

Scalar quasinormal modes of gravastars

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In this study, we present a detailed investigation of gravastars within the framework of scalar-tensor theories, emphasizing both the background and perturbed levels for trivial and nontrivial scalar field. We derive the background equations that govern the equilibrium configurations of gravastars by examining the interaction between matter and the scalar field through a conformal transformation. By selecting two forms of the coupling function, we identify distinct physical characteristics that lead to deviations from the predictions made by general relativity. Our findings indicate that the scalar field significantly influences the compactness parameter. By extending our study to the perturbed level, we demonstrate that quasinormal modes are considerably affected by the presence of the scalar field. We discover new and intriguing quasinormal modes that are absent in general relativity.

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I. INTRODUCTION

Although black holes (BHs), which are considered as a solution to Einstein's equations in the vacuum, are perfectly compatible with gravitational waves [1] and widely accepted by astrophysical researchers, their nature and their physical existence remain among the most questions in theoretical astrophysics. Skepticism regarding their physical reality arises from the exotic solutions of Einstein's equations. Specifically, the presence of a central singularity and an event horizon at the Schwarzschild radius makes the physical properties of classical black hole unknown. The recent observation by the Event Horizon Telescope (ETH) of the supermassive black hole (M87) [2] confirms the existence of such objects and opens new windows for probing the interiors of BHs. It is plausible to expect that horizonless compact objects could serve as alternatives to BHs which have been extensively explored in the literature, for example the Proca stars [3], boson stars [4], and gravastars (gravitational vacuum stars) [5–7].

In the framework of general relativity (GR), gravastars were proposed as a solution to the paradoxes associated with event horizons, where the interior of a black hole is theorized to be de Sitter space. The simplest theoretical construction of gravastars is de Sitter spacetime core, characterized by an equation of state $p = -\rho$, connected

to a Schwarzschild spacetime exterior through a thin shell of perfect-fluid matter. Numerous papers have contributed to the understanding of gravastars within the broader context of exotic compact objects since their introduction by Mazur and Mottola [5]. Instability against radial perturbations was investigated in Ref. [7], where the authors identified a large parameter range that yields stable gravastars. In Ref. [8], the authors computed the axial oscillation modes of gravastars and found not only the absence of instabilities but also concluded that it is possible to distinguish between a gravastar and a black hole through the quasinormal modes (QNMs). Pani *et al.* [9,10] reached similar conclusions by analyzing the polar and axial perturbation equations, taking into account the junction conditions that connect the internal and external regions of gravastars. Their work highlighted the important role of compactness and surface properties, which have significant astrophysical implications for the behavior of these exotic objects.

In this paper, we aim to investigate gravastars in the framework of tensor-scalar field theory due to their potential to probe modified gravitational effects near the surface. These theories are extensions to general relativity where a scalar field is introduced as an extra degree of freedom, which allows us to explore gravitational fields in strong regimes and provides behaviors different from those of GR [11–13]. The scalar field seems to have important astronomical implications in the acceleration of the late Universe [14–16] as well as modifying the features of compact objects such as: black holes [17–19] and neutron stars [20–26]. Moreover, in some subclasses of these theories, an interesting phenomenon called *spontaneous scalarization* [27,28] occurs, in which the scalar field arises

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from a linear tachyonic instability around compact objects [29,30]. The scalarization phenomenon with different models has attracted a lot of attention lately [31–37] for two features. The first is the GR solution that we observe for a trivial scalar field and for a weak gravitational regime. The second is the phase transition from a trivial scalar field to a nontrivial one when the mass (for BHs) or compactness (for neutron stars) exceeds a certain interval. Consequently, depending on the coupling function of the scalar field with the metric, a deviation of the solutions in scalar tensor theories from GR will be noticed during the transition.

Motivated by the scalarization phenomenon, we restrict our attention to the physical characteristics of gravastars with a constant energy density, coupled to the matter field through a conformal transformation [38]. Quasinormal modes, which are intrinsic oscillations characterized by complex frequencies that encode both oscillation and damping, play a pivotal role in gravitational wave spectroscopy, allowing us to extract the gravastars' properties from observed waveforms. Studying QNMs is crucial for understanding the stability and resonant behavior of the scalarization phenomenon, as well as exotic compact objects [39–42]. Furthermore, these modes can be useful to examine the interaction between the scalar field and the energy inside the star and to assess the feasibility of such objects in a universe dominated by dark energy.

The structure of this paper is as follows: In Sec. II, we provide a brief review of the matter distribution of gravastars and the underlying structure of spacetime. Section III introduces the background and perturbation equations that describe our star within the framework of scalar-tensor theory. In Sec. IV, we derive the analytic solution for the perturbed scalar field equation in the interior of the star, assuming a trivial scalar field. Additionally, we compute the QNMs using the continued fraction method. Section V presents a detailed numerical analysis of our findings. Finally, the paper concludes with a summary and discussion.

II. BRIEF REVIEW OF GRAVASTARS IN GENERAL RELATIVITY

We begin by presenting the main layers that define a gravastar. In Ref. [5], the authors divide the star into three regions, each characterized by a distinct equation of state that governs the distribution of matter in spacetime:

- (1) Interior, $r < a$, with $p = -\rho$.
- (2) Thin shell, with a surface tension Θ and a vanishing surface density.
- (3) Exterior, $r > a$, with $p = \rho = 0$.

where p and ρ are the pressure and the energy density of the fluid and a is the radius of the star. In a static spherically symmetric metric,

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.1)$$

the equation of gravitational field for a gravastar are given by

$$\frac{f'(r)}{f(r)} = -8\pi r h(r) \rho + \frac{h(r) - 1}{r}, \quad (2.2)$$

$$\frac{h'(r)}{h(r)} = 8\pi r h(r) \rho + \frac{1 - h(r)}{r}. \quad (2.3)$$

By replacing the equation of state of the interior fluid in the matter conservation equation,

$$p' = -\frac{f'(r)}{2f(r)}(p + \rho), \quad (2.4)$$

give us a constant pressure and, consequently, a constant energy density. Integrating three last equations and imposing the regularity at the center, the solutions reads

$$f(r) = 1/h(r) = \begin{cases} 1 - \frac{8\pi\rho}{3}r^2, & r < a \\ 1 - \frac{r_s}{r}, & r > a \end{cases} \quad (2.5)$$

where the Schwarzschild radius, r_s , must satisfy the condition $r_s < a$ to prevent the existence of an event horizon. When we take the limit $r_s \rightarrow a$ we get the model discussed in Ref. [5]. By imposing the continuity at the surface of the object, we find

$$\rho = \frac{3M}{4\pi a^3}, \quad (2.6)$$

where M is the total mass. However, the first derivative is discontinuous at the surface which is due to the junction conditions [43]

$$[[\nabla_i n_j]] = 8\pi[[\Theta u_i u_j]], \quad \text{with } \{i, j\} = \{t, \theta, \phi\}, \quad (2.7)$$

where $n_\alpha = (0, 1/\sqrt{h(r)}, 0, 0)$ and $u_\alpha = (1/\sqrt{f(r)}, 0, 0, 0)$ are the unite normal vector and the four-velocity vector. The notation $[[A]]$ is the discontinuity of A across the thin shell. Inserting Eq. (2.1) into Eq. (2.7), we get

$$8\pi\Theta a = -\frac{3r_s/a}{2\sqrt{1 - r_s/a}}. \quad (2.8)$$

As we can see a gravastar can be characterized only by its compactness $C = r_s/a$. Using the transformation

$$r \rightarrow a\hat{r}, \quad (2.9)$$

the background solution becomes

$$f(\hat{r}) = 1/h(\hat{r}) = \begin{cases} 1 - C\hat{r}^2, & \hat{r} < 1, \\ 1 - \frac{C}{\hat{r}}, & \hat{r} > 1. \end{cases} \quad (2.10)$$

Therefore, the radius of the star is not important in our analysis since the equations and their solutions are independent of a . As we will see in the next section, this redefinition is going to simplify our work even when extend our model by introducing an extra degree of freedom scalar field.

