymptotically flat black holes and even nonconformal branes. We hope to present results in these directions in the near future.

The rest of the paper is organized as follows. In Sec. II, we first review the supergravity backgrounds of [5], and further exploit their structure to build ten-dimensional equivariantly closed polyforms for the fluxes and on shell actions in type II. The next two sections are dedicated to utilizing these polyforms for two classes of solutions in massive type IIA. In Sec. III we consider a hemisphere HS^4 fibration over a six-dimensional spacetime, giving rise to an extension of Romans supergravity, while Sec. IV focuses on X_6 fibrations over a four-dimensional spacetime where X_6 is the suspension of a five-dimensional Sasaki-Einstein manifold. In both sections, we derive a master formula computing the on shell action for any topology, and further illustrate it for specific examples. Technical details are relegated to various Appendixes.

II. TEN-DIMENSIONAL POLYFORMS

We are interested in computing the (partially) on shell action of supersymmetric solutions of type IIA and type IIB supergravity. In order for the solutions to be well-defined string backgrounds we must also quantize the fluxes. Our goal in this section is to construct equivariantly closed polyforms for the Page fluxes before using these to construct a set of equivariantly closed polyforms for the partially on shell action. A key tool in constructing these polyforms is the classification of supersymmetric solutions using generalized geometry [5,39]. To keep the section self-contained we will first review the necessary conditions for a solution to preserve supersymmetry.

A. Preserving supersymmetry in ten dimensions

In the following we will consider the two type II theories concurrently, and in order to do so we must introduce a little notation. Let g be the metric, ϕ the dilaton, and $H = \mathrm{d}B$ the Neveu-Schwarz (NS)-NS three-form. In addition, using the democratic approach, we consider the polyform of all Ramond-Ramond (RR) fluxes given by

$$F^{\mathrm{RR}} = \sum_k F^{\mathrm{RR}}_k, \tag{2.1}$$

where k runs over all even numbers between 0 to 10 for type IIA and over all odd numbers in this range for type IIB. This democratic formulation doubles up the fluxes, where one must then impose the self-duality constraint

$$F^{\mathrm{RR}} = \star\,\lambda(F^{\mathrm{RR}}), \tag{2.2}$$

where λ acts on a k-form as $\lambda(\omega_k) \equiv (-1)^{\lfloor \frac{k}{2} \rfloor}\omega_k$. Furthermore, we denote by d_H the twisted differential operator defined by

$$\mathrm{d}_H = \mathrm{d} - H \wedge . \tag{2.3}$$

In the democratic formulation the Maxwell equations for the RR fluxes simply become

$$\mathrm{d}_H F^{\mathrm{RR}} = 0. \tag{2.4}$$

Supersymmetric solutions of type IIA and type IIB supergravity admit two Majorana-Weyl Killing spinors ϵ_1 and ϵ_2, which in our conventions satisfy:

$$\begin{aligned} &\Gamma_{11}\epsilon_1 = \epsilon_1, \quad \Gamma_{11}\epsilon_2 = -\epsilon_2 \quad \text{type IIA}, \\ &\Gamma_{11}\epsilon_1 = \epsilon_1, \quad \Gamma_{11}\epsilon_2 = \epsilon_2 \quad \text{type IIB}. \end{aligned} \tag{2.5}$$

Following [5,39,40] one constructs the vector bilinears[1]:

$$K_\alpha = \overline{\epsilon_\alpha}\Gamma^M\epsilon_\alpha\partial_M, \tag{2.6}$$

which are null as follows by the ϵ_α being Majorana-Weyl. It follows that the Killing spinors satisfy the projection conditions,

$$K_\alpha \cdot \epsilon_\alpha = 0, \tag{2.7}$$

with the dot denoting Clifford multiplication. One then constructs the vector field K and one-form $\tilde{K}$ defined as

$$K = \frac{1}{2}(K_1 + K_2), \qquad \tilde{K} = \frac{1}{2}(K_1 - K_2). \tag{2.8}$$

[1]In comparison with [5,39] we have rescaled the bilinear forms by a factor of 32, more in line with the conventions in [40].

Since the K_i are null, it follows that K is either null (if the two K_i are parallel) or otherwise timelike; $K^2 < 0$. One can show that supersymmetry implies

$$\mathrm{d}\tilde{K} = K \lrcorner H. \tag{2.9}$$

Consider now the mixed bispinor

$$\hat{\Phi} = 32\, \epsilon_1 \otimes \overline{\epsilon_2}, \tag{2.10}$$

which satisfies $\hat{\Phi} = \star\lambda(\hat{\Phi})$. One should understand this as a polyform upon application of the Clifford map. If the two Killing spinors have the same chirality then $\hat{\Phi}$ is a sum of odd forms only, whilst if the spinors have opposite chirality then it is a sum of even forms only; therefore

$$\hat{\Phi} = \begin{cases} \hat{\Phi}_0 + \hat{\Phi}_2 + \hat{\Phi}_4 + \hat{\Phi}_6 + \hat{\Phi}_8 + \hat{\Phi}_{10} & \text{type IIA}, \\ \hat{\Phi}_1 + \hat{\Phi}_3 + \hat{\Phi}_5 + \hat{\Phi}_7 + \hat{\Phi}_9 & \text{type IIB}. \end{cases} \tag{2.11}$$

It follows that $\hat{\Phi}$ satisfies

$$(\tilde{K} \wedge + K \lrcorner)\hat{\Phi} = 0. \tag{2.12}$$

One can derive a set of necessary and sufficient conditions for the preservation of supersymmetry. In particular, one finds that K is a Killing vector for the full solution, the Killing spinors are uncharged under it, $\mathcal{L}_K \epsilon_i = 0$, and that one necessarily has

$$\mathrm{d}_H(\mathrm{e}^{-\phi}\hat{\Phi}) = -(\tilde{K} \wedge + K \lrcorner) F^{\mathrm{RR}}. \tag{2.13}$$

There are further conditions that need to be satisfied, known as pairing constraints, and we refer the reader to [5,39] for the full set of necessary and sufficient conditions; however, it turns out that Eqs. (2.9), (2.12), and (2.13) are sufficient for us to construct the equivariantly closed polyforms that we will need.

B. Constructing the polyforms

Having reviewed the necessary background material we will turn to constructing equivariantly closed polyforms, with respect to the Killing vector K, for both type II theories. The key observation is that the supersymmetry conditions outlined above are precisely sufficient to find a completion of all the Page fluxes and the on shell action into equivariantly closed polyforms. This then allows us to appropriately quantize the fluxes and then compute the on shell action. The rest of this section outlines the computation of these polyforms, with a more detailed exposition presented in Appendix A 1.

First recall that in the presence of nontrivial NS-NS flux the Maxwell charges are not conserved, as is evident from (2.4). Instead, the conserved charge is the Page charge. As such we are interested in constructing equivariantly closed polyforms for the Page fluxes. Using the democratic formulation, the Page flux polyform F takes the simple form

$$F = \mathrm{e}^{-B} \wedge F^{\mathrm{RR}}, \tag{2.14}$$

where B is the potential for the NS-NS three-form. From (2.4) it follows that F is closed and therefore each of the terms of the same form degree may serve as the top component of an equivariantly completed polyform.

Since K preserves all of the bosonic fields we have that $\mathcal{L}_K H = 0$. Consequently, as $H = \mathrm{d}B$, we can pick a gauge for B, such that in each patch the Lie derivative of B along the flow generated by K also vanishes, i.e., that $\mathcal{L}_K B = 0$.[2] In this gauge, and using (2.9), one finds

$$0 = \mathcal{L}_K B = \mathrm{d}(\tilde{K} + K \lrcorner B), \tag{2.15}$$

and we may therefore locally introduce a function Υ such that

$$\tilde{K} + K \lrcorner B = \mathrm{d}\Upsilon. \tag{2.16}$$

Using (2.13) and (2.16) we find that K contracted into the Page fluxes is

$$K \lrcorner F = -\mathrm{d}[\Upsilon F + \Phi], \tag{2.17}$$

where we defined the polyform

$$\Phi \equiv \mathrm{e}^{-\phi}\,\mathrm{e}^{-B} \wedge \hat{\Phi}. \tag{2.18}$$

The term in the brackets is the first step in constructing the equivariantly closed polyforms for the Page fluxes. We can then compute K contracted into this polyform and proceed further along the descent (the reader interested in these details should consult Appendix A 1). The final result is a polyform composed of polyforms for the Page fluxes:

$$\Psi^F = F + \sum_m \Psi_m, \tag{2.19}$$

where m runs from 1 to 5 in IIA and 1 to 4 in IIB, while

$$\Psi_m = \frac{(-1)^m}{m!}(\Upsilon^m F + m\Upsilon^{m-1}\Phi). \tag{2.20}$$

This is the central result of this paper and applies to the Page fluxes of both (massive) type IIA and type IIB supergravity under the single assumption of minimal supersymmetry.

[2]The existence of this gauge is a local statement. Indeed, using (2.9) and Cartan's formula gives $\mathcal{L}_K B = \mathrm{d}(K \lrcorner B) + K \lrcorner H = \mathrm{d}(K \lrcorner B + \tilde{K})$. To set this to zero by a gauge transformation $B \to B + \mathrm{d}\Lambda$ one must solve $\mathrm{d}(\mathcal{L}_K \Lambda + K \lrcorner B + \tilde{K}) = 0$, which can always be done locally, but may be obstructed globally.

The equivariantly closed polyform for the Page flux F_k is extracted by taking the sum of the k-form of F, i.e., F_k, the $(k-2)$-form of Ψ_1, the $(k-4)$-form of Ψ_2 and so forth until one ends up with either a zero-form for type IIA or a one-form for type IIB. Explicitly, the polyforms for the first three Page fluxes in massive type IIA are

$$\begin{aligned}\Psi^{F_0} &= F_0,\\ \Psi^{F_2} &= F_2 - (\Upsilon F_0 + \Phi_0),\\ \Psi^{F_4} &= F_4 - (\Upsilon F_2 + \Phi_2) + \frac{1}{2}(\Upsilon^2 F_0 + 2\Upsilon\Phi_0),\end{aligned} \tag{2.21}$$

where we have used the shorthand $\Phi_k = \Phi|_{k\text{-form}}$. The polyforms for the higher-form fluxes, which we also use later, and the ones for type IIB have the same structure and are spelled out explicitly in Appendix A 1. One should then quantize the k-form Page flux as

$$\frac{1}{(2\pi\ell_s)^{k-1}}\int_{\Sigma_k} F_k \in \mathbb{Z}. \tag{2.22}$$

In particular, one can quantize the Romans mass and define

$$n_0 = 2\pi\ell_s F_0. \tag{2.23}$$

We emphasize that these polyforms are valid for any minimally supersymmetric massive type IIA background, and those in Appendix A 1 hold for any minimally supersymmetric type IIB background. Not only will these polyforms be instrumental in quantizing the Page fluxes, they are also sufficient for computing the on shell action to which we now turn.

C. Partially on shell action

The key observable that we wish to compute in this paper is the (partially) on shell string frame action of type II supergravity. For both type II theories, using the democratic formulation one can write a pseudoaction which takes the form

$$S = \frac{1}{(2\pi)^7\ell_s^8}\int \mathrm{d}^{10}x\sqrt{-g}\left[\mathrm{e}^{-2\phi}\left(R + 4|\mathrm{d}\phi|^2 - \frac{1}{2}|H|^2\right) - \frac{1}{4}\sum_k |F_k^{\mathrm{RR}}|^2\right], \tag{2.24}$$

where for a (real) p-form α we define $|\alpha|^2 \equiv \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p}\alpha^{\mu_1\cdots\mu_p}$. By doubling the number of fields we remove the Chern-Simons terms from the action, which are effectively hidden in the last term [41]. Notice that it is the Maxwell fluxes that appear in (2.24), rather than the Page fluxes we have just computed equivariantly closed polyforms for.

The action (2.24) is a pseudoaction in the sense that we are summing over all the RR fluxes but have not imposed the self-duality constraint in (2.2). Imposing this at the level of the action implies that the RR term *vanishes*, much like the usual story in type IIB supergravity for the self-dual five-form F_5. On the other hand, the latter has been addressed recently in the holographic context in [42,43], where it is shown that the correct on shell action for $\mathrm{AdS}_5 \times M_5$ solutions in type IIB supergravity, supported by self-dual five-form flux, is given by splitting the flux into electric and magnetic components. Their result for the on shell action is derived from the Pasti-Sorokin-Tonin (PST) formalism [44], which requires an additional boundary term. Rather than attempting to carry out a similar analysis for type II (using for example[3] [45]), we instead propose that the analogous result holds generally; namely one uses the result (2.24), but separates the RR contributions into electric and magnetic, only including half the terms in the sum. Effectively, the RR terms in (2.24) cancel pairwise on shell, and this prescription means one is choosing to include only half the terms. The inherent ambiguity in the signs in doing this has a physical interpretation; the different choices are related by a Legendre transform that interchanges between electric and magnetic ensembles [46]. For the purposes of the current paper we take this as a conjectured prescription for the on shell action, and show this agrees both with lower-dimensional supergravity results obtained with holographic renormalization (when there is a consistent truncation), and also with field theory results that otherwise have no known gravity dual. We find perfect agreement with all classes of known supergravity solutions and large-N field theory results.

Inserting the dilaton equation of motion and dropping a boundary term (see Appendix A 2), the above prescription for the action for type IIA takes the form

$$S^{\mathrm{IIA}} = \frac{1}{2(2\pi)^7\ell_s^8}\int_{M_{10}}[s_1F_0F_{10} + s_2F_2\wedge F_8 + s_3F_4\wedge F_6], \tag{2.25}$$

and for IIB

$$S^{\mathrm{IIB}} = \frac{1}{2(2\pi)^7\ell_s^8}\int_{M_{10}}[s_1F_1\wedge F_9 + s_2F_3\wedge F_7 + s_3F_5\wedge F_5], \tag{2.26}$$

where the s_i's are the signs depending on the ensemble that we are working in, i.e., which fluxes we held fixed. For all s_i being $+1$ we are in the ensemble where F_0, F_2, and F_4

[3]This can also be made supersymmetric, and we expect the polyforms we have constructed to generalize appropriately. We thank Dmitri Sorokin for discussions on this.

define conserved magnetic charges which we hold fixed in IIA (or F_1, F_3, F_5 in IIB).[4] Other ensembles are then related by the change of sign of the relevant term.

Since the partially on shell action is given by a sum over the wedge products of equivariantly closed forms, we can simply compute the equivariantly closed polyform for the action. It is given by taking the wedge product of the equivariantly closed Page flux polyforms. For massive type IIA we find

$$\Psi = [s_1 F_0 \Psi^{F_{10}} + s_2 \Psi^{F_2} \wedge \Psi^{F_8} + s_3 \Psi^{F_4} \wedge \Psi^{F_6}]. \tag{2.27}$$

We thus obtain a method for both quantizing the fluxes and computing the partially on shell action. In the remainder of this paper we will exhibit the power of this formalism by computing the on shell action of a variety of different classes of solution in type IIA. The IIB action can in principle be evaluated in a similar fashion. However, we postpone this to future work and just discuss the (massive) type IIA theory here.

