

Analysis of the fragmentation function of gluon at next-to-leading order approximation

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We are investigating the behavior of the fragmentation function of a gluon, denoted as $D_g(x, \mu^2)$, where μ represents the observable scale. This function is derived from the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi evolution equations. Our objective is to evolve the fragmentation function of a gluon for heavy-quark-antiquark bound states with large transverse momentum using a Laplace transform technique. This method enables us to calculate numerical solutions for two and four quarkonium states based on the known initial fragmentation function of gluons. We examine both leading-order and higher-order approximations for the fragmentation function of a gluon, $[g \rightarrow T_{nQ}]$, by integrating the evolved fragmentation function of the gluon at the initial scale. In our computations, we utilize the initial scales for T_g^{2c} from Braaten and Yuan [Phys. Rev. Lett. **71**, 1673 (1993)] and for T_g^{4c} and T_g^{4b} from Celiberto *et al.* [Eur. Phys. J. C **84**, 1071 (2024)] and Celiberto and Gatto [Phys. Rev. D **111**, 034037 (2025)], respectively. Through comparing our predictions with existing literature results, we can accurately determine the evolution of the fragmentation function of a gluon at scale μ .

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I. INTRODUCTION

The theory that describes the strong interaction, in which the fundamental degrees of freedom are quarks and gluons, is quantum chromodynamics (QCD). QCD, as the fundamental theory, describes the strong interactions of heavy quarkonia with large transverse momentum P_T . The dominant mechanism for the production of a parton with high transverse momentum is fragmentation, which subsequently decays into the quarkonium state and other partons [1]. A measurement of gluon fragmentation in various three-jet topologies has been performed in the DELPHI experiment [2]. The scaling violation for gluon jets appears to be significantly stronger than for quark fragmentation. The probability for a virtual gluon to decay into a ψ was calculated in Ref. [3], where they studied J/ψ production from gluon jets at the CERN e^+e^- collider, LEP. In Ref. [4], a correct definition of the gluon FF, $D_g^H(z)$, is presented starting from the single gluon incoming state $|g\rangle$ with $Q_\mu^a(x)$, which is the quantum gluon field that fragments into hadron H . The authors demonstrate that

using the non-Abelian field tensor¹ $F_{\mu\nu}^a(x) = \partial_\mu Q_\nu^a(x) - \partial_\nu Q_\mu^a(x) + gf^{abc}Q_\mu^b(x)Q_\nu^c(x)$ instead of $Q_\mu^a(x)$ in the initial state is not in line with the factorization theorem in QCD. They conclude that the accurate definition of the gluon fragmentation function at high energy colliders is derived from the single gluon initial state $|g\rangle$, is gauge invariant, and aligns with the factorization theorem in QCD at high-energy colliders.

The fragmentation functions (FFs) $D_{g,q}^H(x, \mu)$ into a hadron are studied for individual gluon and quark jets in three jet events originating from Z decays using the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations [5–7] by evolving from the initial FF $D_{g,q}^H(x, \mu_0)$. The FFs can be defined as the scattering cross section of the process $A + B \rightarrow H + \text{Jet}$:

$$d\sigma = \sum_{a,b,c} \int_0^1 dx_a \int_0^1 dx_b \int_0^1 dz f_{a/A}(x_a, \mu) f_{b/A}(x_b, \mu) \times d\hat{\sigma}(a + b \rightarrow c + \text{Jet}) D_{H/c}(z, \mu), \quad (1)$$

where a and b are incident partons in the colliding hadrons and $f_{i/A}(i = a, b)$ are the parton distribution functions at the scale μ . The FFs at the initial scale can be found in literature [1,8–10]. In Ref. [1], the authors

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¹Based on the Yang-Mills construction, the coupling between the gluonic field and matter is defined through the covariant derivative $D_\mu = \partial_\mu - igT^a Q_\mu^a$ with $T_{bc}^{(A)a} = -if^{abc}$.

calculated² the FF $D_{g \rightarrow \eta_c}(z, 2m_c)$ for a gluon to split into the $^1S_0$ charmonium state η_c to leading order in $\alpha_s(2m_c)$. In Refs. [8,9], the authors defined the FFs at the initial scale, focusing on the fragmentation of a gluon into the S -wave spin singlet and spin-triplet heavy quarkonia considering the processes $g \rightarrow \eta_c/\eta_b$ at the leading-order (LO) and next-to-leading order (NLO) approximations³ based on the Suzuki model⁴ [10].

In Ref. [11], relativistic correction effects to the $g \rightarrow \eta_c$ FF in the effective-field-theory approach known as non-relativistic QCD (NRQCD) factorization are discussed. In this model, the FF of a gluon fragmenting into the pseudoscalar quarkonium is expanded in terms of the characteristic velocity $q(\bar{q})$ of the quark inside a charmonium denoted as v , by the following form:

$$D_{g \rightarrow \eta_i}(z) = d^{(0)}(z) \langle \mathcal{O}_1^{\eta_i} \rangle + d^{(2)}(z) \frac{\langle \mathcal{P}_1^{\eta_i} \rangle}{m^2} + \mathcal{O}(v^3),$$

$$i = c, b. \quad (2)$$

Here, $d^{(0)}(z)$ and $d^{(2)}(z)$ are the corresponding short-distance coefficient functions. $\mathcal{O}_1^{\eta_i}$ and $\mathcal{P}_1^{\eta_i}$ are color-singlet

²In Feynman gauge, the FF for $g^* \rightarrow \eta_c g$ at the initial scale reads

$$D_{g \rightarrow \eta_c}(z, 2m_c) = \frac{1}{24\pi} \alpha_s(2m_c)^2 \frac{|R(0)|^2}{m_c^3} \times [3z - 2z^2 + 2(1-z) \ln(1-z)],$$

where the value of the S -state wave function at the origin $R(0)$ is determined from the ψ electronic width to be $|R(0)|^2 = (0.8 \text{ GeV})^3$ with $\alpha_s(2m_c) = 0.26$ where $m_c = 1.5 \text{ GeV}$ [11].

³At the LO approximation, FF reads

$$D_{g \rightarrow \eta_i}^{\text{LO}}(z, 2m_i) = \frac{Nz}{F(z, \langle p_T^2 \rangle)} \left[2m_i + \frac{z \langle p_T^2 \rangle}{M(1-z)} + \frac{M(1-z)}{z} \right],$$

$$i = c, b,$$

where $N = \frac{2}{9}(\pi m_c^{7/2} f_M^2 C_F \alpha_s)$ and

$$F(z, \langle p_T^2 \rangle) = \left[\frac{z^2 \langle p_T^2 \rangle + M^2(1-z)^2}{Mz(1-z)} \left(M^2 + \frac{z^2 \langle p_T^2 \rangle}{1-z} \right) \right]^2,$$

where M is the meson mass and $f_M = \sqrt{\frac{12}{M}} |\psi(0)|$ with $m_c = 1.67 \text{ GeV}$ and $m_b = 4.78 \text{ GeV}$. At NLO approximation, the FF reads

$$D_{g \rightarrow \eta_i}^{\text{NLO}}(z, 2m_i) = N(D_{g \rightarrow \eta_i}^{\text{LO}}(z, 2m_i) + D_{g \rightarrow \eta_i}^{\text{Real}}(z, 2m_i)),$$

where N is the normalization constant under the condition $\int_0^1 D_g(z, 2m_i) dz = 1$. The superscript “Real” refers to the contribution of Feynman diagrams including real gluon emissions [8].

⁴In the Suzuki model, the FFs depend on the transverse momentum k_T of the initial parton, which is treated as a phenomenological parameter.

NRQCD production operators.⁵ In Refs. [12–14], the evolution of the gluon FF in the modified leading logarithm approximation (MLLA)⁶ is discussed. The authors present the evolution of the gluon FF at the LO and higher-order approximations. They demonstrate that the higher moments of the evolution obtained from the MLLA are considered spurious higher-order terms in perturbation theory.