III. EQUATIONS OF MOTION OF SCALARIZED GRAVASTAR

In this section, we present the theoretical framework underpinning this study. We will introduce the main principles and assumptions that define the model, detailing the specific variables, parameters, and interactions. Following this, we elaborate on the mathematical formulation of the model, presenting the equations that govern its dynamics and behavior. Starting with the background equations that describe the fundamental, unperturbed state, we then introduce the perturbed equations, which account for deviations from the background state, allowing us to explore linear impact around the background.

A. The model

Now, we propose to study the gravastar presented in the previous section in the presence of a scalar field φ . We consider a class of massless STTs written in the Einstein frame, i.e., that the scalar field is coupled minimally to gravity and has a canonical kinetic term [44]

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi} R - \partial_\mu \varphi \partial^\mu \varphi \right) + S_m(\tilde{g}_{\mu\nu}, \Psi_m) \quad (3.1)$$

where R is the Ricci scalar and the tilde notation is used to express any quantity in Jordan frame. The first part of the action, which describes the spacetime dynamics, is written in the Einstein frame whereas the second part describes the matter field, the metric is in Jordan frame. Alternatively, the above action can be rewritten in terms of the metric $\tilde{g}_{\mu\nu}$, in which case the Einstein-Hilbert term becomes coupled nonminimally to φ , but there is no explicit coupling to the matter sector. S_m is the action of the matter fields Ψ_m . Various Lagrangian formulations relativistic perfect fluids have been proposed. In our paper, we adopt the action Schutz action [45,46]

$$S_m = \int d^4x \sqrt{-\tilde{g}} \tilde{p}, \quad (3.2)$$

where $\tilde{p}$ is the pressure of the matter inside the star in Jordan frame. The variation of S_m with respect to $\tilde{g}_{\mu\nu}$ yields the energy-momentum tensor of matter fields $\tilde{T}^{\mu\nu} = (\tilde{\rho} + \tilde{p})\tilde{u}^\mu\tilde{u}^\nu + \tilde{p}\tilde{g}^{\mu\nu}$, which is covariantly conserved ($\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0$).

Equation (3.2) is said to be written in the Einstein frame, where the scalar field is coupled minimally to gravity via a

conformal transformation and has a canonical kinetic term. In addition, the action (3.2) is related to the action

$$S = \int d^4x \sqrt{-\tilde{g}} (F(\Phi)\tilde{R} + Z(\Phi)\partial_\mu \Phi \partial^\mu \Phi) + S_m(\tilde{g}, \Psi_m). \quad (3.3)$$

The two metrics are linked to each other through the conformal transformation

$$\tilde{g}_{\mu\nu} = A(\varphi)g_{\mu\nu}, \quad (3.4)$$

where $A(\varphi)$ is an arbitrary function of φ . In our model $F(\Phi) = A(\varphi)$ and the scalar fields Φ and φ are related via the equation

$$\left(\frac{d\varphi}{d\Phi} \right)^2 = \frac{3}{4} \left(\frac{d \ln F(\Phi)}{d\Phi} \right)^2 + \frac{Z(\Phi)}{2F(\Phi)}. \quad (3.5)$$

Indeed, the matter interacts with the scalar field via this function, which is expected to alter the metric's behavior and influence the star's characteristics. Without loss of generality, by considering that the value of the scalar field at the present time is zero, one can make a Taylor development of $\alpha(\varphi) = d \log(A(\varphi))/d\varphi$ around 0 as

$$\alpha(\varphi) = \alpha_0 + \beta_0 \varphi^2 + O[\varphi^4], \quad (3.6)$$

where

$$\beta(\varphi) = \frac{d\alpha(\varphi)}{d\varphi}. \quad (3.7)$$

In order to satisfy the constraint made by solar-system experiment $\alpha_0 < 10^{-3}$ [47], we set $\alpha_0 = 0$ [27,48,49]. The constant β_0 is crucial to have spontaneous scalarization [50], where if it is sufficiently negative, we expect an evolution of scalar field which will change the metric solution of GR to an other solution with nontrivial scalar field. In fact, a neutron star becomes unstable when we increase the compactness and it undergoes a phase transition to a scalarized configuration [50,51]. An other characteristic, we should impose is the reflection symmetry $\varphi \rightarrow -\varphi$ to our model, thus it is convenient to choose the following functions

$$M_1: A(\varphi) = e^{\beta\varphi^2}, \quad (3.8)$$

$$M_2: A(\varphi) = (\cosh(\sqrt{6}\beta\varphi))^{\frac{1}{3\beta}}. \quad (3.9)$$

Thus, we determine that $\beta = \beta_0$ and $\alpha = \alpha_0$ in those models. Both models are identical under first- and second-order perturbations, but they differ at higher orders. The first model is the most commonly used in the literature because it is the simplest function that gives the

spontaneous scalarization effect [12,52]. The model M_2 is a good approximation of the action (3.2) equivalent to the action with a massless scalar field Φ nonminimally coupled to gravity

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi} (1 - 8\pi\xi\Phi^2) \tilde{R} - \partial_\mu \Phi \partial^\mu \Phi \right) + \int d^4x \sqrt{-\tilde{g}} \tilde{\rho}. \quad (3.10)$$

where ξ is a real constant called the conformal coupling parameter and

$$\frac{d\varphi}{d\Phi} = 4\pi \frac{\sqrt{1 - 8\pi\xi(1 - 6\xi)\Phi^2}}{1 - 8\pi\xi\Phi^2}, \quad (3.11)$$

$$A(\varphi) = \frac{1}{\sqrt{1 - 8\pi\xi\Phi^2}}. \quad (3.12)$$

At small regime $\varphi \rightarrow 0$ and $\Phi \rightarrow 0$, one can identify that $\xi = \beta/2$ by integrating the above equation. Note that the theory described by (3.10) provides an exponential growth of the scalar field in the presence of gravitational fields produced by relativistic matter through a quantum mechanism [53].

