

Green's functions under magnetic effects in a nontrivial topology

Emerson B. S. Corrêa^{1,2,*}, Michelli S. R. Sarges^{3,†,§} and José A. Helayël-Neto^{4,‡,§}

¹*Faculdade de Física, Universidade Federal do Pará—UFPA, 66075-110, Belém, PA, Brazil*

²*Programa de Pós-Graduação em Física, Universidade Federal do Sul e Sudeste do Pará—Unifesspa, 68505-080, Marabá, PA, Brazil*

³*Independent researcher, Belém, PA, Brazil*

⁴*Centro Brasileiro de Pesquisas Físicas—CBPF, 22290-180, Rio de Janeiro, RJ, Brazil*



(Received 4 September 2025; accepted 26 November 2025; published 15 December 2025)

We compute the propagators of Bose and Dirac fields in four-dimensional Euclidean space in the presence of an external magnetic field, using a hybrid formulation that combines the Ritus and Schwinger methods. This approach yields explicit expressions for both propagators in coordinate and momentum space. By performing a suitable gauge transformation, we eliminate the components of the propagators that break translation invariance. We further derive the Matsubara frequencies of the fields within a toroidal topology. The resulting framework provides a unified mathematical structure that incorporates thermal and finite-volume effects in Green's functions under various boundary conditions, while simultaneously offering an analytically tractable representation of all Landau levels.

DOI: [10.1103/xbmn-1m8m](https://doi.org/10.1103/xbmn-1m8m)

I. INTRODUCTION

Magnetic fields constitute one of the most important physical scenarios to which quantum fields can be subjected. For instance, strong magnetic fields are believed to have been present microseconds after the big bang, during the phase transition from the so-called quark-gluon plasma phase to the hadronic phase in the early Universe [1–6]. Many interesting phenomena arise in the presence of magnetic backgrounds. One example is the chiral magnetic effect, in which an external magnetic field induces an electric current due to a chirality imbalance in the medium [7–9]. Another is magnetic catalysis, where increasing the strength of the magnetic field enhances chiral symmetry breaking [10–14]. Conversely, lattice QCD computations have shown that increasing the magnetic field strength can also lead to chiral symmetry restoration through the reduction of the chiral condensate. This phenomenon, known as inverse magnetic catalysis, was discovered in 2012 and still draws substantial interest [15–21]. Noncentral relativistic heavy-ion collisions generate magnetic fields orthogonal to the reaction plane. The relative motion of the colliding ions

can generate powerful magnetic fields, typically of the order of 10^{15} – 10^{16} -Tesla [22]. For comparison, the pion mass $m_\pi = 0.138$ GeV corresponds to a magnetic field of order $eB \sim m_\pi^2 \sim 10^{14}$ Tesla.

Green's functions determine essential properties of the corresponding quantum system under study, such as its energy spectrum and probability amplitudes [23]. In particular, within the framework of quantum field theory (QFT), Green's functions are indispensable for describing quantum fields interacting with an external electromagnetic background, as in pair-production rate analyses [24,25]. As a fundamental mathematical tool for capturing the effects of interactions on particle propagation, several methods have been developed to compute Green's functions (or Feynman propagators) [24–36]. Furthermore, there is growing interest in determining Green's functions at finite temperature, given their applicability to a wide range of phenomena, including the influence of thermal baths in high-energy heavy-ion collisions, phase transitions, and symmetry breaking or restoration processes [37–40].

In this thermal context, the Matsubara formalism has served as a robust tool in theoretical physics since the pioneering work of T. Matsubara in 1955 [41]. In his paper, Matsubara introduced a new method to compute the thermal Green's function of a statistical system by extensively applying the operator techniques commonly used in QFT. He then applied this formalism to describe an electron-phonon system, obtaining the thermal Green's functions for both bosons and fermions. The thermal distributions for these fields can be derived from the Kubo-Martin-Schwinger (KMS) conditions [42,43], which impose periodic boundary conditions on the imaginary-time

*Contact author: ecorreae@ufpa.br

†Contact author: michellissarges@gmail.com

‡Contact author: helayel@cbpf.br

§These authors contributed equally to this work.

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coordinate for bosons, and antiperiodic ones for fermions. As a direct consequence of this periodicity (or antiperiodicity) in the imaginary-time coordinate τ , discrete frequency modes emerge: $\omega_{n_\tau} = (2n_\tau)\pi/\beta$, for bosons or $\omega_{n_\tau} = (2n_\tau + 1)\pi/\beta$, for fermions, where $n_\tau \in \mathbb{Z}$ and β is the inverse temperature of the system [44].

In 1980, the authors of Ref. [45] extended the Matsubara formalism to include one spatial coordinate (current called of generalized Matsubara formalism). This development marked the starting point for QFT on nontrivial topologies and made it possible to account for finite-size effects in the quantum fields. However, the physical criterion prescribed by the KMS conditions for the imaginary-time coordinate does not necessarily apply to spatial coordinates. At this stage, one has the freedom to choose the type of boundary condition to impose along the compactified spatial direction. For instance, the authors of Ref. [46] employed periodic boundary conditions for bosonic systems in $(D - 1)$ spatial coordinates, while adopting antiperiodic boundary conditions for their fermionic counterparts. Notwithstanding, in Ref. [47], phase diagrams were obtained using the effective potential technique in lower-dimensional systems, considering both periodic and antiperiodic boundary conditions over the same compactified spatial coordinate for fermions. The literature contains several studies that explore periodic, antiperiodic, and even twisted boundary conditions along spatial coordinates, regardless of whether the quantum field carries integer or half-integer spin [48–51]. When a magnetic field $\mathbf{B}$ permeates a charged quantum field, quantized energy levels known as Landau levels arise. In the lowest Landau level, a genuine dimensional reduction occurs, changing the system from $1 + 3 \rightarrow 1 + 1$ dimensions [52–56]. However, if the system occupies higher Landau levels, even though the system is still in four dimensions, $\mathbf{B}$ reduces the number of independent momentum integrations. Thus, two spatial momentum components become hidden, according to the fact that the transverse motion is quantized and no longer corresponds to continuous degrees of freedom. Mathematically,

$$\int \frac{d^D p}{(2\pi)^D} \xrightarrow{\mathbf{B}} \sum_{\ell=0}^{\infty} \frac{|eB|}{2\pi} \int \frac{d^{(D-2)} p}{(2\pi)^{(D-2)}}.$$

Consequently, a charged system that originally involved four Euclidean momentum integrations over (p_τ, p_x, p_y, p_z) is reduced to only two independent momentum integrations, (p_τ, p_z) , for a magnetic field oriented along the z direction. Due to this apparent reduction, a QFT formulated on a toroidal topology and analyzed via the generalized Matsubara formalism effectively possesses only a two-dimensional momentum space (a “film,” heated under $\mathbf{B}$), as implemented in Refs. [57–68].