In particular, we will focus on two classes of solutions; hemisphere HS^4 fibrations over M_6 and S^6 fibrations over M_4. For these two classes the signs in the action are, respectively, $\{s_1 = 1, s_2 = 1, s_3 = +1\}$ and $\{s_1 = 1, s_2 = 1, s_3 = -1\}$. The physical difference between the two classes is clear; in the first class F_4 is a magnetic flux which we hold fixed, whilst in the second F_6 defines a magnetic charge that we hold fixed. The change in ensemble precisely flips the sign s_3. For concreteness let us write the zero-form part of the action polyform for each of these

$$\begin{aligned}\Psi_0^{\text{elec}} &= -\frac{2}{15}\Upsilon^3(F_0^2\Upsilon^2 + 5F_0\Upsilon\Phi_0 + 5\Phi_0^2),\\ \Psi_0^{\text{mag}} &= \frac{1}{30}\Upsilon^3(F_0^2\Upsilon^2 + 5F_0\Upsilon\Phi_0 + 10\Phi_0^2).\end{aligned} \tag{2.28}$$

Similar expressions hold for the higher-form parts. Moreover, turning off the Romans mass these expressions simply reduce to

$$\Psi_0^{\text{elec}} = -\frac{2}{3}\Upsilon^3\Phi_0^2, \qquad \Psi_0^{\text{mag}} = \frac{1}{3}\Upsilon^3\Phi_0^2, \tag{2.29}$$

which are then "obviously" related by electromagnetic duality. Indeed, the two results are related by a Legendre transform, schematically $-2/3 + 1 = 1/3$. Using the embedding of the massless theory into 11d supergravity as spelt out in [39] these results should allow for the computation of both the on shell action and flux quantization in M-theory.

[4]In particular we always have $s_1 = +1$ for IIA and $s_3 = +1$ for IIB.

As a final comment, in order to compare with field theory results we are really interested in computing the Euclidean on shell action rather than the Lorentzian one. As such we need to perform a Wick rotation of the theory. This carries some subtleties in the properties of the spinors used to define the bilinears, where in particular there is no Majorana condition in ten-dimensional Euclidean space. One can follow the logic in [47,48] and complexify the theory working on a "holomorphic" slice and then restricting the theory to Euclidean signature. The Killing vector K is then taken to be complex in the Euclidean theory. Furthermore, we will use that this prescription multiplies the Lorentzian on shell action by an overall $-\mathrm{i}$ and to distinguish the two we will denote the Euclidean (partially) on shell action as $I = -\mathrm{i}S$. In summary, we will use localization to evaluate

$$I = \frac{-\mathrm{i}}{2(2\pi)^7 \ell_s^8}\int_{M_{10}} \Psi. \tag{2.30}$$

Before we move on to using our results, let us quickly summarize this section. Given any minimally supersymmetric type II background, the Page fluxes admit the universal equivariantly closed completion given in (2.19) and (2.20). Using these one should quantize the fluxes over all compact cycles and use them to impose the homology relations of the spacetime under consideration.[5] One can then use this to compute the partially on shell action (2.30) with the equivariantly closed polyform given in (2.27). The action depends on a set of signs which are related to the ensemble in which one is computing.

III. 10 = 6 + 4: ROMANS SUPERGRAVITY COUPLED TO A VECTOR

As a first application of our polyforms we will consider solutions of Euclidean massive type IIA supergravity which are a four-dimensional hemisphere bundle[6] over a six-dimensional base M_6. When the bundle is taken to be trivial and M_6 is Euclidean AdS_6 this is the topology of the famous Brandhüber-Oz solution [50]. In this section we will consider M_6 to have a more general topology than just AdS_6 and allow for a general HS^4 fibration over M_6. We derive a master formula (3.4), depending on the topology and global data of a putative solution, which can be written in terms of a function which we interpret as the prepotential of 6d $\mathrm{U}(1)^2$ gauged supergravity. This generalizes the

[5]This step of imposing all of the homology relations is not strictly necessary and typically follows from extremization of the action subject to the flux quantization conditions (this was first pointed out in [30]). The benefit of imposing all homology relations is that it typically simplifies the final extremization problem.

[6]This is the only solution which admits an AdS_6 vacuum solution in (massive) type IIA [49].

results of [26], which considered the Abelian Romans theory. We also expect to be able to derive our results directly from a six-dimensional $U(1)^2$ gauged supergravity theory, which couples the minimal Romans theory studied in [26] to an additional vector multiplet. This theory should be a consistent truncation of massive type IIA. However, note that we have no need for such a consistent truncation here—we derive formulas by directly localizing in ten dimensions.

Consider the following embedding of the hemisphere in five dimensions

$$HS^4 \subset \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \mathbb{R}_{\geq 0}, \tag{3.1}$$

with $|z_1|^2 + |z_2|^2 + x^2 = 1$, where $z_i \in \mathbb{C}_i$ are complex coordinates and $x \in \mathbb{R}_{\geq 0}$ is a polar coordinate on the hemisphere HS^4. The latter has boundary $\partial HS^4 = \{x = 0\} \cong S^3$. One can define two two-dimensional hemispheres $HS_i^2 = \{z_i = 0\}$ within HS^4 which are trivial in relative homology $H_2(HS^4, S^3)$. We fiber HS^4 over M_6 using the $U(1)^2$ action on $\mathbb{C}_1 \oplus \mathbb{C}_2$ via two complex line bundles $\mathcal{L}_i$ over M_6 with Chern classes $c_1(\mathcal{L}_i) \in H^2(M_6, \mathbb{Z})$ for $i = 1, 2$.

We define the projection $\pi\colon M_{10} \to M_6$ from the total space of the bundle to the base. One can then project the Killing vector K down to the base M_6 so that $\xi \equiv \pi_* K$ is a vector field on M_6. Over a fixed point of ξ on M_6, we decompose the action of K on the normal space in M_{10} as

$$K = \xi + \sum_{i=1}^{2} b_i \partial_{\varphi_i}, \qquad \xi = \sum_a \varepsilon_a \partial_{\psi_a}, \tag{3.2}$$

where ∂_{φ_i} rotates $\mathcal{L}_i$ and we assume that $b_i \neq 0$ so that the fixed points of K are then the fixed points of ξ on M_6 at the pole of HS^4. The fixed point in M_6 can be part of a fixed submanifold of dimension 0, 2 or 4, and the index a correspondingly takes values in $\{1, 2, 3\}$, $\{1, 2\}$ or $\{1\}$, respectively, such that ∂_{ψ_a} rotates the line bundle L_a in the normal space in M_6 over the fixed-point set. The weights ε_a, b_i at different fixed points are determined from the equivariant action of K on M_{10}. In particular, they depend on the fixed point p and to be precise we should write b_i^p but will often suppress the index so as to not clutter formulas. Since the Killing spinors are not charged under K it follows that the weights at a fixed point must satisfy the constraint

$$\sum_a \sigma_a \varepsilon_a + \sum_{i=1}^{2} b_i = 0, \tag{3.3}$$

where σ_a are signs depending on the properties of the spinor at the fixed point; see Appendix B.

One can now analyze the contribution of the different fixed-point sets to the on shell action. The result of this analysis is the general master formula

$$\begin{aligned} I = \frac{8}{27} F_{S^5} \Bigg\{ & \sum_{\mathrm{p}} \mp \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \mathcal{F} + \sum_{\Sigma} \frac{1}{\varepsilon_1 \varepsilon_2} \left[\sum_{i=1}^{2} p_i \partial_{b_i} \mathcal{F} - \sum_{a=1}^{2} \left(\frac{m_a}{\varepsilon_a} \right) \mathcal{F} \right] \\ & + \sum_{B_4} \mp \frac{1}{\varepsilon} \left[\frac{1}{2} \sum_{i,j=1}^{2} \langle p_i, p_j \rangle \partial_{b_i b_j}^2 \mathcal{F} - \frac{1}{\varepsilon} \sum_{i=1}^{2} \langle p_i, m \rangle \partial_{b_i} \mathcal{F} + \frac{1}{\varepsilon^2} \langle m, m \rangle \mathcal{F} \right] \Bigg\}, \end{aligned} \tag{3.4}$$

where

$$\mathcal{F} \equiv \mathcal{F}(b_1, b_2) = (b_1 b_2)^{3/2} \tag{3.5}$$

is the prepotential on HS^4 and F_{S^5} is the S^5 free energy

$$F_{S^5} = -\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{n_0}}. \tag{3.6}$$

The parameters appearing are just the weights, ε_a, b_i (which we emphasize depend on the fixed point and have a suppressed superscript) for the action of K and the (associated) Chern classes m_a, p_i for normal bundles inside M_6, or the HS^4 fibers, respectively. The $\langle , \rangle$ in the final line denotes the intersection pairing $H^2(B_4, \mathbb{Z}) \times H^2(B_4, \mathbb{Z}) \to \mathbb{Z}$. In the following, we will derive this result by looking in turn at isolated fixed points p of ξ on M_6, fixed surfaces $\Sigma = \Sigma_g$, and fixed four-manifolds B_4 giving rise to the various contributions in (3.4). By way of exposition we present explicit applications of this formula for different topologies realizing each of the terms above. We take the Page fluxes (and their equivariant extensions) to represent relative classes in $H^*(M_{10}, \partial)$ in what follows, where their restriction being zero on the D8/O8 boundary ∂ (an $S^3 \subset HS^4$ bundle over M_6) is a boundary condition. We note this is satisfied for uplifted solutions to the Romans theory.

A. Isolated fixed points

Consider a fixed point $\mathrm{p} \in M_6$ of ξ and consider the copy of HS_i^2 lying above it in the fiber. On one hand the flux of F_2 through such a relative cycle is trivial, since $[F_2] \in H^2(M_{10}, \partial)$, and on the other hand, it can be

evaluated using equivariant localization. Combining these observations, and taking Φ_0 to vanish on the boundary, immediately gives

$$0 = \int_{HS_i^2} F_2 = -\frac{2\pi}{b_i}\Psi_0^{F_2}|_{\mathrm{p}} = -\frac{2\pi}{b_i}(F_0\,\Upsilon + \Phi_0)|_{\mathrm{p}}$$
$$\Rightarrow F_0\,\Upsilon|_{\mathrm{p}} = -\Phi_0|_{\mathrm{p}}. \tag{3.7}$$

Next we consider the flux of F_4 through the hemisphere, again at the fixed point[7] p

$$N \equiv \frac{1}{(2\pi\ell_s)^3}\int_{HS^4} F_4 = \frac{1}{(2\pi\ell_s)^3}\frac{(2\pi)^2}{b_1 b_2}\left(F_0\,\frac{1}{2}\Upsilon^2 + \Phi_0\,\Upsilon\right)\Big|_{\mathrm{p}}, \tag{3.8}$$

and solving using (3.7) gives

$$\Upsilon|_{\mathrm{p}} = \pm\frac{2\sqrt{2}\pi \mathrm{i}\ell_s^2}{\sqrt{n_0}}\sqrt{b_1 b_2}\,N^{1/2}. \tag{3.9}$$

This is enough to turn to the evaluation of the action. If $\mathrm{p}\in M_6$ is an isolated fixed point, with weights ε_a, $a = 1, 2, 3$, for K on the normal space, then we can immediately localize

$$I_{\mathrm{p}} = \frac{\mathrm{i}}{2(2\pi)^7\ell_s^8}\frac{(2\pi)^5}{\varepsilon_1\varepsilon_2\varepsilon_3 b_1 b_2}\Psi_0^{\mathrm{elec}}|_{\mathrm{p}}$$
$$= \mp\frac{8\sqrt{2}\pi}{15\sqrt{n_0}}\frac{(b_1 b_2)^{3/2}}{\varepsilon_1\varepsilon_2\varepsilon_3}N^{5/2}. \tag{3.10}$$

Here the sign is that in (3.9). Note that scaling all five weights by the same constant leaves both (3.3) and the action contribution (3.10) invariant, as it should, so this choice of normalization is irrelevant.

To obtain the on shell free energy for EAdS_6, with boundary five-sphere, one should extremize the above subject to the sum of the weights vanishing. Doing this one obtains (3.6). This is the free energy of the dual five-dimensional SCFT on S^5. We may finally rewrite the off shell action result for an arbitrary isolated fixed point in the form

$$I_{\mathrm{p}} = \mp\frac{8}{27}F_{S^5}\frac{\mathcal{F}(b_1^{\mathrm{p}}, b_2^{\mathrm{p}})}{\varepsilon_1^{\mathrm{p}}\varepsilon_2^{\mathrm{p}}\varepsilon_3^{\mathrm{p}}}, \tag{3.11}$$

where we reestablish the superscript p as a reminder that the weights in general depend on the fixed point.

Spindle. As an example, we can consider the case $M_6 = \mathbb{R}_4 \times \Sigma$, where HS^4 is fibered over the spindle $\Sigma \equiv \mathbb{WCP}^1_{[n_+,n_-]}$. The hemisphere directions $\mathbb{R}_1^2 \oplus \mathbb{R}_2^2 = \mathbb{C}_1 \oplus \mathbb{C}_2$ are fibered as $\mathcal{O}(-p_1) \oplus \mathcal{O}(-p_2) \to \Sigma$, where the twist and antitwist are $p_1 + p_2 = n_+ + \sigma n_-$ with $\sigma = \pm 1$. Here the poles of Σ are denoted with $\pm$ (not to be confused with the above sign) and are fixed under the action of ξ. The weights at the two fixed points are then $(\varepsilon_1, \varepsilon_2, \mp\,\varepsilon/n_\pm, b_1^\pm, b_2^\pm)$, with ε the weight of ξ in the spindle direction. The twisting precisely means that

$$b_i^+ - b_i^- = \frac{p_i}{n_+ n_-}\varepsilon, \tag{3.12}$$

while taking $\sigma_a = 1$, the charge condition (3.3) gives

$$\varepsilon_1 + \varepsilon_2 + b_1^+ + b_2^+ - \frac{\varepsilon}{n_+} = 0,$$
$$\varepsilon_1 + \varepsilon_2 + b_1^- + b_2^- + \sigma\frac{\varepsilon}{n_-} = 0. \tag{3.13}$$

Introducing appropriate orbifold factors in (3.11) and recalling $\varepsilon_3 = \mp\,\varepsilon/n_\pm$, the total action then reads

$$I = -\frac{8}{27}F_{S^5}\frac{(b_1^+ b_2^+)^{3/2} - \sigma(b_1^- b_2^-)^{3/2}}{\varepsilon_1\varepsilon_2\varepsilon}. \tag{3.14}$$

In particular, the EAdS_4 case has $\varepsilon_1 = \varepsilon_2 = -\frac{1}{2}$ and the action matches the spindle solution in [20].[8] This can be viewed as an $\mathrm{EAdS}_4 \times X_6$ solution with the internal space $X_6 \equiv HS^4 \ltimes \Sigma$, or as a spindle solution to Romans theory coupled to an additional flavor vector multiplet (which allows $p_1 \neq p_2$).

Squashed sphere. The above analysis allows us to similarly recover the squashed sphere $S_b^3 \times S_\varepsilon^2$ field theory result of [51], with the squashing parameter

$$b \equiv \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}, \qquad Q^2 \equiv \frac{(\varepsilon_1 + \varepsilon_2)^2}{4\varepsilon_1\varepsilon_2}. \tag{3.15}$$

We set $n_+ = 1 = n_-$ so that we have a sphere and look at the twist case $\sigma = +1$. The variables are related in the following way:

$$\mathfrak{t}_i = p_i, \qquad \Delta_i^\pm = -\frac{2(b_i^\pm \mp \frac{\varepsilon}{2}p_i)}{\varepsilon_1 + \varepsilon_2}, \qquad \tilde{\varepsilon} = \frac{2\varepsilon}{\varepsilon_1 + \varepsilon_2}. \tag{3.16}$$

Then $\mathfrak{t}_1 + \mathfrak{t}_2 = 2$, $\Delta_1^\pm + \Delta_2^\pm = 2$, and $\Delta_i^+ - \Delta_i^- = 0$, which matches the field theory variables. We can drop the $\pm$ superscripts, and write $\Delta_i^\pm \equiv \Delta_i$. Note the rescaling of ε is required to match the correct twisting condition, where $\Delta_i \pm \frac{\tilde{\varepsilon}}{2}p_i$ appear in the two blocks.

