

Generalized definition of the isothermal compressibility in (2+1)-flavor QCD

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We introduce a generalized definition of the isothermal compressibility (κ_{T,σ_Q^2}) calculable by keeping net conserved charge fluctuations rather than total number densities constant. We present lattice QCD results for this isothermal compressibility, expressed in terms of fluctuations of conserved charges that are related to baryon (B), electric charge (Q), and strangeness (S) quantum numbers. This generalized isothermal compressibility is compared with hadron resonance gas model calculations as well as with heavy-ion collision data obtained at RHIC and the LHC. We find $\kappa_{T,\sigma_Q^2} = 13.8(1.3) \text{ fm}^3/\text{GeV}$ at $T_{\text{pc},0} = 156.5(1.5) \text{ MeV}$ and $\hat{\mu}_B = 0$. This finding is consistent with the rescaled result of the ALICE Collaboration, where we replaced the number of charged hadrons (N_{ch}) by the total number of hadrons (N_{tot}) at freeze-out. Normalizing this result with the QCD pressure (P) we find that the isothermal compressibility on the pseudocritical line stays close to that of an *ideal gas*, i.e., $P\kappa_{T,\sigma_Q^2} \simeq 1$.

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I. INTRODUCTION

A central goal of theoretical and experimental studies of strong-interaction matter is to understand its different phases and the properties of matter in these phases. Thermal properties of strong-interaction matter [1,2], e.g., the equation of state, velocity of sound or its viscosity, are widely studied experimentally in heavy-ion collisions (HIC) and theoretically in the framework of the theory of strong interactions, quantum chromodynamics (QCD). The small shear and bulk viscosities, determined in HIC and calculated in lattice QCD simulations [3–5], led to the conclusion that this matter is an “almost perfect fluid” [6]. We will show here that in the vicinity of the pseudocritical temperature, which corresponds to the freeze-out temperature in HIC, the isothermal compressibility stays close to that of an ideal gas. We note that the discussion of a generalized compressibility, presented here in the context of QCD, is of relevance also for the discussion of material properties in any field theoretic context, see e.g., [7].

The current phenomenological discussion [8–10] and experimental determination [11–13] of the isothermal compressibility (κ_T) of strong-interaction matter relies on

a statistical physics definition [8,14,15], which is valid for systems in which particle number can be considered as a well-defined conserved quantity,

$$\kappa_T = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_{T,N} = \frac{1}{n^2} \frac{\partial^2 P}{\partial \mu^2} = \frac{1}{T^4 n^2} \frac{\partial^2 \hat{P}}{\partial \hat{\mu}^2}, \quad (1)$$

with pressure P , spatial volume V , temperature T , $n = N/V$ denoting the particle number density, and μ being the chemical potential coupling to the number operator. Here and in the following we use the notation $\hat{O} \equiv OT^{-k}$ with $k \in \mathbb{Z}$ chosen so that $\hat{O}$ is dimensionless. The application of such an approach to the analysis of properties of strong-interaction matter, created in HIC at the time of hadronization (freeze-out), makes use of the fact that relativistic hadron resonance gas (HRG) models provide a rather good description of the thermodynamics of QCD in the vicinity of the freeze-out temperature.

The concept of well-defined particle number densities, however, cannot be carried over straightforwardly to a first-principle, QCD-based calculation of the isothermal compressibility of matter, where the conserved charges are related to differences of particle and antiparticle numbers rather than total particle numbers, i.e., net-baryon number (N_B), net-electric charge (N_Q) and net-strangeness (N_S). In fact, defining κ_T by keeping net-number densities rather than total number densities fixed amounts to replacing in Eq. (1) the total number density n by a net-charge number density n_Q or n_B . In the limit of vanishing chemical

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potential this clearly leads to a divergent isothermal compressibility which easily is verified in noninteracting HRG model calculations and explicitly shown in lattice QCD calculations [16].