The FF of gluon into spin-singlet P-wave quarkonium was developed by Collins and Soper using the velocity expansion in the NRQCD factorization approach. This FF was also applied to $B_c^{(*)}$ within the NRQCD factorization framework at the lowest order in velocity expansion and strong coupling constant. These developments are discussed in Refs. [15,16], respectively. Analytical solutions for DGLAP evolution equations for FFs of mesons and protons based on the Laplace transform method are discussed in Refs. [17–19]. The symmetry breaking of the sea quark fragmentation functions according to their mass ratio is studied in Ref. [17]. The cross section of meson production via quark fragmentation is discussed in Refs. [17,20].

Recently, the inclusive production of fully charmed tetraquarks in high-energy proton collisions has been considered in Refs. [21–23]. The first exotic hadron, the $X(3872)$, was observed in 2003 by Belle at KEKB [24]. Then, in 2020, the LHCb experiment observed the $X(6900)$ from the J/ψ -pair production in the mass region between 6.2 and 7.4 GeV/c^2 , covering the masses of states composed of four charm quarks [25]. It can be considered as a tetraquark or pentaquark. In the lowest Fock state, the T_{4c} tetraquark state is represented as $|cc\bar{c}\bar{c}\rangle$. The production mechanisms are based on the leading-twist fragmentation of a single parton into the observed hadron. The FF of a gluon to T_{2c} and T_{4c} states within a potential quark

⁵For the pseudoscalar quarkonium η_c , we have

$$\mathcal{O}_1^{\eta_c} = 2N_c, \quad \mathcal{P}_1^{\eta_c} = 2N_c(E^2 - m^2),$$

where $N_c = 3$. The corresponding short-distance coefficient functions accurate through order- v^2 , are defined at the LO approximation by the following forms:

$$d^{(0)}(z) = \frac{\alpha_s^2}{4N_c^2 m^3} [3z - 2z^2 + 2(1-z) \ln(1-z)],$$

$$d^{(2)}(z) = -\frac{11}{6} d^{(0)}(z).$$

Additionally, in Ref. [11], the $g \rightarrow \eta_c$ FF modified from the inclusive η_c production from Higgs boson decay $h \rightarrow g^* g \rightarrow \eta_c + gg$ is defined.

⁶The MLLA is a systematic improvement over an earlier approximation known as the double logarithmic approximation, which involves resumming leading logarithms by summing tree-level diagrams where outgoing gluons are strongly ordered in their angles of emission.

becomes dominant over direct short-distance production when the transverse momentum exceeds $\lesssim 10$ and 20 GeV, respectively [26]. The fully charmed tetraquark, T_{4c} , is known to have quantum numbers J^{PC} of 0^{++} for the ground state and 2^{++} for the radial excitation. In Ref. [22], the tetraquark collinear fragmentation functions of $T_{4c}(0^{++})$ and $T_{4c}(2^{++})$ are analyzed for bound states. This analysis focuses on single-parton collinear fragmentation in a variable-flavor number scheme (VFNS). The gluon fragmentation into a color-singlet S-wave $T_{4c}(J^{PC})$ for only the lowest-order contributions in the constituent quarks' relative velocity can be described by the equation

$$D_g^{T_{4c}(J^{PC})}(z, 4m_c) = \frac{1}{m_c^9} \sum_{[n]} \mathcal{D}_g^{(J^{PC})}(z, [n]) \langle \mathcal{O}^{T_{4c}(J^{PC})}([n]) \rangle. \quad (3)$$

Here $m_c = 1.5$ GeV, and $\mathcal{D}_g^{(J^{PC})}(z, [n])$ represents the short-distance coefficient for the $[g \rightarrow (cc\bar{c}\bar{c})]$ perturbative transition. $\langle \mathcal{O}^{T_{4c}(J^{PC})}([n]) \rangle$ denotes the color composite of the long-distance matrix elements (LDMEs) that illustrate the nonperturbative hadronization of the $T_{4c}(J^{PC})$ state. The quantity of $\mathcal{D}_g^{(J^{PC})}(z, [n])$ for the composite quantum number $[n]$ is determined using the color diquark-antidiquark basis to decompose a color-singlet tetraquark state. The quantity $\langle \mathcal{O}^{T_{4c}(J^{PC})}([n]) \rangle$ represents the nonperturbative contribution to the FF at the initial scale. This is obtained by calculating the radial wave functions at the origin using potential models. In Ref. [23], bottomoniumlike states in proton collisions in the semi-inclusive hadroproduction of doubly bottomed tetraquarks ($X_{b\bar{b}q\bar{q}}$) as well as fully bottomed ones (T_{4b}) are discussed. The explicit form of the $g \rightarrow T_{4b}$ at the initial scale $\mu = 4m_b$ with $m_b = 4.9$ GeV is the same as for $g \rightarrow T_{4c}$ (i.e., Appendix A). The values of the LDMEs of $T_{4b}(0^{++})$ and $T_{4b}(2^{++})$ fully bottomed states evaluated within the Coulomb potential diquark model are defined by the following form [23,26]:

$$\langle \mathcal{O}^{T_{4b}(J^{PC})}([n]) \rangle \approx 400 \langle \mathcal{O}^{T_{4c}(J^{PC})}([n]) \rangle. \quad (4)$$

The functions $\mathcal{D}_g^{(J^{PC})}(z, [n])$ and the values of $\langle \mathcal{O}^{T_{4c}(J^{PC})}([n]) \rangle$ are presented in Appendix A.

In this article, we investigate the behavior of the fragmentation function of a gluon that addresses the inclusive production in high-energy proton scattering at the bound states T_{2c} , T_{4c} , and T_{4b} into the fragmentation function of a gluon at the initial scale using Laplace transform techniques. We utilize the Laplace transform technique to solve the DGLAP evolution equation for $D_g(z, \mu)$ by employing the $D_g(z, \mu_0)$ at leading order (LO) and next-to-leading order (NLO) approximations.

In the next section, we will summarize the Laplace transform used to obtain an analytical solution for the

fragmentation function of a gluon at both the LO and NLO approximations. Following that, in Sec. III, we will present the numerical results and compare them with the existing FFs for the $[g \rightarrow T_{nQ}]$ fragmentation mechanism. The conclusions will be provided in Sec. IV. Additional information on fragmentation functions at the initial scales for the fully charmed tetraquarks and splitting functions in s space can be found in the Appendix A.

II. FRAGMENTATION FUNCTIONS

The production of hadrons is not a perturbative process; therefore, we cannot compute the FFs from perturbation theory. We consider the change in FF $D_i(z, \mu)$ when the scale is increased from μ to $\mu + d\mu$ by the DGLAP equations:

$$\frac{\partial D_i^{T_{nQ}}(z, \mu)}{\partial \ln(\mu)} = \sum_j D_j^{T_{nQ}}(z, \mu) \otimes P_{ji}(z, a_s(\mu)), \quad i, j = g, q, \quad (5)$$

where $D_s(z, \mu^2) = \sum_i (D_{q_i} + D_{\bar{q}_i})(z, \mu)$ and $D_g(z, \mu)$ are the singlet and gluon fragmentation functions, respectively. The symbol $\otimes$ denotes convolution according to the usual prescription. In a FF, z represents the fraction of a parton momentum carried by a produced parton. The splitting functions P_{ji} can be computed perturbatively in the strong coupling constant:

$$P_{ji}(z, a_s(\mu)) = \sum_{k=0} a_s^{k+1} P_{ji}^{(k)}(z), \quad a_s = \frac{\alpha_s}{4\pi}, \quad (6)$$

where the lowest-order functions $P_{ji}^{(0)}(z)$ are the same as those in deep inelastic scattering but the higher-order terms are different [27] (please refer to Appendix B). The FF of the gluon can be considered from the gluon-to-gluon splitting only via the DGLAP equation as

$$\frac{\partial D_g^{T_{nQ}}(z, \mu)}{\partial \ln(\mu)} = D_g^{T_{nQ}}(z, \mu) \otimes \left[a_s P_{gg}^{(0)}(z, \mu) + a_s^2 P_{gg}^{(1)}(z, \mu) \right], \quad (7)$$

where the splitting functions at the LO and NLO approximations are defined in Appendix B.