[7]Note that these hemispheres are homologous as classes in $H_4(M_{10}, \partial)$ (taking M_6 to be connected) so that N is independent of p.

[8]We also recognize (3.13) as precisely Eq. (4.23) in [20] correcting the appearance of the twist/antitwist choice σ at the $-$ pole.

B. Fixed surfaces

Now let $\Sigma = \Sigma_g \subset M_6$ be a connected, compact surface that is fixed under ξ. This has normal bundle $N_{\Sigma\hookrightarrow M_6} = L_1 \oplus L_2$ in M_6, where we may write $L_a = \mathcal{O}(-m_a)$, $a = 1, 2$ for $m_a \in \mathbb{Z}$. Similarly, the hemisphere line bundles restrict to $\mathcal{L}_i|_\Sigma = \mathcal{O}(-p_i)$, $i = 1, 2$, for $p_i \in \mathbb{Z}$. The total normal bundle of $\Sigma \subset M_{10}$ is then $N_{\Sigma\hookrightarrow M_{10}} = L_1 \oplus L_2 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$. Equivalently,

$$m_a \equiv -\int_\Sigma c_1(L_a), \qquad p_i \equiv -\int_\Sigma c_1(\mathcal{L}_i). \tag{3.17}$$

We denote the weights on L_a by ε_a, $a = 1, 2$, and the weights on the disc directions in the hemisphere by b_i, $i = 1, 2$, as before. Note these depend on the particular connected two-dimensional fixed-point set Σ, but we suppress this in the notation. The weights are related through the charge constraint (3.3). The latter extends to surfaces to further impose that the sum of the first Chern classes of the relevant line bundles (including the one on Σ, which is equal to its Euler characteristic χ_Σ) vanishes, such that (see Appendix B)

$$m_1 + m_2 + p_1 + p_2 = \chi_\Sigma. \tag{3.18}$$

We start by quantizing the fluxes through the various cycles. Integrating F_2 through the HS_i^2 fibers over a point on Σ gives the same condition as for an isolated fixed point in (3.7). Likewise integrating F_4 through the HS^4 over a point on Σ gives again (3.9). However, we now have additional four-cycles to consider, which are HS_i^2 fibered over Σ, and we denote these by C_i. On one hand, the flux through these cycles is quantized and can be computed using localization,

$$\begin{aligned} N_i &\equiv \frac{1}{(2\pi\ell_s)^3}\int_{C_i} F_4 \\ &= \frac{1}{(2\pi\ell_s)^3}\int_{\Sigma^*}\left[\frac{2\pi}{b_i}\Psi_2^{F_4} + \frac{(2\pi)^2}{b_i^2}\Psi_0^{F_4} p_i\right], \end{aligned} \tag{3.19}$$

where Σ^* is the fixed copy of Σ at the pole $*$ of the HS_i^2. Note that the flux of F_2 through Σ enters the first term, as well as Φ_2, using (2.21). On the other hand, we have the homology relations $N_1 = -p_2 N$, $N_2 = -p_1 N$, as reviewed in Appendix C. Combining these observations one can isolate

$$\begin{aligned} \int_{\Sigma^*}\Phi_2 = &\mp \frac{2\sqrt{2}\mathrm{i}\sqrt{b_1 b_2}\ell_s^2\pi}{\sqrt{n_0}} N^{1/2} \\ &\times \int_{\Sigma^*} F_2 + 4\pi^2\ell_s^3(b_1 p_2 + b_2 p_1)N. \end{aligned} \tag{3.20}$$

The localized action is

$$\begin{aligned} I_\Sigma = &\frac{\mathrm{i}}{2(2\pi)^7\ell_s^8}\frac{(2\pi)^4}{\varepsilon_1\varepsilon_2 b_1 b_2} \\ &\times \int_{\Sigma^*}\left[\Psi_2 + 2\pi\Psi_0\left(\sum_{i=1}^2\frac{p_i}{b_i} + \sum_{a=1}^2\frac{m_a}{\varepsilon_a}\right)\right]. \end{aligned} \tag{3.21}$$

Inserting the previous flux relations gives

$$I_\Sigma = \frac{8}{27}F_{S^5}\frac{1}{\varepsilon_1\varepsilon_2}\left[\sum_{i=1}^2 p_i\partial_{b_i}\mathcal{F}(b_1,b_2) - \mathcal{F}(b_1,b_2)\sum_{a=1}^2\frac{m_a}{\varepsilon_a}\right], \tag{3.22}$$

where F_{S^5}, given in (3.6), is used to eliminate N and recall that $\mathcal{F} = (b_1 b_2)^{3/2}$. Note that, unlike in the fixed-point case, the dependence on the choice of sign in (3.9) drops out of the surface contribution.

Black brane. Consider $M_6 = \mathbb{R}_4 \ltimes \Sigma_g$, with the HS^4 also fibered over Σ_g. This corresponds to an asymptotically AdS_6 black brane, with horizon the Riemann surface Σ_g, magnetic charges p_1 and p_2 and the normal bundle twisting parameters m_1, m_2. Such a setup was previously studied in [26] for $b_1 = b_2$, i.e., the Romans theory. On the other hand, turning off the m's and setting $\varepsilon_1 = \varepsilon_2 = -\frac{1}{2}$, this is the $\mathrm{EAdS}_4 \times X_6$ solution with $X_6 = HS^4 \ltimes \Sigma_g$ studied in [20]. The charge constraints (3.3) and (3.18) (with $\sigma_i = 1$) read

$$\varepsilon_1 + \varepsilon_2 + b_1 + b_2 = 0, \qquad m_1 + m_2 + p_1 + p_2 = \chi(\Sigma_g). \tag{3.23}$$

Note that for $m_1 = m_2 = 0$, the latter is simply the Calabi-Yau condition for X_6.

Taking the two weights b_i equal and redefining the weights ϵ_i as

$$\epsilon_i \equiv -2w_i\frac{b_1}{w_1 + w_2} \tag{3.24}$$

trivializes the weight constraint and the on shell action becomes

$$\begin{aligned} I = &\frac{F_{S^5}}{27 w_1^2 w_2^2}(w_1 + w_2)^2[3w_1 w_2\chi(\Sigma_g) \\ &+ m_1 w_2(w_2 - 2w_1) + m_2 w_1(w_1 - 2w_2)], \end{aligned} \tag{3.25}$$

which agrees with the results in [26] [see Sec. (5.1) thereof].

C. Fixed B_4

Finally, let $B_4 \subset M_6$ be a fixed connected, compact four-manifold under the action of ξ. This has normal bundle $N_{B_4\hookrightarrow M_6} = L$ a complex line bundle (on fixing an

orientation), with Chern class $c_1(L) \in H^2(B_4, \mathbb{Z})$. We pick a basis of two-cycles $\Gamma_\alpha \subset B_4$ which form a basis for the free part of $H_2(B_4, \mathbb{Z})$, with Poincaré duals $\Psi_\alpha = \mathrm{PD}[\Gamma_\alpha] \in H^2(B_4, \mathbb{R})$. Given a closed[9] two-form such as $\Phi_2|_{B_4}$, we may write

$$\phi^\alpha \equiv \int_{\Gamma_\alpha} \Phi_2 \Rightarrow [\Phi_2] = \sum_{\alpha=1}^{\mathrm{b}_2} \hat{\phi}^\alpha \Psi_\alpha \in H^2(B_4, \mathbb{R}), \tag{3.26}$$

where $\mathrm{b}_2 = \dim H_2(B_4, \mathbb{R})$, and

$$\phi^\alpha = \int_{B_4} \Phi_2 \wedge \Psi_\alpha = \sum_{\beta=1}^{\mathrm{b}_2} Q_{\alpha\beta} \hat{\phi}^\beta, \tag{3.27}$$

where $Q_{\alpha\beta} \equiv \int_{B_4} \Psi_\alpha \wedge \Psi_\beta$ is the intersection matrix. This is an integer-valued unimodular symmetric matrix, with inverse $I = Q^{-1}$. Then, for example, one has

$$\int_{B_4} \Phi_2 \wedge \Phi_2 = \sum_{\alpha,\beta=1}^{\mathrm{b}_2} I_{\alpha\beta} \phi^\alpha \phi^\beta \equiv \langle \phi, \phi \rangle. \tag{3.28}$$

We similarly introduce

$$k^\alpha \equiv \frac{1}{2\pi \ell_s} \int_{\Gamma_\alpha} F_2, \quad m^\alpha \equiv -\int_{\Gamma_\alpha} c_1(L), \quad p_i^\alpha \equiv -\int_{\Gamma_\alpha} c_1(\mathcal{L}_i). \tag{3.29}$$

Using our two-cycle basis Γ_α on B_4 we can define four-cycles C_i^α which are HS_i^2 fibered over Γ_α. Note that these cycles satisfy the homology relations $N_1^\alpha = -p_2^\alpha N$, $N_2^\alpha = -p_1^\alpha N$. The argument is the same as for fixed Σ, where now we simply apply this to $\Sigma = \Gamma_\alpha$. Then as before we can solve the analog of (3.26) to give

$$\phi^\alpha = \mp \frac{4\sqrt{2}\mathrm{i}\sqrt{b_1 b_2}\ell_s^3 \pi^2}{\sqrt{n_0}} k^\alpha N^{1/2} + 4\pi^2 \ell_s^3 (b_1 p_2^\alpha + b_2 p_1^\alpha) N. \tag{3.30}$$

There is the copy of B_4 at the pole of HS^4, which we denote B_4^*. We then have the homology relation

$$N^* \equiv \frac{1}{(2\pi\ell_s)^3} \int_{B_4^*} F_4 = \langle p_1, p_2 \rangle N, \tag{3.31}$$

which is proven similarly to the S^4 over B_4 case (see Appendix C).

The localized action is then

$$I_{B_4} = \frac{\mathrm{i}}{2(2\pi)^7 \ell_s^8} \frac{(2\pi)^3}{\varepsilon\, b_1 b_2} \left\{ \int_{B_4^*} \left[\Psi_4 - 2\pi \Psi_2 \wedge \left(\frac{c_1(L)}{\varepsilon} + \sum_{i=1}^{2} \frac{c_1(\mathcal{L}_i)}{b_i} \right) \right] + (2\pi)^2 \Psi_0 \left(\frac{\langle p_1, p_1 \rangle}{b_1^2} + \frac{\langle p_1, p_2 \rangle}{b_1 b_2} + \frac{\langle p_2, p_2 \rangle}{b_2^2} + \frac{\langle m, m \rangle}{\varepsilon^2} + \sum_{i=1}^{2} \frac{\langle m, p_i \rangle}{\varepsilon\, b_i} \right) \right\}. \tag{3.32}$$

After substituting in all the forms and the above formulae, we obtain the simple result

$$I_{B_4} = \mp \frac{1}{\varepsilon} \left[\frac{1}{2} \sum_{i,j=1}^{2} \langle p_i, p_j \rangle \partial_{b_i b_j}^2 \mathcal{F}(b_1, b_2) - \frac{1}{\varepsilon} \sum_{i=1}^{2} \langle p_i, m \rangle \partial_{b_i} \mathcal{F}(b_1, b_2) + \frac{1}{\varepsilon^2} \langle m, m \rangle \mathcal{F}(b_1, b_2) \right] \frac{8}{27} F_{S^5} \pm \frac{2\sqrt{2}\pi}{3\varepsilon} \sqrt{b_1 b_2} \langle k, k \rangle \frac{N^{3/2}}{n_0^{3/2}}. \tag{3.33}$$

Note that the F_2 flux term k^α appears as an $N^{3/2}$ correction, and so is not seen at leading order in the large-N limit (assuming that the F_2 flux is order 1 at large N). It would be interesting to investigate such terms in more detail. Such corrections are

[9]The zero-form parts of all RR equivariant forms are constant on a given connected component of the fixed-point set, which implies that Υ, Φ_0 are constant. Then similarly one can deduce that Φ_2 is closed from equivariant closure and K being zero on a fixed-point set.

common in geometries where one includes these types of fluxes (see e.g., Ref. [52]); however, there may also be other terms contributing at this order, such as higher-derivative corrections.

Product of Riemann surfaces. As an example of the above formula consider B_4 to be the product of two Riemann surfaces: $B_4 = \Sigma_{g_1} \times \Sigma_{g_2}$. For such a B_4 there are two two-cycles, represented by copies of the Riemann surface at a point on the other one. As such the intersection matrix takes the simple form

$$I_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\beta}. \tag{3.34}$$

Then the bracket gives

$$\langle p_i, p_j \rangle = p_i^{(1)} p_j^{(2)} + p_i^{(2)} p_j^{(1)}. \tag{3.35}$$

The condition for the spinors to be well-defined, (B21), implies that the line bundles over the Riemann surfaces are constrained to satisfy

$$\pm\chi(\Sigma_{g_\alpha}) = \sigma_1 p_1^\alpha + \sigma_2 p_2^\alpha + \sigma_3 m^\alpha. \tag{3.36}$$

One can now use this to expand out each of the terms in (3.33) and use (3.36) to eliminate one of the first Chern classes in favor of the Euler characteristic to get a more explicit, and substantially longer, result.

Setting $m = 0$ so that $M_6 = \Sigma_{g_1} \times \Sigma_{g_2} \times \mathbb{R}^2$ is simply a product, using (3.33) and taking the upper sign with $\sigma_1 = \sigma_2 = +1$ in (3.36) we find

$$I = \frac{2\sqrt{2}\pi}{5\sqrt{n_0}\,\varepsilon} \frac{\left[b_2(b_2 p_1^{(2)} + 3b_1 p_2^{(2)}) p_1^{(1)} + b_1(b_1 p_2^{(2)} + 3b_2 p_1^{(2)}) p_2^{(1)}\right]}{(b_1 b_2)^{1/2}} N^{5/2}. \tag{3.37}$$

We may compare this to the large-N field theory result (3.105) for $\log Z$ derived in [53–56] for the dual USp($2N$) Seiberg theory on $\Sigma_{g_1} \times \Sigma_{g_2} \times S^1 = \partial M_6$. Identifying our variables with theirs via $p_i^{(1)} = \mathfrak{s}_i$, $p_i^{(2)} = \mathfrak{t}_i$ and $b_i = \Delta_i$, our R-charge constraint $b_1 + b_2 = -\varepsilon$ matches their $\Delta_1 + \Delta_2 = 2\pi$ provided we normalize the weight $\varepsilon = -2\pi$ on the $\mathbb{R}^2$ factor [with the Chern numbers satisfying (3.36)]. The result (3.37) then perfectly matches the field theory result for $-\log Z$.