In this article we will discuss a generalized version of the isothermal compressibility which does not explicitly rely on the particle content of a thermal medium but makes use of conserved charges and their fluctuations that are accessible in QCD calculations as well as measurements in HIC. We consider the isothermal compressibility of strong-interaction matter at fixed value of the ratio of net-electric charge to net-baryon number, $r_Q = N_Q/N_B$, and fixed net-strangeness to net-baryon number ratio, $r_S = N_S/N_B$. Furthermore, the partial derivative of volume with respect to pressure is taken, keeping a function (f_Q) of the net-electric charge fixed,

$$\kappa_{T,f_Q,r_Q,r_S} = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_{T,f_Q,r_Q,r_S}. \quad (2)$$

In particular we consider the cases of fixed net-electric charge, $f_Q = V \partial P / \partial \mu_Q \equiv N_Q$, and fixed net-electric charge fluctuations, $f_Q = VT \partial^2 P / \partial \mu_Q^2 \equiv \sigma_Q^2$. The latter choice is motivated by the observation that total particle numbers in a noninteracting HRG are well-approximated by net-charge fluctuations and that HRG models provide a good description of lattice QCD calculations close to the freeze-out temperature. Of course, constraints other than fixing χ_2^Q are possible, if these are found to be appropriate in another context.

However, as pointed out, the former choice leads to a divergent compressibility at vanishing chemical potentials even in HRG model calculations.

II. ISOTHERMAL COMPRESSIBILITY IN HRG MODELS

We briefly recapitulate here the calculation of the isothermal compressibility $\kappa_{T,\vec{N}}$ in noninteracting HRG models, which is performed for a fixed number of particles and antiparticles ($\vec{N}$) to circumvent the problem of divergences at $\mu = 0$, and which is used also in experimental determinations of $\kappa_{T,\vec{N}}$. Calculations of $\kappa_{T,\vec{N}}$ in the framework of HRG models take advantage of the fact that in a grand canonical prescription of the HRG model the pressure and other thermodynamic observables are explicitly given in terms of sums of contributions from different particle species. Although the grand canonical ensemble generally is defined as a function of T , V , and conserved-charge chemical potentials $\vec{\mu} = (\mu_B, \mu_Q, \mu_S)$,

$$P = \lim_{V \rightarrow \infty} \frac{T}{V} \ln \mathcal{Z}(T, V, \vec{\mu}), \quad (3)$$

in a noninteracting HRG model, one may project onto different particle and antiparticle sectors by introducing a

chemical potential, μ_i , for each particle species i , which we set to zero after taking appropriate derivatives, i.e., we introduce $\mu_{H,i} = B\mu_B + Q\mu_Q + S\mu_S + \mu_i$ for each hadron with quantum numbers (B, Q, S) . With this one obtains particle numbers and fluctuations in each sector [8,12],

$$N_i = V \left. \frac{\partial P}{\partial \mu_i} \right|_{\mu_i=0} = V n_i = VT^3 \hat{n}_i, \quad (4)$$

$$\sigma_i^2 = T \left. \frac{\partial N_i}{\partial \mu_i} \right|_{\mu_i=0} = VT \frac{\partial^2 P}{\partial \mu_i^2} = VT^3 \hat{\chi}_2^i. \quad (5)$$

The inverse of the isothermal compressibility in a non-interacting HRG is then obtained as [12]

$$\kappa_{T,\vec{N}}^{-1} = -V \left(\frac{\partial P}{\partial V} \right)_{T,\vec{N}} = T^4 \sum_{i \in \text{hadrons}} \frac{\hat{n}_i}{\omega_i}, \quad (6)$$

where $\omega_i = \sigma_i^2 / N_i$ are the scaled variances.

In a noninteracting HRG, Boltzmann-statistics is a good approximation for all baryons. As a consequence, for all baryons $\omega_i = 1$ for all T and $\vec{\mu}$. For mesons appreciable deviations from unity only arise for pions and kaons. In all other cases $\omega_i \simeq 1$ to better than 0.2%.