In the following, we will utilize the method developed in detail in [28–33] to determine the FF of gluon using a Laplace-transform method. We now express the DGLAP evolution equation for the FF of gluon [i.e., Eq. (7)] in terms of the variables $v = \ln(1/z)$ and μ along with the notation $\hat{D}_g^{T_{nQ}}(v, \mu) \equiv D_g^{T_{nQ}}(e^{-v}, \mu)$, as follows:

$$\begin{aligned} \frac{\partial \hat{D}_g^{T_{nQ}}(v, \mu)}{\partial \ln(\mu)} &= \int_0^v dw \left[a_s \hat{P}_{gg}^{(0)}(v-w) + a_s^2 \hat{P}_{gg}^{(1)}(v-w) \right] \\ &\times \hat{D}_g^{T_{nQ}}(w, \mu), \end{aligned} \quad (8)$$

where $\hat{P}_{gg}(v) \equiv e^{-v} P_{gg}(e^{-v})$. We define the Laplace transform of a function $\hat{P}_{gg}(v)$ as $\Phi_g(s)$, where

$$\Phi_g(s) \equiv \mathcal{L}[P_{gg}(v); s] = \int_0^\infty P_{gg}(v) e^{-sv} dv, \quad (9)$$

and the Laplace transform of a convolution of functions is defined as

$$\mathcal{L}\left[\int_0^v \hat{D}_g^{T_{nQ}}(w) \hat{P}_{gg}(v-w) dw; s\right] = \Phi_g(s) \times \mathcal{D}_g^{T_{nQ}}(s), \quad (10)$$

where $\mathcal{D}_g^{T_{nQ}}(s, \mu) = \mathcal{L}[\hat{D}_g^{T_{nQ}}(v, \mu); s]$. By utilizing the property that the Laplace transform of a convolution is equal to the product of the Laplace transforms of the individual factors, we can simplify Eq. (8) in the s space,

$$\frac{\partial \mathcal{D}_g^{T_{nQ}}(s, \mu)}{\partial \ln(\mu)} = [a_s \Phi_g^{(0)}(s) + a_s^2 \Phi_g^{(1)}(s)] \mathcal{D}_g^{T_{nQ}}(s, \mu), \quad (11)$$

where the function $\Phi_g^{(1)}(s)$ is defined in the s space in Appendix C.

(i) LO analysis: In the LO analysis, Eq. (11) can be rewritten as follows:

$$\frac{\partial \mathcal{D}_g^{T_{nQ}}(s, \mu)}{\partial \ln(\mu)} = a_s \Phi_g^{(0)}(s) \mathcal{D}_g^{T_{nQ}}(s, \mu), \quad (12)$$

where the solution to the equation above is as follows [34]:

$$\mathcal{D}_g^{T_{nQ}}(s, \tau) = k_{gg}^{(0)}(s, \tau) \mathcal{D}_{0g}^{T_{nQ}}(s), \quad (13)$$

where

$$k_{gg}^{(0)}(s, \tau) \equiv e^{2\Phi_g^{(0)}(s)} \left[\cosh\left(\frac{\tau}{2} \Phi_g^{(0)}(s)\right) + \sinh\left(\frac{\tau}{2} \Phi_g^{(0)}(s)\right) \right], \quad (14)$$

with

$$\Phi_g^{(0)}(s) = 12 \left(\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} - \frac{1}{s+3} - \psi(s+1) - \gamma_E \right) + \frac{33-2n_f}{3}, \quad (15)$$

where ψ is the digamma function and $\gamma_E = 0.5772156\dots$ is Euler's constant, and

$$\tau(\mu, \mu_0) = \int_{q=\mu_0}^{q=\mu} \frac{\alpha_s(q^2)}{4\pi} d \ln q^2. \quad (16)$$

By applying the inverse Laplace transform of a product to the convolution of functions, we have

$$\begin{aligned} \mathcal{L}^{-1}[\mathcal{D}_g^{T_{nQ}}(s, \tau); v] &= \hat{D}_g^{T_{nQ}}(v, \tau) \\ &= \mathcal{L}^{-1}[k_{gg}^{(0)}(s, \tau) \times \mathcal{D}_{0g}^{T_{nQ}}(s); v] \\ &= \int_0^\infty \hat{K}_{GG}^{(0)}(v-w, \tau) \hat{D}_{0g}^{T_{nQ}}(w) dw, \end{aligned} \quad (17)$$

where $\hat{K}_{GG}^{(0)}(v, \tau) \equiv \mathcal{L}^{-1}[k_{gg}^{(0)}(s, \tau); v]$ and $\hat{D}_{0g}^{T_{nQ}}(v)$ represents the FF of a gluon at the initial scale. Transforming the equation above back into the x space, we find the FF at the scale μ , into the initial scale μ_0 and the lowest-order contributions to the FFs (please refer to Appendix A), by

$$D_g^{T_{nQ}}(z, \mu) = \int_z^1 \frac{d\xi}{\xi} K_{GG}^{(0)}\left(\ln\left(\frac{z}{\xi}\right), \tau(\mu, \mu_0)\right) D_{0g}^{T_{nQ}}(\xi), \quad (18)$$

(ii) NLO analysis: In the NLO analysis, the dependence of the NLO terms on the right-hand side of Eq. (11) on τ preserves a second Laplace transform into U space [35]. This can be represented as where

$$\begin{aligned} \mathcal{G}_g^{T_{nQ}}(s, U) &= \mathcal{L}[\mathcal{D}_g^{T_{nQ}}(s, \tau); U], \\ \mathcal{L}\left[\frac{\partial \mathcal{D}_g^{T_{nQ}}(s, \tau)}{\partial \tau}; U\right] &= U \mathcal{G}_g^{T_{nQ}}(s, U) - \mathcal{D}_{0g}^{T_{nQ}}(s). \end{aligned} \quad (19)$$

In U space, where s is a parameter in U space, the equation becomes

$$\begin{aligned} U \mathcal{G}_g^{T_{nQ}}(s, U) - \mathcal{D}_{0g}^{T_{nQ}}(s) \\ = \Phi_g^{(0)}(s) \mathcal{G}_g^{T_{nQ}}(s, U) + \Phi_g^{(1)}(s) \mathcal{L}[a_s(\tau) \mathcal{D}_g^{T_{nQ}}(s, \tau); U]. \end{aligned} \quad (20)$$

Here, $a_s(\tau) = \frac{\alpha_s(\tau)}{4\pi}$ with an approximation given by $a_s(\tau) \approx a_0 + a_1 e^{-b_1 \tau}$, where the constants are detailed in Ref. [35] as $a_0 = 0.0037$, $a_1 = 0.025$, and $b_1 = 10.7$. This equation can be rewritten in the following form:

$$\begin{aligned} [U - \Phi_g^{(0)}(s) - a_0 \Phi_g^{(1)}(s)] \mathcal{G}_g^{T_{nQ}}(s, U) \\ = \mathcal{D}_{0g}^{T_{nQ}}(s) + a_1 \Phi_g^{(1)}(s) \mathcal{G}_g^{T_{nQ}}(s, U + b_1), \end{aligned} \quad (21)$$

where $\mathcal{L}[a_s(\tau) \mathcal{D}_g^{T_{nQ}}(s, \tau); U] = a_0 \mathcal{G}_g^{T_{nQ}}(s, U) + a_1 \mathcal{G}_g^{T_{nQ}}(s, U + b_1)$. The solution to Eq. (21) is obtained in the following manner:

$$\begin{aligned} \mathcal{G}_g^{T_{nQ}}(s, U) &= \frac{1}{[U - \Phi_g^{(0)}(s) - a_0 \Phi_g^{(1)}(s)]} \\ &\times \left[\mathcal{D}_{0g}^{T_{nQ}}(s) + a_1 \Phi_g^{(1)}(s) \mathcal{G}_g^{T_{nQ}}(s, U + b_1) \right]. \end{aligned} \quad (22)$$