B. The equations of motion

To derive the background field equations, we insert the metric (2.5) and the scalar field $\varphi \equiv \varphi(r)$ in the action (3.2), and then we vary the resulting action with respect to f , h , and φ . Doing so, we get

$$\delta g_{\mu\nu} = \begin{pmatrix} f(r)H_0^{lm} & \sqrt{f(r)}\sqrt{h(r)}H_1^{lm} & 0 & 0 \\ \sqrt{f(r)}\sqrt{h(r)}H_1^{lm} & h(r)H_2^{lm} & r\sqrt{h}H_5^{lm}\partial_\theta Y_{lm} & r\sin\theta\sqrt{h}H_5^{lm}\partial_\phi Y_{lm} \\ 0 & r\sqrt{h}H_5^{lm}\partial_\theta Y_{lm} & 0 & 0 \\ 0 & r\sin\theta\sqrt{h}H_5^{lm}\partial_\phi Y_{lm} & 0 & 0 \end{pmatrix}. \quad (3.17)$$

Here we chose the uniform curvature gauge. H_0^{lm} , H_1^{lm} , H_2^{lm} , and H_5^{lm} depend on the coordinates t and r , and $Y_l^m(\theta, \phi)$ are the spherical harmonic functions. For simplicity, we drop the (sub/super)script lm from the perturbed

$$\frac{h'(r)}{h(r)} = \frac{1-h(r)}{r} + 8\pi r \varphi'(r)^2 + 8\pi r h(r) A(\varphi(r))^2 \tilde{\rho}, \quad (3.13)$$

$$\frac{f'(r)}{f(r)} = \frac{h(r)-1}{r} + 8\pi r \varphi'(r)^2 - 8\pi r h(r) A(\varphi(r))^2 \tilde{\rho}, \quad (3.14)$$

$$\varphi''(r) = \varphi'(r) \left(\frac{h'(r)}{2h(r)} - \frac{f'(r)}{2f(r)} - \frac{2}{r} \right) + h(r) A(\varphi(r)) A_\varphi(\varphi(r)) \tilde{\rho}. \quad (3.15)$$

where the prime denotes the derivative with respect to the coordinate r and the subscript φ the derivative with respect to φ . $\tilde{\rho}$ is the energy density of the fluid in Jordan frame. By examining the above equations, we can see that the metric (2.5) is obtained for a vanishing scalar field, is secured by the requirement

$$A_\varphi(0) = 0. \quad (3.16)$$

The coupling functions of the models M_1 and M_2 satisfy this condition, which means that GR gravastar is a solution within scalar tensor theories gravity. It is difficult to obtain the general analytic expressions of the metric and scalar field, but one can integrate the equations numerically and thus obtain the characteristics of the star.

Now, we investigate the perturbed equations of the first order by considering the perturbations $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$ and $\varphi = \varphi(r) + \delta\varphi(t, r)$, where $g_{\mu\nu}^{(0)}$ is the background metric and

quantities, since they will not contribute to the perturbed equations. By considering the second order expansion of the total action (3.2) and after integrating by parts, it follows that

$$\begin{aligned} \delta^2 S^{\text{polar}} = & \int dr dt (H_0(a_1\delta\varphi' + La_2H_5' + La_3H_2' + a_4\delta\varphi + a_5H_5 + a_6H_2) + La_7H_1^2 \\ & + H_1(a_8\delta\dot{\varphi} + La_9\dot{H}_5 + a_{10}\dot{H}_2) + a_{11}H_2^2 + H_2(a_{12}\delta\varphi' + a_{13}\delta\varphi + La_{14}H_5) \\ & + La_{15}H_5^2 + La_{16}\dot{H}_5^2 + La_{17}H_5\delta\varphi + e_1\delta\varphi^2 + e_2\delta\varphi'^2 + e_3\delta\varphi^2). \end{aligned} \quad (3.18)$$

where $L = l(l+1)$ and the coefficients a_i and e_i are given in the Appendix. To eliminate the nondynamical variables, we must distinct three cases depending on the integer l .

1. The case $l \geq 2$

We vary (3.18) with respect to H_0 and H_1 , respectively, to obtain

$$a_1\delta\varphi' + La_2H_5' + a_3H_2' + a_4\delta\varphi + a_5H_5 + a_6H_2 = 0, \quad (3.19)$$

$$2La_7H_1 + a_8\delta\dot{\varphi} + La_9\dot{H}_5 + a_{10}\dot{H}_2 = 0, \quad (3.20)$$

and we also define the combination [54–57]

$$\psi = La_2H_5 + a_3H_2. \quad (3.21)$$

The last three equations are solved with respect to H_1 , H_2 , and H_5 . Then, substituting the obtained solutions in (3.18), the time derivative of H_5 and H_2 are eliminated from the action. Therefore, the only functions left in (3.18) are ψ and $\delta\varphi$. After long calculations, we arrive to

$$\delta^2 S = \int dr dt (\dot{\chi}' \mathbf{K} \dot{\chi} + \chi' \mathbf{G} \chi' + \chi' \mathbf{L} \chi' + \chi' \mathbf{M} \chi), \quad (3.22)$$

with

$$\chi' = \{\psi, \delta\varphi\}, \quad (3.23)$$

and $\mathbf{K}$, $\mathbf{G}$, and $\mathbf{M}$ are 2×2 matrices, where their expression are complicated, but they are reduced considerably in the case $\varphi = 0$ and under the condition $A_\varphi(0) = 0$.

2. The case $l=0$

If we impose $l = 0$ in the action (3.18), we must reduce the degrees of freedom by choosing the gauge $H_0 = 0$ (or $H_1 = 0$). Therefore, the action (3.18) is reduced to

$$\begin{aligned} \delta^2 S = \int dr dt & (H_1(a_8\delta\dot{\varphi} + a_{10}\dot{H}_2) + a_{11}H_2^2 \\ & + H_2(a_{12}\delta\varphi' + a_{13}\delta\varphi) + e_1\delta\varphi^2 \\ & + e_2\delta\varphi'^2 + e_3\delta\dot{\varphi}^2). \end{aligned} \quad (3.24)$$

The equation, which is derived from the variation of this action with regard to H_1 , can be solved for $\dot{H}_2$. By substituting the result back into the action, we reformulate it as

$$\delta^2 S = \int dr dt (e_3\delta\varphi^2 + e_2\delta\varphi'^2 + e_1\delta\dot{\varphi}^2), \quad (3.25)$$

with

$$\tilde{e}_3 = \tilde{e}_3 + \frac{a_{11}a_8^2}{a_{10}^2} - \frac{a_{11}a_8}{a_{10}} + \frac{d}{dr} \left[\frac{a_{12}a_8}{a_{10}} \right]. \quad (3.26)$$

When $\varphi = 0$ the equation of the perturbed scalar field is a particular case of the equation derived from (3.22), by taking $l = 0$.

3. The case $l=1$

For $l = 1$, it is necessary to constrain the degree of freedom by establishing the gauge condition $\delta\varphi = 0$. Following the same steps for the case $l \geq 2$, we obtain

$$\delta^2 S = \int dr dt (\tilde{a}_3\delta\psi^2 + \tilde{a}_2\delta\psi'^2 + \tilde{a}_1\delta\dot{\psi}^2). \quad (3.27)$$

We calculate that the total action vanish when φ is constant. Therefore, in this paper, we will not take into consideration the case $l = 1$.

