

Electromagnetic structure of axial-vector mesons and implications for the muon $g - 2$

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The electromagnetic structure of axial-vector mesons is investigated via elastic and two-photon transition form factors (TFFs). To this end, we employ a framework based on the Dyson-Schwinger and Bethe-Salpeter equations within a contact interaction (CI) model. This largely algebraic approach exposes the role of symmetries and their breaking in a transparent manner, and has proven successful in describing anomaly-sensitive processes, including pseudoscalar to two-photon TFFs, $\gamma \rightarrow 3\pi$, and vector-to-pseudoscalar radiative decays. Restricting our analysis to the lowest-lying axial-vector meson multiplet, $A = \{a_1, f_1, f'_1\}$, we also explore the corresponding light-by-light contribution to the muon anomalous magnetic moment. This contribution is underestimated in the CI model mainly due to the normalization of the TFFs. After applying a phenomenologically motivated rescaling to the TFFs, a process required given the limitations of the CI model, we obtain $a_\mu^A = 8.16_{-1.02}^{+1.22} \times 10^{-11}$, consistent with contemporary assessments.

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I. INTRODUCTION

The anomalous magnetic moment of the muon, commonly dubbed muon $g - 2$ or simply a_μ , remains one of the most compelling observables in modern physics. Following the most recent analysis of the Muon $g-2$ experiment at Fermilab, the empirical value of this quantity has been determined as $a_\mu^{\text{Exp}} = 116592071.5(14.5) \times 10^{-11}$ [1]. Meanwhile, the Muon $g-2$ Theory Initiative has reached a new consensus, establishing $a_\mu^{\text{SM}} = 116592033(62) \times 10^{-11}$ as the Standard Model (SM) prediction [2]. Contrary to the situation that persisted for decades, these latest values suggest that experiment and SM are not in conflict. However, neither for the new physics enthusiasts nor for the most conservative, this is not the end of the story. The intricate nature of the nuclear strong interactions, described within the SM by quantum chromodynamics (QCD), poses a huge challenge in determining the hadronic contributions to a_μ [2–4]. These components, divided into hadronic vacuum polarization (HVP) and hadronic light-by-light (HLbL), dominate the uncertainty in a_μ^{SM} , preventing its theoretical prediction from reaching the

same level of precision as its experimental counterpart. Axial-vector (AV) mesons, while subdominant compared to pseudoscalars [5–7], remain a key source of uncertainty in the HLbL sector, to a great extent because of the limited experimental knowledge of their relevant transition form factors (TFFs) [8–10]. Axial-vector mesons are also essential for enforcing short-distance QCD constraints in HLbL scattering [11–16], and the need to consider an extended tower of excitations is becoming increasingly evident, e.g., [17–19]. This work is therefore devoted to the study of the two-photon to AV meson TFFs and their contribution to a_μ .

It is well-known that the construction of the AV to two-photon transition amplitude presents a number of challenges [17,20–25]. First, as a spin-1 hadron, its Lorentz structure is naturally more intricate than in the pseudoscalar case. Symmetry requirements and gauge invariance, among others, add to this complexity. Moreover, the Landau–Yang theorem forbids the decay of a spin-1 particle into two on-shell photons, [26,27]. Consequently, while the on-shell amplitude vanishes, AV meson TFFs are probed and defined for configurations with at least one off-shell photon, where the choice of tensor basis is no longer irrelevant [17]. Because of this limitation, experimental extractions of these form factors remain challenging, and robust theoretical descriptions become indispensable. Furthermore, while vector mesons have been extensively investigated (e.g., Refs. [28–36]), axial-vector mesons remain much less explored. The analysis of the structural similarities and differences between these correlated systems, i.e., parity partners, is particularly valuable for probing dynamical

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chiral symmetry breaking (DCSB). This mechanism strongly shapes the internal structure of hadrons [37,38], making its proper understanding essential for accurate Standard Model predictions. In that regard, the exploration of the electromagnetic structure of the AV mesons, as revealed by the corresponding elastic electromagnetic form factors (EFFs), also deserves consideration.

Our approach to QCD is based upon a combination of the Dyson-Schwinger (DS) and Bethe-Salpeter (BS) equations [39–41]. This formalism captures essential characteristics of QCD, such as confinement and DCSB, and has been applied to a wide range of quantities characterizing the hadronic structure, including distribution functions and form factors (see, e.g., [37,38]). Concerning a_μ , both pseudoscalar pole and box contributions have been investigated [42–46]. More recently, the axial-vector and scalar mesons contributions were discussed in Ref. [24]. Herein we employ a primarily algebraic framework that makes the origin of symmetries and anomalies transparent. Specifically, we work within the contact interaction (CI) model presented in Ref. [31], which has proven successful in describing anomaly-sensitive processes such as $\{\pi, \eta, \eta'\} \rightarrow \gamma\gamma$, $\gamma \rightarrow 3\pi$, and vector-to-pseudoscalar radiative decays [47–49]. Note from a DS-BS equations perspective, the description of AV mesons is rather challenging. Among other issues, typical truncations such as the well-known rainbow-ladder (RL) approximation underestimate the mass splittings between parity partners, with the $m_{a_1} - m_\rho$ splitting being a prime example [50–54]. This shortcoming largely arises from the absence of certain effects in the interaction kernels, such as spin-orbit repulsion. In addition to systematic rescalings of the RL results, [24], improved kernels have been proposed [53–55]. Nonetheless, as anticipated, here we adopt a simpler and more straightforward approach to clarify the role of the AV meson mass in the elastic and transition form factors, and its implications. This constitutes a necessary step for future, more sophisticated evaluations.

The manuscript is organized as follows. We first present the employed formalism, including the definitions of the matrix elements, their associated form factors, and the essentials of their computation within the CI model. Next, we provide numerical results for the produced mass spectrum, elastic and transition form factors, and the corresponding contribution a_μ . We capitalize on the role of the AV meson mass on determining these quantities. Finally, we summarize our findings and outline directions for future related works.

II. FORMALISM

A. Matrix elements and form factors

Let us first recall the matrix element that characterizes the coupling of a photon to an axial-vector meson [56]

$$\Lambda_{\mu\nu\rho}(K, Q) = \sum_{j=1}^3 T_{\mu\nu\rho}^j(K, Q) G_j(Q^2) \quad (1)$$

$$\begin{aligned} T_{\mu\nu\rho}^1(K, Q) &= 2K_\mu P_{\nu\alpha}^T(p_i) P_{\alpha\rho}^T(p_f), \\ T_{\mu\nu\rho}^2(K, Q) &= \left[Q_\nu - p_\nu^i \frac{Q^2}{2m_A^2} \right] P_{\mu\rho}^T(p_f) \\ &\quad - \left[Q_\rho + p_\rho^f \frac{Q^2}{2m_A^2} \right] P_{\mu\nu}^T(p_i), \end{aligned} \quad (2)$$

$$T_{\mu\nu\rho}^3(K, Q) = \frac{K_\mu}{m_A^2} \left[Q_\nu - p_\nu^i \frac{Q^2}{2m_A^2} \right] \left[Q_\rho + p_\rho^f \frac{Q^2}{2m_A^2} \right]. \quad (3)$$

Here $P_{\mu\nu}^T(p) = \delta_{\mu\nu} - p_\mu p_\nu / p^2$ and the kinematics is defined as follows: $p_i = K - Q/2$ and $p_f = K + Q/2$ denote the incoming and outgoing meson momenta, respectively, and $Q = p_f - p_i$ is the photon momentum. The on-shell conditions, $p_i^2 = p_f^2 = -m_A^2$ (m_A the mass of the AV meson), imply $K \cdot Q = 0$ and $K^2 = -m_A^2 - Q^2/4$. In addition, the preservation of Ward-Green-Takahashi identities impose [31]

$$p_\nu^i \Lambda_{\mu\nu\rho} = p_\rho^f \Lambda_{\mu\nu\rho} = Q_\mu \Lambda_{\mu\nu\rho} = 0. \quad (4)$$

The electric (G_E), magnetic (G_M) and quadrupole (G_Q) form factors can be constructed in terms of $G_{1,2,3}(Q^2)$ as follows [$\eta = Q^2/(4m_A^2)$]

$$G_E(Q^2) = G_1(Q^2) + \frac{2}{3} \eta G_Q(Q^2), \quad (5a)$$

$$G_M(Q^2) = -G_2(Q^2), \quad (5b)$$

$$G_Q(Q^2) = G_1(Q^2) + G_2(Q^2) + G_3(Q^2)[1 + \eta]. \quad (5c)$$

On its part, the AV meson to two-photons transition amplitude, $\mathcal{A}_{\mu\nu\rho}$, can be parametrized as

$$\mathcal{A}_{\mu\nu\rho}(Q, Q') = \sum_i T_{\mu\nu\rho}^i(Q, Q') f_i(Q^2, Q'^2), \quad (6)$$

where Q, Q' are the photon momenta, and f_i are Lorentz scalar functions. The tensors T^i might be expressed rather generally as follows [24]:

$$\begin{aligned} T_{\mu\nu\rho}^1 &= \epsilon_{\mu\nu\rho} Q, & T_{\mu\nu\rho}^2 &= \epsilon_{\mu\nu\rho} Q', \\ T_{\mu\nu\rho}^3 &= Q_\mu \epsilon_{\nu\rho} Q Q', & T_{\mu\nu\rho}^4 &= Q'_\mu \epsilon_{\nu\rho} Q Q', \\ T_{\mu\nu\rho}^5 &= Q_\nu \epsilon_{\mu\rho} Q Q', & T_{\mu\nu\rho}^6 &= Q'_\nu \epsilon_{\mu\rho} Q Q', \\ T_{\mu\nu\rho}^7 &= Q_\rho \epsilon_{\mu\nu} Q Q', & T_{\mu\nu\rho}^8 &= Q'_\rho \epsilon_{\mu\nu} Q Q'. \end{aligned} \quad (7)$$

The momentum appearing in the index of Levi-Civita symbol is understood as a contraction. Note that the last two structures can be written in terms of the rest according to the Schouten identities ($X = \{Q, Q'\}$)

$$-X_\rho \epsilon_{\rho Q' \mu \nu} = X_\mu \epsilon_{\nu \rho Q Q'} + X_\nu \epsilon_{\rho Q Q' \mu} + X \cdot Q \epsilon_{Q' \mu \nu \rho} + X \cdot Q' \epsilon_{\mu \nu \rho Q}. \quad (8)$$

Bose-symmetry implies $\mathcal{A}_{\mu\nu\rho}(Q, Q') = \mathcal{A}_{\mu\nu\rho}(-Q', -Q)$ and, according to the gauge invariance and transversality of the AV meson,

$$Q_\mu \mathcal{A}_{\mu\nu\rho} = 0, \quad Q'_\nu \mathcal{A}_{\mu\nu\rho} = 0, \quad P_\rho \mathcal{A}_{\mu\nu\rho} = 0. \quad (9)$$

Here $P = Q - Q'$, such that the on-shell condition $P^2 = -m_A^2$ entails $2Q \cdot Q' = Q^2 + Q'^2 + m_A^2$. The constraints in Eq. (9) relate the scalar functions f_i as follows:

$$f_2 + Q^2 f_3 + Q \cdot Q' f_4 = 0, \quad (10a)$$

$$f_1 + Q \cdot Q' f_5 + Q'^2 f_6 = 0, \quad (10b)$$

$$f_1 + f_2 + Q \cdot P f_7 + Q' \cdot P f_8 = 0. \quad (10c)$$

A consequence of these identities is that, at the end, there are only three independent Lorentz structures and corresponding scalar functions for the amplitude $\mathcal{A}_{\mu\nu\rho}$. Accordingly, we adopt the form factors $F_{1,2,3}$ defined in Ref. [24]. The associated tensor basis is detailed therein and provided here in Appendix A. Its connection to the scalar functions f_i is as follows:

$$F_1(Q^2, Q'^2) = (f_3 + f_4 + f_5 + f_6) m_A^2, \quad (11a)$$

$$F_2(Q^2, Q'^2) = \frac{2m_A^4}{Q'^2 - Q^2} (f_3 - f_4 + f_5 - f_6), \quad (11b)$$

$$F_3(Q^2, Q'^2) = \frac{m_A^4}{Q'^2 - Q^2} (f_4 - f_5 - f_7 - f_8). \quad (11c)$$

Having established the structure of $\Lambda_{\mu\nu\rho}$ and $\mathcal{A}_{\mu\nu\rho}$, we now discuss the evaluation of the corresponding form factors within our symmetry-preserving scheme.