The two momentum integrals that appear to be lost due to the magnetic-field effects in the propagators can, in fact, be recovered through careful algebraic manipulation and

the use of integral representations of special functions, as demonstrated in scalar [30,69] and fermionic [28,29,55] contexts. The main purpose of this article is to employ a hybrid approach, here referred to as the Ritus-Schwinger method, to obtain the Green’s functions for both Bose and Dirac fields without reducing the number of momentum integrals, while simultaneously establishing the generalized Matsubara formalism for the translationally invariant components of the propagators. This is accomplished by applying an appropriate gauge transformation to the computed Green’s functions.

The paper is organized as follows. In Sec. II, we compute the Green’s function of a spinless field in the presence of a constant and homogeneous magnetic field oriented along the z axis, using a combination of Ritus’s eigenfunction method and Schwinger’s proper-time approach. As we show, our resulting expressions for the Green’s functions are exact. In Sec. III, we extend this hybrid method to evaluate the electron propagator under the same external field. In Sec. IV, a gauge transformation is applied to remove the Schwinger phase, which would otherwise break the translational invariance of the propagation functions. Section V is devoted to establishing the expressions for QFT on a toroidal topology, without the usual reduction in the number of spatial integrals typically induced by the external magnetic field. We present our conclusions and final remarks in Sec. VI. Throughout the paper, we adopt natural units, $c = \hbar = k_B = 1$ and work in four-dimensional Euclidean space.

II. BOSE FIELD PROPAGATOR UNDER A MAGNETIC FIELD

A. Ritus’s method and the scalar propagator

We begin by calculating the propagator of a scalar field in four-dimensional Euclidean space in the presence of a uniform, homogeneous external magnetic field $\vec{B}$ oriented along the z axis, considering the bulk (infinite-space) case. The scalar field in this magnetic background satisfies the following differential equation

$$(-D_\mu^2 + m_0^2)\Phi(u) = 0,$$

where $u \equiv u^\rho = (\tau, x, y, z)$, $D_\mu = \partial_\mu - ieA_\mu^{\text{ext}}$, and e denotes the boson charge. We adopt the gauge $A_\mu^{\text{ext}} = (0, 0, xB, 0)$, with B a constant, so that the corresponding external magnetic field is given by $\vec{\nabla} \times \vec{A}^{\text{ext}} = B\hat{z}$.

The Green’s function of the scalar field satisfies the equation

$$(D_\mu^2 - m_0^2)G(u, u', A) = -\delta^4(u - u'). \quad (1)$$

To determine the Green’s function, we employ the eigenfunction method, commonly known as Ritus’s method. This approach relies on the existence of a

complete set of eigenfunctions $E_p(u)$ of the operator D_μ^2 . In brief, the method is based on the eigenvalue equation [26,27,32,34–36]

$$D_\mu^2 E_p = -p^2 E_p, \quad (2)$$

where E_p are the Ritus' eigenfunctions, and

$$D_\mu^2 = \nabla_{4D}^2 - 2i\omega x \partial_y - \omega^2 x^2, \quad (3)$$

where $\omega \equiv eB > 0$ denotes the cyclotron frequency. Once the complete set of eigenfunctions E_p is determined, the Green's function can be expanded as

$$G(u, u', A) = \sum_p d p E_p(u) \tilde{G}(p, A) E_p^*(u'). \quad (4)$$

Since, by construction,

$$\sum_p d p E_p(u) E_p^*(u') = \delta^4(u - u'), \quad (5)$$

we obtain the scalar propagator in momentum space by employing Eqs. (1), (2), and (4)

$$\tilde{G}(p, A) = \frac{1}{p^2 + m_0^2}. \quad (6)$$

Thus, once the eigenfunctions $E_p(u)$ and the corresponding eigenvalues p^2 are determined, the propagator of the spinless field in the presence of an external magnetic field is given by Eq. (4) with $\tilde{G}(p, A)$ specified by Eq. (6).

We solve the eigenvalue equation using the standard ansatz [30]

$$E_p = C \exp[i(p_\tau \tau + p_z z + \omega \xi y)] X(x). \quad (7)$$

By applying the ansatz to the eigenvalue equation and the Klein-Gordon operator in Eq. (3), we obtain the differential equation for $X(x)$,

$$\left[\frac{d^2}{dx^2} - \omega^2(x - \xi)^2 + (p^2 - p_\tau^2 - p_z^2) \right] X(x) = 0. \quad (8)$$

It is well known that the differential Hermite equation admits finite solutions only for discrete values of the constant term, such that $(p^2 - p_\tau^2 - p_z^2) \equiv \omega(2\ell + 1)$, where ℓ is a non-negative integer. This condition ensures that Eq. (8) possesses finite solutions.