IV. 10 = 4 + 6: D2-D8 CHERN-SIMONS THEORIES

We now turn to a second application of our results, where the geometry is a fibration of the suspension $X_6 = \mathcal{S}\mathrm{SE}_5$ over a four-dimensional base M_4 with SE_5 a five-dimensional Sasaki-Einstein manifold. When M_4 is Euclidean AdS_4 this describes 3d $\mathcal{N} = 2$ SCFTs arising from D2-branes probing the suspension of the Sasaki-Einstein manifold in the presence of D8-branes [57,58]. From the field theory perspective, the choice of SE_5 determines the matter content of the theory, while the choice of the four-dimensional spacetime means placing the dual theory on the background ∂M_4.

In this section we will focus on the S^5 case, and relegate performing the explicit computations for different Sasaki-Einstein manifolds ($T^{1,1}$ in particular) to Appendix D. The resultant master formula for each Sasaki-Einstein space depends only on the definition of a prepotential given by the off shell Sasakian volume to the power $-2/3$. For the moment we will keep the discussion general before specializing to the S^5 case.

Let us first restrict to the toric Sasaki-Einstein case which allows us to introduce a basis for the $\mathrm{U}(1)^3$ action given by ∂_{φ_i}, $i = 1, 2, 3$. For each of these directions we may introduce a complex line bundle $\mathcal{L}_i$ with which we can fiber over the base M_4. Let us introduce the projection $\pi\colon M_{10} \to M_4$ from the bundle to the base. Projecting the vector field K to the base we define $\xi \equiv \pi_* K$ which is a vector field on M_4. We may then decompose the ten-dimensional Killing vector as

$$K = \xi + \sum_{i=1}^{3} b_i \partial_{\varphi_i}, \qquad \xi = \sum_a \varepsilon_a \partial_{\psi_a}, \tag{4.1}$$

where a takes values $\{1, 2\}$ for isolated fixed points in M_4 and $\{1\}$ for bolts in M_4. As in the previous section it follows that the fixed points of K are the fixed points of ξ in M_4 at the two poles of the suspension.

Following Appendix B, the fact that the Killing spinor is uncharged under K fixes the weights at the fixed point to satisfy[10]

$$\sum_a \sigma_a \varepsilon_a + \sum_{i=1}^{3} b_i = 0, \tag{4.2}$$

[10]We fix all of the signs for the weights on the S^5 to be 1 at each fixed point of M_4.

where σ_a are signs which are fixed at each fixed point. Furthermore, when considering the weights of the suspension the product of the weights (independent of the number of weights used to compute a cycle) picks up an overall sign between the north and south pole. We will take the sign to be positive at the north pole (when computing using toric geometry) and negative at the south pole. Note that the weights in the external spacetime are the same at the north and south pole.

For this class of theory it follows that the partially on shell action is given by the following master formula:

$$I = \frac{9}{4} F_{S^3} \left\{ \sum_{\mathrm{p}} \frac{1}{\varepsilon_1 \varepsilon_2} \mathcal{F} + \sum_{\Sigma} \frac{1}{\varepsilon} \left[\sum_{i=1}^{3} p_i \partial_{b_i} \mathcal{F} - \left(\frac{m}{\varepsilon} \right) \mathcal{F} \right] \right\}, \tag{4.3}$$

where the prepotential is a homogeneous degree two function of the weights b_i, $i = 1, 2, 3$ determined by the choice of SE_5 in terms of the Sasakian volume. For the five-sphere the prepotential is

$$\mathcal{F}_{S^5} = (b_1 b_2 b_3)^{2/3}, \tag{4.4}$$

and F_{S^3} is the free energy of the CFT dual to the S^5 suspension placed on S^3

$$F_{S^3} = -\frac{2^{1/3}\pi}{5 \times 3^{5/6}} (3\mathrm{i} + \sqrt{3}) n_0^{1/3} N^{5/3}. \tag{4.5}$$

Note in particular that F_{S^3} is complex and it is a nontrivial result that our formalism recovers both the real and imaginary part.

In the remainder of this section we prove the master formula (4.3) for $X_6 = \mathcal{S}S^5 = S^6$ and then generalize the analysis to other SE_5 suspensions in Appendix D. The logic is similar to the previous section. We first quantize the fluxes and impose the homology relations before substituting this into the action and extremizing over the free parameters. As in the previous section we first recover the contribution of fixed points before turning to the fixed surface contributions. We provide examples to further illustrate both types of contributions, which recover and extend known results in the literature.

A. Isolated fixed points

Let us first quantize the various fluxes through the various cycles. There are three $S_i^2 \subset S^6$ and similarly three $S_i^4 \subset S^6$. These are all invariant under K and trivial in homology, such that using localization

$$\begin{aligned} 0 &= \int_{S_i^2} F^2 = \frac{2\pi}{b_i} \left[(F_0 \Upsilon_N + \Phi_0^N) - (F_0 \Upsilon_S + \Phi_0^S) \right], \\ 0 &= \int_{S_i^4} F^4 = \frac{(2\pi)^2}{b_j b_k} \frac{1}{2} \Big[(F_0 \Upsilon_N^2 + 2\Upsilon_N \Phi_0^N) \\ &\quad - (F_0 \Upsilon_S^2 + 2\Upsilon_S \Phi_0^S) \Big], \end{aligned} \tag{4.6}$$

from which we obtain

$$\Phi_0^N = -\Phi_0^S = -\frac{1}{2} F_0 (\Upsilon_N - \Upsilon_S). \tag{4.7}$$

We now turn to the quantization of F_6 through X_6, which counts the number of D2-branes

$$\begin{aligned} N &= \frac{1}{(2\pi \ell_s)^5} \frac{(2\pi)^3}{b_1 b_2 b_3} \frac{1}{3!} \Big[(F_0 \Upsilon_N^3 + 3\Upsilon_N^2 \Phi_0^N) \\ &\quad - (F_0 \Upsilon_S^3 + 3\Upsilon_S^2 \Phi_0^S) \Big] \\ &= \frac{1}{b_1 b_2 b_3} \frac{n_0}{96 \ell_s^6 \pi^3} (\Upsilon_N - \Upsilon_S)^3. \end{aligned} \tag{4.8}$$

Inserting our previous result for $\Phi_0^{N,S}$, we can isolate an expression for Υ_N. The flux being cubic in Υ_N, it admits three roots

$$\Upsilon_N = -2^{5/3} \mathrm{e}^{2\pi \mathrm{i} n_N/3} \pi \ell_s^2 (3 b_1 b_2 b_3 n_0^{-1} N)^{1/3} + \Upsilon_S, \tag{4.9}$$

where $n_N = \{0, 1, 2\}$ and labels the three distinct roots.[11]

We have now all the ingredients to evaluate the action. Localization gives

$$I_{\mathrm{p}} = \frac{\mathrm{i}}{2(2\pi)^7 \ell_s^8} \frac{(2\pi)^5}{\varepsilon_1 \varepsilon_2 b_1 b_2 b_3} (\Psi_0^{\mathrm{mag}}|_N - \Psi_0^{\mathrm{mag}}|_S), \tag{4.10}$$

where we insert the expressions for $\Phi_0^{N,S}$ and Υ_N obtained above. The explicit expression is not particularly enlightening as it stands, however extremizing over the remaining free parameter Υ_S we find that the critical point fixes

$$\Upsilon_S = -\Upsilon_N. \tag{4.11}$$

Inserting this back into the action gives the extremized result

$$I_{\mathrm{p}} = (-1)^{n_N} \mathrm{i} \mathrm{e}^{\mathrm{i}\pi n_N/3} \frac{3^{5/3}\pi}{2^{2/3}5} n_0^{1/3} \frac{(b_1 b_2 b_3)^{2/3}}{\varepsilon_1 \varepsilon_2} N^{5/3}. \tag{4.12}$$

Note that this is an off shell result and there is still an extremization over the weights to perform. In order to obtain

[11]The root with $n_N = 0$ leads to a purely imaginary action and should be ignored.

a result with positive real part one should take $n_N = 1$ and the result is

$$I_{\rm p} = \frac{3^{7/6}(3-\mathrm{i}\sqrt{3})\pi}{2^{5/3}\times 5}\frac{(b_1b_2b_3)^{2/3}}{\epsilon_1\epsilon_2}n_0^{1/3}N^{5/3}. \tag{4.13}$$

On the other hand the choice $n_N = 2$ switches the sign of the real part, playing the same role as the $\pm$ sign in the fixed-point contribution (3.11) we obtained in the previous section.

In the case of EAdS_4, there is a single fixed point and the extremization over the weights sets them to all be equal, such that (4.12) recovers the sphere free energy (4.5). In terms of this quantity and of the prepotential (4.4), the fixed-point contribution to the action can be rewritten as

$$I_{\rm p} = \frac{9}{4}F_{S^3}\frac{\mathcal{F}(b_i^{\rm p})}{\varepsilon_1^{\rm p}\varepsilon_2^{\rm p}}. \tag{4.14}$$

This is the gravitational block term in [59].

Spindle. In particular we can consider an accelerating black hole, whose horizon is a spindle $\mathbb{WCP}^1_{[n_+,n_-]}$. The spacetime has topology $\mathrm{AdS}_2\times\Sigma$ and there are two pairs of fixed points; the two poles of the suspensions (N, S) at the two poles of the spindle ($\pm$). Using that the on shell action in (4.13) is homogeneous of degree zero under rescaling all of the weights we may choose to set the weight on AdS_2 to be $\varepsilon_1 = -1$. After the rescaling the weights on the spindle poles are $\varepsilon_2^{\pm} \equiv \mp\,\varepsilon/n_\pm$ and the charge condition (4.2) then reads

$$-\frac{\varepsilon}{n_+}+\sum_{i=1}^3 b_i^+ = 1, \qquad \sigma\frac{\varepsilon}{n_-}+\sum_{i=1}^3 b_i^- = 1, \tag{4.15}$$

where $\sigma = \pm 1$ for the twist/antitwist. We can directly use (4.14) at the two poles of the spindle (and introducing the appropriate orbifold factor) to obtain

$$I = \frac{9}{4}F_{S^3}\frac{\mathcal{F}(b_i^+)-\sigma\mathcal{F}(b_i^-)}{\varepsilon}. \tag{4.16}$$

Note that upon extremization this generalizes the result of the universal spindle black hole in [60] which uplifts the spindle black hole solution of four-dimensional minimal gauged supergravity on the suspension. This is an example of a solution which would not currently be possible to study without our ten-dimensional method.

B. Fixed surfaces

Now we consider the contribution coming from a fixed surface Σ of ξ. The fixed-point set then consists of two bolts; the copies of the fixed surface at the two poles of the suspension. The bolt has a normal line bundle $L = \mathcal{O}(m)$ in M_4, while the fibration of the internal space over the fixed surface is characterized by the integers p_i, $i = 1, 2, 3$ defined as

$$\begin{aligned}\int_{\Sigma^N} c_1(L) &= \int_{\Sigma^S} c_1(L) \equiv m\\ \int_{\Sigma^N} c_1(\mathcal{L}_i) &= -\int_{\Sigma^S} c_1(\mathcal{L}_i) \equiv p_i.\end{aligned} \tag{4.17}$$

Recall that to each fixed point we have a set of signs σ depending on the eigenvalues of the spin generators of the spinors at the fixed point. Let σ_4 be the eigenvalue for the copy of $\mathbb{C}$ inside M_4 at the bolt. This is fixed for both poles. On the other hand the signs for the suspension should flip sign going from the north pole to the south. We may fix the signs at the north pole to all be $+1$ without loss of generality. Then the condition for the spinor to be well-defined imposes the condition:

$$\sigma_4[m+\chi(\Sigma)] = p_1+p_2+p_3. \tag{4.18}$$

In addition we have that the weights satisfy

$$\sigma_4\varepsilon+\sum_{i=1}^3 b_i = 0. \tag{4.19}$$

Having fixed the geometry let us quantize the fluxes. The cycles discussed in the previous section are still present, but there are now additional ones involving the fixed surface. As in the previous section the two-form and four-form fluxes through the compact cycles in the S^6 are again trivial

$$\int_{S_i^2}F^2 = 0, \quad \int_{S_i^4}F^4 = 0 \Rightarrow \Phi_0^N = -\Phi_0^S = -\frac{1}{2}F_0(\Upsilon_N-\Upsilon_S), \tag{4.20}$$

while the F_6 flux through the S^6 remains unchanged

$$N = \frac{1}{(2\pi\ell_s)^5}\int_{X_6}F_6 = \frac{1}{b_1b_2b_3}\frac{n_0}{96\ell_s^6\pi^3}(\Upsilon_N-\Upsilon_S)^3. \tag{4.21}$$

Once again we obtain an expression for Υ_N as one of the cubic roots of this flux, giving (4.9).

We now turn to the "new" cycles involving the bolt. We have the two-cycle which is the bolt at the north and south pole of S^6 denoted by $[\Sigma^N]$ and $[\Sigma^S]$, respectively. We also have four-cycles which are given by the linearly embedded spheres inside S^6 fibered over the Riemann surface. These cycles are trivial in homology and the flux through them should vanish. Finally we have new six-cycles given by the linearly embedded four-spheres fibered over the Riemann surface. For the S_i^4 defined by $z_i = 0$ inside S^6, the six-cycle satisfies $[S_i^4 \ltimes \Sigma] = p_i[S^6]$ and we have a total of three such six-cycles.