B. Impulse approximation

To evaluate the elastic and transition form factors, we shall consider the so called impulse approximation, wherein the external probe couples directly to a single valence constituent while the others remain as spectators. Therefore, $\Lambda_{\mu\nu\rho}$ reads

$$\Lambda_{\mu\nu\rho}(Q^2) = N_c \text{tr}_D \int_l \Gamma_\rho^A(-p_f) S(l + p_f) \times i \Gamma_\mu^\gamma(Q) S(l + p_i) \Gamma_\nu^A(p_i) S(l). \quad (12)$$

An analogous expression holds for the transition amplitude $\mathcal{A}_{\mu\nu\rho}$

$$\mathcal{A}_{\mu\nu\rho} = 2N_c f \text{tr}_D \int_l S\left(l + \frac{P}{2}\right) \Gamma_\rho^A(P) S\left(l - \frac{P}{2}\right) \times \Gamma_\mu^\gamma(-Q) S\left(l + \frac{Q + Q'}{2}\right) \Gamma_\nu^\gamma(Q'). \quad (13)$$

Here $\int_l := \int d^4l / (2\pi)^4$ is a Poincaré invariant four-momentum integral. The symbol tr_D indicates a trace over Dirac space, $N_c = 3$ is the number of colors, and N_{cf} arises from the color and flavor traces, such that

$$N_{cf}^{a_1} = \frac{1}{\sqrt{2}}, \quad N_{cf}^{f_1} = \frac{5}{3\sqrt{2}}, \quad N_{cf}^{f'_1} = \frac{1}{3}. \quad (14)$$

The rest of the components in Eqs. (12) and (13) are defined as usual: $S(p)$ is the fully dressed quark propagator, while Γ_ρ^A and $\Gamma_{\mu,\nu}^\gamma$ denote the AV meson's Bethe-Salpeter amplitude (BSA) and quark-photon vertex (QPV), respectively. These pieces are obtained self-consistently within the CI framework presented below. For simplicity, we have omitted the flavor indices.

C. Contact interaction: Highlights

To evaluate the elastic and transition form factors, Eqs. (12) and (13), we shall employ the symmetry-preserving CI model described in Ref. [31], which incorporate components beyond the typical RL approximation [57,58].

The first element to be described is the DS equation for the quark propagator, namely the gap equation. This reads

$$S^{-1}(p) = i\gamma \cdot p + m + \frac{4}{3m_G^2} \int_q \gamma_\mu S(q) \gamma_\mu. \quad (15)$$

Here m is the current quark mass, and m_G sets the mass scale governing the strength of the interaction. The above integral is divergent, which reflects the fact that the CI model is nonrenormalizable and thus requires the implementation of a regularization scheme. We adopt the symmetry-preserving approach presented in [59], and summarized in Appendix B. This procedure preserves essential symmetries, including the identities of Eq. (10), and the gauge invariance of the $\mathcal{A}_{\mu\nu\rho}$ transition amplitude. The so-called gauge symmetry consistency relations are fulfilled as well, [60,61], and special care is taken with the γ_5 matrix, whose treatment is particularly subtle [62–66]. Irrespective of the scheme, any consistent regularization produces the following form for the dressed quark propagator:

$$S^{-1}(p) = i\gamma \cdot p + M. \quad (16)$$

Evidently, the mass function M turns out to be independent of the quark momentum p , adopting the role of a constituent quark mass.

The regularization procedure employed herein incorporates infrared and ultraviolet proper-time regulators,

$\tau_{ir} := \Lambda_{ir}^{-1}$ and $\tau_{uv} := \Lambda_{uv}^{-1}$, respectively. The former, $\Lambda_{ir} \approx \Lambda_{\text{QCD}}$, ensures the absence of quark production thresholds, producing a picture compatible with confinement; $\Lambda_{uv} \approx m_p$ ($\sim$ proton mass) is irremovable and fixes the scale of all dimensionful quantities. It is well established that the presence of a finite cutoff in effective field theories can lead to only partial realization of the expected anomalies [67]. However, it has been suggested that higher-order corrections designed to emulate the underlying suppressed dynamics can fix this issue [68]. Following these ideas, we employ the modified rainbow-ladder (MRL) truncation developed in Ref. [31]. Within this framework, an anomalous magnetic moment (AMM) term is dynamically generated in the QPV. The infrared strength of such is governed by DCSB [69,70]. Consequently, it has been shown to restore the chiral anomaly associated to the $\pi^0 \rightarrow \gamma\gamma$ and $\gamma \rightarrow 3\pi$ processes [47,48], effectively capturing the short-distance dynamics left out by the ultraviolet cutoff. Its robustness has also been tested in the $\eta(\eta') \rightarrow \gamma\gamma$ process and the vector to pseudoscalar radiative decays [49].

Taking all these observations into account, we express the homogeneous BS equation for an arbitrary meson H as follows:

$$\Gamma_H(Q) = -\frac{4}{3m_G^2} \int_q \gamma_\alpha S(q) \Gamma_H(Q) S(q-Q) \gamma_\alpha + \frac{4\xi}{3m_G^2} \int_q \tilde{\Gamma}_j S(q) \Gamma_H(Q) S(q-Q) \tilde{\Gamma}_j. \quad (17)$$

The first line in the above equation stems from the RL truncation. The second line contains the beyond RL contribution, defined by the tensors $\tilde{\Gamma}_j = \{I_4, \gamma_5, \frac{i}{\sqrt{6}} \sigma_{\alpha\beta}\}$, and weighted by the dimensionless parameter ξ . As can be noted, the meson BSA depends solely on the meson's total momentum Q (with $Q^2 = -m_H^2$). There is no dependence on the relative momentum between the valence quark and antiquark. This, together with the momentum-independent mass function, is an essential characteristic of the CI.

The general tensor decomposition of the BSA is determined by the meson's quantum numbers. For example, for the AV meson

$$\Gamma_\mu^A(Q) = \gamma_5 \gamma_\mu^T E_A(Q), \quad (18)$$

In this case, the combination of Eqs. (17) and (18) yields ($\bar{\alpha} = 1 - \alpha$)

$$1 = \mathcal{K}(P^2 = -m_A^2),$$

$$\mathcal{K}(P^2) = -\frac{16}{3m_G^2} \int_0^1 d\alpha (M^2 + \alpha\bar{\alpha}P^2) I_0(M^2 + \alpha\bar{\alpha}P^2), \quad (19)$$

from which one can identify the AV meson mass m_A . The resulting amplitudes must be then normalized according to the canonical condition, [71], such that

$$\frac{1}{E_A^2} = -\frac{9m_G^2}{2} \frac{d\mathcal{K}(P^2)}{dP^2} \Big|_{P^2 = -m_A^2} \quad (20)$$

Within the present scheme, the QPV is obtained from the corresponding inhomogeneous BS equation

$$\Gamma_\mu(Q) = \gamma_\mu - \frac{4}{3m_G^2} \int_q \gamma_\alpha S(q) \Gamma_\mu(Q) S(q-Q) \gamma_\alpha + \frac{4\xi}{3m_G^2} \int_q \tilde{\Gamma}_j S(q) \Gamma_\mu(Q) S(q-Q) \tilde{\Gamma}_j. \quad (21)$$

The structure acquired by the QPV is

$$\Gamma_\mu(Q) = \gamma_\mu^L \hat{f}_L(Q^2) + \gamma_\mu^T \hat{f}_T(Q^2) + \frac{\sigma_{\mu\nu} Q_\nu}{M} \hat{f}_A(Q^2), \quad (22)$$

where $\gamma_\mu^T = \gamma_\mu - \frac{\not{Q}\gamma_\mu}{Q^2}$, $\gamma_\mu^L = \gamma_\mu - \gamma_\mu^T$. Plugging it into Eq. (21), it can be readily shown that $\hat{f}_L(Q^2) = 1$. The remaining part of the vertex has a similar structure to that of the vector meson BSA, therefore both $\hat{f}_{T,A}(Q^2)$ exhibit a vector meson pole in the timelike axis ($Q^2 = -m_V^2$). Moreover, the AMM-related term $\hat{f}_A(Q^2)$ is zero when $\xi = 0$, reflecting that its existence emerges from contributions beyond RL. Additional characteristics of the QPV dressing functions are examined in Refs. [31,49].

III. RESULTS AND DISCUSSION

A. Parameters and mass spectrum

Being focused on the lightest AV mesons, namely a_1, f_1, f_1' , we adopt CI model parameters typical for the light-sector [31,59]. First, the parameters defining the CI model are fixed as

$$m_G = 0.132 \text{ GeV}, \quad \Lambda_{ir} = 0.240 \text{ GeV},$$

$$\Lambda_{uv} = 0.905 \text{ GeV}. \quad (23)$$

In addition, $\xi = 0.151$ is chosen to ensure the fulfillment of the chiral anomaly [47,48]. Note the variation of ξ only affects the vector channels, e.g. the vector meson masses. Assuming isospin symmetry $m_u = m_d$, and seeking to reproduce the pion and kaon masses, we set $m_{u/d} = 0.007 \text{ GeV}$ and $m_s = 0.170 \text{ GeV}$. This setup yields the constituent quark and meson masses collected in Table I.