Therefore, the solutions in the x variable, which encompass all Landau levels ℓ , are given by

$$X_\ell(x) = (\omega)^{1/4} h_\ell[\sqrt{\omega}(x - \xi)], \quad (9)$$

where the functions $h_\ell[\sqrt{\omega}(x - \xi)]$ are the Hermite functions, defined as

$$h_\ell(s) \equiv \frac{1}{\sqrt{2^\ell \ell! \sqrt{\pi}}} \exp(-s^2/2) H_\ell(s). \quad (10)$$

Thanks to the properties of the Hermite polynomials $H_\ell(s)$, the functions $h_\ell(s)$ form an orthonormal set [70]

$$\begin{cases} \int_{-\infty}^{\infty} h_\ell(s) h_{\ell'}(s) ds, & = \delta_{\ell, \ell'}, \\ \sum_{\ell=0}^{\infty} h_\ell(s) h_\ell(s') & = \delta(s - s'). \end{cases} \quad (11)$$

By combining the ansatz in Eq. (7) for $E_p(u)$, the second relation given in Eq. (11), and the complex exponential representation of the Dirac delta function, one readily verifies the validity of Eq. (5). With the normalization constant $|C| = 1/\sqrt{(2\pi)^3}$, the Ritus eigenfunctions take the form

$$E_p(u) = \frac{\omega^{1/4}}{\sqrt{(2\pi)^3}} \exp[i(p_\tau \tau + p_z z + \omega \xi y)] h_\ell[\sqrt{\omega}(x - \xi)]. \quad (12)$$

From Eqs. (12) and (11), it is straightforward to show that the Ritus eigenfunctions satisfy the orthogonality relation

$$\begin{aligned} \int_{-\infty}^{+\infty} du E_p(u) E_{p'}^*(u) &= \delta(p_\tau - p'_\tau) \delta[\omega(\xi - \xi')] \\ &\quad \times \delta(p_z - p'_z) \delta_{\ell, \ell'}, \quad \ell = 0, 1, 2, \dots \end{aligned}$$

The Green's function of the spin-zero field propagator under the magnetic field then takes the form

$$G(u, u', A) = \sum_{\ell=0}^{\infty} \int d p_\tau d p_z d \xi E_p(u) \tilde{G}(p, A) E_p^*(u'), \quad (13)$$

where $E_p(u)$ is given by Eq. (12), and $\tilde{G}(p, A)$ is provided by Eq. (6), with the discrete values of p^2 given by

$$p^2 = p_\tau^2 + p_z^2 + \omega(2\ell + 1), \quad \ell = 0, 1, 2, 3, \dots \quad (14)$$

Physically, the quantization of the orbital angular momentum is manifested through the Landau levels ℓ .

B. Deriving the scalar propagator in the proper-time representation

Let us now obtain the scalar propagator using Schwinger's proper-time representation, denoted by the parameter S . To begin, we employ the operator identity

$$\mathcal{O}^{-1} = \int_0^\infty dS \exp(-S\mathcal{O}). \quad (15)$$

Applying this identity to Eq. (13), with $\mathcal{O} \equiv [p_\tau^2 + p_z^2 + \omega(2\ell + 1) + m_0^2]$, we obtain

$$G(u, u', A) = \int_0^\infty dS \int \frac{dp_\tau}{2\pi} \frac{dp_z}{2\pi} \frac{(\omega)^{3/2}}{2\pi} \exp \{i[p_\tau(\tau - \tau') + p_z(z - z')]\} \\ \times \exp[-S(p_\tau^2 + p_z^2 + m_0^2)] \mathcal{I}, \quad (16)$$

where $\mathcal{I}$ is defined by

$$\mathcal{I} \equiv \sum_{\ell=0}^{\infty} \exp[-S\omega(2\ell + 1)] \int_{-\infty}^{+\infty} d\xi \exp \{i[\omega\xi(y - y')]\} h_\ell[\sqrt{\omega}(x - \xi)] h_\ell[\sqrt{\omega}(x' - \xi)].$$

Considering the definition of the Hermite functions in Eq. (10), and after completing the square in the variable ξ , we get

$$\mathcal{I} = \sum_{\ell=0}^{\infty} \exp[-S\omega(2\ell + 1)] \frac{1}{2^\ell \ell! \sqrt{\pi}} \int_{-\infty}^{+\infty} d\xi \exp(\mathcal{D}) H_\ell[\sqrt{\omega}(x - \xi)] H_\ell[\sqrt{\omega}(x' - \xi)], \quad (17)$$

where $\mathcal{D}$ is a shorthand notation for

$$\mathcal{D} = -\omega \left\{ \left[\xi - \mathcal{A} \right]^2 + \frac{1}{4} [(x - x')^2 + (y - y')^2] - \frac{i}{2} (x + x')(y - y') \right\}; \quad \mathcal{A} \equiv \frac{(x + x') + i(y - y')}{2}.$$

Under the change of variable $\xi \rightarrow -\xi$, Eq. (17) becomes

$$\mathcal{I} = \sum_{\ell=0}^{\infty} \exp[-S\omega(2\ell + 1)] \frac{1}{2^\ell \ell! \sqrt{\pi}} \exp \left\{ i \frac{\omega}{2} [(x + x')(y - y')] \right\} \\ \times \exp \left\{ -\frac{\omega}{4} [(x - x')^2 + (y - y')^2] \right\} \int_{-\infty}^{+\infty} d\xi \exp[-\omega(\xi + \mathcal{A})^2] \\ \times H_\ell[\sqrt{\omega}(\xi + x)] H_\ell[\sqrt{\omega}(\xi + x')]. \quad (18)$$

By making the variable change $\Theta = \sqrt{\omega}(\xi + \mathcal{A})$, we obtain

$$\mathcal{I} = \sum_{\ell=0}^{\infty} \exp[-S\omega(2\ell + 1)] \frac{1}{\sqrt{\omega} 2^\ell \ell! \sqrt{\pi}} \exp \left\{ i \frac{\omega}{2} [(x + x')(y - y')] \right\} \\ \times \exp \left\{ -\frac{\omega}{4} [(x - x')^2 + (y - y')^2] \right\} \int_{-\infty}^{+\infty} d\Theta \exp(-\Theta^2) \\ \times H_\ell[\Theta + \sqrt{\omega}(x - \mathcal{A})] H_\ell[\Theta + \sqrt{\omega}(x' - \mathcal{A})]. \quad (19)$$