Because the terms in $\mathcal{G}_g^{T_{nQ}}(s, U)$ decrease rapidly as $1/|U|$ for $|U| \rightarrow \infty$, we can now apply the Laplace inverse from U space to τ space, as

$$\mathcal{D}_g^{T_{nQ}}(s, \tau) = k_{gg}^{(1)}(a_1, b_1, s, \tau) \mathcal{D}_{0g}^{T_{nQ}}(s). \quad (23)$$

The function $k_{gg}^{(1)}(a_1, b_1, s, \tau)$ can be expressed as a power series in the NLO expansion parameter a_1 . The coefficients of this series are analytic functions of s and τ , and can be represented by the following form [17]:

$$\begin{aligned} k_{gg}^{(1)}(a_1, b_1, s, \tau) &= \left\{ e^{-\frac{1}{2}b_1\tau} \left[b_1((3\Phi_g - b_1^2)e^{\tau(b_1+\Phi_g)} \right. \right. \\ &\quad + 4a_1 e^{\frac{1}{2}\tau(b_1+\Phi_g)} \left(\Phi_g^4 \cosh\left(\frac{1}{2}\tau\Phi_g\right) \sinh\left(\frac{1}{2}\tau b_1\right) \right. \\ &\quad \left. \left. - \Phi_g^2(b_1^2 - \Phi_g^2) \sinh\left(\frac{1}{2}\tau\Phi_g\right) \right) \right] \right\} \\ &\quad / (2b_1\Phi_g(\Phi_g^2 - b_1^2)), \end{aligned} \quad (24)$$

with Φ_g defined as $\Phi_g \equiv \Phi_g^{(0)}(s) + a_0 \Phi_g^{(1)}(s)$. After applying the inverse Laplace transform for s to v spaces and transforming back into x space, we find that

$$D_g^{T_{nQ}}(z, \mu) = \int_z^1 \frac{d\xi}{\xi} K_{GG}^{(1)}\left(\ln\left(\frac{z}{\xi}\right), \tau(\mu, \mu_0)\right) D_{0g}^{T_{nQ}}(\xi), \quad (25)$$

where $K_{GG}^{(1)}(v, \tau) \equiv \mathcal{L}^{-1}[k_{gg}^{(1)}(a_1, b_1, s, \tau); v]$. So, we have an explicit solution for the FF of a gluon at the LO and NLO approximations, which can be evaluated with the numerical accuracy of $D_{0g}^{T_{nQ}}(z)$ known for $[g \rightarrow T_{2Q}]$ and $[g \rightarrow T_{4Q}]$. With an analytical FF of a gluon at the initial scale, one can numerically extract the FF of a gluon at any desired z and μ^2 values.

III. RESULTS AND DISCUSSIONS

With the explicit form of gluon fragmentation in the channel leading to the $g \rightarrow T_{nQ}$ at the initial scale inputs, we can now extract numerical results for $\mu > \mu_0$. This is done using the FF of a gluon at initial scales of $\mu_0 = 2m_c$, $4m_c$, and $4m_b$ as reported in Refs. [1,22]. The QCD parameter Λ for active flavors has been determined [36] based on $\alpha_s(M_z^2) = 0.1166$ as

$$\begin{aligned} \text{LO: } \Lambda(n_f = 3) &= 0.1368 \text{ GeV}, & \Lambda(n_f = 4) &= 0.1368 \text{ GeV}, & \Lambda(n_f = 5) &= 0.8080 \text{ GeV}, \\ \text{NLO: } \Lambda(n_f = 3) &= 0.3472 \text{ GeV}, & \Lambda(n_f = 4) &= 0.2840 \text{ GeV}, & \Lambda(n_f = 5) &= 0.1957 \text{ GeV}. \end{aligned} \quad (26)$$

In Fig. 1, the results for the FF of a gluon in the channel $g \rightarrow T_{2c}$ within the LO approximation are shown and compared with the results of Ref. [1] at $\mu = 20m_c$. To demonstrate the effects of evolution, we evolve the FF at the scale $\mu = 20m_c$ with respect to the initial FF $D_{g \rightarrow \eta_c}(z, \mu = 2m_c)$. The z dependence of $D_{g \rightarrow \eta_c}(z, \mu)$ at the initial scale $\mu = 2m_c$ and the evolved scale $\mu = 20m_c$ at the LO approximation is shown in Fig. 1. Additionally, in Fig. 2, the results for the FF of a gluon in the channel $g \rightarrow T_{2c}$ within the NLO approximation are shown and compared with the results of Ref. [1] at $\mu = 20m_c$ as shown in Fig. 1. The difference between the results in the LO and NLO approximations with the Braaten [1] results are shown in the lower panel of Figs. 1 and 2, respectively. As observed, the gluon fragmentation at NLO accuracy using the Laplace transform is closer to the result in Ref. [1] than the calculations at LO in a wide range of z . These differences indicate that our results are in good agreement

with other models in the low- z region. In the following, we apply the NLO corrections to the calculations of the gluon fragmentation functions.

In Fig. 3, the evolution of the fragmentation function $D_{g \rightarrow \eta_c}(z, \mu)$ by Laplace transform at the NLO approximation is shown over a wide range of $\mu = 10\text{--}80m_c$. We observe that the point $z \simeq 0.4$ is independent of the evolution. The fragmentation function $D_{g \rightarrow \eta_c}(z, \mu)$ increases as μ increases and z decreases (i.e., $z \lesssim 0.4$). It decreases as μ and z increase (i.e., $z \gtrsim 0.4$). In Figs. 4 and 5, the gluon fragmentation channels to the $T_{4c}(0^{++})$ and $T_{4c}(2^{++})$ states as a function of the variable z using the Laplace transform method are presented at the NLO approximation and compared with those in Ref. [22] at the scale $\mu = 5m_c$. The results are evaluated based on the gluon fragmentation at the initial scale $\mu = 4m_c$ as defined in Eqs. (A9) and (A10) in Appendix A, respectively. The difference between the results in the NLO approximation

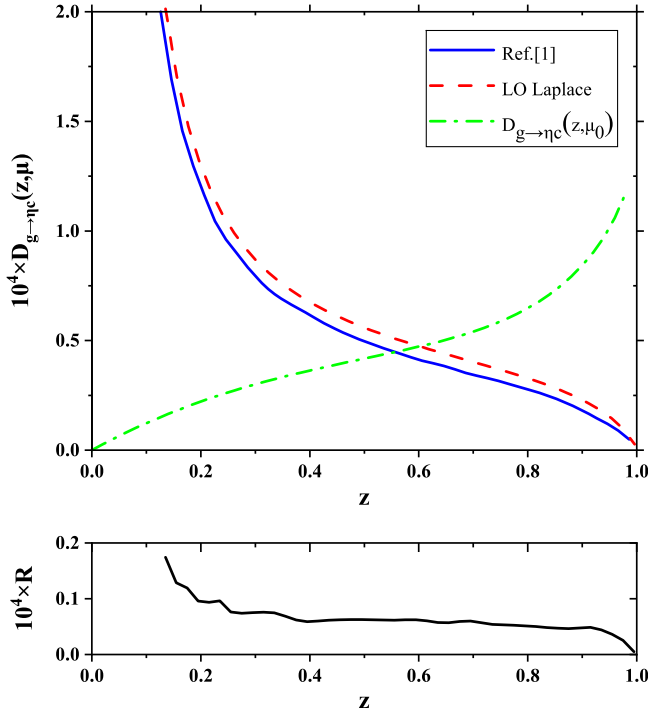


FIG. 1. Evolution of the fragmentation function $D_{g \rightarrow \eta_c}(z, \mu)$ as a function of the variable z is shown through Laplace transform for $\mu = 20m_c$ at the LO approximation (red-dashed curve), compared with the FF in Ref. [1] (blue-solid curve). The green-dash-dotted curve represents the FF of a gluon at the initial scale $\mu = 2m_c$ according to Ref. [1]. The difference $R = D_{g \rightarrow \eta_c}(\text{Laplace-Bratten})(z, 20m_c)$ between the results in the LO approximation and the Braaten results [1] is shown in the lower panel over a wide range of z .