It is important to highlight that the beyond RL contribution favors a better description of the vector channels, and so the QPV, but it does not contribute at all to the axial-vector

TABLE I. Quark and meson masses (in GeV).

m_u	m_s	M_u	M_s	m_π	m_K	m_ρ	m_ϕ
0.007	0.170	0.368	0.533	0.140	0.499	0.879	1.058

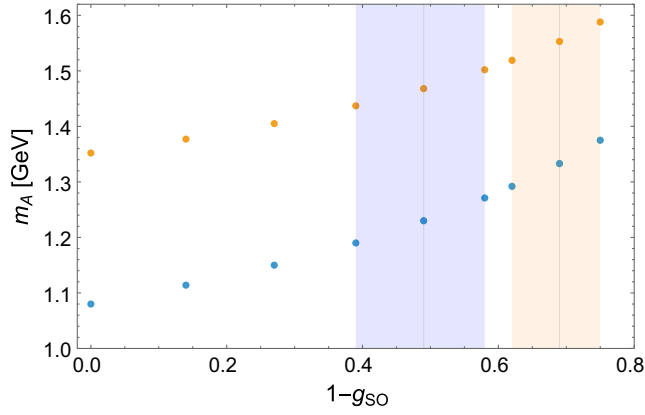


FIG. 1. Computed m_{a_1} (blue) and $m_{f'_1}$ (orange) masses as a function of the spin-orbit repulsion parameter g_{SO} . The blue and orange bands point out to $m_{a_1} = 1.230(40)$ GeV and $m_{a_1} = 1.333(41)$ GeV, corresponding to the empirical values of m_{a_1} and $m_{a_1} - m_\rho$, respectively [73].

case [31]. Accordingly, AV mesons are described solely by the RL truncation, which is known to fail because it omits spin-orbit repulsion effects expected to be relevant in states with non-negligible components of orbital angular momentum [50–54]. Among other issues, this results in underestimated AV meson masses. A typical solution within the CI framework is to multiply the RL kernel by a dimensionless coupling $g_{SO}^2 \leq 1$ in these channels [50–52,72]. A smaller g_{SO} translates into a larger mass, while the original result is clearly recovered with $g_{SO} = 1$.

The impact of spin-orbit coupling on m_A is illustrated in Fig. 1, with the relevant numerical values summarized in Table II. We consider three cases: g_{SO} fixed to either produce the experimental $m_{a_1} - m_\rho$ mass-splitting (case I) or the empirical value m_{a_1} (case II), [73]; and $1 - g_{SO} = 0$, corresponding to the omission of spin-orbit repulsion effects (case O). These three cases will be used later for the analysis of the form factors and their contribution to a_μ . Note, due to isospin symmetry ($m_u = m_d$) and no mixing between the f_1 and f'_1 states, the a_1 and f_1 mesons become mass degenerate while the f'_1 is described as a purely $s\bar{s}$ state.

B. Elastic form factors

Let us consider the positively charged state a_1^+ and, for the purposes of the discussion, the f_1 and f'_1 mesons are likewise treated as if they were positively charged. The same applies to the ρ^+ meson and its $s\bar{s}$ counterpart, the ϕ meson. Within the present scheme, a_1 and f_1 are thus identical and the resulting EFFs are the same. These, together with the f'_1 case, are depicted in Fig. 2. It is evident that, within the range of variation for the AV meson masses in Table II, the EFFs exhibit little sensitivity.

The $Q^2 \rightarrow 0$ limit of the EFFs, Eq. (5), defines the charge, magnetic, and quadrupole moments

TABLE II. Axial-vector meson masses (in GeV), and the spin-orbit coupling g_{SO} . The bold quantities indicate the experimental values used to determine g_{SO} [73].

	m_{a_1}	$m_{f'_1}$	$m_{a_1} - m_\rho$	$1 - g_{SO}$
Case-I	1.333(41)	1.553(35)	0.452(41)	0.69(7)
Case-II	1.230(41)	1.469(31)	0.351(40)	0.49(10)
Case-O	1.080	1.352	0.200	0

$$G_E(0) = 1, \quad G_M(0) = \mu, \quad G_Q(0) = \mathcal{Q}. \quad (24)$$

The produced values are collected in Table III, in addition to the corresponding ones for the vector mesons ρ and ϕ . The magnetic moments are essentially the same for the AV mesons under examination, $\mu \approx 2.13$, and somewhat smaller than those of their vector meson counterparts. The f'_1 quadrupole moment exhibits a $\sim 25\%$ reduction in magnitude with respect to a_1 . This pattern is repeated in the vector meson case and, in fact, we find $\mathcal{Q}_{a_1} \approx \mathcal{Q}_\rho$ and $\mathcal{Q}_{f'_1} \approx \mathcal{Q}_\phi$.

It is also revealed by Fig. 2 that, despite being somewhat compatible within errors, the f'_1 form factors exhibit a slightly less pronounced large- Q^2 falloff. This can be quantified through the corresponding radii, defined as

$$r_i^2 = -6 \frac{1}{G_i(0)} \frac{\partial G_i(Q^2)}{\partial Q^2} \Big|_{Q^2 \rightarrow 0}. \quad (25)$$

The results of this evaluation are presented in Table III. Unsurprisingly, for all radii shown, we find $r_i^{f'_1} \lesssim r_i^{a_1}$. Moreover, as with the magnetic and quadrupole moments, $r_i^{a_1} \approx r_i^\rho$ and $r_i^{f'_1} \approx r_i^\phi$. Although the vector-meson case is not shown explicitly, all these quantities reflect the similarity between the EFFs of vector and AV mesons in the low- Q^2 region.

Turning our attention to $G_E(Q^2)$, we observe a sign flip near $Q^2 \approx 11$ and 13 GeV² for the a_1 and f'_1 states, respectively. This zero-crossing reflects the predicted large- Q^2 behavior of spin-1 mesons, [74,75], and arises from the destructive interference between the $G_{1,2,3}$ form factors defining G_E , Eq. (5a). However, the zero-crossing occurs much more rapidly for the ρ and ϕ : at around $Q^2 = 3.5$ and 5 GeV², respectively. The specific values are listed in Table III. When expressing this sign change in terms of $x_c = Q_c^2/m_H^2$, such that $G_E^H(Q_c^2) = 0$, the value of x_c is smaller for the heavier system. This pattern is consistent with the findings reported in Ref. [30]. Thus, although the EFFs of vector and axial-vector mesons are similar in the low-virtuality domain, differences emerge as Q^2 increases. This would result, for instance, into appreciably different charge distributions, [36].

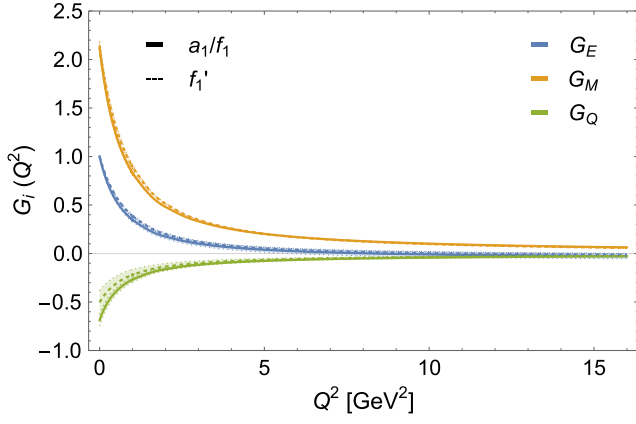


FIG. 2. Axial-vector meson EFFs. The band reflects the variation of m_{a_1} and $m_{f'_1}$ within the whole range considered here (see Table II).

TABLE III. Static properties of axial-vector and vector mesons. Herein $x_c = Q_c^2/m_H^2$, defined by the zero-crossing $G_E^H(Q_c^2) = 0$. The error in the AV case stems from the variation on the corresponding masses (see Table II).

	μ	$\mathcal{Q}$	r_E/fm	r_M/fm	r_Q/fm	x_c
a_1	2.12(5)	-0.69(6)	-0.59(2)	0.57(1)	0.58(1)	6.55(1.61)
f'_1	2.13(6)	-0.50(12)	-0.53(1)	0.52(1)	0.52(1)	5.61(1.43)
ρ	2.22	-0.63	-0.61	0.55	0.56	4.57
ϕ	2.26	-0.53	-0.51	0.47	0.48	4.38

C. Transition form factors

As in the case of EFFs, our present framework produces identical a_1 and f_1 TFFs, differing solely by the flavor-color factors N_{fc} given in Eq. (14). The f'_1 remains a purely $s\bar{s}$ state. Thus, considering first case I, where the experimental value of $m_{a_1} - m_\rho$ is reproduced, in the limit $Q^2, Q'^2 \rightarrow 0$ we obtain:

$$F_{1,2,3}^A(0,0) = 0.554_{-0.012}^{+0.018}, 0.280_{-0.006}^{+0.009}, -0.006_{-0.001}^{+0.001}. \quad (26)$$

Each F_i^A arises from adding the individual contributions of $\{a_1, f_1, f'_1\}$ with their corresponding N_{fc} weights. When we consider case II instead, with the empirical value for m_{a_1} , our exploration leads us to

$$F_{1,2,3}^A(0,0) = 0.532_{-0.004}^{+0.005}, 0.263_{-0.006}^{+0.008}, -0.004_{-0.001}^{+0.001}. \quad (27)$$

The asymmetric errors in both case I and case II account for the experimental uncertainty associated to $m_{a_1} - m_\rho$ and m_{a_1} , respectively. Finally, the omission of spin-orbit repulsion in the interaction kernels, characterized by $m_{a_1} = 1.080$ GeV, yields

$$F_{1,2,3}^A(0,0) = 0.518, 0.230, -0.002. \quad (28)$$

It is worth noting that the magnitude of all F_i^A exhibit a mild increasing trend with m_{a_1} and, in fact, the TFFs in the whole range are just mildly affected by the variation of the AV meson mass. This is a common feature with the corresponding EFFs. The resulting $F_3^A(0,0)$ are practically negligible, despite the fact that it exhibits the largest relative variation among the different setups. By contrast, the variation range is below 20% for $F_2^A(0,0)$, and under 10% for $F_1^A(0,0)$. Therefore, in the remainder of the discussion we will consider an average that accounts for the entire range of m_{a_1} values being considered (see Fig. 1 and Table II). This leads us to the following:

$$F_{1,2,3}^A(0,0) = 0.545(27), 0.259(30), -0.005(2). \quad (29)$$

According to the analysis of Ref. [24], which makes use of data-driven TFF parametrizations from Refs. [9,10], one expects the following constraints:

$$F_{1,2,3}^A(0,0) = \begin{bmatrix} 2.45(35) \\ 1.96(42) \end{bmatrix}, \begin{bmatrix} 2.18(1.17) \\ 0.42(1.49) \end{bmatrix}, \begin{bmatrix} -0.74(84) \\ -0.33(84) \end{bmatrix}. \quad (30)$$

Clearly, our expectations for $F_{1,2,3}^A(0,0)$ are underestimated, with the latter being particularly notable. It can be thus anticipated that the resulting contribution to a_μ will be smaller than expected. Increasing $\xi = 0.151 \rightarrow 0.38$ in the BS kernel, Eq. (17), allows one to reproduce the physical ρ -meson mass without affecting that of the AV meson. This modification enhances $F_i(0,0)$ by approximately 50%, 80%, and 600% for $i = 1, 2, 3$, respectively. This would certainly bring our expectations closer to the constraints in Eq. (30). On the other hand, in a RL-like setup ($\xi = 0$, $m_\rho = 0.929$ GeV), the situation would be much worse. We choose to retain our preferred value of $\xi = 0.151$, as its determination is dictated by the chiral anomaly, [47,48]. Another contributing factor for this shortcoming arises from the CI model itself, where axial-vector mesons are described by a single BSA and vector mesons by two. This approach oversimplifies the rich tensor structure that characterizes the BSAs of these systems, [76]. In an ideal scenario, one would use a truncation that captures both symmetry requirements (e.g., the chiral anomaly) and the properties of the vector and axial-vector mesons, guaranteeing that the AV TFFs fulfill the low-energy constraints.