IV. EXACT SOLUTIONS FOR VANISHING SCALAR FIELD

In this section, we consider the case where $\varphi = 0$, which corresponds to gravastars in the limit of GR case under the condition (3.16) and thus one can study the first order perturbed scalar field independently of the metric. Since, in our work, we are interested by the scalar field propagation, we will only study scalar field's equation. Varying the action (3.22) with respect to $\delta\varphi$ and using the Fourier transform $\delta\varphi \rightarrow \int dt e^{i\omega t} \delta\varphi/r$, we obtain

$$\frac{\partial^2 \delta\varphi}{\partial r^{*2}} + (\omega^2 - V(r))\delta\varphi = 0, \quad (4.1)$$

where r^* is the tortoise coordinate and $V(r)$ is the potential which has the form

$$V(r) = \frac{f\rho(\kappa A_{\varphi\varphi}(0) - 1)}{\kappa} + \frac{f(h(l(l+1)+1) - 1)}{hr^2}, \quad (4.2)$$

where we have chosen the conformal function such that $A(0) = 1$. For both models M_1 and M_2 , we remarque that $2\beta = A_{\varphi\varphi}(0)$, which means they are indistinguishable for the vanishing scalar field case. However, both models differ from the GR case where the constant β will play an important role to have a deviation from GR. The explicit form of the potential $V(r)$ is given by

$$V(\hat{r}) = \begin{cases} \frac{l(l+1)}{\hat{r}^2} - 2C^2\hat{r}^2(3\beta\kappa - 1) + C(l(l+1) + 2 - 6\beta\kappa), & \hat{r} < 1, \\ -\frac{C^2}{\hat{r}^4} + \frac{C(1-l(l+1))}{\hat{r}^3} + \frac{l(l+1)}{\hat{r}^2}, & \hat{r} > 1. \end{cases} \quad (4.3)$$

The standard method for deriving the interior solution for perturbed stars is the direct integration of the ordinary differential equation (4.1) from the center of the star to its surface. In the de Sitter interior ($r < 1$), numerical integration is unnecessary, as a regular solution to the perturbation equations may be expressed using hypergeometric functions. To have a physical solution, we require regularity everywhere inside the star. At the center of the star $\delta\varphi$ behaves as

$$\delta\varphi \approx c_1 r^{l+1} + c_2 r^{-l}, \quad (4.4)$$

where c_1 and c_2 are constant of integration. The scalar field does not have a singularity when $l = 0$, but it does when $l > 0$. By imposing $c_2 = 0$, the analytic solution of Eq. (4.1) is

$$\delta\varphi = b(C\hat{r})^{l+1}(1 - C\hat{r}^2)^{-\frac{l\hat{\omega}}{2\sqrt{C}}} \times {}_2F_1\left(m_- - \frac{l\hat{\omega}}{2\sqrt{C}}, m_+ - \frac{l\hat{\omega}}{2\sqrt{C}}, l + \frac{3}{2}, C\hat{r}^2\right) \quad (4.5)$$

where $\hat{\omega} = a\omega$ and b is a constant of integration, ${}_2F_1$ is the hypergeometric function and

$$m_{\pm} = \frac{1}{4}(2l + 3 \pm \sqrt{9 - 24\beta\kappa}). \quad (4.6)$$

Outside the star, the solution of Eq. (4.1) is not evident and we cannot provide an analytic expression for the scalar field; nevertheless, its asymptotic behavior at infinity may be described as a linear combination of an outgoing and an ingoing wave

$$\delta\varphi \approx a_{\text{in}} e^{l\hat{\omega}\hat{r}^*} + a_{\text{out}} e^{-l\hat{\omega}\hat{r}^*}, \quad (4.7)$$

where the rescaled tortoise coordinate $\hat{r}^*$ is expressed as

$$\hat{r}^* = \hat{r} + C \log(\hat{r} - C). \quad (4.8)$$

Given our focus on gravastar quasinormal modes, $\delta\varphi$ must manifest as a purely outgoing wave at infinity, without any ingoing wave component. However, for stable modes $\text{Im}(\hat{\omega}) > 0$, the numerical solution suffer from numerical precision at infinity and we cannot be sure if we have totally eliminated the ingoing wave. To overcome this problem and avoid error coming from ingoing wave, we use continued-fraction method [58].

In a manner that is analogous to the situation of black holes, in which solutions are explored with an asymptotic behavior that simultaneously fulfills an outgoing wave at infinity and an ingoing wave at the horizon, we make use of an outgoing solution that contains a power series around the radius of the star

$$\delta\varphi = e^{-l\hat{\omega}\hat{r}^*} \sum_{i=0}^{i=N} a_i \left(\frac{\hat{r}-1}{\hat{r}}\right)^i, \quad (4.9)$$

where a_i are complex constants and they are related by the following four-term recurrence relation

$$a_2\alpha_1 + a_1\beta_1 + a_0\gamma_1 = 0 \quad n = 1, \quad (4.10)$$

$$a_{n+1}\alpha_n + a_n\beta_n + a_{n-1}\gamma_n + a_{n-2}\delta_n = 0 \quad n > 1. \quad (4.11)$$

with

$$\begin{aligned} \alpha_n &= n(n+1)(1-C), \quad \beta_n = n(n(3C-2)-2l\hat{\omega}), \\ \gamma_n &= n^2 - n - l(l+1) - (3n^2 - 3n + 1)C, \quad \delta_n = (n-1)^2 C. \end{aligned} \quad (4.12)$$

These expressions are true for any star with a Schwarzschild exterior solution and they can written as a three-term recurrence relation [59]

$$a_{n+1}\hat{\alpha}_n + a_n\hat{\beta}_n + a_{n-1}\hat{\gamma}_n = 0 \quad (4.13)$$

where

$$\hat{\alpha}_n = \alpha_n, \quad \hat{\beta}_n = \beta_n - \frac{\alpha_n\beta_{n-1}}{\hat{\gamma}_{n-1}}, \quad \hat{\gamma}_n = \gamma_n - \frac{\delta_n\beta_{n-1}}{\hat{\gamma}_{n-1}}. \quad (4.14)$$

By defining the recurrence relation

$$a_{n+1} = R_n a_n, \quad (4.15)$$

then substituting in (4.13), we obtain

$$R_n = -\frac{\hat{\gamma}_{n+1}}{\hat{\beta}_{n+1} + \hat{\alpha}_{n+1}R_{n+1}}. \quad (4.16)$$