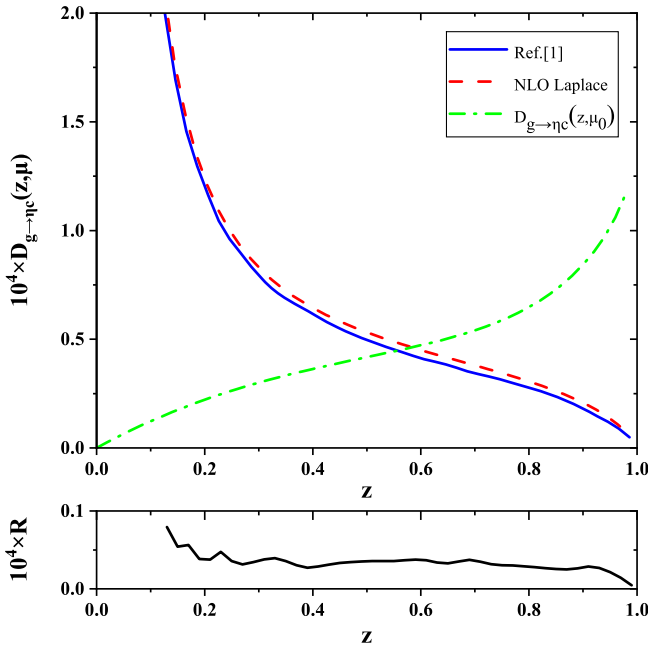


FIG. 2. The same as Fig. 1 at the NLO approximation.

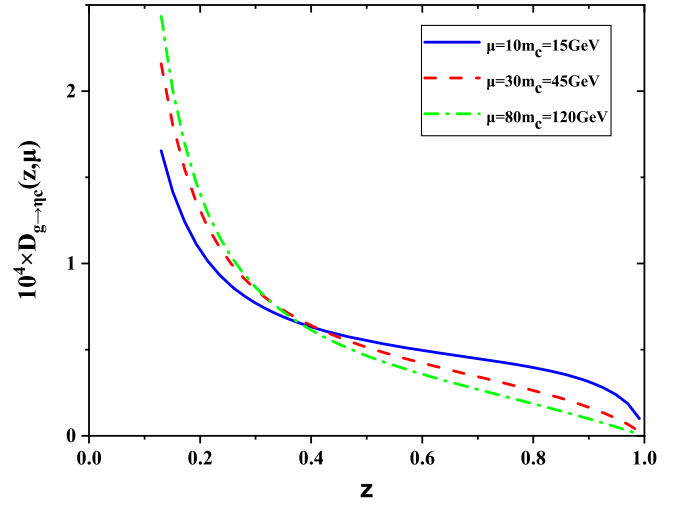


FIG. 3. The fragmentation functions $D_{g \rightarrow \eta_c}(z, \mu)$ are plotted at fixed μ values: $\mu = 10m_c$ (solid-blue curve), $\mu = 30m_c$ (dashed-red curve), and $\mu = 80m_c$ (dash-dotted-green curve). The plot shows the FF as a function of the variable z at the NLO approximation.

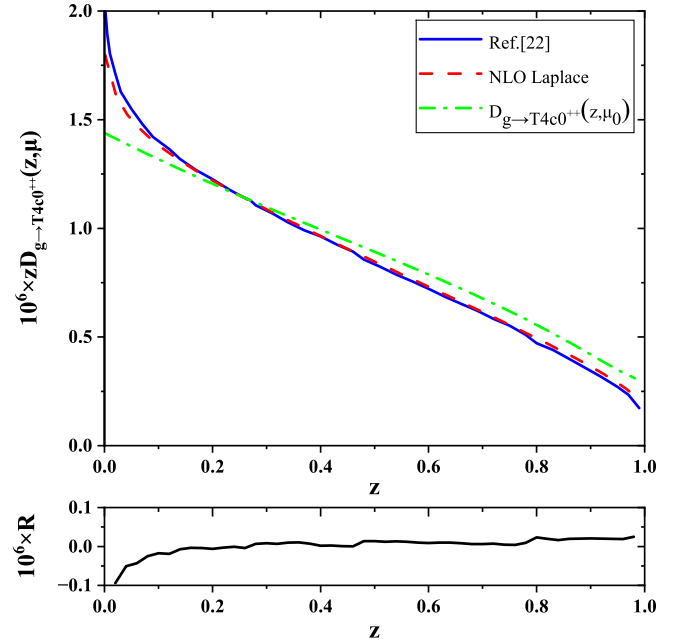


FIG. 4. Evolution of the fragmentation function $D_{g \rightarrow T_{4c}(0^{++})}(z, \mu)$ for the $T_{4c}(0^{++})$ state is shown as a function of the variable z using Laplace transform for $\mu = 5m_c$ at the NLO approximation (red-dashed curve), compared with the FF in Ref. [22] (blue-solid curve). The green-dash-dotted curve represents the FF of a gluon at the initial scale $\mu = 4m_c$ according to Ref. [22] for the $T_{4c}(0^{++})$ state. The difference $R = D_{g \rightarrow T_{4c}(0^{++})}(\text{Laplace-Celiberto})(z, 5m_c)$ between the results in the NLO approximation and the results from Celiberto *et al.* [22] is shown in the lower panel over a wide range of z .

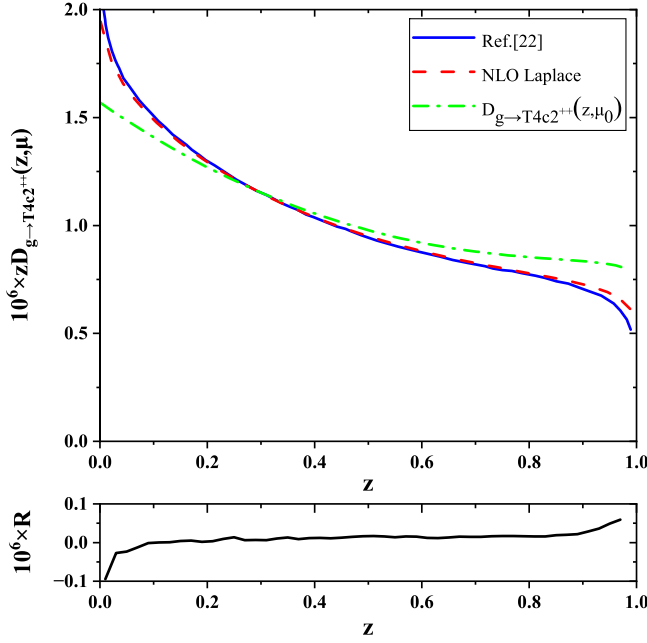


FIG. 5. The same as Fig. 4 for the gluon fragmentation channel to the $T_{4c}(2^{++})$ state.