The photon virtuality dependence of the TFFs is illustrated in Fig. 3, in the symmetric limit $Q^2 = Q'^2$. The $|F_1^A| > |F_2^A| > |F_3^A|$ hierarchy is readily apparent. This ordering persists over a broad range, with all TFFs decreasing in magnitude as Q^2 increase. Notably, this falloff is much sharper than that seen in the CI description of the pion elastic and transition form factors [47,57,58]. To quantify this rate, one may define transition radii r_i^{Ay} in

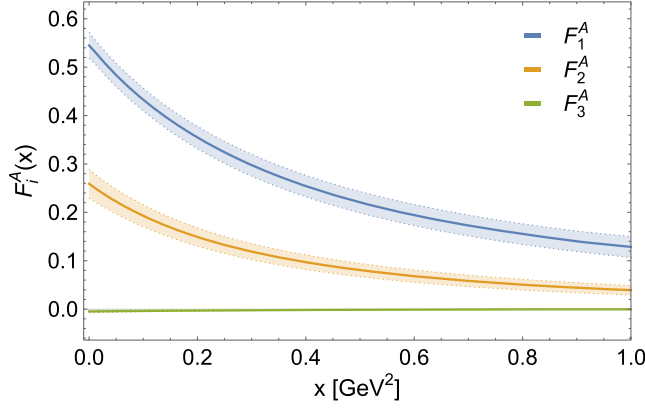


FIG. 3. Axial-vector meson to two-photon TFFs, as a function of $x = (Q^2 + Q'^2)/2$. The band reflects both the variation on m_{a_1} and the kinematic range from the symmetric ($Q^2 = Q'^2$) to the asymmetric ($Q^2 = 0$ or $Q'^2 = 0$) limits.

analogy to Eq. (25), which yield

$$r_{1,2,3}^{A\gamma} = 0.76(1), 0.86(1), 0.95(1) \text{ fm}. \quad (31)$$

The AV TFFs display not only a noticeable decrease with momentum transfer but also the correct asymptotic power-law behavior [23]. On the other hand, by definition, the CI model does not reproduce certain aspects of perturbation theory, including the anomalous dimensions of the form factors. To account for this limitation, we extrapolate our numerical results using the QCD-driven parametrization of Ref. [24]. The explicit representation and related parameters might be found in Appendix C.

Finally, it is worth pointing out that the detailed breakdown of the TFFs into a_1 , f_1 , and f'_1 contributions is omitted here, as a_1 and f_1 possess identical form factors and the f'_1 ones are quite similar in shape. For reference, the f'_1 contribution amounts to about 10%–25% of the whole. Alternatively, assuming that the TFFs of all these mesons are identical, Eq. (14) indicates that f'_1 would account for ~18% of the total.

D. Hadronic light-by-light contributions

For the computation of the HLbL contribution, we consider the master formula presented in [77]

$$\alpha_{\mu}^{\text{HLbL}} = \frac{\alpha_{em}^3}{432\pi^2} \int_{\Omega} \sum_i^{12} T_i(Q_1, Q_2, Q_3) \bar{\Pi}_i(Q_1, Q_2, Q_3), \quad (32)$$

where α_{em} is the QED coupling constant, and

$$\int_{\Omega} := \int_0^{\infty} d\Sigma \Sigma^3 \int_0^1 dr r \sqrt{1-r^2} \int_0^{2\pi} d\phi. \quad (33)$$

The integration variables $\{\Sigma, r, \phi\}$ are related with the (Euclidean) photon virtualities, $Q_{1,2,3}^2$, as follows:

$$\begin{aligned} Q_1^2 &= \frac{\Sigma}{3} \left(1 - \frac{r}{2} \cos \phi - \frac{r}{2} \sqrt{3} \sin \phi \right), \\ Q_2^2 &= \frac{\Sigma}{3} \left(1 - \frac{r}{2} \cos \phi + \frac{r}{2} \sqrt{3} \sin \phi \right), \\ Q_3^2 &= \frac{\Sigma}{3} (1 + r \cos \phi). \end{aligned} \quad (34)$$

The kernel functions T_i entering Eq. (32) are well known. These are of purely kinematic nature and are provided in the Appendix of Ref. [77]. The dynamical component, specific to each type of HLbL contribution, is encoded in the twelve scalar functions $\bar{\Pi}_i$. As discussed in Ref. [78], these can be expressed via six representative functions $\hat{\Pi}_i$. The case of the AV meson contribution is presented in [78], in terms of form factors $\mathcal{F}_i$. When translated into the form factors F_i defined in Eq. (11), these read

$$\begin{aligned} \mathcal{F}_1(Q^2, Q'^2) &= -\frac{Q^2 - Q'^2}{2m_A^2} F_3, \\ \mathcal{F}_2(Q^2, Q'^2) &= \frac{1}{2} \left(F_1 - \frac{Q^2 - Q'^2}{2m_A^2} F_2 \right), \\ \mathcal{F}_3(Q^2, Q'^2) &= -\frac{1}{2} \left(F_1 + \frac{Q^2 - Q'^2}{2m_A^2} F_2 \right). \end{aligned} \quad (35)$$

Numerical integrations are performed using the CUBA library [79]. To calibrate and benchmark our approach, we compute the pion pole and box contributions, where the associated form factors are derived consistently within the CI formalism [47]. By employing Padé-type parametrizations for each form factors [80], both considered at different orders, we obtain

$$\begin{aligned} a_{\mu}^{\pi\text{-box}} &= -21.2(3) \times 10^{-11}, \\ a_{\mu}^{\pi\text{-pole}} &= 94.1(4) \times 10^{-11}. \end{aligned} \quad (36)$$

The uncertainty combines both the parametrization and numerical errors. As can be noted, the produced values are about 30%–50% larger than typical estimates, including DSE evaluations, [42–46], and the 2025 White Paper (WP2) consensus [2]. For instance, $a_{\mu}^{\pi\text{-box}}[\text{WP2}] = -15.9(2) \times 10^{-11}$. This outcome is not unexpected given the fact that the CI not only produces hard form factors but, in these cases, also drives their asymptotic behavior toward a constant [57,58], failing to fulfill the asymptotic expectations [23,81]. Furthermore, the CI model delivers a flat-top valence-quark distribution function, [58], corresponding to one of the physical bounds identified for this system, [82]. The results in Eq. (36) should therefore be interpreted as limiting cases rather than precise estimates for these quantities.

Turning our attention to the AV meson case, and considering solely the lowest-lying multiplet, $\{a_1, f_1, f'_1\}$, we

arrive at the following:

$$a_\mu^A[\text{Case-I}] = 0.77_{-0.08}^{+0.11} \times 10^{-11}, \quad (37a)$$

$$a_\mu^A[\text{Case-II}] = 1.14_{-0.18}^{+0.20} \times 10^{-11}, \quad (37b)$$

$$a_\mu^A[\text{Case-O}] = 2.24 \times 10^{-11}. \quad (37c)$$

Here, the asymmetric error accounts for the variation of g_{SO} to produce the experimental mass splitting $m_{a_1} - m_\rho$ in case I, and the empirical value m_{a_1} in case II. Recall case O corresponds to neglecting the spin-orbit coupling ($g_{SO} = 1$), hence producing a smaller AV mass, $m_{a_1} = 1.08$ GeV (see Table II). The numerical and parametrization error estimates are minimal and shall be omitted in the discussion. The produced values are nevertheless disfavored by recent analyses. For instance, results from holographic models [25], referred to in the WP2 [2], place the contribution of the lowest-lying multiplet at

$$a_\mu^A[\text{HL} - \text{WP2}] = 15.8(1.6) \times 10^{-11}, \quad (38)$$

while the DSE evaluation from [24] favors

$$a_\mu^A[\text{DSE}] = 17.4(4.0)(4.0)(0.5) \times 10^{-11}. \quad (39)$$

This positions our estimates below recent trends. The underestimation is evidently driven by the value of the form factors at $Q^2, Q'^2 \rightarrow 0$, Eq. (29). To address this issue, and in line with Ref. [24], we rescale the TFFs so as to enforce

$$|F_1^A(0, 0)|^2 = \frac{48\Gamma_{\gamma\gamma}}{\pi\alpha_{em}^2 m_A}, \quad (40)$$

with $\Gamma_{\gamma\gamma} = 3.5(6)(5)$ keV for $f_1(1285)$ [83]. Accounting only for the central value of the decay width, we find

$$a_\mu^A[\text{Case I}] = 6.33_{-0.92}^{+1.18} \times 10^{-11}, \quad (41a)$$

$$a_\mu^A[\text{Case II}] = 9.99_{-1.81}^{+2.14} \times 10^{-11}, \quad (41b)$$

$$a_\mu^A[\text{Case O}] = 22.53 \times 10^{-11}. \quad (41c)$$

It is evident that these estimates are closer to those in Eq. (39). However, as exposed by Fig. 4, a_μ^A is in fact highly sensitive to the AV meson mass. Since the variation in the magnitude of the form factors with m_{a_1} is relatively modest, Eq. (29), the $1/m_A$ factors appearing in the HLbL master formula are primarily responsible for this behavior. To mitigate this sensitivity, we compute a_μ^A for a range of m_{a_1} and consider the expectation value

$$\langle a_\mu^A \rangle := \frac{1}{m_1 - m_0} \int_{m_0}^{m_1} dm_{a_1} a_\mu^A(m_{a_1}), \quad (42)$$

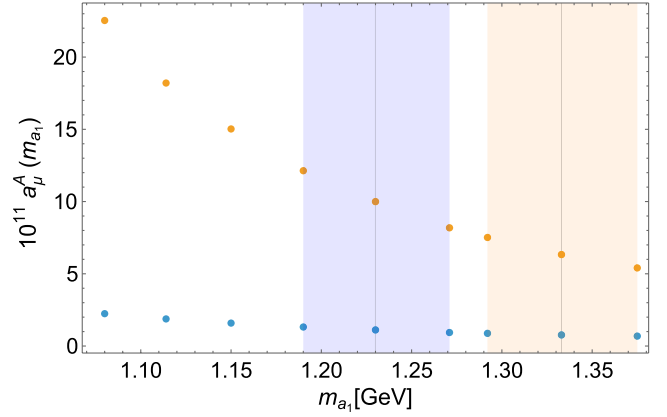


FIG. 4. AV meson mass dependence on a_μ^A , obtained from the original (blue) and rescaled (orange) TFFs. The blue and orange bands point out to $m_{a_1} = 1.230(40)$ GeV and $m_{a_1} = 1.333(41)$ GeV, corresponding to the empirical values of m_{a_1} and $m_{a_1} - m_\rho$, respectively [73].

such that $m_0 = 1.080$ GeV and $m_1 = 1.374$ GeV (the smallest and largest m_{a_1} being considered). Defining the standard deviation in the usual way, we thus find:

$$a_\mu^A = 11.30(4.71) \times 10^{-11}. \quad (43)$$

With a somewhat conservative uncertainty, this evaluation is consistent with current trends. Conversely, averaging only the physically motivated cases I and II yields

$$a_\mu^A = 8.16_{-1.02}^{+1.22} \times 10^{-11}. \quad (44)$$

Although intended mainly for illustration, this result will be taken as our final assessment. It is important to recall, however, that it is based on TFFs whose normalization is underestimated and has been rescaled to reproduce the decay width in Eq. (40). The shortcoming is naturally attributable to the limitations of the CI model, including momentum-independent quark mass functions and the oversimplified structure of the meson BSA. However, the most fundamental cause of the problem is the current inadequacy of the interaction kernels in capturing the properties of AV mesons. As pointed out before, such kernels should also comply with both symmetry and phenomenological requirements [53,54].