Finally, by employing the well-known relation between Hermite and associated Laguerre polynomials,

$$\int_{-\infty}^{+\infty} d\Theta \exp(-\Theta^2) H_\ell(\Theta + a) H_{\ell'}(\Theta + a') = 2^{\ell'} \sqrt{\pi} \ell! (a')^{\ell-\ell'} L_\ell^{\ell'-\ell}(-2aa'),$$

we get

$$\mathcal{I} = \frac{1}{\sqrt{\omega}} \exp \left\{ i \frac{\omega}{2} [(x + x')(y - y')] \right\} \exp \left\{ -\frac{\omega}{4} [(x - x')^2 + (y - y')^2] \right\} \\ \times \sum_{\ell=0}^{\infty} \exp[-S\omega(2\ell + 1)] L_\ell^0[-2\omega(x - \mathcal{A})(x' - \mathcal{A})]. \quad (20)$$

It can be readily shown that

$$(x - \mathcal{A})(x' - \mathcal{A}) = -\frac{1}{4}[(x - x')^2 + (y - y')^2] \equiv -\frac{1}{4}[(\mathbf{r} - \mathbf{r}')_{\perp}^2], \quad (21)$$

which yields

$$\mathcal{I} = \frac{1}{\sqrt{\omega}} \exp[ie\Phi(\mathbf{r}_{\perp}, \mathbf{r}'_{\perp})] \sum_{\ell=0}^{\infty} \exp[-S\omega(2\ell+1)] \exp\left\{-\frac{\omega}{4}[(\mathbf{r} - \mathbf{r}')_{\perp}^2]\right\} L_{\ell}\left[\frac{\omega}{2}(\mathbf{r} - \mathbf{r}')_{\perp}^2\right], \quad (22)$$

where we define the so-called Schwinger phase as

$$\Phi(\mathbf{r}_{\perp}, \mathbf{r}'_{\perp}) \equiv \frac{B}{2}[(x + x')(y - y')]. \quad (23)$$

Let us now rewrite Eq. (22) in terms of the momentum coordinates. In this context, we make use of the integral representation of a Gaussian multiplied by Laguerre polynomials, namely

$$\frac{(\alpha - \beta)^n}{\alpha^{n+1}} \exp(-\mathcal{Y}^2/2\alpha) L_n\left[\frac{\beta\mathcal{Y}^2}{2\alpha(\beta - \alpha)}\right] = \int_0^{\infty} d\mathcal{X} \mathcal{X} \exp(-\alpha\mathcal{X}^2/2) L_n(\beta\mathcal{X}^2/2) J_0(\mathcal{X}\mathcal{Y}), \quad (24)$$

where $\mathcal{Y} > 0$, $\Re(\alpha) > 0$ and J_0 is the Bessel function [see Ref. [71], equation ET II 13(4)a].

We can rewrite Eq. (24) in a more convenient form by introducing the definitions $\mathcal{X} \equiv p_{\perp} = \sqrt{p_x^2 + p_y^2}$, $\mathcal{Y}^2 \equiv (\mathbf{r} - \mathbf{r}')_{\perp}^2$, and $\alpha = \beta/2 \equiv 2/\omega$. The resulting expression is

$$\exp\left\{-\frac{\omega}{4}[(\mathbf{r} - \mathbf{r}')_{\perp}^2]\right\} L_{\ell}\left[\frac{\omega}{2}(\mathbf{r} - \mathbf{r}')_{\perp}^2\right] = \frac{2(-1)^{\ell}}{\omega} \int_0^{\infty} dp_{\perp} p_{\perp} \exp(-p_{\perp}^2/\omega) L_{\ell}(2p_{\perp}^2/\omega) J_0(p_{\perp} \sqrt{(\mathbf{r} - \mathbf{r}')_{\perp}^2}).$$

By substituting this result into Eq. (22), we encounter

$$\mathcal{I} = \exp[ie\Phi(\mathbf{r}_{\perp}, \mathbf{r}'_{\perp})] \left(\frac{2\exp(-S\omega)}{\sqrt{\omega^3}}\right) \int_0^{\infty} dp_{\perp} p_{\perp} \exp(-p_{\perp}^2/\omega) J_0(p_{\perp} \sqrt{(\mathbf{r} - \mathbf{r}')_{\perp}^2}) \sum_{\ell=0}^{\infty} [-\exp(-2S\omega)]^{\ell} L_{\ell}(2p_{\perp}^2/\omega). \quad (25)$$

We now proceed to sum over all Landau levels. The generating function for the Laguerre polynomials is given by [see Ref. [71], Eq. EH II 189(17), MO 109]

$$\sum_{\ell=0}^{\infty} \mathcal{Z}^{\ell} L_{\ell}^{\epsilon}(\mathcal{Q}) = \frac{1}{(1 - \mathcal{Z})^{1+\epsilon}} \exp\left[\frac{\mathcal{Q}\mathcal{Z}}{(\mathcal{Z} - 1)}\right],$$

for $|\mathcal{Z}| < 1$. In the present case, we set $\mathcal{Z} \equiv -\exp[-2\omega S]$ and $\mathcal{Q} \equiv 2p_{\perp}^2/\omega$. Consequently, the summation over ℓ becomes (for $\epsilon = 0$)

$$\sum_{\ell=0}^{\infty} [-\exp(-2\omega S)]^{\ell} L_{\ell}(2p_{\perp}^2/\omega) = \frac{\exp(\omega S)}{2 \cosh(\omega S)} \exp\left\{\frac{p_{\perp}^2}{\omega} \left[\frac{2}{1 + \exp(2\omega S)}\right]\right\}. \quad (26)$$

Upon substituting Eq. (26) into Eq. (25), we obtain

$$\mathcal{I} = \frac{\exp[ie\Phi(\mathbf{r}_{\perp}, \mathbf{r}'_{\perp})]}{\sqrt{\omega^3} \cosh(\omega S)} \int_0^{\infty} dp_{\perp} p_{\perp} J_0(p_{\perp} \sqrt{(\mathbf{r} - \mathbf{r}')_{\perp}^2}) \exp\left[-\frac{p_{\perp}^2}{\omega} \tanh(\omega S)\right]. \quad (27)$$