The homology relations of the four-cycles $[S_i^2 \ltimes \Sigma]$ impose

$$0=\int_{S_i^2\ltimes\Sigma}F_4=\frac{2\pi}{b_i}\left[\int_{\Sigma^N}\left(\Psi_2^{F_4}-\frac{2\pi}{b_i}\Psi_0^{F_4}c_1(\mathcal{L}_i)\right)-\int_{\Sigma^S}\left(\Psi_2^{F_4}+\frac{2\pi}{b_i}\Psi_0^{F_4}c_1(\mathcal{L}_i)\right)\right], \tag{4.22}$$

which after some simplification and using (4.20) gives the simple condition

$$\int_{\Sigma^N}(\Upsilon_N F_2+\Phi_2)=\int_{\Sigma^S}(\Upsilon_S F_2+\Phi_2). \tag{4.23}$$

Since the two copies of Σ are homologous, $[\Sigma^N]=[\Sigma^S]$, and F_2 is closed we have

$$\frac{1}{2\pi\ell_s}\int_{\Sigma^N}F_2=\frac{1}{2\pi\ell_s}\int_{\Sigma^S}F_2\equiv M. \tag{4.24}$$

Then we may use this to eliminate the integral of Φ_2 over Σ^N as[12]

$$\int_{\Sigma^N}\Phi_2=\int_{\Sigma^S}\Phi_2+(\Upsilon_S-\Upsilon_N)M. \tag{4.25}$$

Finally the six-cycle homology relations impose

$$p_iN=\frac{1}{(2\pi\ell_s)^5}\int_{S_i^4\ltimes\Sigma}F_6 \tag{4.26}$$

$$=\frac{1}{(2\pi\ell_s)^5}\frac{2\pi}{b_jb_k}\left[\int_{\Sigma^N}\Psi_2^{F_6}-\int_{\Sigma^S}\Psi_2^{F_6}-(2\pi\Psi_{0,N}^{F_6}-2\pi\Psi_{0,S}^{F_6})\left(\frac{p_j}{b_j}+\frac{p_k}{b_k}\right)\right], \tag{4.27}$$

for $i=1,2,3$ with i,j,k all distinct in the above formula. We may solve these as

$$\int_{\Sigma^S}\Phi_2=\frac{\pi(\Upsilon_N-\Upsilon_S)}{6}\left[6\ell_sM+F_0(\Upsilon_N-\Upsilon_S)\sum_{i=1}^3\frac{p_i}{b_i}\right]. \tag{4.28}$$

This exhausts the homology relations and we can now compute the action:

$$I_\Sigma=\frac{-\mathrm{i}}{2(2\pi)^7\ell_s^8}\frac{(2\pi)^4}{\varepsilon b_1b_2b_3}\times\left[\int_{\Sigma^N}\Psi_2-\int_{\Sigma^S}\Psi_2-(2\pi\Psi_{0,N}-2\pi\Psi_{0,S})\left(\frac{m}{\varepsilon}+\sum_{i=1}^3\frac{p_i}{b_i}\right)\right]. \tag{4.29}$$

In addition to the weights, the action still depends on Υ_S which should be extremized over. Performing the extremization for Υ_S we have two possible options, either $m=0$ (which forces $M=0$ and the dependence in Υ_S drops out) or for $m\neq 0$ Υ_S satisfies

$$\Upsilon_S=-\Upsilon_N-\frac{4\pi\ell_s^2M\varepsilon}{mn_0}. \tag{4.30}$$

For this second, more general solution, we find

$$I_\Sigma=\frac{-\mathrm{i}e^{10\pi\mathrm{i}n_N/3}3^{5/3}\pi}{2^{2/3}\times 5}\times\frac{1}{\varepsilon}\left[(b_1b_2b_3)^{2/3}\frac{m}{\varepsilon}-\sum_{i=1}^3p_i\partial_{b_i}(b_1b_2b_3)^{2/3}\right]n_0^{1/3}N^{5/3}+\frac{\mathrm{i}\pi M^2N}{mn_0}. \tag{4.31}$$

Note that setting $m=M=0$ this recovers the $m=0$ branch above which just drops the final term. In terms of the S^3 free energy (for $m\neq 0$) we find

$$I_\Sigma=\frac{9}{4}F_{S^3}\frac{1}{\varepsilon}\left[\sum_{i=1}^3p_i\partial_{b_i}\mathcal{F}(b_i)-\mathcal{F}(b_i)\frac{m}{\varepsilon}\right]+\frac{\mathrm{i}\pi M^2N}{mn_0}, \tag{4.32}$$

where $\mathcal{F}(b_i)=(b_1b_2b_3)^{2/3}$ as in Eq. (4.4). Note that the term involving M is purely imaginary and assuming that M does not scale with N is subleading in the large N limit. Notice that this has the same scaling as the subleading terms appearing in [61], [cf. Eq. (2.38) for example]. Similarly to the previous section, there can be additional subleading contributions arising from higher-derivative corrections and perturbative one-loop contributions.

Black bolt. To exemplify our results consider taking M_4 to be a black saddle with Riemann surface horizon and let m define the twisting of the thermal circle over the Riemann surface making it into a "black bolt", [62–64]. From the perspective of the four-dimensional black hole the p_i are the magnetic charges of three U(1) gauge fields. We fix $\sigma_4=-1$ in the following so that (4.18) and (4.19) read

$$\chi(\Sigma_g)+m+\sum_{i=1}^3p_i=0,\qquad -\varepsilon+\sum_{i=1}^3b_i=0. \tag{4.33}$$

It is useful to make the redefinition

$$b_i=\varepsilon\Delta_i, \tag{4.34}$$

so that the action reads[13]:

[12]Since Φ_2 is not closed generically these need not be equal and the difference is parametrized by M.

[13]We have set the two-form flux term to vanish.

$$I_{\text{black saddle}} = \frac{9}{4} F_{S^3} \left[\sum_{i=1}^{3} p_i \partial_{\Delta_i} \mathcal{F}(\Delta) - m\mathcal{F}(\Delta) \right], \tag{4.35}$$

with $\mathcal{F}(\Delta) = (\Delta_1\Delta_2\Delta_3)^{2/3}$.

V. CONCLUSION

In this paper we have constructed equivariantly closed polyforms for the Page fluxes and the on shell action of both type II supergravity theories directly in ten dimensions. Our construction relies only on the universal supersymmetry conditions of generalized geometry [5] and therefore applies to arbitrary minimally supersymmetric type IIA and type IIB backgrounds admitting a pair of Killing spinors. This provides a unified localization framework in which both flux quantization and the on shell action can be computed for arbitrary backgrounds.

We illustrated the formalism for two large classes of massive type IIA solutions. The first was a four-dimensional hemisphere bundle over a six-dimensional external spacetime which generalizes the localization of Romans supergravity in [26] to the $\mathrm{U}(1)^2$ theory. We derived the master formula (3.4) for the on shell action for a general topology with arbitrary number of isolated fixed points, fixed surfaces and four-dimensional fixed submanifolds. Similarly, we analyzed the suspension of a toric Sasaki-Einstein five-manifold fibered over a four-dimensional spacetime, again obtaining a master formula for the on shell action in (4.3). The master formula depends on a prepotential which encodes the dependence on the fibers. In particular the prepotential is a power of the Sasakian volume of the Sasaki-Einstein manifold. In each case the localized action is expressed entirely in terms of the equivariant weights and the topology of the fixed-point sets and makes no assumptions about a consistent truncation on the fiber.

One of the main advantages of the ten-dimensional perspective is that the localization computation completely bypasses the need for a consistent truncation. This represents a substantial simplification of the problem. Indeed, for a general Sasaki-Einstein suspension there is no known consistent truncation containing three vector fields and so obtaining these results otherwise seems prohibitively difficult. Furthermore, for some questions, one needs to work in ten dimensions and a gauged supergravity description is not sufficient. For example, to compute the on shell action of probe branes one needs to work directly in ten dimensions to understand the embedding of the brane in the geometry [9]. Moreover, one could consider solutions where the external spacetime has punctures, i.e., a punctured Riemann surface. One can resolve these singularities, but this however requires the ten-dimensional embedding and the gauged supergravity description is insufficient; see, for example, Refs. [30,65].

There are several natural directions for future work. First, while we have focused on massive type IIA, the construction of the equivariant polyforms is equally valid in type IIB, where it would be interesting to derive analogous master localization formulas. Second, in the massless limit our expressions should admit a direct interpretation in 11-dimensional supergravity, providing a route towards localization in M-theory. Third, though we have considered geometries which have an asymptotically AdS factor the localization works equally well for other geometries. One could, for example, consider the on shell action of non-conformal branes or asymptotically flat black holes. In these cases one needs to be more careful with boundary terms, which for the asymptotically AdS spacetimes we have considered (conjecturally) cancel. Fourth, we only exploited the generalized geometrical structure to build the polyforms. However, this perspective should also help to gain further insight into the structure underlying our results, for example using the generalized equivariance of [66].[14] Finally, our analysis suggests that subleading contributions, arising from fluxes through resolution cycles, can also be incorporated systematically; see Ref. [67] for an example where such fluxes are incorporated in field theory. Understanding the physical interpretation of these corrections, as well as extending the formalism to more general geometries and flux backgrounds, would further clarify the scope of ten-dimensional localization.

ACKNOWLEDGMENTS

We thank Florian Gaar, Pietro Benetti Genolini, Chris Hull, Carlos Núñez, Tomás Ortín, Dmitri Sorokin, and Peter West for discussions. We thank the Centro de Ciencias de Benasque for hospitality in the final stages of this work. J. F. S. is supported in part by STFC Grant No. ST/X000761/1.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: DETAILS OF DERIVATIONS

In this appendix we present a more explicit derivation of the polyforms for the Page fluxes used in the main text, before giving the explicit polyforms in type IIB.

1. Obtaining the polyforms

First consider the Maxwell equations in the democratic formalism (2.4) which we may rewrite in the form

[14]We thank the anonymous referee for bringing this to our attention.

$$\mathrm{d}_H F^{\mathrm{RR}} = 0 \Leftrightarrow \mathrm{d}(\mathrm{e}^{-B} \wedge F^{\mathrm{RR}}) = 0 \Leftrightarrow \mathrm{d}F = 0, \tag{A1}$$

where the Page flux F is defined in (2.14). Considering the contraction of K into the Page flux, one finds

$$K \lrcorner F = -(K \lrcorner B + \tilde{K}) \wedge F - \mathrm{d}(\mathrm{e}^{-\phi}\, \mathrm{e}^{-B}\, \hat{\Phi}). \tag{A2}$$

Using the definition of Υ in Eq. (2.16) and the definition of Φ in (2.18) we find

$$K \lrcorner F = \mathrm{d}[-(\Upsilon F + \Phi)] \equiv \mathrm{d}\Psi_1. \tag{A3}$$

This is the first step in constructing equivariantly closed forms with respect to K. Considering the contraction of K into Φ, we have

$$K \lrcorner \Phi = -(K \lrcorner B) \wedge \Phi - \tilde{K} \wedge \Phi = -\mathrm{d}\Upsilon \wedge \Phi, \tag{A4}$$

and therefore

$$\begin{aligned} K \lrcorner \Psi_1 &= \Upsilon \mathrm{d}(\Upsilon F + \Phi) + \mathrm{d}\Upsilon \wedge \Phi \\ &= \frac{1}{2}\mathrm{d}(\Upsilon^2 F + 2\Upsilon\, \Phi) \equiv \mathrm{d}\Psi_2. \end{aligned} \tag{A5}$$

Continuing in this manner we find that the full tower of equivariantly closed polyforms for the Page fluxes is given by

$$\Psi^F = \sum_{m=0} \Psi_m, \tag{A6}$$

where

$$\Psi_m = \frac{(-1)^m}{m!}(\Upsilon^m F + m\Upsilon^{m-1}\Phi). \tag{A7}$$

We have therefore constructed a polyform of equivariantly closed polyforms for all of the Page fluxes in both type II theories. For massive type IIA these read

$$\begin{aligned}
\Psi^{F_0} &= F_0, \\
\Psi^{F_2} &= F_2 - (\Upsilon F_0 + \Phi_0), \\
\Psi^{F_4} &= F_4 - (\Upsilon F_2 + \Phi_2) + \frac{1}{2}(\Upsilon^2 F_0 + 2\Upsilon\Phi_0), \\
\Psi^{F_6} &= F_6 - (\Upsilon F_4 + \Phi_4) + \frac{1}{2}(\Upsilon^2 F_2 + 2\Upsilon\Phi_2) - \frac{1}{3!}(\Upsilon^3 F_0 + 3\Upsilon^2\Phi_0), \\
\Psi^{F_8} &= F_8 - (\Upsilon F_6 + \Phi_6) + \frac{1}{2}(\Upsilon^2 F_4 + 2\Upsilon\Phi_4) - \frac{1}{3!}(\Upsilon^3 F_2 + 3\Upsilon^2\Phi_2) + \frac{1}{4!}(\Upsilon^4 F_0 + 4\Upsilon^3\Phi_0), \\
\Psi^{F_{10}} &= F_{10} - (\Upsilon F_8 + \Phi_8) + \frac{1}{2}(\Upsilon^2 F_6 + 2\Upsilon\Phi_6) - \frac{1}{3!}(\Upsilon^3 F_4 + 3\Upsilon^2\Phi_4) + \frac{1}{4!}(\Upsilon^4 F_2 + 4\Upsilon^3\Phi_2) - \frac{1}{5!}(\Upsilon^5 F_0 + 5\Upsilon^4\Phi_0).
\end{aligned} \tag{A8}$$

For type IIB one finds that the equivariantly closed polyforms for each of the fluxes are

$$\Psi^{F_1} = F_1, \tag{A9}$$

$$\begin{aligned}
\Psi^{F_3} &= F_3 - (\Upsilon F_1 + \Phi_1), \\
\Psi^{F_5} &= F_5 - (\Upsilon F_3 + \Phi_3) + \frac{1}{2}(\Upsilon^2 F_1 + 2\Upsilon\Phi_1), \\
\Psi^{F_7} &= F_7 - (\Upsilon F_5 + \Phi_5) + \frac{1}{2}(\Upsilon^2 F_3 + 2\Upsilon\Phi_3) - \frac{1}{3!}(\Upsilon^3 F_1 + 3\Upsilon^2\Phi_1), \\
\Psi^{F_9} &= F_9 - (\Upsilon F_7 + \Phi_7) + \frac{1}{2}(\Upsilon^2 F_5 + 2\Upsilon\Phi_5) - \frac{1}{3!}(\Upsilon^3 F_3 + 3\Upsilon^2\Phi_3) + \frac{1}{4!}(\Upsilon^4 F_1 + 4\Upsilon^3\Phi_1).
\end{aligned}$$

2. Putting the action partially on shell

Starting from the pseudoaction (2.24), the equation of motion for the dilaton is

$$R - \frac{1}{2}|H|^2 = 4|\mathrm{d}\phi|^2 - 4\nabla^2\phi, \tag{A10}$$

and we may go partially on shell by eliminating the Ricci scalar. The partially on shell action becomes

$$\begin{aligned} S &= -\frac{1}{4(2\pi)^7\ell_s^8}\int_{M_{10}}\sum_k F_k^{\mathrm{RR}}\wedge\star F_k^{\mathrm{RR}} + \frac{1}{(2\pi)^7\ell_s^8}\int_{M_{10}}\mathrm{d}^{10}x\sqrt{-g}\,\nabla^2(\mathrm{e}^{-2\phi}) \\ &= -\frac{1}{4(2\pi)^7\ell_s^8}\int_{M_{10}}\mp\langle F^{\mathrm{RR}},F^{\mathrm{RR}}\rangle + \frac{1}{(2\pi)^7\ell_s^8}\int_{\partial M_{10}}\mathrm{d}^9x\sqrt{-h}\,n^\mu\nabla_\mu(\mathrm{e}^{-2\phi}), \end{aligned} \tag{A11}$$

where n^μ is the normal to the boundary, h the induced metric on the boundary and $\langle\cdot,\cdot\rangle$ denotes the Mukai pairing. To write the Mukai pairing we have used the self-duality condition (2.2) with $\star\lambda(F) = \mp\lambda(\star F)$ in IIA/IIB, respectively. This should then be interpreted as a formal rewriting; we will come back to this momentarily. Observe that the full NS-NS sector is a total derivative term for the partially on shell action and in the following we will neglect such boundary terms.[15]

We still have a little work to relate this to our polyforms in the previous section. Fortunately for any two-form ω the Mukai pairing satisfies

$$\langle \mathrm{e}^\omega G, \mathrm{e}^\omega\tilde{G}\rangle = \langle G,\tilde{G}\rangle, \tag{A12}$$

and therefore we may replace the Maxwell fluxes F^{RR} with the Page fluxes F in the above. Our final result, ignoring boundary contributions, is then

$$S = -\frac{1}{4(2\pi)^7\ell_s^8}\int_{M_{10}}\mp\langle F,F\rangle, \tag{A13}$$

where the $-$ sign is for type IIA and the $+$ sign for type IIB. In this form the partially on shell action can be equivariantly completed using our previous results, provided we interpret (A13) suitably using a choice of electric/magnetic polarization so as to obtain a nonzero result. This is discussed in the main text.