IV. SUMMARY AND SCOPE

For the lowest-lying axial-vector meson multiplet, $\{a_1, f_1, f_1'\}$, we computed the elastic and two photon TFFs, and corresponding contributions to the HLbL piece of a_μ . The form factors are defined in a symmetry-preserving basis and obtained via the impulse approximation, with each element therein derived within a CI framework of the DS and BS equations. By extending the usual RL truncation, this approach improves the treatment of anomalous processes (e.g., those involving the chiral anomaly) and provides a

more accurate description of vector channels. This results in vector meson masses closer to their empirical values, an improved description of the QPV, and a more realistic Q^2 dependence of the TFFs. For the AV channels, the present approach is supplemented by an effective coupling that allows emulating the spin-orbit repulsion effects neglected in the interaction kernels. The variation of this coupling, g_{SO} , translates into the changes in the mass of the AV meson. Consequently, the dependence of the form factors and a_μ on the AV meson mass is also studied.

In the present context, the a_1 and f_1 mesons are mass-degenerate, while the f'_1 is a purely $s\bar{s}$ state. Therefore, the form factors associated with the a_1 and f_1 are identical, up to differences in the color-flavor factors in the case of the TFFs. The EFFs show little sensitivity to variations in the AV meson mass, such that the form factors associated with the f'_1 decrease only slightly more slowly than those of the a_1 . In the $Q^2 \rightarrow 0$ region, which defines the associated radii, magnetic and quadrupole moments, the EFFs of the AV mesons are, within uncertainties, fully consistent with those of their corresponding vector mesons, i.e. their parity partners. Nevertheless, as the photon virtuality grows, differences emerge. A notable outcome is the position of the zero-crossing of the form factor $G_E(Q^2)$, which is located in the range $Q^2 \approx 3.5\text{--}5 \text{ GeV}^2$ for the light vector mesons, but much further in the AV mesons case (above 10 GeV^2). This could result in a noticeably different charge distribution. These features will be explored in more detail in a future work.

A first approach to the TFFs reveal that their magnitudes are underestimated. In particular, in the soft limit $Q^2, Q'^2 \rightarrow 0$, this effect appears moderately for F_2 , more clearly for F_1 , and most strongly for F_3 , which is negative and practically negligible over the entire range of photon virtualities. These patterns will eventually translate into a rather small value of a_μ^Λ . On the other hand, note that the ρ -meson mass predicted by our scheme is about $m_\rho = 0.879 \text{ GeV}$. Bringing it closer to the empirical value by increasing the parameter ξ , which characterizes the beyond-RL contribution in the interaction kernels, significantly improves the TFF normalization. The RL limit ($\xi = 0$) yields the opposite result, as in that case $m_\rho = 0.929 \text{ GeV}$ adopts its largest value. Given the fact that ξ is set by the chiral anomaly, we keep it fixed in our analysis. Adding to this drawback, there is also the fact that, within the present CI model, the vector and axial-vector mesons are described by two and one BSAs, respectively. This is an evident oversimplification of their otherwise rich structures. Altogether, the need for interaction kernels that are robust enough to satisfy both symmetry constraints and phenomenological consistency is evident. One positive aspect is the Q^2 evolution of the TFFs. Unlike typical CI model results, the produced form factors show a more

visible drop, with larger transition radii and a correct power-law fall-off in the asymptotic limits.

Regarding a_μ^Λ , we first observe that the initial values obtained lie noticeably below recent estimates. This stems from an underestimation of the magnitude of the TFFs. A significant sensitivity to the AV meson mass is also observed. It is therefore clear that a precise description of the AV mesons, starting with their masses, is essential. From the perspective of the DS and BS equations, this is already a highly challenging task. In this context, CI-based studies are valuable, as they provide a clear and transparent way to disentangle the role of each component, while maintaining the symmetry requirements and key nonperturbative features of the theory. After rescaling the TFFs according to the empirical estimate of the two-photon to f_1 decay, our final result becomes consistent with recent estimates. This emphasizes once again that the main limitation of the present analysis lies in the normalization of the calculated TFFs. The rescaling procedure is therefore necessary not only because of the drawbacks of the CI model, but also, more generally, because of those inherent to the interaction kernels commonly used in DS and BS equations studies. As we have established, these issues could be resolved within a consistent scheme that simultaneously provides a good description of vector and axial-vector channels. Consequently, the present exploration constitutes a necessary step toward the refinement of subsequent assessments. Within this same framework, we are also in position to analyze the process $\gamma^*\pi \rightarrow \gamma\gamma$, which plays a role in the data-driven determination of a_μ .

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DATA AVAILABILITY

The data supporting the findings of this article are available within the article itself.

APPENDIX A: ALTERNATE DECOMPOSITION

The amplitude decomposition for $\mathcal{A}_{\mu\nu\rho}$ suggested in Ref. [24] reads

$$\mathcal{A}_{\mu\nu\rho}(Q, Q') = m_A \sum_{i=1}^6 \tau_{\mu\nu\rho}^i(Q, Q') F_i(Q^2, Q'^2), \quad (\text{A1})$$

where

$$\begin{aligned}
m_A^3 \tau_{\mu\nu\rho}^1 &= \frac{1}{2} (\epsilon_{\mu\nu\alpha Q} \delta_{\alpha\nu}^T(Q') - \delta_{\mu\alpha}^T(Q) \epsilon_{\nu\rho\alpha Q'}), \\
m_A^5 \tau_{\mu\nu\rho}^2 &= \frac{\omega}{2} (\epsilon_{\mu\nu\alpha Q} \delta_{\alpha\nu}^T(Q') + \delta_{\mu\alpha}^T(Q) \epsilon_{\nu\rho\alpha Q'}), \\
m_A^5 \tau_{\mu\nu\rho}^3 &= 2\omega \epsilon_{\mu\nu Q Q'} \Sigma_\rho, \\
m_A^3 \tau_{\mu\nu\rho}^4 &= \epsilon_{\mu\nu Q Q'} P_\rho, \\
m_A \tau_{\mu\nu\rho}^5 &= \epsilon_{\mu\nu\rho\Sigma}, \\
m_A^3 \tau_{\mu\nu\rho}^6 &= \omega \epsilon_{\mu\nu\rho P},
\end{aligned} \tag{A2}$$

with $\delta_{\mu\nu}^T(X) = X^2 \delta_{\mu\nu} - X_\mu X_\nu$ and $\Sigma = (Q + Q')/2$. The transversality of the AV meson and gauge invariance reduce the number of nontrivial scalar functions $F_i(Q^2, Q'^2)$. In particular, $F_{4,5,6} = 0$ for all kinematics.

APPENDIX B: SYMMETRY PRESERVING REGULARIZATION

1. Loop integrals

Due to the nonrenormalizable nature of the CI, any loop integral must be regularized in a Poincare invariant manner. Herein we adopt the symmetry preserving regularization scheme described in Ref. [59], which expresses any one loop integral with a class of the so-called irreducible loop integrals (ILIs) introduced in Ref. [60]

$$\begin{aligned}
I_{-2\alpha}(\mathcal{M}^2) &= \int_q \frac{1}{(q^2 + \mathcal{M}^2)^{\alpha+2}}, \\
I_{-2\alpha}^{\mu\nu}(\mathcal{M}^2) &= \int_q \frac{q_\mu q_\nu}{(q^2 + \mathcal{M}^2)^{\alpha+3}}, \\
I_{-2\alpha}^{\mu\nu\rho\sigma}(\mathcal{M}^2) &= \int_q \frac{q_\mu q_\nu q_\rho q_\sigma}{(q^2 + \mathcal{M}^2)^{\alpha+4}}.
\end{aligned} \tag{B1}$$

These ILIs are regularized through the Schwinger's proper time method. For the scalar type ILIs

$$I_{-2\alpha R}(\mathcal{M}^2) = \frac{1}{16\pi^2} \frac{\Gamma[\alpha, \tau_{uv}^2 \mathcal{M}^2] - \Gamma[\alpha, \tau_{ir}^2 \mathcal{M}^2]}{\mathcal{M}^{2\alpha} \Gamma(\alpha+2)}, \tag{B2}$$

where $\tau_{uv,ir}$ are ultraviolet (UV) and infrared (IR) regulators, respectively, and $\Gamma(n, z)$ is the incomplete gamma function. The label R stands for regularized integrals and will be suppressed for simplicity. For the tensor type ILIs, it is proved that they can be related with the scalar type ones through

$$I_{-2\alpha R}^{\mu\nu}(\mathcal{M}^2) = \frac{\Gamma(\alpha+2)}{2\Gamma(\alpha+3)} \delta_{\mu\nu} I_{-2\alpha R}(\mathcal{M}^2), \tag{B3}$$

$$I_{-2\alpha R}^{\mu\nu\rho\sigma}(\mathcal{M}^2) = \frac{\Gamma(\alpha+2)}{4\Gamma(\alpha+4)} S_{\mu\nu\rho\sigma} I_{-2\alpha R}(\mathcal{M}^2). \tag{B4}$$

These relations are precisely the so-called consistency conditions of gauge symmetry in Refs. [60,61], which are independent of regularization and are necessary for preserving the gauge invariance of theories.