In order to make contact with experimental determinations of $\kappa_{T,\vec{N}}$, which is based on a measurement of charged particle yields and fluctuations, we separate in Eq. (6) contributions from charged and neutral hadrons,

$$\kappa_{T,\vec{N}}^{-1} = T^4 \sum_{i \in \text{charged}} \frac{\hat{n}_i}{\omega_i} + T^4 \sum_{i \in \text{neutral}} \frac{\hat{n}_i}{\omega_i}. \quad (7)$$

As in HRG model calculations the scaled variances for different particle species vary little, introducing a common scaled variance for all charged (ω_{ch}), neutral (ω_0), or even all hadrons (ω_{tot}) provides a good approximation,

$$\kappa_{T,\vec{N}}^{-1} \simeq T^4 \frac{\hat{n}_{\text{ch}}}{\omega_{\text{ch}}} + T^4 \frac{\hat{n}_0}{\omega_0} \simeq T^4 \frac{n_{\text{tot}}}{\omega_{\text{tot}}}, \quad (8)$$

where we introduced

$$\hat{n}_X = \sum_{i \in X} \hat{n}_i, \quad \omega_X = \frac{\sum_{i \in X} \sigma_i^2}{\sum_{i \in X} N_i}, \quad X = \text{ch}, 0, \text{tot} \quad (9)$$

for charged, neutral, and total number of hadrons, respectively.

We note that the determination of $\kappa_{T,\vec{N}}$ from HIC on particle yields and fluctuations [11–13] is based only on the first term in the middle expression of Eq. (8), i.e., the contribution of neutral hadrons to the compressibility is neglected. To the extent that the total number of hadrons in HRG model calculations as well as in heavy-ion collisions is almost a factor of two larger than the number of charged

TABLE I. Summary of charged and neutral particle multiplicities and scaled variance in a noninteracting HRG at $T = 156.5$ MeV.

Hadrons	$\hat{n}_X$	ω_X
<i>Mesons</i>		
Charged [$X = \text{ch}$]	0.3448	1.0667
Neutral [$X = 0$]	0.3059	1.0418
<i>Baryons</i>		
Charged [$X = \text{ch}$]	0.0726	1.0
Neutral [$X = 0$]	0.0549	1.0
<i>Combined</i>		
Charged [$X = \text{ch}$]	0.4174	1.0551
Neutral [$X = 0$]	0.3608	1.0354
<i>All hadrons</i>	0.7782	1.0460

hadrons [13,17] this would reduce $\kappa_{T,\vec{N}}$ by almost a factor 2.

In Table I we give values for $\hat{n}_X$ and ω_X at the pseudocritical temperature, $T_{\text{pc},0} = 156.5$ MeV and $\mu_B = 0$ for a noninteracting HRG.¹ Using also the QMHRG2020 result for the pressure, $P = 0.7630T_{\text{pc},0}^4$, we thus obtain

$$\kappa_{T,\vec{N}}T^4 = 1.3441, \quad P\kappa_{T,\vec{N}} = 1.0256, \quad (10)$$

which is close to the isothermal compressibility of a medium obeying the ideal gas equation of state (EOS)² ($PV = NkT$), i.e., $\kappa_{T,N} = 1/P$.

III. GENERALIZED ISOTHERMAL COMPRESSIBILITY

The HRG model calculation discussed above of course has the disadvantage that it explicitly relies on the introduction of pointlike, noninteracting particles as fundamental (conserved charge) degrees of freedom. That is not possible in QCD, and even in the HRG context, it ignores that the truly conserved charges are B , Q , S and eventually also the quantum numbers of heavier quarks. Rather than defining the isothermal compressibility by using $\partial V/\partial P$ taken at fixed $(T, \vec{N})$ we thus introduce a generalized version of the isothermal compressibility for strangeness-neutral ($r_S = 0$) $(2+1)$ -flavor QCD with fixed r_Q ,

$$\kappa_{T,f_Q,r_Q,0}T^4 = -\frac{1}{f_Q\hat{n}_B} \left(\frac{\partial f_Q}{\partial \hat{\mu}_B} k_B + \frac{\partial f_Q}{\partial \hat{\mu}_Q} k_Q + \frac{\partial f_Q}{\partial \hat{\mu}_S} k_S \right), \quad (11)$$

¹All HRG model calculations shown in this work are obtained by using the QMHRG2020 hadron list [18].