The normal frequencies $\hat{\omega}$ are those values for which the series in (4.9) exists and is finite, i.e., a_n must be non-null and converge absolutely. For $n = 0$, we have

$$\frac{a_1}{a_0} = -\frac{\hat{\gamma}_1}{\hat{\beta}_1 - \frac{\hat{\alpha}_1\hat{\gamma}_2}{\hat{\beta}_2 - \frac{\hat{\alpha}_2\hat{\gamma}_3}{\hat{\beta}_3 - \frac{\hat{\alpha}_3\hat{\gamma}_4}{\hat{\beta}_4 - \dots}}}}. \quad (4.17)$$

where the left side of the above equation is determined from the continuity conditions at radius $[[\delta\varphi]] = 0$ and $[[\delta\varphi']] = 0$, as follow

$$\begin{aligned} \frac{a_1}{a_0} &= 1 + l + \frac{l\hat{\omega}}{1 - \sqrt{C}} + C \left(m_+ - \frac{l\hat{\omega}}{2\sqrt{C}}\right) \left(m_- - \frac{l\hat{\omega}}{2\sqrt{C}}\right) \\ &\times \frac{{}_2F_1(1 + m_- - \frac{l\hat{\omega}}{2\sqrt{C}}, 1 + m_+ - \frac{l\hat{\omega}}{2\sqrt{C}}, l + \frac{5}{2}, C)}{(3 + 2l){}_2F_1(m_- - \frac{l\hat{\omega}}{2\sqrt{C}}, m_+ - \frac{l\hat{\omega}}{2\sqrt{C}}, l + \frac{3}{2}, C)}. \end{aligned} \quad (4.18)$$

The nontrivial solutions of the Eq. (4.13) exist only if Eq. (4.18) is satisfied, where this is true for discrete value of $\hat{\omega}_n$. The identification of the QNMs, for given values of the gravastars compactness C and of the angular momentum integer l , has been reduced to a numerical problem, which is resolved using a root-finding algorithm in the complex plane. Since for any value chosen for $|R_N| < 1$ will introduce a small error, which turns out to be negligible when N is chosen to be large enough, we require the condition $R_N = 0$ when for large n , in practice we choose $n = 20$ [60]. Our results are showed in Fig. 1, where we compare the QNMs of our model with those in general relativity for different values of the compactness C and the integer l .

In fact, we vary the compactness from $C = 1/1000$ to $C = 1$ for three values of the constant β , where β is in the interval where spontaneous scalarization phenomena occurs. The values of the quasi normal modes collapse to the value l as $C \rightarrow 1$, i.e., when the gravastar most closely approximates a BH for $l = 0$ and $l = 2$. In this limit, we see that the real part is very small compared to the imaginary part where the solution (4.18) is not valid. Which means the QNMs of gravastars do not reduce to QNMs of black holes in both frameworks of general relativity and scalar tensor theories. The same conclusion is found in Refs. [61,62]. This can be seen by studying the interior solution in the limit $C \rightarrow 1$ and $r \rightarrow 1$, i.e.,

$$\delta\varphi \approx \frac{e^{I\hat{\omega}r^*}\Gamma(I\hat{\omega})}{\Gamma(m_+ + I\hat{\omega}/2)\Gamma(m_- + I\hat{\omega}/2)} + \frac{e^{-I\hat{\omega}r^*}\Gamma(-I\hat{\omega})}{\Gamma(m_- - I\hat{\omega}/2)\Gamma(m_+ - I\hat{\omega}/2)}. \quad (4.19)$$

We see that the scalar field has ingoing and outgoing waves which are presented by the first and the second terms, respectively. If we wish to have only outgoing term at the exterior of the stars the first must vanish which happens at the poles of the gamma function in the denominator. Hence, the QNMs in the high compactness limit, for all values of l , are

$$\hat{\omega}_{n'} = I2(n' + m_+), \quad n' = \{0, 1, 2, 3, \dots\} \quad (4.20)$$

which are purely imaginary number. Because of the term $\Gamma(I\omega)$ in numerator, these modes do not exist in when $\beta = 0$, i.e., the general relativity limit. The same result is obtained in Ref. [63]. We remark that these frequencies are similar to the spectrum of QNMs for Schwarzschild black hole in high-damping regime [64,65], where in this later the real part is a nonzero constant. These type of quasinormal modes are interpreted in the literature as frequencies of excited levels of a black holes behaving as a damped harmonic oscillator [66]. Like in the previous works [64–67], we observe an equally spaced spectrum which will allow us to examine the propriety of quantum black holes in semiclassical frame work via Bhor's correspondence. A transition from an excited black to a black hole might be an absorption of an energy equivalent to $\Delta M = \hbar(\omega_n^{(0)} - \omega_{n-1}^{(0)}) = 4\pi T_H$, where $T_H = \hbar/(8\pi M)$ is Hawking temperature [68] and $\omega_n^{(0)} = \sqrt{(\text{Re}[\omega_n^{(0)}])^2 + (\text{Im}[\omega_n^{(0)}])^2}$ [69,70]. If we follow the steps in these references, a gravastar in the limit of a black hole may also absorb energy as

$$\Delta M = 8\pi T_H, \quad (4.21)$$

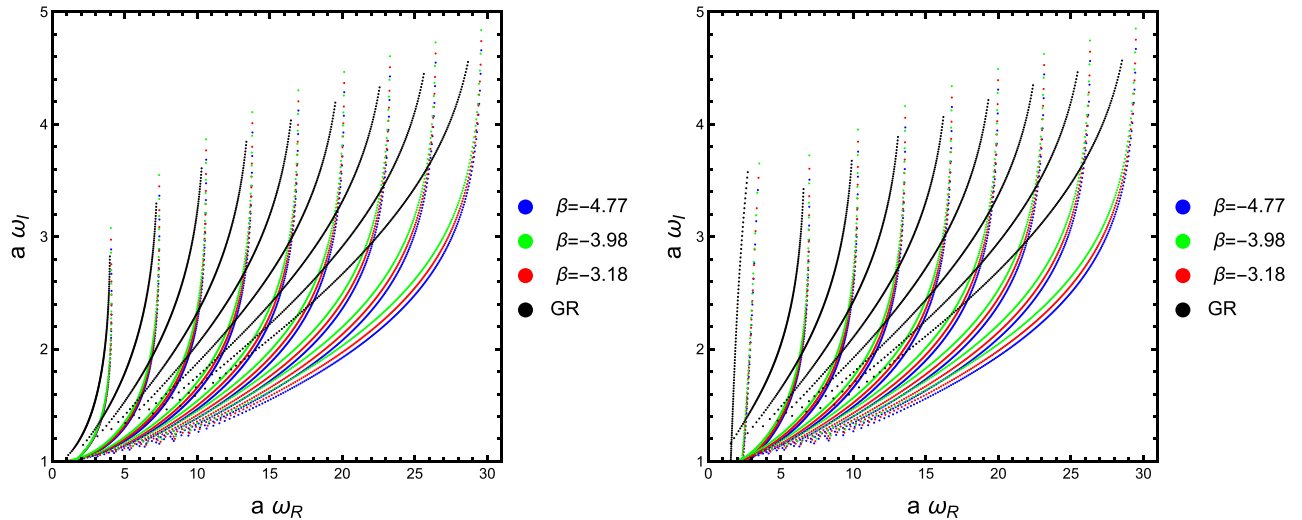


FIG. 1. The first nine quasinormal modes of gravastars in scalar-tensor theories and general relativity for the cases $l = 0$ (left panel) and $l = 2$ (right panel).

which differs black hole by a factor of 2. And thus, according to second law of black hole thermodynamics the variation of the gravastar's era ΔA generated by ΔM is

$$\Delta A = -16\pi \quad (4.22)$$

once again the variation of the era is different from those we get in a black hole by a factor of 2. Note that this result is considerably different from Hod's conjuctor because we used Maggiore [68] rather than Hod's one [65].