Finally, we employ the integral representation of the Bessel function J_0

$$J_0(p_{\perp} \sqrt{(\mathbf{r} - \mathbf{r}')_{\perp}^2}) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \exp[ip_{\perp} \sqrt{(\mathbf{r} - \mathbf{r}')_{\perp}^2} \cos \varphi], \quad (28)$$

which allows us to express Eq. (27) as

$$\mathcal{I} = 2\pi \frac{\exp[ie\Phi(\mathbf{r}_\perp, \mathbf{r}'_\perp)]}{\sqrt{\omega^3} \cosh(\omega S)} \int \frac{dp_x}{2\pi} \frac{dp_y}{2\pi} \exp\{i[p_x(x-x') + p_y(y-y')]\} \exp\left[-(p_x^2 + p_y^2) \frac{\tanh(\omega S)}{\omega}\right], \quad (29)$$

where we have transformed the integral from cylindrical to Cartesian momentum coordinates.

Having determined $\mathcal{I}$, we can substitute Eq. (29) into Eq. (16), yielding the spinless propagator in the presence of a constant external magnetic field in the proper-time representation

$$G(u, u', A) = \exp[ie\Phi(\mathbf{r}_\perp, \mathbf{r}'_\perp)] \int_0^\infty dS \int \frac{d^4 p}{(2\pi)^4} \exp[ip_\mu(u-u')_\mu] \times \frac{1}{\cosh(\omega S)} \exp\left\{-S\left[p_\tau^2 + p_z^2 + (p_x^2 + p_y^2) \frac{\tanh(\omega S)}{\omega S} + m_0^2\right]\right\}. \quad (30)$$

Equation (30) was derived using an alternative approach in Ref. [30] (by considering the kinetic term of the Hamiltonian) and in Minkowski spacetime in Ref. [69]. The Feynman propagator for the spin-zero field, accounting for all Landau levels as given by Eq. (30), reduces exactly to the free propagator in the limit $\omega \rightarrow 0$.

III. DIRAC FIELD PROPAGATOR UNDER A MAGNETIC FIELD

A. Ritus's method and the fermion propagator

The Dirac equation in four-dimensional Euclidean space under a magnetic field oriented along the z direction is given by

$$(\mathcal{D} + im_0)\Psi(u) = 0.$$

Here, we again use the minimal coupling $D_\mu = \partial_\mu - ieA_\mu^{\text{ext}}$ and $A_\mu^{\text{ext}} = (0, 0, xB, 0)$. The Green's function in this fermionic case is then written as

$$(\mathcal{D} + im_0)\mathbb{S}(u, u', A) = \mathbb{I}_4 \delta^4(u - u'). \quad (31)$$

We use the gamma matrices in Euclidean space as defined in [35]

$$\gamma_0 = -i \begin{pmatrix} 0_2 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0_2 \end{pmatrix}; \quad \gamma_j = \begin{pmatrix} 0_2 & -\sigma_j \\ \sigma_j & 0_2 \end{pmatrix}, \quad j = 1, 2, 3, \quad (32)$$

where are σ_j the Pauli matrices. In this representation, we have

$$\mathcal{D} = \gamma_\mu D_\mu; \quad \{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}.$$

The commutation relation $[\mathcal{D}^2, \mathcal{D}] = 0$ allows us to look for eigenfunctions of the quadratic Dirac operator. Consequently, we can rewrite Eq. (31) as

$$(\mathcal{D}^2 + m_0^2)\mathbb{G}(u, u', A) = \mathbb{I}_4 \delta^4(u - u'), \quad (33)$$

where $\mathbb{G}(u, u', A)$ is an auxiliary propagator, related to the electron propagator $\mathbb{S}(u, u', A)$ by

$$\mathbb{S}(u, u', A) \equiv (\mathcal{D} - im_0)\mathbb{G}(u, u', A). \quad (34)$$

Therefore, we first compute the quantity $\mathbb{G}(u, u', A)$, and then apply the operator $(\mathcal{D} - im_0)$ over it to obtain the fermionic propagator $\mathbb{S}(u, u', A)$. From the definitions above, it is straightforward to show that

$$\mathcal{D}^2 = \frac{1}{4}(ie[\gamma_\mu, \gamma_\nu]F_{\mu\nu} - 4D_\mu^2),$$

where D_μ^2 is the same operator defined in Eq. (3). For a purely magnetic background, we have $F_{12} = -F_{21} = B$. The Dirac matrices defined in Eq. (32) allow us to write

$$\mathcal{D}^2 = \omega(\mathbb{I}_2 \otimes \sigma_z) - D_\mu^2, \quad (35)$$

being

$$(\mathbb{I}_2 \otimes \sigma_z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (36)$$

As in the bosonic case, once we find the eigenfunctions $\mathbb{E}_p$ of the quadratic Dirac operator, we can express the auxiliary propagator $\mathbb{G}$ as an expansion in terms of these eigenfunctions, namely

$$\mathbb{G}(u, u', A) = \sum_p dp \mathbb{E}_p(u) \tilde{\mathbb{G}}(p, A) \mathbb{E}_p^\dagger(u'). \quad (37)$$

From Eq. (35), we observe that the only difference between the scalar propagator G and the auxiliary fermion propagator $\mathbb{G}$ is the spin term $(\mathbb{I}_2 \otimes \sigma_z)$. Thus, it is sufficient to introduce the following matrix ansatz

$$\mathbb{E}_p(u) = E_p(u)\mathbf{\Omega}_{\sigma_B}, \quad (38)$$

where the matrix $\mathbf{\Omega}_{\sigma_B}$ is diagonal,

$$\mathbf{\Omega}_{\sigma_B} = \text{diag}(\delta_{1,\sigma_B}; \delta_{-1,\sigma_B}; \delta_{1,\sigma_B}; \delta_{-1,\sigma_B}), \quad (39)$$

and $E_p(u)$ are given by Eq. (12).