²In noninteracting HRG models, significant deviations from the ideal gas EOS only arise from pions and kaons, which need to be treated as Bose gases. All other hadrons can be treated in Boltzmann approximation for which a ideal gas EOS is a good approximation.

where we keep a function f_Q of the net-electric charge Q fixed. The partial derivatives on the right-hand side are understood to be taken at fixed T , V as well as keeping the remaining two chemical potentials fixed, and

$$k_B = -(1 + (2r_Q - 1)A)/D, \quad (12)$$

$$k_S = (\hat{\chi}_{11}^{BS}/\hat{\chi}_2^S - (2r_Q - 1)A)/D, \quad (13)$$

$$k_Q = 2(2r_Q - 1)A/D, \quad (14)$$

with $D = 1 - (2r_Q - 1)^2A$, and

$$A = \frac{\hat{\chi}_2^S \hat{\chi}_{11}^{BQ} - \hat{\chi}_{11}^{BS} \hat{\chi}_{11}^{QS}}{\hat{\chi}_2^S (\hat{\chi}_{11}^{BQ} - 2\hat{\chi}_2^Q + \hat{\chi}_{11}^{QS})}. \quad (15)$$

Here the dimensionless generalized susceptibilities [19] are defined as

$$\hat{\chi}_{ijk}^{BQS}(T, V, \vec{\mu}) = \frac{1}{VT^3} \frac{\partial^{i+j+k} \ln \mathcal{Z}(T, V, \vec{\mu})}{\partial \hat{\mu}_B^i \partial \hat{\mu}_Q^j \partial \hat{\mu}_S^k}. \quad (16)$$

Using results for second-order cumulants, given in [18], we find $A = -0.071(2)$ in QCD and $A = -0.070$ in HRG. Differences in the isothermal compressibility obtained in the isospin symmetric limit, $r_Q = 1/2$, and in the case of interest in HIC experiments, $r_Q = 0.4$, respectively, thus are smaller than 1.5%. In the following we therefore discuss only the isospin symmetric case $r_Q = 1/2$. In this case Eq. (11) reduces to³

$$\kappa_{T,f_Q}T^4 = \frac{1}{f_Q\hat{n}_B} \left(\frac{\partial f_Q}{\partial \hat{\mu}_B} - \frac{\hat{\chi}_{11}^{BS}}{\hat{\chi}_2^S} \frac{\partial f_Q}{\partial \hat{\mu}_S} \right). \quad (17)$$

We discuss the cases of fixed net-electric charge,⁴ $f_Q = N_Q$, and fixed electric charge fluctuations, $f_Q = \sigma_Q^2 = VT^3 \hat{\chi}_2^Q$. For these two cases we find, $f_Q = N_Q$:

$$\kappa_{T,N_Q}T^4 = \frac{\hat{\chi}_{11}^{BQ}}{\hat{n}_B\hat{n}_Q} \left(1 - \frac{\hat{\chi}_{11}^{QS}\hat{\chi}_{11}^{BS}}{\hat{\chi}_{11}^{BQ}\hat{\chi}_2^S} \right), \quad (18)$$

$$f_Q = \sigma_Q^2 = VT^3 \hat{\chi}_2^Q:$$

³In the following we suppress the subscripts for $r_Q = 1/2$ and $r_S = 0$.

⁴Lattice QCD results for the isothermal compressibility at fixed net-baryon number have been presented in [16]. They clearly show the strong divergence appearing in κ_{T,N_Q} as well as κ_{T,N_B} in the limit of vanishing chemical potentials.

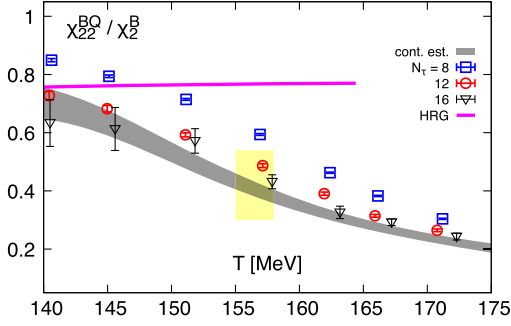


FIG. 1. Ratio of fourth-order cumulant of electric charge and baryon number correlations and the second-order cumulant of baryon number fluctuations for $\vec{\mu} = 0$ versus T . Shown are results from (2 + 1)-flavor QCD and HRG model calculations. The gray band shows the weighted average of continuum limit extrapolations obtained with and without including the $N_\tau = 8$ dataset.