V. NUMERICAL ANALYSIS

In the last section, we derived the solution of a gravastar with a trivial scalar field where most of the solutions were analytic, i.e., they are reduced to general relativity. It is interesting to look into the physical characteristics of a nonvanishing scalar field at the background level and at the first-order perturbation. To do so, unlike in the case $\varphi = 0$, we must give a form to the coupling function A and we will consider the functions (3.8) as examples.

A. Background solutions

The presence of a nontrivial scalar field makes the resolution of the differential equations (3.14), (3.13), and (3.15) requires a numerical algorithm. The stability of the numerical solutions are ensured by boundary conditions at the center of the star

$$h(0) = 1 \quad \text{and} \quad \varphi'(0) = 0, \quad (5.1)$$

where the second condition ensures the regularity of both scalar fields Φ and φ at the center of the star. Additionally, it guarantees the regularity of metric the Jordan frame at the center of the star, which is due to the fact that the metric in Jordan frame is related to that in the Einstein frame by a conformal transformation defined in terms of scalar field. However, these conditions are not sufficient to get

asymptotically flatness at infinity which requires the boundary conditions

$$f(\infty) = 1 \quad \text{and} \quad \varphi(\infty) = 0, \quad (5.2)$$

and thus the metric in Einstein frame and Jordan frame become indistinguishable. In order to get the last condition we integrate the Eqs. (3.14), (3.13), and (3.15) using the specified starting condition

$$\varphi(0) = \varphi_c, \quad (5.3)$$

where φ_c is a real constant. In fact, using the transformation (2.9), we integrate the background equation from $\hat{r} = 0$ to $\hat{r} = 1$ with $\rho = \text{const}$ and then from $\hat{r} = 1$ to $\hat{r} = \infty$ in the vacuum by taking into account the junction condition at $\hat{r} = 1$ for the coupling functions (3.8). During the integration, we need to give a numerical value to the energy density ρ and the constant coupling β while φ_c is chosen to satisfy (5.2).

In the presence of the scalar field, the compactness C cannot be obtained directly from the definition (2.10), since we do not have an analytic expression of the metric. In this case, the behavior of the metric is

$$\lim_{\hat{r} \rightarrow \infty} h(\hat{r}) = 1 - \frac{C}{\hat{r}}, \quad (5.4)$$

which is equivalent to GR. Unlike in GR, the compactness of the star obtained from the above equation is different from the compactness derived from the energy density which is confirmed in the right of Fig. 2 for both models M_1 and M_2 . In Fig. 2, we show the variation of the energy density of the star and central value of the scalar field with respect to compactness, where we can see that φ_c is equal to zero in interval $[0, C_c]$, where C_c depends on the value of the constant β . Note the form of the coupling function do not contribute to the value of C_c because the scalar field

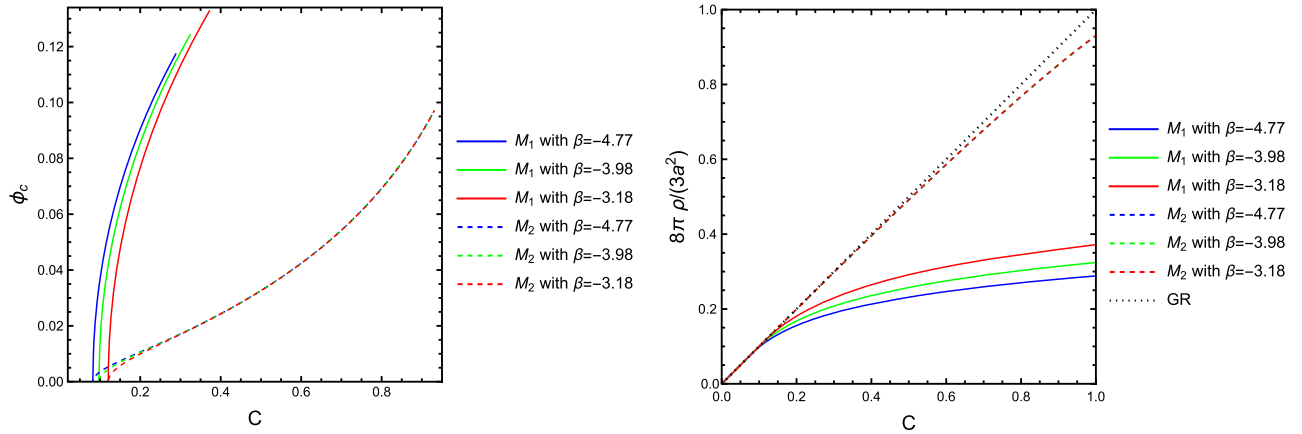


FIG. 2. The variation of scalar field at the center of star (left) and the energy density (right) as function of the compactness.

around this values is small and thus the both functions are identical. Thus, in this interval the solutions are reduced to GR. By increasing the compactness, the central value of the scalar field is no longer zero and starts to increase at is illustrated in the left of Fig. 2. For larger gravastar energy density in the presence of nontrivial scalar field leads to a decrease of the gravastar compactness and central value of the scalar field.

The variation of φ_c and ρ as function of C depends on the value of β for both models M_1 and M_2 , but one can notice that the relations between these constants vary quantitatively. First, the relation $C - \rho$ in model M_2 start to deviate from GR at high compactness while in model M_1 is distinguishable from GR when φ_c is not equal to zero. Second, the constant β play an important role in the relations $C - \rho$ and $C - \varphi_c$ where by increasing β the deviation from GR increase in the interval where φ_c is not

vanished. But, in the model M_2 we see that β does not effect the $\varphi_c - C$ only in small region close to C_c . We note that $C \approx \rho$ in M_2 at low compactness but significant deviation appears when $C \rightarrow 1$. Finally, it is expected that these differences will have impacts at the perturbed level which we will see in the next subsection.