2. Transition amplitude: Scalar functions

The structure of $\mathcal{A}_{\mu\nu\rho}$ demands the evaluation of a chiral trace, i.e., that containing a γ_5 matrix. Regularizing such traces entails well-known subtleties [62–64]. Herein we adopt the process described in Ref. [48] to handle the chiral trace and express it in terms of scalar ILIs. For this purpose, the transition amplitude is first regularized employing dimensional regularization within the Breitenlohner-Maison/'t Hooft-Veltman scheme [65,66]. Among others, this process avoids ambiguities from the Schouten identity. The dimensionally regularized scalar ILIs are then mapped onto their counterparts in proper-time regularization. The resulting form factors can be expressed as follows:

$$f_i = 2N_{cf} E_A \sum_{X,Y} f_i^{XY}(Q^2, Q'^2) \hat{f}_X(Q^2) \hat{f}_Y(Q'^2), \tag{B5}$$

with $X, Y = \{L, T, A\}$ and $i = 1, \dots, 8$. The functions $\hat{f}_{X,Y}$ are those that appear in the QPV, Eq. (22). Meanwhile, after regularization, the functions f_i^{XY} can be written as

$$f_2^{TT'} = \int_\Omega -4((3u_2 - 1)I_0(\omega) + \mathcal{A}I_{-2}(\omega)), \tag{B6a}$$

$$\begin{aligned}
f_2^{TA'} &= \int_\Omega 8(2u_1 u_2 (m_A^2 + Q^2 + Q'^2) \\
&\quad + Q^2 u_1 (2u_1 - 3) + 2Q'^2 u_2^2 - Q'^2 u_2) I_{-2}(\omega),
\end{aligned} \tag{B6b}$$

$$f_2^{AT'} = \int_\Omega 8(u_2 (m_A^2 + Q^2 + Q'^2) - Q^2) I_{-2}(\omega), \tag{B6c}$$

$$f_2^{AA'} = \int_\Omega 4(Q^2 u_1 I_0(\omega) - \mathcal{B}I_{-2}(\omega))/M^2, \tag{B6d}$$

$$f_3^{LT'} = \int_\Omega -4((3u_2 - 1)I_0(\omega) + \mathcal{C}I_{-2}(\omega))/Q^2, \tag{B6e}$$

$$f_3^{LA'} = \int_\Omega 8(Q^2(1 - 2u_1)u_1 + Q'^2 u_2(2u_2 - 1)) I_{-2}(\omega)/Q^2, \tag{B6f}$$

$$f_3^{TT'} = \int_\Omega 4((3u_2 - 1)I_0(\omega) + \mathcal{D}I_{-2}(\omega))/Q^2, \tag{B6g}$$

$$f_3^{TA'} = \int_\Omega 8(Q^2(3 - 2u_1)u_1 + Q'^2(1 - 2u_2)u_2) I_{-2}(\omega)/Q^2, \tag{B6h}$$

$$f_3^{AT'} = \int_\Omega 8I_{-2}(\omega), \tag{B6i}$$

$$f_3^{AA'} = \int_{\Omega} -4(u_1 I_0(\omega) - \mathcal{E} I_{-2}(\omega))/M^2, \quad (\text{B6j})$$

$$f_4^{TT'} = \int_{\Omega} -16u_1 u_2 I_{-2}(\omega), \quad (\text{B6k})$$

$$f_4^{TA'} = \int_{\Omega} -32u_1 u_2 I_{-2}(\omega), \quad (\text{B6l})$$

$$f_4^{AT'} = \int_{\Omega} -16u_2 I_{-2}(\omega), \quad (\text{B6m})$$

$$f_4^{AA'} = \int_{\Omega} 16u_2(-2M^2 + u_1 u_2(m_A^2 + Q'^2) + Q^2 u_1(2u_1 + u_2 - 1))I_{-2}(\omega)/M^2, \quad (\text{B6n})$$

$$f_7^{TT'} = \int_{\Omega} -4((3u_1 + 3u_2 - 2)I_0(\omega) + \mathcal{F}I_{-2}(\omega))/m_A^2, \quad (\text{B6o})$$

$$f_7^{TA'} = \int_{\Omega} 8(m_A^2 u_1(2u_2 + 1) + 2Q^2 u_1(u_1 + u_2 - 1) + Q'^2(2u_2 + 1)(u_1 + u_2 - 1))I_{-2}(\omega)/m_A^2, \quad (\text{B6p})$$

$$f_7^{AT'} = \int_{\Omega} 8(m_A^2(2u_1 + 1)u_2 + Q^2(2u_1 + 1)(u_1 + u_2 - 1) + 2Q'^2 u_2(u_1 + u_2 - 1))I_{-2}(\omega)/m_A^2, \quad (\text{B6q})$$

$$f_7^{AA'} = \int_{\Omega} 4((Q^2 u_1 + Q'^2 u_2)I_0(\omega) + \mathcal{G}I_{-2}(\omega))/(M^2 m_A^2), \quad (\text{B6r})$$

where $\int_{\Omega} = \int_0^1 du_1 \int_0^{1-u_1} du_2 = \int_0^1 du_2 \int_0^{1-u_2} du_1$, $\omega = M^2 - m_A^2 u_1 u_2 - (u_1 + u_2 - 1)(Q^2 u_1 + Q'^2 u_2)$ and

$$\mathcal{A} = 2((u_2 - 1)u_2(2m_A^2 u_1 + Q'^2(2u_1 + 2u_2 - 1)) + Q^2 u_1(2u_1(u_2 - 1) + u_2(2u_2 - 3) + 2)), \quad (\text{B7a})$$

$$\mathcal{B} = 2(u_1(Q^2 u_2(2m_A^2(u_1 + u_2 - 1) + Q'^2(2u_1 - 1)) + u_2^2(m_A^2 + Q'^2)^2 + Q^4(2u_1^2 + u_1(2u_2 - 3) + (u_2 - 1)^2)) - 2M^2(u_2(m_A^2 + Q^2 + Q'^2) - Q'^2)), \quad (\text{B7b})$$

$$\mathcal{C} = 2u_2(m_A^2 u_1(2u_2 - 1) + 2Q^2 u_1(u_1 + u_2 - 1) + Q'^2(2u_2 - 1)(u_1 + u_2 - 1)), \quad (\text{B7c})$$

$$\mathcal{D} = 2(u_2(2u_2 - 1)(m_A^2 u_1 + Q'^2(u_1 + u_2 - 1)) + 2Q^2 u_1(u_1(u_2 - 1) + u_2(u_2 - 1) + 1)), \quad (\text{B7d})$$

$$\mathcal{E} = 2(2M^2 + u_1(-u_2(m_A^2 + 2Q'^2 u_2) + Q^2 u_1(2u_1 - 3) - Q^2 u_2 + Q'^2)), \quad (\text{B7e})$$

$$\mathcal{F} = 2(2m_A^2 u_1 u_2(u_1 + u_2 - 2) + Q^2 u_1(u_1 + u_2 - 1)(2u_1 + 2u_2 - 3) + Q'^2 u_2(u_1 + u_2 - 1)(2u_1 + 2u_2 - 3)), \quad (\text{B7f})$$

$$\mathcal{G} = 2(2M^2(m_A^2(u_1 + u_2) + Q'^2(u_1 + u_2 - 1) + Q'^2(u_1 + u_2 - 1)) - 2Q^2 u_1 u_2(m_A^2(2u_1 + u_2 - 1) + Q'^2(u_1 + u_2 - 1)) - u_2(m_A^4 u_1(u_1 + u_2) + 2m_A^2 Q'^2 u_1(u_1 + 2u_2 - 1) + Q'^4(u_1 + u_2 - 1)(u_1 + 2u_2 - 1)) - Q^4 u_1(u_1 + u_2 - 1)(2u_1 + u_2 - 1)). \quad (\text{B7g})$$

For remaining scalar functions, considering the symmetry of the Feynman parameters, i.e., that the interchange $u_1 \leftrightarrow u_2$ does not effect the Feynman parameter integration, one finds

$$f_1(Q^2, Q'^2) = f_2(Q'^2, Q^2), \quad (\text{B8a})$$

$$f_5(Q^2, Q'^2) = f_4(Q'^2, Q^2), \quad (\text{B8b})$$

$$f_6(Q^2, Q'^2) = f_3(Q'^2, Q^2), \quad (\text{B8c})$$

$$f_7(Q^2, Q'^2) = f_7(Q'^2, Q^2). \quad (\text{B8d})$$

In addition, regardless of the choice of ξ , one finds the relation

$$f_8(Q^2, Q'^2) = -f_7(Q^2, Q'^2), \quad (\text{B9})$$

and if one turns off the AMM channel, i.e., $\xi = 0$, one obtains

$$f_4(Q^2, Q'^2) = f_5(Q^2, Q'^2), \quad (\text{B10})$$

which leads to $F_3(Q^2, Q'^2) = 0$. That is, the form factor $F_3(Q^2, Q'^2)$ is generated by the AMM effect.

With the above expressions, one can demonstrate the gauge invariance of the amplitude $\mathcal{A}_{\mu\nu\rho}$ by showing that the scalar functions fulfill the relations in Eqs. (10a)–(10c). This verification is straightforward and the key steps given below.

3. Transition amplitude: Gauge invariance

To prove these relations, consider the following integral

$$\mathcal{I}_1 = \int_{\Omega} \partial_{u_2}(4u_2 I_0(\omega)) - \partial_{u_1}(4u_1 I_0(\omega)) \quad (\text{B11})$$

treat the undifferentiated Feynman parameter as constant, one obtains

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^1 du_1 (4u_2 I_0(\omega)|_{u_2=1-u_1} - 4u_2 I_0(\omega)|_{u_2=0}) \\
&\quad - \int_0^1 du_2 (4u_1 I_0(\omega)|_{u_1=1-u_2} - 4u_1 I_0(\omega)|_{u_1=0}) \\
&= \int_0^1 du_1 4(1-u_1) I_0(M^2 - m_A^2 u_1(1-u_1)) \\
&\quad - \int_0^1 du_2 4(1-u_2) I_0(M^2 - m_A^2 u_2(1-u_2)) \\
&= 0
\end{aligned} \tag{B12}$$

While directly computing the partial derivative

$$\begin{aligned}
\mathcal{I}_1 &= \int_{\Omega} 4(I_0(\omega) + u_2 \partial_{u_2} I_0(\omega) - I_0(\omega) - u_1 \partial_{u_1} I_0(\omega)) \\
&= \int_{\Omega} 8(u_1(1-2u_1)Q^2 - u_2(1-2u_2)Q^2) I_{-2}(\omega).
\end{aligned} \tag{B13}$$

For the evaluation of the last line, we use the relation

$$I_{-2(\alpha+1)}(\mathcal{M}^2) = -\frac{1}{(\alpha+2)} \frac{d}{d\mathcal{M}^2} I_{-2\alpha}(\mathcal{M}^2), \tag{B14}$$

which is valid under the proper time regularization Eq. (B2). Therefore, one has

$$0 = \int_{\Omega} 8(u_1(1-2u_1)Q^2 - u_2(1-2u_2)Q^2) I_{-2}(\omega) \tag{B15}$$

Similarly, by considering

$$\mathcal{I}_2 = \int_{\Omega} \partial_{u_2} (4(1-u_2)u_2 I_0(\omega)) - \partial_{u_1} (4u_1 u_2 I_0(\omega)), \tag{B16}$$

one arrives at another identity

$$0 = \mathcal{I}_2 = \int_{\Omega} 4((1-3u_2)I_0(\omega) - CI_{-2}(\omega)). \tag{B17}$$

Note that Eqs. (B15) and (B17) leads to vanishing $f_3^{LT'}$ and $f_3^{LA'}$, which indicates that the longitudinal part of the QPV does not contribute to the transition amplitude, as expected. Plugging in the regularized form factors f_i into Eqs. (10a)–(10c), one finds that Eq. (10a) becomes

$$2N_{cf} E_A(f_L(Q^2)f_T(Q^2)\mathcal{I}_2 + f_L(Q^2)f_A(Q^2)\mathcal{I}_1) = 0. \tag{B18}$$

On its part, Eq. (10b) can be verified by just changing Q^2 and Q'^2 in Eq. (10a). For the remaining piece, Eq. (10c),

we see that it trivially vanishes when inserting the form factors f_i into it. This of course can be expected from the transverse structure of the AV meson BSA. At this stage, we have proved that the gauge invariance of the transition amplitude. It should be noted that, within our formalism and regularization, gauge invariance is preserved for arbitrary quark masses, AV meson masses, and the dressing functions of the QPV, even if these functions are not obtained from the corresponding equations.