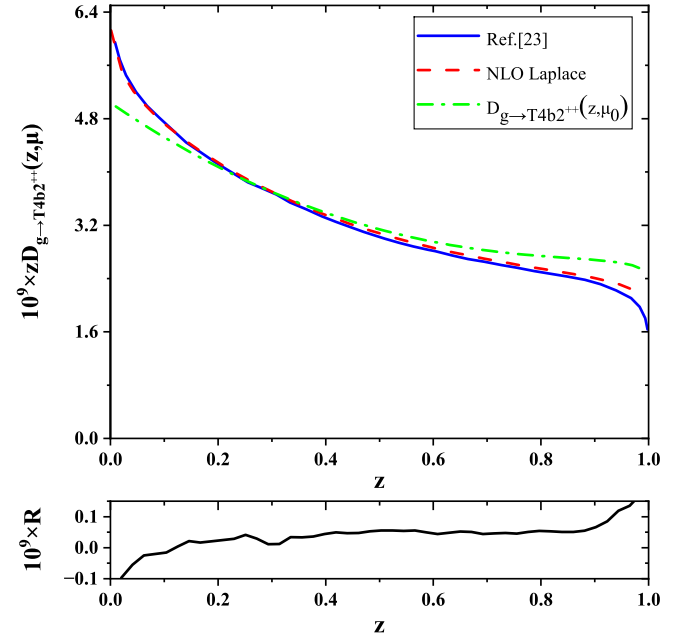


FIG. 7. The same as Fig. 6 for the gluon fragmentation channel to the $T_{4b}(2^{++})$ state.

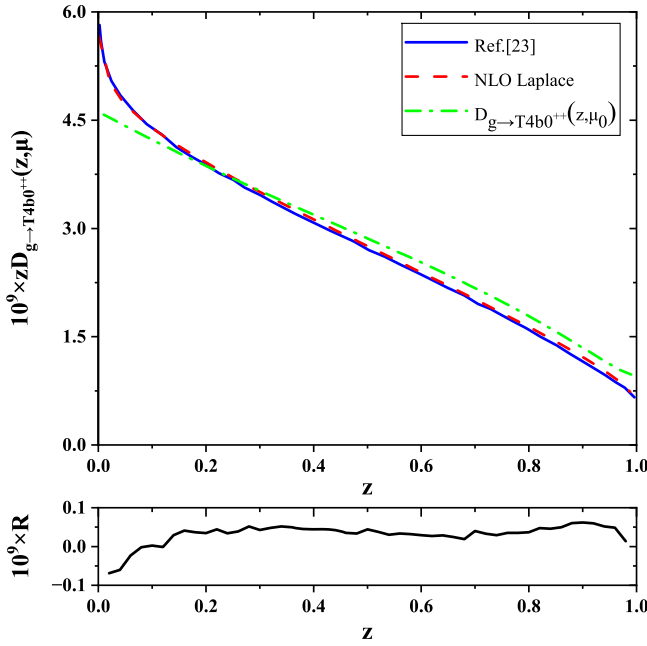


FIG. 6. Evolution of the fragmentation function $D_{g \to T_{4b}(0^{++})}(z, \mu)$ for the $T_{4b}(0^{++})$ state is shown as a function of the variable z using Laplace transform for $\mu = 5m_b$ at the NLO approximation (red-dashed curve), compared with the FF in Ref. [23] (blue-solid curve). The green-dash-dotted curve represents the FF of a gluon at the initial scale $\mu = 4m_b$ according to Ref. [23] for the $T_{4b}(0^{++})$ state. The difference $R = D_{g \to T_{4b}(0^{++})}(\text{Laplace-Celiberto})(z, 5m_b)$ between the results in the NLO approximation and the results from Celiberto *et al.* [23] is shown in the lower panel over a wide range of z .

for the gluon fragmentation channels to the $T_{4c}(0^{++})$ and $T_{4c}(2^{++})$ states compared to the results by Celiberto *et al.* [22] are shown in the lower panel of Figs. 4 and 5, respectively.

In Figs. 6 and 7, the gluon fragmentation channels to the $T_{4b}(0^{++})$ and $T_{4b}(2^{++})$ states as a function of the variable z by Laplace transform are presented at the NLO approximation and compared with those in Ref. [23] at the scale $\mu = 5m_c$. The results are evaluated according to the gluon

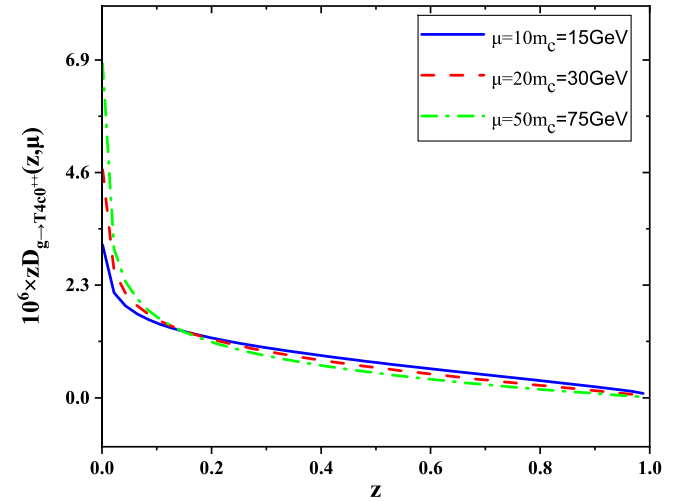


FIG. 8. The evolution of the gluon fragmentation channel to the $T_{4c}(0^{++})$ state as a function of the variable z for $\mu = 10m_c$ (blue-solid curve), $\mu = 20m_c$ (red-dashed curve), and $\mu = 50m_c$ (green-dash-dotted curve) is presented at the NLO approximation.

fragmentation at the initial scale $\mu = 4m_b$ as defined in Eqs. (A1) and (A2) in Appendix A, respectively.

In Figs. 4–7, we present a comparison between the FF of a gluon, $g \rightarrow T_{4Q}$, in Laplace transform for the ground state (0^{++}) and the radial excitation (2^{++}), which shows correct behavior compared to the results in Refs. [22,23]. For clarity, in Fig. 8 we display the z dependence of our $D_g^{T_{4c}(0^{++})}$ using the Laplace evolution method at various scales $\mu = 10, 20$, and $50m_c$, which can also be applied to the radial excitations $T_{4c}(2^{++})$ for fully charmed tetraquarks and fully bottomed tetraquarks. The evolution of the fragmentation function $D_{g \rightarrow \eta_c}(z, \mu)$ through Laplace transform at the NLO approximation is illustrated in Fig. 8 across a wide range of $\mu = 10, 20$, and $50m_c$. We notice that the point $z \simeq 0.2$ remains constant throughout the evolution. In comparison to Fig. 3, this point decreases as the fragmentation process of the gluon transitions from $[g \rightarrow T_{2Q}]$ to $[g \rightarrow T_{4Q}]$. The fragmentation function $D_{g \rightarrow \eta_c}(z, \mu)$ increases as μ increases and z decreases (i.e., $z \lesssim 0.2$) and decreases as both μ and z increase (i.e., $z \gtrsim 0.2$).

IV. CONCLUSION

We have introduced a method based on the Laplace transform technique to analyze the evolution of the fragmentation mechanisms $[g \rightarrow T_{2Q}]$ and $[g \rightarrow T_{4Q}]$ at both the LO and NLO approximations. The gluon fragmentation function in the $[g \rightarrow T_{2c}]$ channel is calculated for the fragmentation of a gluon splitting into an S -wave charmonium state starting from $\mu = 2m_c$. The evolution of the

fragmentation function for the semi-inclusive detection of fully charmed and bottomed tetraquarks is then examined, including bound states of $T_{4Q}(0^{++})$ and their $T_{4Q}(2^{++})$ radial excitations. This analysis is based on single-parton collinear fragmentation in a VFNS and relies on the parametrization of the fragmentation functions within a kinematical region characterized by moderate and large transverse momenta at the initial scales. We find that the Laplace transform method, used in the evolution of the fragmentation function at the NLO approximation, provides correct behaviors across a wide range of z and μ scales.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A