$$\kappa_{T,\sigma_Q^2} T^4 = \frac{\hat{\chi}_{12}^{BQ}}{\hat{n}_B \hat{\chi}_2^Q} \left(1 - \frac{\hat{\chi}_{21}^{QS} \hat{\chi}_{11}^{BS}}{\hat{\chi}_{12}^{BQ} \hat{\chi}_2^S} \right) = \kappa_T^{(0)} (1 + \delta\kappa_T) T^4. \quad (19)$$

We note that $\kappa_{T,N_Q} \sim \hat{n}_Q^{-2}$ diverges in the limit of vanishing chemical potentials, while the appearance of ratios of first- and third-order cumulants in Eq. (19) ensures that the isothermal compressibility at fixed $f_Q = VT^3 \hat{\chi}_2^Q$ has a finite limit for vanishing chemical potentials.

The conserved charge cumulants, $\hat{\chi}_{ijk}^{BQS}$, introduced in Eq. (16), are obtained as Taylor series in terms of cumulants calculated at vanishing chemical potential, which we denote in the following by $\chi_{ijk}^{BQS} \equiv \hat{\chi}_{ijk}^{BQS}(T, \vec{0})$ (i.e., without a “hat”) [19,20]. The leading-order expansion terms are $\mathcal{O}(\hat{\mu}_B)$ and $\mathcal{O}(\hat{\mu}_B^0)$ for $i + j + k$ odd and even, respectively. The prefactor $\kappa_T^{(0)}$ in Eq. (19) may be then be written as

$$\kappa_T^{(0)} T^4 = \frac{\hat{\chi}_{12}^{BQ}}{\hat{n}_B \hat{\chi}_2^Q} = \frac{\chi_{22}^{BQ}}{\chi_2^B \chi_2^Q} \frac{1 - \frac{\chi_{12}^{BQS} \chi_{11}^{BS}}{\chi_{22}^{BQ} \chi_2^S} + \mathcal{O}(\vec{\mu}^2)}{1 - \frac{(\chi_{11}^{BS})^2}{\chi_2^B \chi_2^S} + \mathcal{O}(\vec{\mu}^2)}. \quad (20)$$

It gives the dominant contribution to κ_{T,σ_Q^2} . It indeed reflects the intuitive understanding that the isothermal compressibility is inversely proportional to the number of particles in a thermal, strong-interaction medium. In Fig. 1 we show the ratio χ_{22}^{BQ}/χ_2^B . At $T_{pc,0} = (156.5 \pm 1.5)$ MeV [21] its value is close to 0.5, i.e., $(\chi_{22}^{BQ}/\chi_2^B)|_{T_{pc,0}} = 0.41(3)(4)$, the second error reflecting the error on $T_{pc,0}$. At $T_{pc,0}$ the second-order net-electric charge cumulants in (2 + 1)-flavor QCD and HRG model calculations differ by less than 10%. As χ_2^Q in noninteracting HRG models is proportional to the total number of charged particles, which represent about half of the total number of particles at $T_{pc,0}$ (see Table I), we see that $\kappa_T^{(0)}$ is a good proxy for the inverse number of hadrons

TABLE II. Leading-order isothermal compressibility $\kappa_T^{(0)}$ obtained with Eq. (20), the correction term $\delta\kappa_T$, and the total isothermal compressibility κ_{T,σ_Q^2} multiplied with T^4 evaluated at $T_{pc,0}$ and $\hat{\mu}_B = 0$ for (2 + 1)-flavor QCD and HRG. The last column gives κ_{T,σ_Q^2} at $T_{pc,0}$.