B. Scalar quasinormal modes

In order to derive the quasi normal modes for nonvanishing scalar field, we should integrate the Eq. (4.1) numerically because the scalar field and the metric do not have analytic expression in all spacetime. Here, we considered the case $l = 0$ where the perturbed metric and the perturbed scalar field are decoupled. In the case $\varphi \neq 0$, the Schrödinger-like differential equation has the same form as (4.1) with the following potential

$$V(\hat{r}) = \begin{cases} f\left(\rho\left(\frac{A^2 - 4A\hat{r}A_{\varphi}\varphi'}{\kappa} - \frac{2A^2\hat{r}^2\varphi'^2}{\kappa^2} - A_{\varphi}^2 - AA_{\varphi\varphi}\right) + \frac{1-h}{h\hat{r}^2} + \frac{2\varphi'^2}{\kappa}\right), & \hat{r} < 1, \\ f\left(\frac{1-h}{h\hat{r}^2} + \frac{2\varphi'^2}{\kappa}\right), & \hat{r} > 1. \end{cases} \quad (5.5)$$

As an example, we plot $V(\hat{r})$ in Fig. 3 for different values of β using the model M_1 . The method used in the last section to derive QNMs cannot be applied in this case due to the numerical nature of the solutions for the background metric and the scalar field. Instead, the WKB approximation approach, as elucidated in Ref. [41], where the author identified novel families of QNMs for relativistic neutron stars, can be

employed to resolve our equation. Outside the star, the WKB approximation for the Eq. (4.1) has the standard form

$$\delta\varphi(\hat{\omega}, \hat{r}) = \frac{1}{Q(r)^{1/2}} (a(\omega)e^{-I \int^{r^*} Q(r)dr} + b(\omega)e^{I \int^{r^*} Q(r)dr}), \quad (5.6)$$

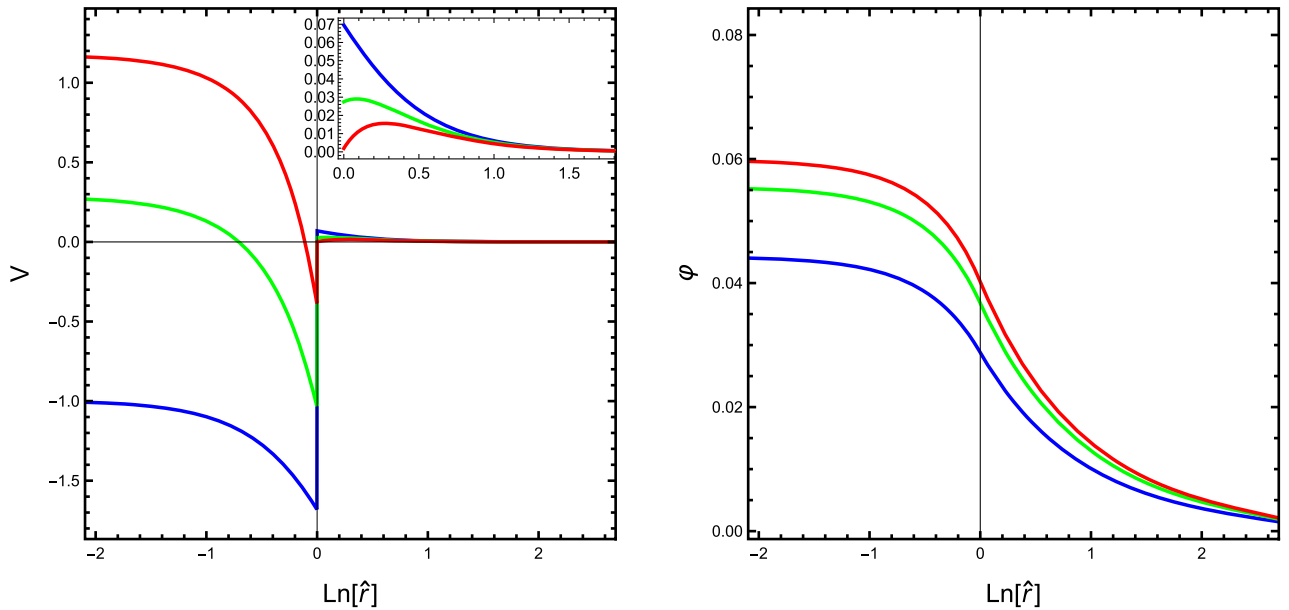


FIG. 3. The variation of the potential V (in the right graph) and the scalar field (in the left graph) φ with respect to the radial coordinate r for different values of $\beta = \{-4.77(\text{red line}), -3.98(\text{green line}), -3.13(\text{blue line})\}$.

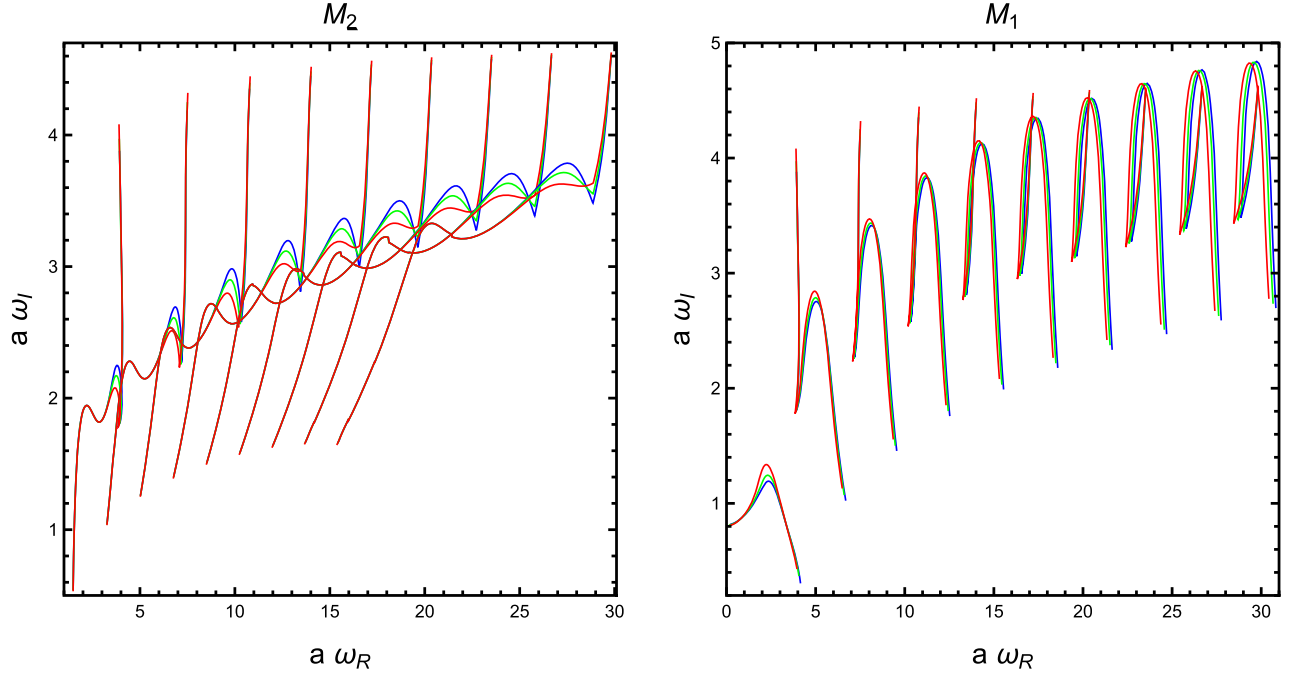


FIG. 4. The first few quasinormal modes of scalar field perturbation equation for $\varphi \neq 0$ and for $l = 0$. In the left panel we QNMs of model M_2 is plotted for $\beta = \{-4.77(\text{blue}), -3.93(\text{green}), -3.18(\text{red})\}$. In the right panel we do the same for model M_1 .