We note that the matrix $\mathbb{E}_p(u)$ satisfies the relation

$$(\mathbb{I}_2 \otimes \sigma_z)\mathbb{E}_p(u) = \mathbb{E}_p(u)(\mathbb{I}_2 \otimes \sigma_z). \quad (40)$$

Using Eqs. (38), (39), and (5), along with the relation $(\mathbf{\Omega}_{\sigma_B} \mathbf{\Omega}_{\sigma_B}^T) = \mathbb{I}_4$, we straightforwardly find that

$$\oint dp \mathbb{E}_p(u) \mathbb{E}_p^\dagger(u') = \mathbb{I}_4 \delta^4(u - u'). \quad (41)$$

By performing steps analogous to those in the bosonic case, i.e., substituting Eqs. (35), (37), (38), and (2) into

Eq. (33) and using the Dirac-delta representation from Eq. (41) along with the property stated in Eq. (40), we obtain the electron propagator in momentum space, which has the following matrix structure

$$\tilde{\mathbb{G}}(p, A) = \frac{1}{[p^2 + m_0^2 + \omega(\mathbb{I}_2 \otimes \sigma_z)]}, \quad (42)$$

where p^2 is defined in Eq. (14).

B. Deriving the fermionic propagator in the proper-time representation

In this subsection, we apply the Schwinger trick given by Eq. (15) and perform the summation over the Landau levels ℓ . The computations follow the same steps that lead to Eq. (30), with the exception of the spin term. The resulting expression is

$$\begin{aligned} \mathbb{G}(u, u', A) = & \exp[ie\Phi(\mathbf{r}_\perp, \mathbf{r}'_\perp)] \int_0^\infty dS \int \frac{d^4 p}{(2\pi)^4} \exp[ip_\mu(u - u')_\mu] \frac{1}{\cosh(\omega S)} \\ & \times \exp\left\{-S\left[p_\tau^2 + p_z^2 + (p_x^2 + p_y^2) \frac{\tanh(\omega S)}{\omega S} + m_0^2 + \omega(\mathbb{I}_2 \otimes \sigma_z)\right]\right\}, \end{aligned} \quad (43)$$

where Schwinger's phase is defined in Eq. (23).

Finally, let us apply the operator $(\not{D} - im_0)$ to Eq. (43) in order to obtain the fermionic propagator $\mathbb{S}(u, u', A)$, as defined in Eq. (34). The application is straightforward and yields

$$\begin{aligned} \mathbb{S}(u, u', A) = & \exp[ie\Phi(\mathbf{r}_\perp, \mathbf{r}'_\perp)] \int_0^\infty dS \int \frac{d^4 p}{(2\pi)^4} \left\{ i\not{p} - im_0 + i\frac{\omega}{2}[\gamma_x(y - y') - \gamma_y(x - x')] \right\} \\ & \times \frac{\exp[ip_\mu(u - u')_\mu]}{\cosh(\omega S)} \exp\left\{-S\left[p_\tau^2 + p_z^2 + (p_x^2 + p_y^2) \frac{\tanh(\omega S)}{\omega S} + m_0^2 + \omega(\mathbb{I}_2 \otimes \sigma_z)\right]\right\}. \end{aligned} \quad (44)$$

However, we can write the typical terms in Eq. (44) as

$$\frac{d}{dp_x} \{ \exp[ip_x(x - x')] \exp[-p_x^2 \tanh(\omega S)/\omega] \} = \left[i(x - x') - \frac{2p_x}{\omega} \tanh(\omega S) \right] \{ \exp[ip_x(x - x')] \exp[-p_x^2 \tanh(\omega S)/\omega] \}. \quad (45)$$

By integrating Eq. (45) term by term over the variable $-\infty \leq p_x \leq +\infty$, and taking into account the decaying factor $\exp(-p_x^2)$ at the boundaries, we obtain

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{dp_x}{2\pi} [i(x - x')] \{ \exp[ip_x(x - x')] \exp[-p_x^2 \tanh(\omega S)/\omega] \} \\ & = \int_{-\infty}^{+\infty} \frac{dp_x}{2\pi} \frac{2p_x}{\omega} \tanh(\omega S) \{ \exp[ip_x(x - x')] \exp[-p_x^2 \tanh(\omega S)/\omega] \}. \end{aligned} \quad (46)$$

Similarly, for the y coordinate,

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{dp_y}{2\pi} [i(y - y')] \{ \exp[ip_y(y - y')] \exp[-p_y^2 \tanh(\omega S)/\omega] \} \\ & = \int_{-\infty}^{+\infty} \frac{dp_y}{2\pi} \frac{2p_y}{\omega} \tanh(\omega S) \{ \exp[ip_y(y - y')] \exp[-p_y^2 \tanh(\omega S)/\omega] \}. \end{aligned} \quad (47)$$

Substituting Eqs. (46) and (47) into Eq. (44), we get

$$\begin{aligned} \mathbb{S}(u, u', A) = & \exp[ie\Phi(\mathbf{r}_\perp, \mathbf{r}'_\perp)] \int_0^\infty dS \int \frac{d^4 p}{(2\pi)^4} \{i\not{p} - im_0 + [\gamma_x p_y - \gamma_y p_x] \tanh(\omega S)\} \\ & \times \frac{\exp[ip_\mu(u - u')_\mu]}{\cosh(\omega S)} \exp\left\{-S\left[p_\tau^2 + p_z^2 + (p_x^2 + p_y^2) \frac{\tanh(\omega S)}{\omega S} + m_0^2 + \omega(\mathbb{I}_2 \otimes \sigma_z)\right]\right\}. \end{aligned} \quad (48)$$

Using the series expansion for the spin term,

$$\exp[-\omega S(\mathbb{I}_2 \otimes \sigma_z)] = \sum_{n=0}^{\infty} \frac{[-\omega S(\mathbb{I}_2 \otimes \sigma_z)]^n}{n!},$$

we obtain

$$\exp[-\omega S(\mathbb{I}_2 \otimes \sigma_z)] = \mathbb{I}_4 \left[1 + \frac{(\omega S)^2}{2!} + \frac{(\omega S)^4}{4!} + \dots\right] - (\mathbb{I}_2 \otimes \sigma_z) \left[(\omega S) + \frac{(\omega S)^3}{3!} + \frac{(\omega S)^5}{5!} + \dots\right],$$

which yields

$$\exp[-\omega S(\mathbb{I}_2 \otimes \sigma_z)] = \mathbb{I}_4 \cosh(\omega S) - (\mathbb{I}_2 \otimes \sigma_z) \sinh(\omega S). \quad (49)$$