	$\kappa_T^{(0)} T^4$	$\delta\kappa_T$	$\kappa_{T,\sigma_Q^2} T^4$	κ_{T,σ_Q^2} [fm ³ /GeV]
QCD	1.078(65)	0.0036(41)	1.08(10)	13.8(1.3)
HRG	1.713	0.0022	1.716	21.93

at $T_{pc,0}$, i.e., at freeze-out in heavy-ion collisions. Results for $\kappa_T^{(0)}$ obtained in (2 + 1)-flavor QCD and a noninteracting HRG model at $T_{pc,0}$ are given in Table II. The difference in both results mainly arises from the difference in χ_{22}^{BQ}/χ_2^B shown in Fig. 1. This difference may be traced back to the contribution of the doubly charged Δ -resonance to net-electric charge fluctuations. Like all other hadrons the Δ -resonance is, of course, treated as a stable resonance in noninteracting HRG models, but is known to be strongly modified in a strong-interaction medium described by QCD [22]. We will further elaborate on this in the following.

IV. ISOTHERMAL COMPRESSIBILITY IN (2 + 1)-FLAVOR QCD

We evaluate the isothermal compressibility at fixed net-electric charge fluctuation in isospin symmetric, strangeness-neutral matter, introduced in Eq. (19), using a sixth-order Taylor series for P/T^4 [20,23]. With this we obtain Taylor series for the first-, second-, and third-order cumulants appearing in the expression for κ_{T,σ_Q^2} . Figure 2 shows results for the temperature dependence of $\kappa_{T,\sigma_Q^2} T^4$ at $\vec{\mu} = \vec{0}$ and as function of $\hat{\mu}_B$ for several values of the temperature. We note that κ_{T,σ_Q^2} drops rapidly with increasing temperature as well as increasing baryon chemical potential. With increasing $\hat{\mu}_B$ the pseudocritical temperature in (2 + 1)-flavor QCD decreases,

$$T_{pc}(\hat{\mu}_B) = T_{pc,0}(1 - \kappa_2 \hat{\mu}_B^2 + \mathcal{O}(\hat{\mu}_B^4)). \quad (21)$$

We use $T_{pc,0} = (156.5 \pm 1.5)$ MeV and $\kappa_2 = 0.012(4)$, from Ref. [21], which is consistent within errors with Refs. [24,25]. As a consequence we find that κ_{T,σ_Q^2} , evaluated on the pseudocritical line, has only a weak $\hat{\mu}_B$ -dependence.

As pointed out in Eq. (10) for all T the isothermal compressibility of an ideal gas is proportional to the inverse of the pressure. It thus is reasonable to express κ_{T,σ_Q^2} in units of the pressure P , which has been calculated in (2 + 1)-flavor QCD for vanishing [26] as well as nonvanishing [23] values of the chemical potentials. On the pseudocritical line we obtain in next-to-leading order,

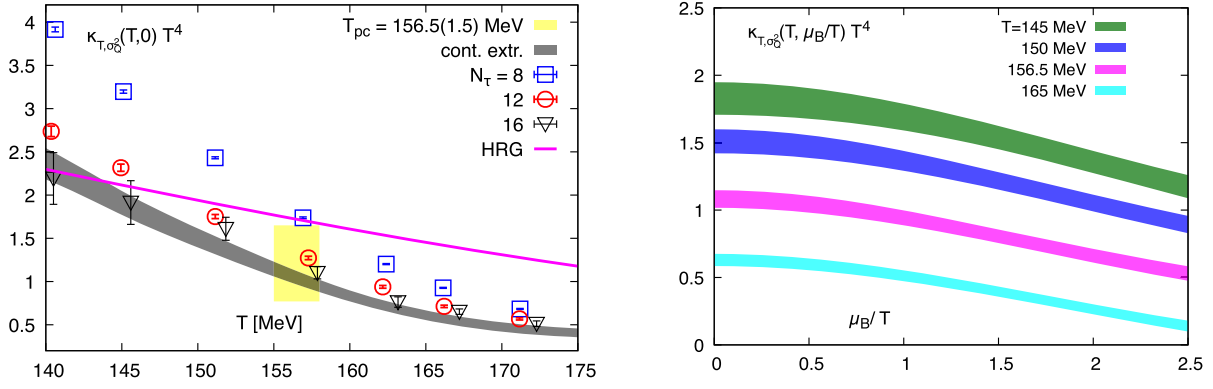


FIG. 2. Isothermal compressibility of a strangeness-neutral, isospin-symmetric matter for $\hat{\mu}_B = 0$ (left) and as function of $\hat{\mu}_B$ at several values of the temperature (right). The gray band in the left-hand figure shows the weighted average of continuum limit extrapolations obtained with and without including the $N_\tau = 8$ dataset. Results for $\hat{\mu}_B > 0$ use Taylor series for the pressure up to sixth order, which has been shown previously by us [20] to remain reliable for $\mu_B/T \leq 2$ within the temperature range $130 \leq T \leq 170$ MeV.