where $Q(r) = \sqrt{\hat{\omega}^2 - V(r)}$. And the interior solution $\delta\varphi_{\text{int}}$ is found from integrating inside the star and imposing the regularity condition $\delta\varphi \approx \hat{r}$. The junction conditions impose the continuity of the scalar field and its derivative through the surface with the interior of the star. By doing so, we get the ratio

$$\frac{b(\hat{\omega})}{a(\hat{\omega})} = e^{-2I} \int^1 Q(r) dr \frac{Q(1)^2 + \left(\frac{\delta\varphi'_{\text{int}}(1)}{\delta\varphi_{\text{int}}(1)} + \frac{Q'(1)}{2Q(1)}\right)^2}{\left(Q(1) + I\left(\frac{\delta\varphi'_{\text{int}}(1)}{\delta\varphi_{\text{int}}(1)} + \frac{Q'(1)}{2Q(1)}\right)\right)^2}. \quad (5.7)$$

in order to produce a pure outgoing wave function, this ratio must equal zero which happens only for certain discrete values of $\hat{\omega}$. Nevertheless, this approach breaks down when $M\omega$ is very large and suffers from numerical errors when $M\omega \approx 1$ [41]. In order to overcome these issues, we employ the mixed numerical/WKB approach, where a numerical integration from infinity to the star's radius is performed as well as the approximate relationship between the outgoing $\delta\varphi_-$ and ingoing $\delta\varphi_+$ waves functions,

$$\frac{\delta\varphi'_+}{\delta\varphi_+} + \frac{\delta\varphi'_-}{\delta\varphi_-} = -\frac{Q'_+}{Q_+}, \quad (5.8)$$

is used. From the continuity conditions, it follows that

$$\frac{\delta\varphi'_+}{\delta\varphi_+} + \frac{\delta\varphi'_{\text{int}}}{\delta\varphi_{\text{int}}} = -\frac{Q'_+}{Q_+}. \quad (5.9)$$

The function $\delta\varphi_+$ is obtained by integrating inward from a large value of $\hat{r}$ to the stellar radius, using the ingoing wave

function as the initial condition. This choice ensures that represents a purely ingoing wave. In contrast, obtaining $\delta\varphi_-$ from an outgoing wave function specified at infinity as the initial condition is more challenging, as ensuring its purity is not straightforward.

To find the QNM frequencies, we fix the compactness C , the form of the coupling function and the constant β , and then we search for the frequencies ω that verify the equation (5.9). Doing so, for each compactness, we get a complex frequency where its real and imaginary components are illustrated in Fig. 4. Varying the compactness from $C = 0.001$ to $C = 0.999$, we compute the $\omega_R - \omega_I$ relation shown in Fig. 4 for both models M_1 and M_2 . In both models, we observe two parts in the plots for each frequency, where the first part is identical to GR and the second is not. This is due to manifestation of the scalar field which is the main reason for this deviation. In both models, we observe that the plots have two distinct regions for each frequency. The first region is indistinguishable from the predictions of GR, while the second deviates considerably, which is caused principally by the presence of the scalar field.

VI. CONCLUSION

In this present work, we have investigated the dynamics of gravitational and scalar field in gravastars, where the conformal transformation has been utilized to couple the matter field with the metric. A numerical and analytic analysis has been performed to describe the background and the scalar field. The scalar field perturbation equation

was analyzed for the trivial and nontrivial solutions. For the trivial case, we computed the quasinormal modes using the continued fraction method, while for the nontrivial case, we employed the WKB approximation. We have provided key insights into the behavior of quasinormal modes and how the internal characteristics of gravastars affects them.

Our results have showed us that the scalar field and the metric are significantly influenced by the choice of the form of the coupling function A with β considered as free parameter that control the interaction between different fields. This leads to distinct observable signatures, highlighting the important role of the coupling function in shaping the physical feature of gravastars. We found that the gravastars is indistinguishable from GR case at small compactness and it starts to deviate from GR at high compactness. We found also that the central value of the scalar field (the scalar field is not equal to zero in all the spacetime) is present when the compactness is bigger than a particular compactness. This behavior is found in neutron stars and black holes [13].

Through quasinormal modes, we expanded our understanding of the exotic compact object in alternative theories of gravity by studying the scalar field. The quasinormal frequencies provided us new behaviors by varying the compactness for both cases trivial and nontrivial solutions.

For the trivial solution, the form of the function A does not impact the equations of perturbation and thus the QNMs. However, when the scalar field is present we have observed new features of QNMs where they have qualitative differences which depends on A . In fact, the model M_1 deviates considerably from GR and we found a mode which is absent in GR and in model M_2 .

In summary, our work contributes to the understanding of the characteristic of gravastars within the framework of scalar tensor-theories, providing important insights into the configuration of these astrophysical objects. Extending our study to axial and polar perturbations will gives us more information about the interior of gravastars. Finally, this study provides a path toward modeling and understanding gravastars in scalar-tensor theories, motivating us for further theoretical and observational exploration.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: COEFFICIENTS

The coefficients that appears in $\delta^2 S$ are given by

$$\begin{aligned}
 a_1 &= \frac{\sqrt{f}r^2\varphi'}{\sqrt{h}}, & a_2 &= \frac{\kappa}{2}r\sqrt{f}, & a_3 &= -\frac{\kappa}{2\sqrt{h}}\sqrt{f}r, & a_4 &= C_\varphi C\sqrt{f}r^2\sqrt{h}\rho, & a_5 &= \kappa\sqrt{f}, \\
 a_6 &= -\frac{1}{4}\sqrt{f}\sqrt{h}\kappa L - \frac{1}{2}\sqrt{f}(\sqrt{h}\kappa - hr^2\rho), & a_7 &= \frac{\kappa}{4}\sqrt{f}\sqrt{h}, & a_8 &= -2r^2\varphi', \\
 a_9 &= \kappa r, & a_{10} &= -\frac{\kappa}{2}\sqrt{h}r, & a_{11} &= \frac{1}{4}\sqrt{f}\sqrt{h}(\kappa - \sqrt{h}\rho r^2), & a_{12} &= \frac{\sqrt{f}r^2\varphi'}{\sqrt{h}}, \\
 a_{13} &= -CC_\varphi\sqrt{f}\sqrt{h}r^2\rho, & a_{14} &= -\frac{\kappa(rf' + 2f)}{4\sqrt{f}}, & a_{15} &= \frac{\kappa}{2}\sqrt{f}\sqrt{h}, & a_{16} &= \frac{\sqrt{h}\kappa r^2}{4\sqrt{f}}, \\
 a_{17} &= 2\sqrt{f}r\varphi', & e_2 &= -e_3 = -\frac{\sqrt{f}r^2}{\sqrt{h}}, & e_1 &= -\sqrt{f}\sqrt{h}(L + \rho r^2(CC_{\varphi\varphi} + C_\varphi^2)).
 \end{aligned} \tag{A1}$$

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