The lowest-order contributions to the FF of a gluon describe the perturbative part of the $[g \rightarrow T_{4c}(0^{++})]$ and $[g \rightarrow T_{4c}(2^{++})]$. The composite quantum number $[n]$, which runs over the combinations $[3, 3]$, $[6, 6]$, $[3, 6]$, and $[6, 3]$, is defined by the following forms, respectively [22]:

$$D_g^{T_{4c}(0^{++})}(z, 4m_c) = \frac{\text{GeV}^9}{m_c^9} \left[0.0347 \mathcal{D}_g^{(0^{++})}(z, [3, 3]) + 0.0128 \mathcal{D}_g^{(0^{++})}(z, [6, 6]) + 0.0422 \mathcal{D}_g^{(0^{++})}(z, [3, 6]) \right], \quad (\text{A1})$$

and

$$D_g^{T_{4c}(2^{++})}(z, 4m_c) = \frac{\text{GeV}^9}{m_c^9} \left[0.072 \mathcal{D}_g^{(2^{++})}(z, [3, 3]) \right], \quad (\text{A2})$$

with $m_c = 1.5$ GeV being the mass of the charm quark. The short-distance coefficients read

$$\begin{aligned} D_g^{(0^{++})}(z, [3, 3]) = & \frac{\pi^2 \alpha_s^4(4m_c)}{497664z(2-z)^2(3-z)} \left[186624 - 430272z + 511072z^2 - 425814z^3 + 217337z^4 - 61915z^5 \right. \\ & + 7466z^6 - 42(1-z)(2-z)(3-z)(-144 + 634z - 358z^2 + 70z^3) \ln(1-z) \\ & + 36(2-z)(3-z)(144 - 634z + 749z^2 - 364z^3 + 74z^4) \ln\left(1 - \frac{z}{2}\right) \\ & \left. + 12(2-z)(3-z)(72 - 362z + 361z^2 - 136z^3 + 23z^4) \ln\left(1 - \frac{z}{3}\right) \right], \end{aligned} \quad (\text{A3})$$

$$\begin{aligned}
D_g^{(0^{++})}(z, [6, 6]) = & \frac{\pi^2 \alpha_s^4(4m_c)}{331776z(2-z)^2(3-z)} \left[186624 - 430272z + 617824z^2 - 634902z^3 + 374489z^4 - 115387z^5 \right. \\
& + 14387z^6 - 6(1-z)(2-z)(3-z)(-144 - 2166z + 1015z^2 + 70z^3) \ln(1-z) \\
& - 156(2-z)(3-z)(144 - 1242z + 1693z^2 - 876z^3 + 170z^4) \ln\left(1 - \frac{z}{2}\right) + 300(2-z)(3-z)(72 \\
& \left. - 714z + 953z^2 - 472z^3 + 87z^4) \ln\left(1 - \frac{z}{3}\right) \right], \quad (A4)
\end{aligned}$$

$$\begin{aligned}
D_g^{(0^{++})}(z, [3, 6]) = & \frac{\pi^2 \alpha_s^4(4m_c)}{165888z(2-z)^2(3-z)} \left[186624 - 430272z + 490720z^2 - 394422z^3 + 199529z^4 - 57447z^5 \right. \\
& + 7082z^6 + 6(1-z)(2-z)(3-z)(-432 + 3302z - 1855z^2 + 210z^3) \ln(1-z) \\
& - 12(2-z)(3-z)(720 - 2258z + 2329z^2 - 1052z^3 + 226z^4) \ln\left(1 - \frac{z}{2}\right) \\
& \left. + 12(2-z)(3-z)(936 - 4882z + 4989z^2 - 1936z^3 + 331z^4) \ln\left(1 - \frac{z}{3}\right) \right], \quad (A5)
\end{aligned}$$

$$\begin{aligned}
D_g^{(2^{++})}(z, [3, 3]) = & \frac{\pi^2 \alpha_s^4(4m_c)}{622080z^2(2-z)^2(3-z)} \left[(46656 - 490536z + 1162552z^2 - 1156308z^3 + 595421z^4 - 170578z^5 \right. \\
& + 21212z^6)2z + 3(1-z)(2-z)(3-z)(-20304 - 31788z)(1296 + 1044z + 73036z^2 - 36574z^3 \\
& + 7975z^4) \ln(1-z) + 33(2-z)(3-z)(1296 + 25)(-9224z^2 + 9598z^3 - 3943z^4 \\
& \left. + 725z^5) \ln\left(1 - \frac{z}{3}\right) \right], \quad (A6)
\end{aligned}$$

$$D_g^{(2^{++})}(z, [6, 6]) = 0, \quad (A7)$$

$$D_g^{(2^{++})}(z, [3, 6]) = 0. \quad (A8)$$

The lowest-order contributions to the FF of a gluon describe the perturbative part of the $[g \rightarrow T_{4b}(0^{++})]$ and $[g \rightarrow T_{4b}(2^{++})]$ processes. They are defined by the following forms, respectively [22]:

$$D_g^{T_{4b}(0^{++})}(z, 4m_b) = \frac{\text{GeV}^9}{m_b^9} \left[13.88 \mathcal{D}_g^{(0^{++})}(z, [3, 3]) + 5.12 \mathcal{D}_g^{(0^{++})}(z, [6, 6]) + 16.88 \mathcal{D}_g^{(0^{++})}(z, [3, 6]) \right], \quad (A9)$$

and

$$D_g^{T_{4b}(2^{++})}(z, 4m_b) = \frac{\text{GeV}^9}{m_b^9} \left[28.8 \mathcal{D}_g^{(2^{++})}(z, [3, 3]) \right], \quad (A10)$$

with $m_b = 4.9$ GeV being the mass of the bottom quark.

APPENDIX B

The splitting functions at the LO and NLO approximations read

$$P_{gg}^{(0)}(z) = 12 \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \frac{33-2n_f}{3} \delta(1-z), \quad (B1)$$

where the plus function is defined as

$$\int_x^1 \frac{f(z)}{(1-z)_+} dz = \int_x^1 \frac{f(z) - f(1)}{1-z} dz + f(1) \ln(1-x) \quad (\text{B2})$$

$$\begin{aligned} P_{gg}^{(1)}(z) = & 2C_F T_f \left\{ -4 + 12z - \frac{164}{9} z^2 + \left(10 + 14z + \frac{16}{3} z^2 + \frac{16}{3z} \right) \ln(z) + \frac{92}{9z} + 2(1+z) \ln^2(z) \right\} \\ & + 2C_A T_f \left\{ 2 - 2z + \frac{26}{9} \left(z^2 - \frac{1}{z} \right) - \frac{4}{3} (1+z) \ln(z) - \left(\frac{20}{9} + \frac{8}{3} \ln z \right) P_{gg}(z) \right\} \\ & + 2C_A^2 \left\{ \frac{27}{2} (1-z) + \frac{67}{9} \left(z^2 - \frac{1}{z} \right) + \left(\frac{11}{3} - \frac{25}{3} z - \frac{44}{3z} \right) \ln(z) - 4(1+z) \ln^2(z) + 2P_{gg}(-z) S_2(z) \right. \\ & \left. + \left[4 \ln(z) \ln(1-z) - 3 \ln^2(z) + \frac{22}{3} \ln(z) - \frac{\pi^2}{3} + \frac{67}{9} \right] P_{gg}(z) \right\}, \end{aligned} \quad (\text{B3})$$

where for the specific case of SU(3) we have

$$C_A = 3, \quad C_F = 4/3, \quad T_R = 1/2. \quad (\text{B4})$$

Here $P_{gg}(z) = 1/(1-z) + 1/z + z(1-z) - 2$ and $S_2(z) = \int_{z/(1+z)}^{1/(1+z)} dy \ln[(1-y)/y]/y = -2\text{Li}_2(-z) + (1/2) \ln^2(z) - 2 \ln(z) \ln(1+z) - \pi^2/6$.