APPENDIX B: SPINOR PROJECTION CONDITION

In this appendix we want to analyze the constraints imposed on the weights and various Chern classes by the existence of spinors on the background.[16] Since our examples in this paper are in type IIA we will focus on this only here and leave the type IIB analysis to future work. In this section in particular we need to be careful with the Wick rotation that we have imposed. Though Majorana-Weyl spinors exist in ten-dimensional Lorentzian spacetime they do not exist in the Euclidean counterpart. It is then natural to give up the Majorana condition and keep two complex spinors ϵ_i. One now needs to make a choice in the bilinears we have constructed. In the original Lorentzian theory we have

$$\hat{\Phi} = 32\,\epsilon_1\otimes\overline{\epsilon_2} = 32\,\epsilon_1\otimes(\epsilon_2^T C), \tag{B1}$$

with C the Clifford algebra intertwiner satisfying $\Gamma_A^T = -C\,\Gamma_A C^{-1}$. In the Euclidean theory it is then natural to work with $\hat{\Phi} = 32\,\epsilon_1\otimes(\epsilon_2^T C)$ working on a "holomorphic slice" of the theory, see for example Refs. [47,48].

Let us introduce a frame which is regular on the fixed-point set and assume that the normal bundle decomposes as the direct sum of copies of $\mathbb{C}$. We may introduce rectangular coordinates (x_I, y_I) on each copy such that the Killing vector field takes the form

$$K = \sum_{I=1}^k b_I(x_I\partial_{y_I} - y_I\partial_{x_I}). \tag{B2}$$

In this frame for a $10-2k$-dimensional fixed-point set one can write

$$(\mathrm{d}K)_{AB} = 2\left[\begin{pmatrix}0 & b_1\\ -b_1 & 0\end{pmatrix}\oplus\ldots\oplus\begin{pmatrix}0 & b_k\\ -b_k & 0\end{pmatrix}\oplus\ldots\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right]_{AB}, \tag{B3}$$

where the b_I are the weights of the action. Recall that the two spinors have vanishing Lie derivative along K and this should hold in the Euclidean theory too, thus we have $\mathcal{L}_K\epsilon_i = 0$. At a fixed point we have

$$\begin{aligned} 0 = \mathcal{L}_K\epsilon_\alpha|_{\mathrm{f.p}} &= K^M\nabla_M\epsilon_\alpha + \frac{1}{8}(\mathrm{d}K)_{MN}\Gamma^{MN}\epsilon_\alpha|_{\mathrm{f.p}} \\ &= \frac{1}{8}(\mathrm{d}K)_{MN}\Gamma^{MN}\epsilon_\alpha|_{\mathrm{f.p}} \\ &= -\frac{\mathrm{i}}{2}\sum_{I=1}^k b_I J_I\epsilon_\alpha|_{\mathrm{f.p}}, \end{aligned} \tag{B4}$$

where we introduced the commuting spinor generators adapted to the frame:

$$J_I = \mathrm{i}\Gamma^{2I-1}\Gamma^{2I}, \qquad J_I^2 = 1. \tag{B5}$$

[15]These boundary terms may be important in other setups. Their analysis is beyond the scope of this paper, but we intend to come back to these contributions in future work.

[16]The analysis here closely follows the analysis in [26].

We can decompose the spinors at the fixed points which gives us a set of signs

$$J_I\epsilon_\alpha \equiv \sigma_I^{(\alpha)}\epsilon_\alpha, \tag{B6}$$

with $\sigma_I^{(\alpha)} = \pm 1$ and the condition becomes

$$\sum_{I=1}^{k} b_I \sigma_I^{(\alpha)}\epsilon_\alpha|_{\text{f.p}} = 0. \tag{B7}$$

There are *a priori* two sets of signs for each of the two spinors, but they in fact turn out to be related. To see this recall that in type IIA the polyform Φ has only even-dimensional forms, whilst in type IIB Φ contains only odd-dimensional forms. We have that the spin generators satisfy

$$J_I^T C = -C J_I, \tag{B8}$$

and therefore for the zero-form part of Φ in type IIA we have

$$\Phi_0 = \epsilon_2^T C\,\epsilon_1 = \epsilon_2^T C J_I^2 \epsilon_1 = -\sigma_I^{(1)}\sigma_I^{(2)}\Phi_0. \tag{B9}$$

At a fixed point, obeying the projection condition (B6), it follows that all polyforms in Φ are proportional to Φ_0 and moreover that if $\Phi_0 = 0$ then either $\epsilon_1 = 0$ or $\epsilon_2 = 0$. The spinors should be nowhere vanishing and therefore at a fixed point we have that Φ_0 is nonvanishing. Since Φ_0 is nonzero at the fixed points we have that $\sigma_I^{(1)} = -\sigma_I^{(2)} \equiv \sigma_I$ for every direction in the normal bundle and we may choose, without loss of generality, to work with the signs for the spinor ϵ_1. Notice that we must have $\prod_{I=1}^{5}\sigma_I = 1$ in order for the spinors to have the correct chirality. Then, for massive type IIA, the weights at a $(5-k)$-dimensional fixed point satisfy

$$\sum_{I=1}^{k} \sigma_I b_I = 0. \tag{B10}$$

For the type IIB case the discussion is a little more subtle since one is necessarily dealing with forms and the basis is not invariant under the rotation and so we postpone this for future work.

Isolated fixed point. For isolated fixed points we have therefore shown that the weights necessarily satisfy the constraint

$$\sum_{I=1}^{5} \sigma_I b_I = 0. \tag{B11}$$

Two-dimensional fixed-point sets. Let us now consider restricting the spinors to a connected two-dimensional component of the fixed-point set, $\mathscr{F}_2$. If we choose an orientation on $\mathscr{F}_2$ this forces $\mathscr{F}_2$ to be a Riemann surface, classified by its genuss[17] $\mathscr{F}_2 = \Sigma_g$. Complex line bundles over the Riemann surface are uniquely fixed up to isomorphism by the first Chern number of the bundle, known as the degree, and as such we can unambiguously use the notation $\mathcal{O}(n)$ for the line bundle of degree n over $\mathscr{F}_2 = \Sigma_g$. The tangent space splits as

$$\begin{aligned} TM_{10}|_{\mathscr{F}_2} &= T\Sigma_g \oplus \mathcal{N}\Sigma_g = T\Sigma_g \oplus \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_4 \\ &= \mathcal{O}(\chi(\Sigma_g)) \oplus \mathcal{O}(-p_1) \oplus \ldots \oplus \mathcal{O}(-p_4), \end{aligned} \tag{B12}$$

with $-p_I = \int_{\Sigma_g} c_1(\mathcal{L}_I)$, $I = 1, \ldots, 4$ and $\chi(\Sigma_g) = 2(1-g)$ the Euler characteristic of the Riemann surface. The chiral spinors satisfying the projection condition (B6) are sections of the complex line bundles

$$\mathcal{O}\left[\pm\frac{1}{2}\left(-\sum_{I=1}^{4}\sigma_I p_I + \sigma_{\Sigma_g}\chi(\Sigma_g)\right)\right], \tag{B13}$$

where $\sigma_{\Sigma_g} = \prod_{I=1}^{4}\sigma_I$. These two spinors should both be nowhere zero[18] which implies that they both define nowhere-zero sections of the complex line bundle (B13). This is only possible if the line bundle is trivial. Therefore we necessarily find that for a two-dimensional fixed-point set we have

$$\sum_{I=1}^{4}\sigma_I p_I = \sigma_{\Sigma_g}\chi(\Sigma_g). \tag{B14}$$

Four-dimensional fixed-point set. Consider now a four-dimensional fixed-point set and let us denote such a connected component as B_4. In this case the tangent space decomposes as

$$TM_{10}|_{B_4} = TB_4 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3. \tag{B15}$$

The three weights satisfy

$$\sum_{I=1}^{3} b_I \sigma_I = 0. \tag{B16}$$

Let $\mathcal{S}_{B_4}^{\pm}$ be the two spinor bundles on B_4; then the ten-dimensional spinors satisfying the projection condition (B6) are sections of the rank-two spinor bundle

$$\mathcal{E} = \mathcal{S}_{B_4}^{\eta} \otimes \mathcal{L}_1^{\pm\sigma_1/2} \otimes \mathcal{L}_2^{\pm\sigma_2/2} \otimes \mathcal{L}_3^{\pm\sigma_3/2}, \tag{B17}$$

[17]One can also consider punctures but we will restrict to smooth spaces here.

[18]The spinors satisfy first-order differential equations, where a standard argument shows that if they vanish at a point they will vanish identically. But then all forms and fluxes vanish.

where $\eta = \pm$ is the chirality of the spinor on B_4, and is not correlated with the signs in the exponents but rather the product of the σ's; $\prod_{I=1}^{3}\sigma_I = \eta$. As before, since the spinors are nowhere-vanishing sections of a bundle we have that

$$\int_{B_4} c_2(E) = 0. \tag{B18}$$

The spinor spans a trivial line subbundle of $\mathcal{E}$ and therefore we have $\mathcal{E} \cong \mathcal{L}_\epsilon \oplus \mathcal{L}_\perp$. Let us now assume that B_4 admits a U(2) structure, for example induced by supersymmetry. In terms of this almost complex structure the spinor bundles on B_4 are given by

$$\mathcal{S}^+_{B_4} = K_{B_4}^{1/2} \oplus K_{B_4}^{-1/2}, \qquad \mathcal{S}^-_{B_4} = K_{B_4}^{1/2} \otimes \Lambda^{(0,1)}(B_4), \tag{B19}$$

where K_{B_4} is the canonical line bundle and $\Lambda^{(0,1)}$ denotes the set of (0,1)-forms on B_4. With our assumption of a U(2) structure on B_4, $\mathcal{S}^+_{B_4}$ decomposes into the sum of line bundles; however, in general $\mathcal{S}^-_{B_4}$ does not decompose this way unless $\Lambda^{(0,1)}$ does (see Ref. [26] for some further discussion). Therefore in the case of a positive chirality spinor on B_4 we have

$$\begin{aligned}\mathcal{E} = &\left(K_{B_4}^{1/2} \otimes \mathcal{L}_1^{\sigma_1/2} \otimes \mathcal{L}_2^{\sigma_2/2} \otimes \mathcal{L}_3^{\sigma_3/2}\right) \\ &\oplus \left(K_{B_4}^{-1/2} \otimes \mathcal{L}_1^{\sigma_1/2} \otimes \mathcal{L}_2^{\sigma_2/2} \otimes \mathcal{L}_3^{\sigma_3/2}\right)\end{aligned} \tag{B20}$$

and one of these line bundles is necessarily trivial. Therefore we have that

$$\pm c_1(K_{B_4}) = \sum_{I=1}^{3} \sigma_I c_1(\mathcal{L}_I), \tag{B21}$$

with the $\pm$ sign the choice of canonical or anticanonical bundle.

Six-dimensional fixed-point sets. We now analyze a six-dimensional fixed-point set, B_6. The tangent space decomposes as

$$TM_{10}|_{B_6} = TB_6 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2. \tag{B22}$$

The Killing vector rotates the two line bundles and the weights necessarily satisfy

$$\sigma_1 b_1 + \sigma_2 b_2 = 0. \tag{B23}$$

The spinors satisfying (B6) are now sections of the rank 4 bundle

$$\mathcal{E} = S_{B_6}^{\pm\sigma_1\sigma_2} \otimes \mathcal{L}_1^{\pm\sigma_1/2} \otimes \mathcal{L}_2^{\pm\sigma_2/2}. \tag{B24}$$

Notice that necessarily one of the spinors involves a positive chiral spinor on B_6 and the other involves a spinor of the other chirality. Assuming that B_6 admits an almost complex structure, with K the canonical bundle, the positive and negative spin bundles on B_6 decompose as

$$\begin{aligned}S^+_{B_6} &= K^{1/2} \otimes (\Lambda^{(0,0)} \oplus \Lambda^{(0,2)}), \\ S^-_{B_6} &= K^{1/2} \otimes (\Lambda^{(0,1)} \oplus \Lambda^{(0,3)}).\end{aligned} \tag{B25}$$

Observing that $\Lambda^{(0,3)} = \overline{K_{B_6}}$, i.e., the anticanonical bundle, we have that the spinors preserving the $U(3)$ structure on B_6 are nowhere-vanishing sections of

$$K^{\pm\sigma_1\sigma_2/2} \otimes \mathcal{L}_1^{\pm\sigma_1/2} \otimes \mathcal{L}_2^{\pm\sigma_2/2}, \tag{B26}$$

and therefore we find that we necessarily have

$$\sigma_1\sigma_2 c_1(K_{B_6}) + \sum_{I=1}^{2} \sigma_I c_1(\mathcal{L}_I) = 0. \tag{B27}$$

There may be more general choices involving spinors in $\Lambda^{(0,1)}$ and $\Lambda^{(0,2)}$, in analogy with the discussion in the four-dimensional fixed-point set example.

Eight-dimensional fixed-point sets. There is an immediate obstruction to having an eight-dimensional fixed point; the invariance of the Killing spinor under K implies that the solitary weight must vanish. This then implies that the action is trivial in the normal direction and thus there is no fixed point in this case. A similar result holds for four-dimensional ungauged supergravity, for precisely the same reason.[19]

APPENDIX C: HOMOLOGY CONSIDERATIONS

In this appendix we prove the homology relations used in Sec. III. The extension of these arguments to the relations in Sec. IV is straightforward.

The manifold M_{10} has a "boundary" which is $\partial HS^4 \to M_6$. We denote this by $\partial \subset M_{10}$, noting that M_6 itself will have a (conformal) boundary which is different.[20] Then in what follows we will take the RR Page forms,

$$F_{2n} \in H^{2n}(M_{10}, \partial), \tag{C1}$$

in particular meaning they have zero restriction to the D8/O8 boundary ∂, so $F_{2n}|_\partial = 0$. One can regard this as a choice of boundary conditions, where we note this is

[19]Notice that in the minimal AdS_2 classification in [68] the bilinear K is tangent to the AdS_2 factor. In Poincaré coordinates $ds^2_{AdS_2} = -\rho^2 dt^2 + d\rho^2/\rho^2$ this Killing vector bilinear is $K = \partial_t$, under which the corresponding Killing spinor is indeed uncharged. Note that ∂_t has no fixed points (in the Wick rotation), so this is not in contradiction with the above comments.

[20]Thus, ∂ itself has a boundary, namely $S^3 \to \partial M_6$.

satisfied for uplifts of the minimal Romans theory [the RR forms in (2.1) of [26] are all uplifts of Romans fields that are multiplied by functions that go to zero on ∂HS^4]. We will need also the stronger condition that also the lower degree parts of equivariantly closed forms for these fluxes are zero; concretely, that Υ and the bilinear form Φ are zero when restricted to ∂. To see this one should solve the ten-dimensional killing spinor equations near a supersymmetric D8/O8 boundary. Using the results in [49] we can see that indeed the spinors vanish on the D8/O8 locus and therefore both Υ and Φ do too. We will assume that

$$\Psi^{F_{2n}} \in H_K(M_{10}, \partial), \tag{C2}$$

with (C1) then a corollary of this, where H_K denotes equivariant cohomology for d_K.

As a first consequence, it is immediate from (C2) that there is no flux of the Page RR two-form F_2 through the copies of HS_i^2 at the fixed points. Indeed, since $H^2(HS^4, S^3) = 0$, necessarily $F_2 = \mathrm{d}\omega$ is exact restricted to a fiber HS^4 where $\omega|_{S^3} = 0$, but then $\int_{HS_i^2} F_2 = \int_{S_i^1} \omega = 0$ where $S_i^1 \subset S^3$ lies on the boundary.