APPENDIX C: PARAMETRIZATIONS

Let us consider the representation for the AV TFFs introduced in Ref. [24]

$$F_i^A(x, w) = G_i^A(x) H_i^A(x, w). \tag{C1}$$

Here $x = (Q^2 + Q'^2)/(2m_V)$ and $w = (Q^2 - Q'^2)/(Q^2 + Q'^2)$, with $|w| \leq 1$. Clearly, $w = 0$ corresponds to the symmetric limit, whereas $w = \pm 1$ stem from either $Q^2 = 0$ or $Q'^2 = 0$. The functions $G_i^A(x)$ and $H_i^A(x, w)$ are defined as follows:

TABLE IV. Fit parameters for the $\{a_i, f_i, f'_i\}$ components of the AV TFFs, according to the representation in Eq. (C1).

Case I						
	$F_1^{a_1+f_1}$	$F_2^{a_1+f_1}$	$F_3^{a_1+f_1}$	$F_1^{f'_1}$	$F_2^{f'_1}$	$F_3^{f'_1}$
a_i	0.486	0.249	-0.005	0.068	0.031	-0.001
b_i	0.955	0.944	-0.016	0.124	0.127	-0.003
c_i	4.344	0.437	-0.013	0.136	0.055	-0.003
d_i	17.800	2.363	0.797	11.317	3.082	1.414
e_i	174.01	0.0	0.0	25.971	0.442	0.117
μ_i	1.434	1.552	...	1.396	1.464	...
λ_i	4.235	5.813	...	4.553	6.331	...
Case II						
	$F_1^{a_1+f_1}$	$F_2^{a_1+f_1}$	$F_3^{a_1+f_1}$	$F_1^{f'_1}$	$F_2^{f'_1}$	$F_3^{f'_1}$
a_i	0.465	0.231	-0.003	0.067	0.032	-0.001
b_i	0.916	0.802	-0.010	0.123	0.100	-0.001
c_i	8.733	0.443	-0.098	0.136	0.057	-0.002
d_i	14.378	2.131	1.055	8.698	2.318	2.934
e_i	296.089	0.0	0.188	19.650	0.0	0.0
μ_i	1.498	1.598	...	1.48	1.524	...
λ_i	3.916	5.711	...	4.619	5.752	...
Case O						
	$F_1^{a_1+f_1}$	$F_2^{a_1+f_1}$	$F_3^{a_1+f_1}$	$F_1^{f'_1}$	$F_2^{f'_1}$	$F_3^{f'_1}$
a_i	0.448	0.196	-0.002	0.070	0.032	-0.001
b_i	0.894	0.683	-0.006	0.131	0.106	-0.002
c_i	2.886	0.425	-0.173	0.141	0.070	-0.002
d_i	8.876	1.904	2.251	6.099	1.862	1.052
e_i	59.960	0.0	57.629	12.685	0.0	0.301
μ_i	1.534	1.186	...	1.497	1.525	...
λ_i	3.605	5.421	...	3.846	5.682	...

$$\begin{aligned}
G_i^A(x) &= \frac{1}{(1+x)^{2\nu_i}} \left(a_i + b_i x + c_i x^{\nu_i} \frac{x+d_i}{x+e_i} \right), & H_1^A(x, w) &= 1 + \sum_{n=1}^N \frac{x^{\mu_1}}{x^{\mu_1} + \lambda_1 n} \frac{w^{2n}}{1 + \frac{2n}{3}}, \\
H_2^A(x, w) &= 1 + \sum_{n=1}^N \frac{x^{\mu_2}}{x^{\mu_2} + \lambda_2 n} \frac{(1+n)w^{2n}}{(1 + \frac{2n}{3})(1 + \frac{2n}{5})}, & H_3^A(x, w) &= 1.
\end{aligned} \tag{C2}$$

The profile of $G_i^A(x)$ governs the power-law falloff, so we adopt $\nu_1 = 2$ and $\nu_{2,3} = 3$. The functions $H_i^A(x, w)$ encode the asymptotic logarithms in the limit $N \rightarrow \infty$. For convenience, we set $N = 100$. Recall, in our case, $m_{\rho, \phi} = 0.879, 1.058$ GeV. The rest of the parameters are collected in Table IV.

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- [1] D.P. Aguillard *et al.*, Measurement of the positive muon anomalous magnetic moment to 127 ppb, *Phys. Rev. Lett.* **135**, 101802 (2025).
 - [2] R. Aliberti *et al.*, The anomalous magnetic moment of the muon in the Standard Model: An update, *Phys. Rept.* **1143**, 1 (2025).
 - [3] T. Aoyama *et al.*, The anomalous magnetic moment of the muon in the standard model, *Phys. Rep.* **887**, 1 (2020).
 - [4] G. Colangelo *et al.*, Prospects for precise predictions of a_μ in the standard model, p. 3, 2022, <https://lss.fnal.gov/archive/2022/conf/fermilab-conf-22-236-t.pdf>.
 - [5] Martin Hoferichter, Peter Stoffer, and Maximilian Zillinger, Complete dispersive evaluation of the hadronic light-by-light contribution to muon g-2, *Phys. Rev. Lett.* **134**, 061902 (2025).
 - [6] Martin Hoferichter, Peter Stoffer, and Maximilian Zillinger, Dispersion relation for hadronic light-by-light scattering: Subleading contributions, *J. High Energy Phys.* **02** (2025) 121.
 - [7] Gilberto Colangelo, Martin Hoferichter, Massimiliano Procura, and Peter Stoffer, Dispersion relation for hadronic light-by-light scattering: Theoretical foundations, *J. High Energy Phys.* **09** (2015) 074.
 - [8] Antoni Szczurek, Production of axial-vector mesons at e^+e^- collisions with double-tagging as a way to constrain the axial meson light-by-light contribution to the muon g-2 and the hyperfine splitting of muonic hydrogen, *Phys. Rev. D* **102**, 113015 (2020).
 - [9] Martin Hoferichter, Bastian Kubis, and Marvin Zanke, Axial-vector transition form factors and $e^+e^- \rightarrow f_1 \pi^+ \pi^-$, *J. High Energy Phys.* **08** (2023) 209.
 - [10] Marvin Zanke, Martin Hoferichter, and Bastian Kubis, On the transition form factors of the axial-vector resonance $f_1(1285)$ and its decay into e^+e^- , *J. High Energy Phys.* **07** (2021) 106.
 - [11] Kirill Melnikov and Arkady Vainshtein, Hadronic light-by-light scattering contribution to the muon anomalous magnetic moment revisited, *Phys. Rev. D* **70**, 113006 (2004).
 - [12] Jan Lüdtkke, Massimiliano Procura, and Peter Stoffer, Dispersion relations for the hadronic VVA correlator, *J. High Energy Phys.* **04** (2025) 130.
 - [13] Johan Bijnens, Nils Hermansson-Truedsson, and Antonio Rodríguez-Sánchez, Short-distance constraints for the HLbL contribution to the muon anomalous magnetic moment, *Phys. Lett. B* **798**, 134994 (2019).
 - [14] Johan Bijnens, Nils Hermansson-Truedsson, Laetitia Laub, and Antonio Rodríguez-Sánchez, Short-distance HLbL contributions to the muon anomalous magnetic moment beyond perturbation theory, *J. High Energy Phys.* **10** (2020) 203.
 - [15] Johan Bijnens, Nils Hermansson-Truedsson, Laetitia Laub, and Antonio Rodríguez-Sánchez, The two-loop perturbative correction to the $(g-2)_\mu$ HLbL at short distances, *J. High Energy Phys.* **04** (2021) 240.
 - [16] Johan Bijnens, Nils Hermansson-Truedsson, and Antonio Rodríguez-Sánchez, Constraints on the hadronic light-by-light in the Melnikov-Vainshtein regime, *J. High Energy Phys.* **02** (2023) 167.
 - [17] Pablo Roig and Pablo Sanchez-Puertas, Axial-vector exchange contribution to the hadronic light-by-light piece of the muon anomalous magnetic moment, *Phys. Rev. D* **101**, 074019 (2020).
 - [18] Jonas Mager, Luigi Cappiello, Josef Leutgeb, and Anton Rebhan, Longitudinal short-distance constraints on hadronic light-by-light scattering and tensor-meson contributions to the muon g-2, *Phys. Rev. Lett.* **135**, 091901 (2025).
 - [19] Luigi Cappiello, Josef Leutgeb, Jonas Mager, and Anton Rebhan, Tensor meson transition form factors in holographic QCD and the muon $g-2$, *J. High Energy Phys.* **07** (2025) 033.
 - [20] Johann H. Kuhn, Jean Kaplan, and El Ghali Oudrhiri Safiani, Electromagnetic annihilation of e^+e^- into quarkonium states with even charge conjugation, *Nucl. Phys. B* **157**, 125 (1979).
 - [21] A.S. Rudenko, $f_1(1285) \rightarrow e^+e^-$ decay and direct f_1 production in e^+e^- collisions, *Phys. Rev. D* **96**, 076004 (2017).
 - [22] Pere Masjuan, Pablo Roig, and Pablo Sanchez-Puertas, The interplay of transverse degrees of freedom and axial-vector mesons with short-distance constraints in $g-2$, *J. Phys. G* **49**, 015002 (2022).