APPENDIX C

The Laplace transform of the gluon-gluon splitting function, denoted by $\Phi_g^{(1)}$ at the NLO approximation, reads [17]

$$\begin{aligned} \Phi_g^{(1)}(s) = & C_F T_f \left\{ -\frac{16}{3s^2} + \frac{92}{9s} + \frac{4}{(1+s)^3} - \frac{10}{(1+s)^2} - \frac{4}{1+s} + \frac{4}{(2+s)^3} - \frac{14}{(2+s)^2} + \frac{12}{2+s} - \frac{16}{3(3+s)^2} - \frac{164}{9(3+s)} \right\} \\ & + C_A T_f \left\{ \frac{8}{3s^2} - \frac{46}{9s} - \frac{4}{(1+s)^2} + \frac{58}{9(1+s)} + \frac{4}{(2+s)^2} - \frac{38}{9(2+s)} - \frac{8}{3(3+s)^2} + \frac{46}{9(3+s)} + \frac{8}{3} \psi'(1+s) \right\} \\ & + C_A^2 \left\{ -\frac{8}{s^3} + \frac{22}{3s^2} + \frac{2}{(1+s)^3} + \frac{11}{(1+s)^2} + \frac{4.4407}{1+s} - \frac{17.9984}{(2+s)^3} + \frac{1}{(2+s)^2} - \frac{6.9024}{2+s} - \frac{\pi^2}{3(2+s)} + \frac{5.9702}{(3+s)^3} \right. \\ & + \frac{22}{3(3+s)^2} - \frac{6.7917}{3+s} + \frac{\pi^2}{3(3+s)} - \frac{1.801}{(4+s)^3} - \frac{3.5389}{4+s} + \frac{1.3242}{(5+s)^3} + \frac{1.2736}{5+s} - \frac{0.6348}{(6+s)^3} - \frac{5.6479}{6+s} + \frac{0.1398}{(7+s)^3} \\ & + \frac{9.2228}{7+s} - \frac{7.6863}{8+s} + \frac{6.5284}{9+s} - \frac{3.1431}{10+s} + \frac{0.9985}{11+s} - \frac{0.1537}{12+s} - \frac{67}{9} H_s + \frac{\pi^2}{3} H_s - \frac{1}{s^2} \left\{ \ln(16) - 2\psi\left(1 + \frac{s}{2}\right) \right. \\ & + 2\psi\left(\frac{1+s}{2}\right) + s\psi'\left(1 + \frac{s}{2}\right) - s\psi'\left(\frac{1+s}{2}\right) \left. \right\} + 2 \left\{ \frac{4}{(1+s)^3} - \frac{\ln(4)}{(1+s)^2} - \frac{\psi(1 + \frac{s}{2})}{(1+s)^2} + \frac{\psi(\frac{1+s}{2})}{(1+s)^2} \right. \\ & + \frac{\psi'(1 + \frac{s}{2})}{2(1+s)} - \frac{\psi'(\frac{1+s}{2})}{2(1+s)} \left. \right\} - \frac{1.9992}{(2+s)^3} \left\{ \frac{16}{(1+s)^2} + \frac{12s}{(1+s)^2} + (2+s) \ln(16) - 2(2+s) \psi\left(1 + \frac{s}{2}\right) \right. \\ & + 2(2+s) \psi\left(\frac{1+s}{2}\right) + (2+s)^2 \psi'\left(1 + \frac{s}{2}\right) - (2+s)^2 \psi'\left(\frac{1+s}{2}\right) \left. \right\} + \frac{0.0149}{(1+s)^3} \left\{ \frac{164}{(1+s)^2(2+s)^2} \right. \\ & + \frac{284s}{(1+s)^2(2+s)^2} + \frac{188s^2}{(1+s)^2(2+s)^2} + \frac{60s^3}{(1+s)^2(2+s)^2} + \frac{8s^4}{(1+s)^2(2+s)^2} - 4(3+s) \ln(2) \\ & - 2(3+s) \psi\left(1 + \frac{s}{2}\right) + 2(3+s) \psi\left(\frac{1+s}{2}\right) + (3+s)^2 \psi'\left(1 + \frac{s}{2}\right) - (3+s)^2 \psi'\left(\frac{1+s}{2}\right) \left. \right\} \\ & - \frac{0.9005}{(4+s)^3} \left\{ \frac{2176}{(1+s)^2(2+s)^2(3+s)^2} + \frac{4392s}{(1+s)^2(2+s)^2(3+s)^2} + \frac{3504s^2}{(1+s)^2(2+s)^2(3+s)^2} \right. \\ & + \frac{1408s^3}{(1+s)^2(2+s)^2(3+s)^2} + \frac{288s^4}{(1+s)^2(2+s)^2(3+s)^2} + \frac{24s^5}{(1+s)^2(2+s)^2(3+s)^2} \left. \right\} \end{aligned}$$

$$\begin{aligned}
& + 4(4+s) \ln(2) - 2(4+s) \psi \left(1 + \frac{s}{2} \right) + 2(4+s) \psi \left(\frac{1+s}{2} \right) + (4+s)^2 \psi' \left(1 + \frac{s}{2} \right) - (4+s)^2 \psi' \left(\frac{1+s}{2} \right) \Big\} \\
& - \frac{0.6621}{(5+s)^3} \left\{ \frac{57328}{(1+s)^2(2+s)^2(3+s)^2(4+s)^2} + \frac{146144s}{(1+s)^2(2+s)^2(3+s)^2(4+s)^2} + \frac{162160s^2}{(1+s)^2(2+s)^2(3+s)^2(4+s)^2} \right. \\
& + \frac{103728s^3}{(1+s)^2(2+s)^2(3+s)^2(4+s)^2} + \frac{42144s^4}{(1+s)^2(2+s)^2(3+s)^2(4+s)^2} + \frac{11160s^5}{(1+s)^2(2+s)^2(3+s)^2(4+s)^2} \\
& + \frac{1880s^6}{(1+s)^2(2+s)^2(3+s)^2(4+s)^2} + \frac{184s^7}{(1+s)^2(2+s)^2(3+s)^2(4+s)^2} + \frac{8s^8}{(1+s)^2(2+s)^2(3+s)^2(4+s)^2} \\
& \left. - 4(5+s) \ln(2) - 2(5+s) \psi \left(1 + \frac{s}{2} \right) + 2(5+s) \psi \left(\frac{1+s}{2} \right) + (5+s)^2 \psi' \left(1 + \frac{s}{2} \right) - (5+s)^2 \psi' \left(\frac{1+s}{2} \right) \right\} \\
& + \frac{4}{s^3} (1 + s\gamma_E + s(\psi(s) - s\psi'(1+s))) - \frac{8}{(s+1)^2} \left(\gamma_E + \frac{1}{1+s} + \psi(1+s) - (1+s)\psi'(2+s) \right) \\
& + \frac{4}{(s+2)^2} \left(\gamma_E + \frac{1}{2+s} + \psi(2+s) - (2+s)\psi'(3+s) \right) - \frac{4}{(s+3)^2} \left(\gamma_E + \frac{1}{3+s} + \psi(3+s) - (3+s)\psi'(4+s) \right) \\
& - \frac{0.3174}{(6+s)^2} \left\{ \ln(16) - 2\psi \left(4 + \frac{s}{2} \right) + 2\psi \left(\frac{7+s}{2} \right) + (6+s)\psi' \left(4 + \frac{s}{2} \right) - (6+s)\psi' \left(\frac{7+s}{2} \right) \right\} \\
& + \frac{0.0699}{(7+s)^2} \left\{ \ln(16) + 2\psi \left(4 + \frac{s}{2} \right) - 2\psi \left(\frac{9+s}{2} \right) - (7+s)\psi' \left(4 + \frac{s}{2} \right) + (7+s)\psi' \left(\frac{9+s}{2} \right) \right\} + 4 \left(H_s \psi'(1+s) \right. \\
& \left. - \frac{1}{2} \psi''(1+s) \right) - \frac{22}{3} \psi''(1+s) + 3\psi'''(1+s) \Big\}, \tag{C1}
\end{aligned}$$

where $H_s = \psi(s+1) + \gamma_E$.

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