Next, consider the 6-manifold B_6 which is HS^4 fibered over a two-dimensional manifold Σ, with four-cycles C_i which are HS_i^2 fibered over Σ. These and the fiber HS^4 are classes in $H_4(B_6, \partial, \mathbb{Z})$. Consider a closed four-form $F_4 \in H^4(B_6, \partial)$. Fix, say, C_1, and let the Poincaré dual be Ψ_1, which has support along the fiber of $\mathcal{L}_2$. This satisfies $\Psi_1 = \pi^* c_1(\mathcal{L}_2) \in H^2(B_6)$ by explicit construction of this Poincaré dual, where $\pi : HS^4 \to \Sigma$ is the projection. Then

$$\begin{aligned} N_1 &\equiv \int_{C_1} F_4 = \int_{B_6} F_4 \wedge \Psi_1 \\ &= \int_{B_6} F_4 \wedge \pi^* c_1(\mathcal{L}_2) = N \int_\Sigma c_1(\mathcal{L}_2), \end{aligned} \tag{C3}$$

where $N \equiv \int_{HS^4} F_4$ is the integral of F_4 over the HS^4 fiber.

Finally, consider HS^4 fibered over B_4, and let B_4^* be the copy of B_4 at the pole of HS^4. The Poincaré dual of B_4^* inside the total space B_8 is $\Psi_1 \wedge \Psi_2 = \pi^*[c_1(\mathcal{L}_1) \wedge c_1(\mathcal{L}_2)] \in H^4(B_8)$. Then,

$$\begin{aligned} \int_{B_4^*} F_4 &= \int_{B_8} F_4 \wedge \Psi_1 \wedge \Psi_2 \\ &= \int_{B_4} c_1(\mathcal{L}_1) \wedge c_1(\mathcal{L}_2) \int_{HS^4_{\text{fiber}}} F_4 = \langle p_1, p_2 \rangle N, \end{aligned} \tag{C4}$$

with notation as in the main text.

APPENDIX D: SUSPENSION OVER $T^{1,1}$

In this section we will consider the suspension of $T^{1,1}$. Let $\{\partial_{\varphi_i}\}$ be a basis for the $\mathrm{U}(1)^3$ action on $T^{1,1}$ adapted so that the Killing spinor is charged only under ∂_{φ_1} and uncharged under the remaining two. In this basis $T^{1,1}$ is described by the four toric vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \tag{D1}$$

These define the facets of a three-dimensional cone. The tip of the cone is not smooth, where one can see this by noting that there are four faces meeting at the tip rather than three. One manifestation of the singularity is that the normal bundle of the fixed point does not decompose into the sum of three line bundles, and this makes the localization more subtle. A simple way to proceed is to perform a small resolution of the singular point, blowing up a $\mathbb{CP}^1$ at the tip, and then later collapsing the blow-up. Since this is a crepant resolution this does not change the volume computed using localization [69], and thus the result does not depend on the resolution at the end.

There are two options for the resolution (see Fig. 1), related by a flop, specified by taking either pair of maximal cones for the resolution

$$\begin{aligned} \tau_1^+ &= \{v_1, v_2, v_3\}, \qquad \tau_1^- = \{v_1, v_3, v_4\}, \\ \tau_2^+ &= \{v_1, v_2, v_4\}, \qquad \tau_2^- = \{v_2, v_3, v_4\}. \end{aligned} \tag{D2}$$

To describe the suspension we need to glue two copies of the cone over $T^{1,1}$ together. We cut the (infinite) cones off at some radius and glue the two together, reversing the orientation of one copy and identifying the faces. For this to make sense we must ensure that the glued faces are described by the same toric vector, or more precisely the boundaries of the two cones are the same. The result need not be a (symplectic or complex) toric space in the usual sense. There are two distinct choices: one gives rise to $\mathbb{CP}^3$, whilst the other has the topology of a trivial S^2 bundle over S^4 and thus does not admit a Hamiltonian toric action. We will consider this choice and leave analyzing $\mathbb{CP}^3$ for future work.

1. Same small resolution

For our first example we will consider the same small resolution at both poles of the suspension. Without loss of generality we will take the first resolution in (D2), with the second giving an equivalent result under relabeling. The final result is that $X_6 = S^2 \times S^4$; see Fig. 2.

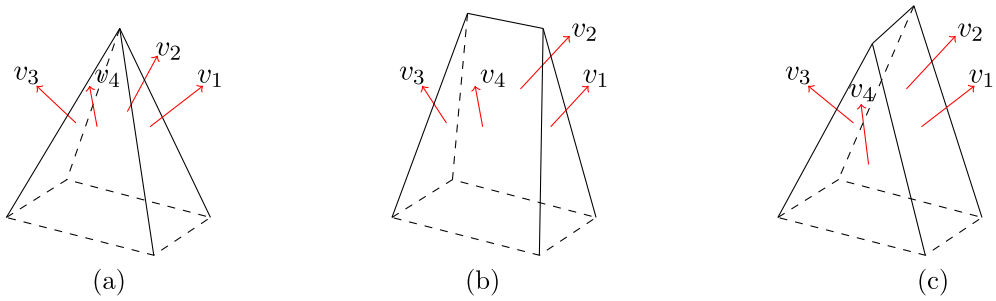


FIG. 1. The cone of $T^{1,1}$ along with the two choices of small resolution. (a) The cone of $T^{1,1}$. (b) The first choice of small resolution. (c) The second choice of small resolution.

Let us consider the various cycles in X_6. We have the two blown-up spheres which are homologous; we label their homology class as $[S^2]$ in the following. We may also consider the two two-spheres, $[S_i^2]$ inside S^4 at a pole of the S^2. These two two-cycles are trivial in homology unlike the class $[S^2]$. There are two copies of the S^4 located at either pole of the S^2. These are homologous and we will denote this class as $[S^4]$. One can also construct the trivial four-cycles from the direct product of the S^2 and one of the trivial $[S_i^2]$ cycles. This gives all the cycles in the geometry.

In order to collapse the blown-up cycle at the end of the computation we must require that the flux through the cycle $[S^2]$ vanishes. It would be interesting to keep this flux, which can be interpreted as turning on a magnetic baryonic flux; however, this makes finding the critical points a lot more challenging, and so for simplicity we set it to zero.

To proceed we first impose the homology relations, starting with the two-cycles. It is useful to first define

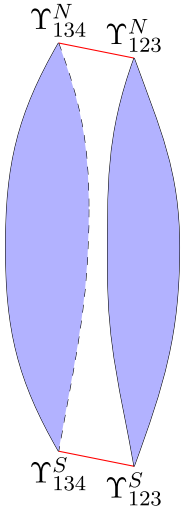


FIG. 2. The "toric" diagram for the same small resolution at both poles. The blue faces are S^4's (not toric) whilst the two red lines are the resolution $\mathbb{CP}^1$'s which are homologous. The black lines denote $S_i^2 \subset S^4$ at the poles of the resolution $\mathbb{CP}^1$, and the white faces are $\mathbb{CP}^1 \times S_i^2$.

$$D_{123} \equiv (b_0 - b_2)(b_2 - b_1)b_1,$$
$$D_{134} \equiv (b_0 - b_1)(b_1 - b_2)b_2, \tag{D3}$$

which are the products of the weights at the fixed points. In terms of these weights the usual $T^{1,1}$ volume is given by

$$\mathrm{vol}(T^{1,1}) = \frac{1}{D_{123}} + \frac{1}{D_{134}} = \frac{b_1}{b_2 b_3 (b_1 - b_2)(b_1 - b_3)}. \tag{D4}$$

Let us also define

$$Z \equiv -\Psi_0^{F_2} = \Phi_0 + F_0 \Upsilon. \tag{D5}$$

Integrating the two-form flux over the trivial two-cycles imposes

$$Z^N_{123} = Z^S_{123}, \qquad Z^N_{134} = Z^S_{134}, \tag{D6}$$

whilst if we set the quantized flux through the resolution $\mathbb{CP}^1$ to be B we have

$$B \equiv \frac{1}{2\pi \ell_s} \int_{\mathbb{CP}^1} F_2 \Leftrightarrow Z^N_{123} - Z^N_{134}$$
$$= \ell_s (b_2 - b_3) B = Z^S_{123} - Z^S_{134}. \tag{D7}$$

In the following we will set $B = 0$ in order to blow down at the very end, which implies that in terms of the Z variables the two-cycle homology relations imply that Z is equal at all the fixed points.

To perform a similar analysis for the four-cycles we first define:

$$\Upsilon^{\pm}_{123} \equiv \frac{1}{\pi \ell_s^2} (\Upsilon^N_{123} \pm \Upsilon^S_{123}), \qquad \Upsilon^{\pm}_{134} \equiv \frac{1}{\pi \ell_s^2} (\Upsilon^N_{134} \pm \Upsilon^S_{134}), \tag{D8}$$

and we define the one independent value of Z to be Z_0, that is,

$$Z_0 = Z^N_{123} = Z^S_{123} = Z^N_{134} = Z^S_{134}. \tag{D9}$$

Then the four-cycle homology relations through the cycles $[S^2 \times S^2_i]$ gives[21]

$$\Upsilon^+_{123} = \frac{4Z_0}{n_0} = \Upsilon^+_{134}, \tag{D10}$$

which also implies that the flux through the S^4 necessarily vanishes.[22] Finally, quantizing the F_6 flux through the internal space we find

$$96(b_2 - b_3)N = n_0[(\Upsilon^-_{134})^3 - (\Upsilon^-_{123})^3], \tag{D11}$$

which we may solve for Υ^-_{123}:

$$\Upsilon^-_{123} = e^{2\pi i n_{123}/3}\left[\frac{96(b_2 - b_3)N + n_0(\Upsilon^+_{134})^3}{n_0}\right]^{1/3}, \tag{D12}$$

where we have included the third root of unity with $n_{123} = \{0, 1, 2\}$.

Up to this point, the analysis above applies to both an isolated fixed point or a bolt. We will first consider the former, dealing with an isolated fixed point before looking at the bolt computation.

Isolated fixed point. The partially on shell action now depends only on the free parameters Z_0, Υ^-_{134} and $\{b_1, b_2, b_3\}$. It is simple to see that the extremization over Z_0 imposes $Z_0 = 0$ whilst the extremization over Υ^-_{134} imposes

$$\Upsilon^-_{134} = e^{\pi i n_{134}/3}\Upsilon^-_{123}. \tag{D13}$$

That is, they are equal up to a sixth root of unity. In order for us to blow down we must take them to be the same. The final result is that the partially on shell action for an isolated fixed point is

$$I = \frac{9}{4}F^{T^{1,1}}_{S^3}\frac{\mathcal{F}}{\epsilon_1\epsilon_2}. \tag{D14}$$

To write the final result we have defined the prepotential

$$\mathcal{F}(b_i) = \left(\frac{\mathrm{vol}_{T^{1,1}}}{\mathrm{vol}_{T^{1,1}}(\vec{b})}\right)^{2/3}, \tag{D15}$$

where $\mathrm{vol}_{T^{1,1}}(b_i)$ is the off shell Sasakian volume and $\mathrm{vol}_{T^{1,1}} = \frac{16\pi^3}{27}$ is the on shell value.

Bolt contribution. Consider now a bolt $\Sigma \subset M_4$. In X_6 we twist each of the three U(1)'s generated by ∂_{φ_i} by a complex line bundle $\mathcal{L}_i$ on Σ and also turn on a normal line bundle L inside M_4 so that the total normal bundle is $L \oplus_{i=1}^3 \mathcal{L}_i \to \Sigma$. Over the four copies of Σ at the fixed points of the internal space we take

$$\int_{\Sigma^N_\pm} c_1(L) = \int_{\Sigma^S_\pm} c_1(L) \equiv m,$$
$$\int_{\Sigma^N_\pm} c_1(\mathcal{L}_i) = -\int_{\Sigma^S_\pm} c_1(\mathcal{L}_i) = p_i, \tag{D16}$$

as in (4.17). The total space is then

$$M_{10} = \mathcal{O}(\{m, \vec{p}\})_{\Sigma_g} \times_{U(1)^4} \mathbb{C} \times X_6, \tag{D17}$$

i.e., we use the $\mathrm{U}(1)^4$ transition functions of $\mathcal{O}(\{m, \vec{p}\})_{\Sigma_g}$ to fiber $\mathbb{C} \times X_6$ over Σ_g.

Since we have a bolt we now have additional two-, four- and six-cycles over which we can quantize the flux. We may turn on a two-form potential through the bolt and as in Sec. IV B we take

$$\frac{1}{2\pi\ell_s}\int_{\Sigma^N_\pm} F_2 = \frac{1}{2\pi\ell_s}\int_{\Sigma^S_\pm} F_2 \equiv M, \tag{D18}$$

where we have used that all two-cycles are homologous $[\Sigma^N_\pm] = [\Sigma^S_\pm]$. Consider now a four-cycle constructed by fibering a (trivial) two-cycle over the bolt. We will need to compute the first Chern class of the normal line bundle to each facet. To do this we need to compute the dual basis for each fixed point on the resolved $T^{1,1}$. For the cones $\sigma^\pm_1 = \{v_1, v_2, v_3\}$ the dual bases are

$$u^+_1 = (1,0,-1), \quad u^+_2 = (0,0,1), \quad u^+_3 = (0,1,1), \tag{D19}$$

$$u^-_1 = (1,-1,0), \quad u^-_3 = (0,0,1), \quad u^-_4 = (0,1,-1). \tag{D20}$$

Therefore on the cone $\tau^\pm$ the normal bundle to the facet defined by the vector v_a is

$$c_1(L_a) = \sum_{i=1}^3 (u^\pm_a)_i c_1(\mathcal{L}_i), \tag{D21}$$

where the result depends on which fixed point one is looking at. Using this we can compute the integral of the four-form flux through the trivial four-cycles, and we find the two constraints:

$$\int_{\Sigma^S_+} \Phi_2 = \int_{\Sigma^N_+} \Phi_2 + 2\pi\ell_s^3 M\Upsilon^-_{123},$$
$$\int_{\Sigma^S_-} \Phi_2 = \int_{\Sigma^N_-} \Phi_2 + 2\pi\ell_s^3 M\Upsilon^-_{134}, \tag{D22}$$

where Υ^-_{123} and Υ^-_{134} are defined in (D8).

[21] Keeping the two-form flux through the resolution $\mathbb{CP}^1$ nontrivial would change the relations here.

[22] This is also true when we do not set $B = 0$ and holds in general.

Finally, we may consider the six-cycles $[\hat{D}_a]$ consisting of the four-cycles $[D_a] \subset X_6$ fibered over the Riemann surface. In the blown-down solution the four-cycles are constructed by fibering the three-cycle divisors of $T^{1,1}$ over an interval. The six-cycles obtained by fibering these four-cycles over the Riemann surface are not independent and instead satisfy[23]

$$\sum_{a=1}^{4} v_a^i [\hat{D}_a] = p_i [X_6]. \tag{D23}$$

Imposing the homology relation allows us to fix

$$\int_{\Sigma_-^N} \Phi_2 = \frac{(b_1-b_2)b_3}{b_2(b_1-b_3)} \int_{\Sigma_+^N} \Phi_2 + \frac{(\Upsilon_{134}^-)^2 \ell_s^3 n_0 \pi^2 \mathrm{vol}(\vec{b})}{12} \left[\frac{(b_2-b_3)b_3(b_1^2-b_2b_3)}{b_1(b_1-b_3)} p_1 + \frac{(b_1-2b_2)(b_2-b_3)b_3}{b_2} p_2 + \frac{(b_1-b_2)(b_1-2b_3)(b_2-b_3)}{b_1-b_3} p_3 \right], \tag{D24}$$

where we have presented the $M=0$ case for simplicity only. Upon inserting this together with the other constraints into the action, it remains to extremize over Z_0 which fixes

$$Z_0 = -\frac{M\varepsilon}{m}, \tag{D25}$$

and the partially on shell action takes the universal form

$$I = \frac{9}{4} F_{S^3}^{T^{1,1}} \left(\sum_{i=1}^{3} p_i \partial_{b_i} \mathcal{F} - \frac{m}{\varepsilon} \mathcal{F} \right) + \frac{\mathrm{i}\pi N M^2}{m n_0}, \tag{D26}$$

where for $m=0$ we must set $M=0$. The prepotential is

$$\mathcal{F} = \left(\frac{\mathrm{vol}_{T^{1,1}}}{\mathrm{vol}_{T^{1,1}}(\vec{b})} \right)^{2/3}, \tag{D27}$$

where $\mathrm{vol}_{T^{1,1}}(\vec{b})$ is the Sasakian volume.