Therefore, after substituting Eq. (49) into Eq. (48), we obtain the electron propagator in the proper-time representation,

$$\begin{aligned} \mathbb{S}(u, u', A) = & \exp[ie\Phi(\mathbf{r}_\perp, \mathbf{r}'_\perp)] \int_0^\infty dS \int \frac{d^4 p}{(2\pi)^4} \{i\not{p} - im_0 + [\gamma_x p_y - \gamma_y p_x] \tanh(\omega S)\} \\ & \times \exp[ip_\mu(u - u')_\mu] \exp\left\{-S\left[p_\tau^2 + p_z^2 + (p_x^2 + p_y^2) \frac{\tanh(\omega S)}{\omega S} + m_0^2\right]\right\} \\ & \times [\mathbb{I}_4 - (\mathbb{I}_2 \otimes \sigma_z) \tanh(\omega S)]. \end{aligned} \quad (50)$$

Equation (50) was also obtained by Miransky and Shovkovy in Eq. (107) of Ref. [55]. We note that in the limit $B \rightarrow 0$, Eq. (50) correctly reduces to the free electron propagator, as expected.

IV. GAUGE TRANSFORMATION

Note that both the scalar and fermion propagators in a magnetic field oriented along the z direction are not translationally invariant in the plane perpendicular to the external field. This is due to the Schwinger phase, $\Phi(\mathbf{r}_\perp, \mathbf{r}'_\perp)$. However, a gauge transformation can be performed to remove the translationally noninvariant part of the Green's functions, as shown below.

We begin by showing that the translationally noninvariant part of the propagators can be obtained by integrating the vector potential along a straight line connecting the initial and final points in the plane perpendicular to the magnetic field, i.e.,

$$\Phi(\mathbf{r}_\perp, \mathbf{r}'_\perp) = \int_{\mathbf{r}'_\perp}^{\mathbf{r}_\perp} \vec{A}(\vec{\xi}) \cdot d\vec{\xi}, \quad (51)$$

where

$$\vec{A}(\vec{\xi}) = (0, B\xi_1, 0) \Rightarrow \vec{\nabla}_{\vec{\xi}} \times \vec{A}(\vec{\xi}) = B\hat{z}.$$

The straight line can be parametrized in terms of a parameter λ (see Chapter 3 of Ref. [72]).

$$\vec{\xi} = (1 - \lambda)\vec{r}'_\perp + \lambda\vec{r}_\perp, \quad \therefore d\vec{\xi} = (\vec{r}_\perp - \vec{r}'_\perp)d\lambda, \quad 0 \leq \lambda \leq 1.$$

Therefore,

$$\begin{aligned} \int_{\mathbf{r}'_\perp}^{\mathbf{r}_\perp} \vec{A}(\vec{\xi}) \cdot d\vec{\xi} &= \int_0^1 B[(1 - \lambda)x' + \lambda x](y - y')d\lambda \\ &= \frac{B}{2}(x + x')(y - y') \equiv \Phi(\mathbf{r}_\perp, \mathbf{r}'_\perp). \end{aligned} \quad (52)$$

We see that Eqs. (30) and (50) can be rewritten using

$$\exp[ie\Phi(\mathbf{r}_\perp, \mathbf{r}'_\perp)] = \exp\left[ie \int_{\mathbf{r}'_\perp}^{\mathbf{r}_\perp} \vec{A}(\vec{\xi}) \cdot d\vec{\xi}\right]. \quad (53)$$

Since we have the freedom to choose an arbitrary function $\Lambda(\vec{\xi})$ that does not affect the external field, we obtain an equivalent expression for the propagator

according to the following prescription

$$\vec{A} \mapsto \vec{A} + \vec{\nabla} \Lambda \Rightarrow \begin{cases} G(u, u', A) \mapsto G(u, u', A + \partial \Lambda), \\ \mathbb{S}(u, u', A) \mapsto \mathbb{S}(u, u', A + \partial \Lambda). \end{cases} \quad (54)$$

Without loss of generality, we can choose the arbitrary function as $\Lambda(\vec{\xi}) = -B(x + x')\xi_2/2$, whose gradient, when

integrated along the straight line, is given by

$$\int_{\mathbf{r}'_{\perp}}^{\mathbf{r}_{\perp}} \vec{\nabla} \Lambda(\vec{\xi}) \cdot d\vec{\xi} = -\frac{B}{2}(x + x')(y - y'). \quad (55)$$

Therefore, under the gauge transformation in Eq. (54) and using Eq. (55), the Green's function for the spinless field in a magnetic background becomes

$$G(u - u', \omega) = \int_0^\infty dS \int \frac{d^4 p}{(2\pi)^4} \exp[ip_\mu(u - u')_\mu] \frac{1}{\cosh(\omega S)} \exp \left\{ -S \left[p_\tau^2 + p_z^2 + (p_x^2 + p_y^2) \frac{\tanh(\omega S)}{\omega S} + m_0^2 \right] \right\}. \quad (56)$$

Analogously, the Green's function of the fermionic field in a magnetic background, taking into account Eqs. (54) and (55), reads

$$\begin{aligned} \mathbb{S}(u - u', \omega) &= \int_0^\infty dS \int \frac{d^4 p}{(2\pi)^4} \{ i\not{p} - im_0 + [\gamma_x p_y - \gamma_y p_x] \tanh(\omega S) \} \\ &\times \exp[ip_\mu(u - u')_\mu] \exp \left\{ -S \left[p_\tau^2 + p_z^2 + (p_x^2 + p_y^2) \frac{\tanh(\omega S)}{\omega S} + m_0^2 \right] \right\} \\ &\times [\mathbb{I}_4 - (\mathbb{I}_2 \otimes \sigma_z) \tanh(\omega S)]. \end{aligned} \quad (57)$$

In addition to the fact that Eqs. (56) and (57) are translationally invariant, this property will be used to establish the generalized Matsubara formalism in the next Section.