$$P\kappa_{T,\sigma^2}|_{T_{pc}(\hat{\mu}_B)} = 0.80(12) + 0.06(1)\hat{\mu}_B^2 + \mathcal{O}(\hat{\mu}_B^4). \quad (22)$$

We thus find that the isothermal compressibility in QCD at the pseudocritical temperature is close to that of an ideal gas, i.e., $\kappa_{T,\sigma_0^2} \simeq 1/P$, as expected when a noninteracting HRG model provides a good description of thermodynamics at the pseudocritical temperature.

In Fig. 3 we show the isothermal compressibility κ_{T,σ_0^2} , evaluated along the pseudocritical line $T_{pc}(\hat{\mu}_B)$. We compare our QCD results for κ_{T,σ_0^2} with those for $\kappa_{T,\bar{N}}$ extracted from heavy-ion collision data obtained by the ALICE Collaboration at $\hat{\mu}_B = 0$ [13] and an analysis of data on particle yields and freeze-out radii [27,28] obtained by the STAR Collaboration.

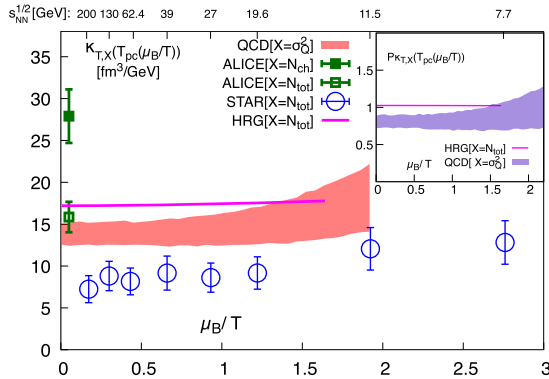


FIG. 3. Isothermal compressibility κ_{T,σ_0^2} (QCD) and $\kappa_{T,\bar{N}}$ (HIC, HRG) along $T_{pc}(\hat{\mu}_B)$ [Eq. (21)] calculated in QCD compared to HIC results. The ALICE[N_{tot}] and STAR[N_{tot}] data have been obtained by taking into account the contribution of neutral hadrons to the total hadron multiplicity as done in [17] [see Eq. (8) and discussion in the text]. In the inset we present $P\kappa_{T,X}$ obtained from QCD ($X = \sigma_0^2$) and HRG ($X = N_{tot}$) calculations along the line $T_{pc}(\hat{\mu}_B)$.

A similar analysis, taking into account also data from fixed target experiments and using a different approach to determine the thermal component of the scaled variance, is given in [12]. The inset of Fig. 3 shows the dimensionless product of isothermal compressibility and pressure, $P\kappa_{T,\sigma^2}$ calculated in $(2+1)$ -flavor QCD and $P\kappa_{T,\bar{N}}$ calculated in a noninteracting HRG model. It is evident that both results are consistent with each other and show only a mild dependence on $\hat{\mu}_B$ as expected from Eq. (22). They stay close to the ideal (Boltzmann) gas value, $P\kappa_{T,\bar{N}} = 1$. These analyses of HIC data are based on results for charged particle yields and multiplicity distributions that are used to calculate the isothermal compressibility $\kappa_{T,\bar{N}}$ at fixed charged-particle numbers using Eq. (8) with $n_0 = 0$, i.e., ignoring the contribution of neutral hadrons. In this case the ALICE Collaboration finds $\omega_{ch} = 1.15(6)$ and $\kappa_{T,\bar{N}} = (27.9 \pm 3.18) \text{ fm}^3/\text{GeV}$ [13]. Taking into account the contribution of neutral hadrons by using Eq. (8) with $\omega_0 = \omega_{ch}$ and $N_{tot} = 2486(146)$ [17] one obtains $\kappa_{T,\bar{N}} = (15.8 \pm 1.3) \text{ fm}^3/\text{GeV}$, which is shown in Fig. 3 by an open square. This rescaled ALICE result for $\kappa_{T,\bar{N}}$ compares well with the QCD result obtained at $\vec{\mu} = \vec{0}$, $\kappa_{T,\sigma_0^2} = 1.08(10)/T_{pc,0}^4 = 13.8(1.3) \text{ fm}^3/\text{GeV}$ as shown in Table II.