2. General toric Sasaki-Einstein manifolds

The localization computation presented above for the suspension of $T^{1,1}$ extends naturally to an arbitrary toric Sasaki-Einstein five-manifold. Let Y_5 be a toric Sasaki-Einstein manifold with Calabi-Yau cone $C(Y_5)$. The cone is specified by a convex lattice polygon, or equivalently by primitive generators

$$v_a = (1, w_a) \in \mathbb{Z}^3, \qquad a = 1, \ldots, d, \tag{D28}$$

where d is the number of facets of the toric diagram.

In general the cone is singular at the tip, as in the $T^{1,1}$ example above. To perform our equivariant localization we choose a simplicial crepant subdivision of the fan, or equivalently a triangulation of the toric diagram using its lattice points. The resulting toric variety is a crepant partial resolution of the cone. We do not require the triangulation to be unimodular, and hence allow orbifold singularities. Different choices of triangulation are related by toric flops.

Each maximal cone

$$\tau_{a_1 a_2 a_3} = \langle v_{a_1}, v_{a_2}, v_{a_3} \rangle \tag{D29}$$

corresponds to an isolated fixed point of the torus action. One can now follow a similar procedure as above. Each maximal cone contributes one fixed point and if the subdivision is not smooth the local geometry at each fixed point is $\mathbb{C}^3/\Gamma$ where $|\Gamma| = |\det(v_{a_1}, v_{a_2}, v_{a_3})|$ and one introduces the corresponding orbifold factor in the BV-AB formula. One sets the Page fluxes through the exceptional divisors introduced by the subdivision to zero and at the very end blows the exceptional divisors back down. Since the subdivision is crepant, the canonical bundle is preserved, and the resulting localization formula depends only on the original Sasaki-Einstein geometry and not on the particular choice of simplicial crepant subdivision.

The end result of this procedure is that the partially on shell action satisfies (4.3) where the prepotential is

$$\mathcal{F} = \left(\frac{\mathrm{vol}_{\mathrm{SE}_5}}{\mathrm{vol}_{\mathrm{SE}_5}(\vec{b})} \right)^{2/3}. \tag{D30}$$

Here the numerator is the on-shell volume and the denominator is the Sasakian volume, with the latter depending on the three R-symmetry parameters b_i.

[23] This can be obtained by performing a similar analysis as in Appendix C. Alternatively, one can use the results in Appendix B of [70] for Sasaki-Einstein fibers and introduce the line interval for the suspension.

[1] J. P. Gauntlett, D. Martelli, S. Pakis, and D. Waldram, G structures and wrapped NS5-branes, Commun. Math. Phys. **247**, 421 (2004).

[2] J. P. Gauntlett, D. Martelli, and D. Waldram, Superstrings with intrinsic torsion, Phys. Rev. D **69**, 086002 (2004).

[3] J. P. Gauntlett and S. Pakis, The geometry of D = 11 Killing spinors, J. High Energy Phys. 04 (2003) 039.

[4] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, Supersymmetric backgrounds from generalized Calabi-Yau manifolds, J. High Energy Phys. 08 (2004) 046.

[5] A. Tomasiello, Generalized structures of ten-dimensional supersymmetric solutions, J. High Energy Phys. 03 (2012) 073.

[6] O. Aharony, F. Benini, O. Mamroud, and E. Milan, A gravity interpretation for the Bethe Ansatz expansion of the $\mathcal{N} = 4$ SYM index, Phys. Rev. D **104**, 086026 (2021).

[7] Y. Chen, M. Heydeman, Y. Wang, and M. Zhang, Probing supersymmetric black holes with surface defects, J. High Energy Phys. 10 (2023) 136.

[8] A. Cabo-Bizet, M. David, and A. González Lezcano, Thermodynamics of black holes with probe D-branes, J. High Energy Phys. 06 (2024) 193.

[9] P. Benetti Genolini, C. Couzens, and A. Lüscher, Probing black holes with equivariant localization, Phys. Rev. D **114**, 066004 (2026).

[10] M. Suh, A gravity interpretation for the complex Euclidean saddles of the ABJM index, arXiv:2605.00987.

[11] G. B. De Luca, A. Gnecchi, G. Lo Monaco, and A. Tomasiello, Holographic duals of 6d RG flows, J. High Energy Phys. 03 (2019) 035.

[12] N. Bobev, D. Cassani, and H. Triendl, Holographic RG flows for four-dimensional $\mathcal{N} = 2$ SCFTs, J. High Energy Phys. 06 (2018) 086.

[13] P. Benetti Genolini, J. P. Gauntlett, and J. Sparks, Equivariant localization in supergravity, Phys. Rev. Lett. **131**, 121602 (2023).

[14] P. Benetti Genolini, J. P. Gauntlett, and J. Sparks, Localizing wrapped M5-branes and gravitational blocks, Phys. Rev. D **108**, L101903 (2023).

[15] P. Benetti Genolini, J. P. Gauntlett, and J. Sparks, Equivariant localization for AdS/CFT, J. High Energy Phys. 02 (2024) 015.

[16] P. Benetti Genolini, J. P. Gauntlett, Y. Jiao, A. Lüscher, and J. Sparks, Localization and attraction, J. High Energy Phys. 05 (2024) 152.

[17] P. Benetti Genolini, J. P. Gauntlett, Y. Jiao, A. Lüscher, and J. Sparks, Localization of the free energy in supergravity, Phys. Rev. Lett. **133**, 141601 (2024).

[18] D. Cassani, A. Ruipérez, and E. Turetta, Localization of the 5D supergravity action and Euclidean saddles for the black hole index, J. High Energy Phys. 12 (2024) 086.

[19] E. Colombo, F. Faedo, D. Martelli, and A. Zaffaroni, Equivariant volume extremization and holography, J. High Energy Phys. 01 (2024) 095.

[20] C. Couzens and A. Lüscher, A geometric dual of F-maximization in massive type IIA, J. High Energy Phys. 08 (2024) 218.

[21] D. Martelli and A. Zaffaroni, Equivariant localization and holography, Lett. Math. Phys. **114**, 15 (2024).

[22] M. Suh, Equivariant localization for wrapped M5-branes and D4-branes, J. High Energy Phys. 09 (2026) 098.

[23] K. Hristov, Equivariant localization and gluing rules in 4d $N = 2$ higher derivative supergravity, in *2024 MATRIX Annals, Part I* (Springer, Cham, 2024).

[24] P. Benetti Genolini, J. P. Gauntlett, Y. Jiao, A. Lüscher, and J. Sparks, Toric gravitational instantons in gauged supergravity, Phys. Rev. D **111**, 046024 (2025).

[25] P. Benetti Genolini, J. P. Gauntlett, Y. Jiao, A. Lüscher, and J. Sparks, Equivariant localization for D = 4 gauged supergravity, J. High Energy Phys. 08 (2025) 211.

[26] C. Couzens, C. M. Gregory, D. Muniz, T. Sieper, and J. Sparks, Localizing Romans supergravity, J. High Energy Phys. 09 (2025) 103.

[27] E. Colombo, V. Dimitrov, D. Martelli, and A. Zaffaroni, Equivariant localization in supergravity in odd dimensions, arXiv:2502.15624.

[28] L. Cassia and K. Hristov, M2-brane partition functions and HD supergravity from equivariant volumes, J. High Energy Phys. 03 (2026) 100.

[29] P. Benetti Genolini, J. P. Gauntlett, Y. Jiao, J. Park, and J. Sparks, Equivariant localization for D = 5 gauged supergravity, J. High Energy Phys. 03 (2026) 080.

[30] C. Couzens, A. Lüscher, and J. Sparks, Localizing punctures in M-theory, J. High Energy Phys. 02 (2026) 180.

[31] E. Colombo, V. Dimitrov, D. Martelli, and A. Zaffaroni, Patch-wise localization with Chern-Simons forms in five dimensional supergravity, arXiv:2511.13824.

[32] J. Park, Localizing AlAdS_5 black holes and the SUSY index on $\mathrm{S}^1 \times \mathrm{M}_3$, J. High Energy Phys. 06 (2026) 107.

[33] K. Hristov, S. Khandelwal, Y. Pang, and G. Tartaglino-Mazzucchelli, Holographic origin of a maximization and higher-derivative $\mathrm{AdS}_5/\mathrm{CFT}_4$, Phys. Rev. Lett. **137**, 081602 (2026).

[34] K. Hristov, Indices of M5 and M2 branes at finite N from equivariant volumes, and a new duality, arXiv:2604.06343.

[35] P. Benetti Genolini, F. Gaar, J. P. Gauntlett, and J. Sparks, Equivariant localization for higher derivative supergravity, Phys. Rev. Lett. **137**, 111602 (2026).

[36] F. Gaar, J. P. Gauntlett, J. Park, and J. Sparks, Superconformal index and localizing higher derivative supergravity, Phys. Rev. D **114**, L021904 (2026).

[37] M. Galli, C. Kennedy, P. Raina, and G. Tartaglino-Mazzucchelli, Superform approach to equivariant localization in supergravity, arXiv:2605.17919.

[38] D. Cassani and E. Turetta, The black hole at the end of the cone: Localizing the anomaly polynomial on toric geometries, arXiv:2606.16986.

[39] A. Legramandi, L. Martucci, and A. Tomasiello, Timelike structures of ten-dimensional supersymmetry, J. High Energy Phys. 04 (2019) 109.

[40] L. Martucci, Electrified branes, J. High Energy Phys. 02 (2012) 097.

[41] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, and A. Van Proeyen, New formulations of $D = 10$ supersymmetry and D8-O8 domain walls, Classical Quantum Gravity **18**, 3359 (2001).

[42] S. A. Kurlyand and A. A. Tseytlin, Type IIB supergravity action on M5 × X5 solutions, Phys. Rev. D **106**, 086017 (2022).

[43] S. Adhikari, J. Hong, C. Joung, and G. Lee, Type IIB supergravity action and holography, J. High Energy Phys. 06 (2026) 271.
[44] P. Pasti, D. P. Sorokin, and M. Tonin, On Lorentz invariant actions for chiral p forms, Phys. Rev. D **55**, 6292 (1997).
[45] K. Mkrtchyan and F. Valach, Democratic actions for type II supergravities, Phys. Rev. D **107**, 066027 (2023).
[46] S. W. Hawking and S. F. Ross, Duality between electric and magnetic black holes, Phys. Rev. D **52**, 5865 (1995).
[47] E. A. Bergshoeff, J. Hartong, A. Ploegh, J. Rosseel, and D. Van den Bleeken, Pseudo-supersymmetry and a tale of alternate realities, J. High Energy Phys. 07 (2007) 067.
[48] E. D'Hoker, M. Gutperle, and C. F. Uhlemann, Exact solutions to complex type IIB supergravity for complex superalgebra F(4) and its real forms, J. High Energy Phys. 07 (2026) 148.
[49] A. Passias, A note on supersymmetric AdS_6 solutions of massive type IIA supergravity, J. High Energy Phys. 01 (2013) 113.
[50] A. Brandhuber and Y. Oz, The D-4 - D-8 brane system and five-dimensional fixed points, Phys. Lett. B **460**, 307 (1999).
[51] S. M. Hosseini, I. Yaakov, and A. Zaffaroni, The joy of factorization at large N: Five-dimensional indices and AdS black holes, J. High Energy Phys. 02 (2022) 097.
[52] M. Honda and K. Okuyama, Exact results on ABJ theory and the refined topological string, J. High Energy Phys. 08 (2014) 148.
[53] S. M. Hosseini, I. Yaakov, and A. Zaffaroni, Topologically twisted indices in five dimensions and holography, J. High Energy Phys. 11 (2018) 119.
[54] M. Suh, Supersymmetric AdS_6 black holes from F(4) gauged supergravity, J. High Energy Phys. 01 (2019) 035.
[55] S. M. Hosseini, K. Hristov, A. Passias, and A. Zaffaroni, 6D attractors and black hole microstates, J. High Energy Phys. 12 (2018) 001.
[56] M. Suh, Supersymmetric AdS_6 black holes from matter coupled $F(4)$ gauged supergravity, J. High Energy Phys. 02 (2019) 108.
[57] M. Fluder and J. Sparks, D2-brane Chern-Simons theories: F-maximization = a-maximization, J. High Energy Phys. 01 (2016) 048.
[58] A. Tomasiello and A. Zaffaroni, Parameter spaces of massive IIA solutions, J. High Energy Phys. 04 (2011) 067.
[59] S. M. Hosseini and A. Zaffaroni, The large N limit of topologically twisted indices: A direct approach, J. High Energy Phys. 12 (2022) 025.
[60] C. Couzens and K. Stemerdink, Universal spindles: D2's on Σ and M5's on $\Sigma \times \mathbb{H}^3$, arXiv:2207.06449.
[61] J. Hong and J. T. Liu, Subleading corrections to the S^3 free energy of necklace quiver theories dual to massive IIA, J. High Energy Phys. 11 (2021) 183.
[62] F. Benini, H. Khachatryan, and E. Milan, Black hole entropy in massive type IIA, Classical Quantum Gravity **35**, 035004 (2018).
[63] F. Azzurli, N. Bobev, P. M. Crichigno, V. S. Min, and A. Zaffaroni, A universal counting of black hole microstates in AdS_4, J. High Energy Phys. 02 (2018) 054.
[64] S. M. Hosseini, K. Hristov, and A. Passias, Holographic microstate counting for AdS_4 black holes in massive IIA supergravity, J. High Energy Phys. 10 (2017) 190.
[65] C. Couzens, A. Lüscher, and J. Sparks, IIB an equivariantly localized puncture, SciPost Phys. **21**, 069 (2026).
[66] S. Hu and B. Uribe, Extended manifolds and extended equivariant cohomology, arXiv:math/0608319.
[67] S. Choi and C. Hwang, Universal 3D Cardy block and black hole entropy, J. High Energy Phys. 03 (2020) 068.
[68] A. Legramandi, N. T. Macpherson, and A. Passias, G-structures for black hole near-horizon geometries, J. High Energy Phys. 06 (2024) 056.
[69] D. Martelli, J. Sparks, and S.-T. Yau, Sasaki-Einstein manifolds and volume minimisation, Commun. Math. Phys. **280**, 611 (2008).
[70] J. P. Gauntlett, D. Martelli, and J. Sparks, Toric geometry and the dual of $\mathcal{I}$-extremization, J. High Energy Phys. 06 (2019) 140.