V. QFT IN A TOROIDAL TOPOLOGY UNDER A MAGNETIC FIELD

In this Section, we consider QFT on a toroidal topology embedded in an external magnetic field. We begin by setting $u' = (0, 0, 0, 0)$ in both the scalar and spinor Green's functions given in Eqs. (56) and (57). This choice does not affect the generality of $G(u - u', \omega)$ and $\mathbb{S}(u - u', \omega)$, but it significantly simplifies the calculations that follow.

As mentioned in the Introduction, the KMS conditions impose periodic boundary conditions on the imaginary-time coordinate τ for bosons

$$G(\tau, x_j, \omega) = +G(\tau + \beta, x_j, \omega), \quad j = 1, 2, 3,$$

and, for fermions, antiperiodic boundary conditions

$$\mathbb{S}(\tau, x_j, \omega) = -\mathbb{S}(\tau + \beta, x_j, \omega), \quad j = 1, 2, 3,$$

where $\beta^{-1} = T$ is the temperature of the heated system.

In terms of momentum coordinates, the corresponding KMS conditions for bosons are given by

$$\begin{aligned} \int \frac{dp_\tau}{2\pi} f(p_\tau, p_j) &\rightarrow \frac{1}{\beta} \sum_{n_\tau=-\infty}^{\infty} f(\omega_{n_\tau}, p_j), \quad j = 1, 2, 3. \\ p_\tau \rightarrow \omega_{n_\tau} &\equiv \frac{\pi}{\beta}(2n_\tau) - i\mu, \quad n_\tau = 0, \pm 1, \pm 2, \dots, \end{aligned} \quad (58)$$

and, for fermions, are given by

$$\begin{aligned} \int \frac{dp_\tau}{2\pi} f(p_\tau, p_j) &\rightarrow \frac{1}{\beta} \sum_{n_\tau=-\infty}^{\infty} f(\omega_{n_\tau}, p_j), \quad j = 1, 2, 3. \\ p_\tau \rightarrow \omega_{n_\tau} &\equiv \frac{\pi}{\beta}(2n_\tau + 1) - i\mu, \quad n_\tau = 0, \pm 1, \pm 2, \dots, \end{aligned} \quad (59)$$

where μ denotes the chemical potential of the system.

To incorporate a nontrivial topology in the system, it is not necessary to restrict ourselves to purely periodic or antiperiodic boundary conditions in the spatial coordinates, depending on whether the system is bosonic or fermionic. In fact, these two types of boundary conditions can be interpolated in a unified way. Consequently, each compactified spatial coordinate can be assigned periodic, antiperiodic, or even twisted boundary conditions. For bosons in a magnetic field, we have

$$G(\tau, x_j, \omega) = \exp(-i\pi\alpha_j) G(\tau, x_j + L_j, \omega), \quad j = 1, 2, 3,$$

and for fermions in a magnetic field, we have

$$\mathbb{S}(\tau, x_j, \omega) = \exp(-i\pi\alpha_j)\mathbb{S}(\tau, x_j + L_j, \omega), \quad j = 1, 2, 3,$$

where $\alpha_j = 0$ corresponds to periodic boundary conditions and $\alpha_j = 1$ to antiperiodic boundary conditions. For $0 < \alpha_j < 1$, the system is subject to twisted boundary conditions.

Therefore, the spatial momenta are prescribed as follows:

$$\int \frac{dp_j}{2\pi} f(p_\tau, p_j) \rightarrow \frac{1}{L_j} \sum_{n_j=-\infty}^{\infty} f(p_\tau, \omega_{n_j}), \quad j = 1, 2, 3.$$

$$p_j \rightarrow \omega_{n_j} \equiv \frac{\pi}{L_j} (2n_j) - i\mu_j; \quad \mu_j = i\frac{\pi}{L_j} \alpha_j, \quad j = 1, 2, 3, \quad (60)$$

where $n_j = 0, \pm 1, \pm 2, \dots$, and L_j is the length of the compactified spatial coordinate x_j .

Equations (56) and (57), together with Equations (58), (59), and (60), define the generalized Matsubara prescription in a magnetic background, without reducing the number of spatial integrals.

VI. COMMENTS AND CONCLUSIONS

We have applied the Ritus-Schwinger method to calculate the Green's functions of Bose and Dirac fields in the presence of a constant and homogeneous external magnetic field aligned along the z axis. Using Ritus eigenfunctions combined with Schwinger's technique, we perform the

summation over all Landau levels to obtain closed-form expressions for the scalar and spinor propagators.

Furthermore, we performed a gauge transformation to eliminate the translation-noninvariant part of the propagators. This allowed us to establish the Matsubara generalized prescription for bosonic and fermionic systems in the presence of an external magnetic field. Notably, our computations do not rely on the additional Ritus condition for the fermionic field used in his 1970s papers, namely: $(\gamma \cdot \Pi)E_p(x) = E_p(x)\gamma \cdot p$.

For the bosonic propagator, our expression differs slightly from that in Ref. [73], due to the use of a different ansatz for the Ritus eigenfunction E_p . We also note that the hybrid method was employed to obtain the fermion propagator in Ref. [74]. However, in that work, the expressions were given in terms of the Dirac operator $(\not{D} + m_0)$ in Minkowski spacetime. The spectrum given in our Eq. (60) for $\alpha_3 = 1$, corresponding to antiperiodic boundary conditions along the z direction, coincides exactly with that in Eq. (5) of Ref. [75].

Moreover, it was not necessary to employ the full machinery developed by Schwinger in his seminal 1951 paper. Thanks to its simplicity, the hybrid method presented here provides an elegant and practical alternative for obtaining Green's functions in external backgrounds with nontrivial topology.

DATA AVAILABILITY

No data were created or analyzed in this study.

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