- [23] Martin Hoferichter and Peter Stoffer, Asymptotic behavior of meson transition form factors, *J. High Energy Phys.* **05** (2020) 159.
- [24] Gernot Eichmann, Christian S. Fischer, Tim Haeuser, and Oliver Regenfelder, Axial-vector and scalar contributions to hadronic light-by-light scattering, *Eur. Phys. J. C* **85**, 445 (2025).
- [25] Josef Leutgeb and Anton Rebhan, Axial vector transition form factors in holographic QCD and their contribution to the anomalous magnetic moment of the muon, *Phys. Rev. D* **101**, 114015 (2020).
- [26] L. D. Landau, On the angular momentum of a system of two photons, *Dokl. Akad. Nauk SSSR* **60**, 207 (1948).
- [27] Chen-Ning Yang, Selection rules for the dematerialization of a particle into two photons, *Phys. Rev.* **77**, 242 (1950).
- [28] Jozef J. Dudek, Robert G. Edwards, and David G. Richards, Radiative transitions in charmonium from lattice QCD, *Phys. Rev. D* **73**, 074507 (2006).
- [29] Benjamin Owen, Waseem Kamleh, Derek Leinweber, Benjamin Menadue, and Selim Mahbub, Light meson form factors at near physical masses, *Phys. Rev. D* **91**, 074503 (2015).
- [30] Yin-Zhen Xu, Daniele Binosi, Zhu-Fang Cui, Bo-Lin Li, Craig D Roberts, Shu-Sheng Xu, and Hong Shi Zong, Elastic electromagnetic form factors of vector mesons, *Phys. Rev. D* **100**, 114038 (2019).
- [31] Zanbin Xing, Khépani Raya, and Lei Chang, Quark anomalous magnetic moment and its effects on the ρ meson properties, *Phys. Rev. D* **104**, 054038 (2021).
- [32] B. Almeida-Zamora, J. J. Cobos-Martínez, A. Bashir, K. Raya, J. Rodríguez-Quintero, and J. Segovia, Light-front wave functions of vector mesons in an algebraic model, *Phys. Rev. D* **107**, 074037 (2023).
- [33] Antonio Rojas and Genaro Toledo, ρ meson magnetic dipole moment from *BABAR* $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$ cross section measurement, *Phys. Rev. D* **110**, 056037 (2024).
- [34] Fernando E. Serna, Roberto Correa da Silva, J. J. Cobos-Martínez, Bruno El-Bennich, and Eduardo Rojas, Distribution amplitudes of heavy mesons and quarkonia on the light front, *Eur. Phys. J. C* **80**, 955 (2020).
- [35] R. J. Hernández-Pinto, L. X. Gutiérrez-Guerrero, M. A. Bedolla, and A. Bashir, Electric, magnetic, and quadrupole form factors and charge radii of vector mesons: From light to heavy sectors in a contact interaction, *Phys. Rev. D* **110**, 114015 (2024).
- [36] Yin-Zhen Xu, Adnan Bashir, Khépani Raya, José Rodríguez-Quintero, and Jorge Segovia, Imaging the charge distributions of flavor-symmetric and -asymmetric mesons, *Phys. Lett. B* **869**, 139864 (2025).
- [37] Khépani Raya, Adnan Bashir, Daniele Binosi, Craig D. Roberts, and José Rodríguez-Quintero, Pseudoscalar mesons and emergent mass, *Few Body Syst.* **65**, 60 (2024).
- [38] Craig D. Roberts, David G. Richards, Tanja Horn, and Lei Chang, Insights into the emergence of mass from studies of pion and kaon structure, *Prog. Part. Nucl. Phys.* **120**, 103883 (2021).
- [39] Craig D. Roberts and Anthony G. Williams, Dyson-Schwinger equations and their application to hadronic physics, *Prog. Part. Nucl. Phys.* **33**, 477 (1994).
- [40] Gernot Eichmann, Helios Sanchis-Alepuz, Richard Williams, Reinhard Alkofer, and Christian S. Fischer, Baryons as relativistic three-quark bound states, *Prog. Part. Nucl. Phys.* **91**, 1 (2016).
- [41] Si-xue Qin and Craig D Roberts, Impressions of the continuum bound state problem in QCD, *Chin. Phys. Lett.* **37**, 121201 (2020).
- [42] Khépani Raya, Adnan Bashir, and Pablo Roig, Contribution of neutral pseudoscalar mesons to a_μ^{HLbL} within a Schwinger-Dyson equations approach to QCD, *Phys. Rev. D* **101**, 074021 (2020).
- [43] Ángel Miramontes, Adnan Bashir, Khépani Raya, and Pablo Roig, Pion and Kaon box contribution to a_μ^{HLbL} , *Phys. Rev. D* **105**, 074013 (2022).
- [44] Angel S. Miramontes, K. Raya, A. Bashir, P. Roig, and G. Paredes-Torres, Radially excited pion: Electromagnetic form factor and the box contribution to the muon's $g-2$, *Chin. Phys.* **49**, 083108 (2025).
- [45] Gernot Eichmann, Christian S. Fischer, Esther Weil, and Richard Williams, Single pseudoscalar meson pole and pion box contributions to the anomalous magnetic moment of the muon, *Phys. Lett. B* **797**, 134855 (2019); **799**, 135029(E) (2019).
- [46] Gernot Eichmann, Christian S. Fischer, and Richard Williams, Kaon-box contribution to the anomalous magnetic moment of the muon, *Phys. Rev. D* **101**, 054015 (2020).
- [47] Hao Dang, Zanbin Xing, M. Atif Sultan, Khépani Raya, and Lei Chang, Chiral anomaly and the pion transition form factor: Beyond the cutoff, *Phys. Rev. D* **108**, 054031 (2023).
- [48] Zanbin Xing, Hao Dang, M. Atif Sultan, Khépani Raya, and Lei Chang, QCD anomalies in electromagnetic processes: A solution to the $\gamma \rightarrow 3\pi$ puzzle, *Phys. Rev. D* **109**, 054028 (2024).
- [49] Yehan Xu, M. Atif Sultan, Khépani Raya, and Lei Chang, Contact interaction treatment of $V \rightarrow P\gamma$ for light-quark mesons, *Phys. Rev. D* **110**, 094036 (2024).
- [50] Ya Lu, Chen Chen, Craig D. Roberts, Jorge Segovia, Shu-Sheng Xu, and Hong-Shi Zong, Parity partners in the baryon resonance spectrum, *Phys. Rev. C* **96**, 015208 (2017).
- [51] Pei-Lin Yin, Zhu-Fang Cui, Craig D Roberts, and Jorge Segovia, Masses of positive- and negative-parity hadron ground-states, including those with heavy quarks, *Eur. Phys. J. C* **81**, 327 (2021).
- [52] L. X. Gutiérrez-Guerrero, G. Paredes-Torres, and A. Bashir, Mesons and baryons: Parity partners, *Phys. Rev. D* **104**, 094013 (2021).
- [53] Si-Xue Qin and Craig D. Roberts, Resolving the Bethe-Salpeter Kernel, *Chin. Phys. Lett.* **38**, 071201 (2021).
- [54] Lei Chang and Craig D. Roberts, Sketching the Bethe-Salpeter kernel, *Phys. Rev. Lett.* **103**, 081601 (2009).
- [55] Lei Chang, Yu-Bin Liu, Khépani Raya, J. Rodríguez-Quintero, and Yi-Bo Yang, Linking continuum and lattice quark mass functions via an effective charge, *Phys. Rev. D* **104**, 094509 (2021).
- [56] Khépani Raya, L. X. Gutiérrez-Guerrero, Adnan Bashir, Lei Chang, Zhu-Fang Cui, Ya Lu, Craig D. Roberts, and Jorge Segovia, Dynamical diquarks in the $\gamma^{(*)}p \rightarrow N(1535)_{\frac{1}{2}}^-$ transition, *Eur. Phys. J. A* **57**, 266 (2021).

- [57] H. L. L. Roberts, C. D. Roberts, A. Bashir, L. X. Gutierrez-Guerrero, and P. C. Tandy, Abelian anomaly and neutral pion production, *Phys. Rev. C* **82**, 065202 (2010).
- [58] L. X. Gutierrez-Guerrero, A. Bashir, I. C. Cloet, and C. D. Roberts, Pion form factor from a contact interaction, *Phys. Rev. C* **81**, 065202 (2010).
- [59] Zanbin Xing and Lei Chang, Symmetry preserving contact interaction treatment of the kaon, *Phys. Rev. D* **107**, 014019 (2023).
- [60] Yue-Liang Wu, Symmetry principle preserving and infinity free regularization and renormalization of quantum field theories and the mass gap, *Int. J. Mod. Phys. A* **18**, 5363 (2003).
- [61] Yue-Liang Wu, Symmetry preserving loop regularization and renormalization of QFTs, *Mod. Phys. Lett. A* **19**, 2191 (2004).
- [62] J. G. Korner, D. Kreimer, and K. Schilcher, A Practicable 5⁻ scheme in dimensional regularization, *Z. Phys. C* **54**, 503 (1992).
- [63] F. Jegerlehner, Facts of life with γ_5 , *Eur. Phys. J. C* **18**, 673 (2001).
- [64] Hermès Bélusca-Maïto, Amon Ilakovac, Paul Kühler, Marija Mađor-Božinović, Dominik Stöckinger, and Matthias Weißwange, Introduction to renormalization theory and chiral gauge theories in dimensional regularization with nonanticommuting γ_5 , *Symmetry* **15**, 622 (2023).
- [65] Gerard 't Hooft and M. J. G. Veltman, Regularization and renormalization of gauge fields, *Nucl. Phys. B* **44**, 189 (1972).
- [66] P. Breitenlohner and D. Maison, Dimensionally renormalized Green's functions for theories with massless particles. I. *Commun. Math. Phys.* **52**, 39 (1977).
- [67] Reinhard Alkofer and H. Reinhardt, The chiral anomaly and electromagnetic pion decay in effective quark models, *arXiv:hep-ph/9212238*.
- [68] G. P. Lepage, How to renormalize the Schrodinger equation, in *8th Jorge Andre Swieca Summer School on Nuclear Physics* (1997), vol. 2, pp. 135–180, <https://arxiv.org/abs/nucl-th/9706029>.
- [69] Lei Chang, Yu-Xin Liu, and Craig D. Roberts, Dressed-quark anomalous magnetic moments, *Phys. Rev. Lett.* **106**, 072001 (2011).
- [70] A. Bashir, R. Bermudez, L. Chang, and C. D. Roberts, Dynamical chiral symmetry breaking and the fermion–gauge-boson vertex, *Phys. Rev. C* **85**, 045205 (2012).
- [71] Noboru Nakanishi, A general survey of the theory of the Bethe-Salpeter equation, *Prog. Theor. Phys. Suppl.* **43**, 1 (1969).
- [72] Chen Chen, Lei Chang, Craig D. Roberts, Shaolong Wan, and David J. Wilson, Spectrum of hadrons with strangeness, *Few Body Syst.* **53**, 293 (2012).
- [73] S. Navas *et al.*, Review of particle physics, *Phys. Rev. D* **110**, 030001 (2024).
- [74] Helmut Haberzettl, Model-independent form-factor constraints for electromagnetic spin-1 currents, *Phys. Rev. D* **100**, 036008 (2019).
- [75] Stanley J. Brodsky and John R. Hiller, Universal properties of the electromagnetic interactions of spin one systems, *Phys. Rev. D* **46**, 2141 (1992).
- [76] Thomas Hilger, M. Gomez-Rocha, and Andreas Krassnigg, Light-quarkonium spectra and orbital-angular-momentum decomposition in a Bethe–Salpeter-equation approach, *Eur. Phys. J. C* **77**, 625 (2017).
- [77] Gilberto Colangelo, Martin Hoferichter, Massimiliano Procura, and Peter Stoffer, Dispersion relation for hadronic light-by-light scattering: Two-pion contributions, *J. High Energy Phys.* **04** (2017) 161.
- [78] Martin Hoferichter, Peter Stoffer, and Maximilian Zillinger, An optimized basis for hadronic light-by-light scattering, *J. High Energy Phys.* **04** (2024) 092.
- [79] T. Hahn, CUBA: A library for multidimensional numerical integration, *Comput. Phys. Commun.* **168**, 78 (2005).
- [80] Pere Masjuan and Pablo Sanchez-Puertas, Pseudoscalar-pole contribution to the $(g_\mu - 2)$: A rational approach, *Phys. Rev. D* **95**, 054026 (2017).
- [81] G. Peter Lepage and Stanley J. Brodsky, Exclusive processes in perturbative quantum chromodynamics, *Phys. Rev. D* **22**, 2157 (1980).
- [82] Ya Lu, Yin-Zhen Xu, Khépani Raya, Craig D. Roberts, and José Rodríguez-Quintero, Pion distribution functions from low-order Mellin moments, *Phys. Lett. B* **850**, 138534 (2024).
- [83] P. Achard *et al.*, $f_1(1285)$ formation in two photon collisions at LEP, *Phys. Lett. B* **526**, 269 (2002).