For $\hat{\mu}_B > 0$ our calculation of κ_{T,σ_0^2} can be directly confronted with the isothermal compressibility $\kappa_{T,\bar{N}}$ at $\hat{\mu}_B > 0$ obtained by us using data of the STAR Collaboration for charged-particle multiplicities and freeze-out volume at various beam energies [27,28]. We use results for STAR particle yields ($\pi^\pm, K^\pm, p, \bar{p}, \Lambda, \bar{\Lambda}$) obtained in the beam energy scan at RHIC as compiled in [29]. With this, we determine the number of charged and total particles in central collisions [17]. The scaled variances, ω_{ch} , at various beam energies have been determined by the PHENIX [11] and ALICE Collaborations [13]. The

thermal component of the scaled variances is found to be close to unity and compatible with HRG model calculations. As shown in Table II for $\mu_B = 0$ one finds in a noninteracting HRG $\omega_{\text{ch}} \simeq \omega_0 \simeq \omega_{\text{tot}} = 1.05$. Also for $\hat{\mu}_B > 0$ we find in HRG model calculations that the scaled variance for charged, neutral as well as total hadron numbers varies only little on the pseudocritical line, $T_{\text{pc}}(\hat{\mu}_B)$. We find $\omega_{\text{tot}} = 1.046$ at $(T_{\text{pc}}, \mu_B) = (156.5, 0)$ MeV which decreases to $\omega_{\text{tot}} = 1.032$ at $(T_{\text{pc}}, \mu_B) = (144.3, 398)$ MeV. We thus use HRG model results for ω_{tot} to obtain $\kappa_{T, \bar{N}}$ at various RHIC beam energies on the pseudocritical line $T_{\text{pc}}(\hat{\mu}_B)$. The results are also shown in Fig. 3. We note that $\kappa_{T, \bar{N}}$, extracted from the STAR data, rises slowly with increasing $\hat{\mu}_B$. This, however, is solely driven by the decrease of N_{tot} with decreasing $\sqrt{s_{NN}}$ and not due to an increase in fluctuations of the particle multiplicities. We thus have no evidence for reaching the vicinity of a critical point where these fluctuations and as such also $\kappa_{T, \bar{N}}$ should diverge.

V. SUMMARY

We introduced and examined the properties of a generalized isothermal compressibility which is suitable for the analysis of compressibility of matter in which the number of particles is not a conserved quantity. We calculated the compressibility of strong-interaction matter keeping temperature and the fluctuations of net-electric charge fixed, which in a noninteracting HRG is a good proxy for total charged particle numbers. We found that this generalized compressibility varies little on the pseudocritical line and stays close to the value $\kappa_{T, \sigma_Q^2} = 13.8(1.3) \text{ fm}^3/\text{GeV}$ obtained at $(T_{\text{pc}}, \mu_B) = (156.5, 0)$ MeV. This finding is

consistent with the rescaled result of the ALICE Collaboration, where we replaced the number of charged hadrons by the total number of hadrons at freeze-out. It suggests that the isothermal compressibility of strong-interaction matter along the pseudocritical transition line, or at the time of hadronization in HIC, stays close to that of an ideal (Boltzmann) gas.

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DATA AVAILABILITY

All data from our calculations, presented in the figures of this paper, can be found in [30]. They have been obtained using the data analysis packages [31,32]. The code for generating the raw datasets (gauge field configurations) is publicly available [33].

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