

Spin alignment of vector mesons in local equilibrium by Zubarev's approach

Shi-Zheng Yang¹, Xin-Qing Xie¹, Shi Pu^{1,2,3}, Jian-Hua Gao⁴ and Qun Wang^{1,5}

¹*Department of Modern Physics and Anhui Center for fundamental Sciences (Theoretical Physics),
University of Science and Technology of China, Anhui 230026, China*

²*Southern Center for Nuclear-Science Theory (SCNT), Institute of Modern Physics,
Chinese Academy of Sciences, Huizhou 516000, Guangdong, China*

³*Shanghai Research Center for Theoretical Nuclear Physics, NSFC and Fudan University,
Shanghai 200438, China*

⁴*Shandong Key Laboratory of Optical Astronomy and Solar-Terrestrial Environment,
School of Space Science and Physics, Shandong University, Weihai, Shandong, 264209, China*

⁵*School of Mechanics and Physics, Anhui University of Science and Technology,
Huainan, Anhui 232001, China*



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We compute the 00 element of the spin density matrix, denoted as ρ_{00} and called the spin alignment, up to the second order of the gradient expansion in local equilibrium by Zubarev's approach. In the first order, we obtain $\rho_{00} = 1/3$, meaning that the contributions from thermal vorticity and shear stress tensor are vanishing. The nonvanishing contributions to $\rho_{00} - 1/3$ appear in the second order of gradients in the Belinfante and canonical cases. We also discuss the properties of the spin density matrix under the time reversal transformation. The effective transport coefficient for the spin alignment induced by the thermal shear stress tensor is T odd in the first order, implying that the first order effect is dissipative.

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I. INTRODUCTION

In noncentral heavy-ion collisions there is a huge orbital angular momentum (OAM) generated in the initial state of two colliding nuclei at relativistic energies. A small fraction of the initial OAM can be transferred into the medium in the form of the spin polarization of quarks (and antiquarks) via spin-orbit couplings, which leads to the global spin polarization of hadrons with respect to the reaction plane [1–4]. Later, Ref. [5] has established a connection between global spin polarization and hydrodynamic vorticity.

The global spin polarization of Λ hyperons (including $\bar{\Lambda}$) has been measured by experiments of STAR [6–10], ALICE [11,12] and HADES [13]. These measurements have been successfully described by a variety of phenomenological models [5,14–25]. In contrast, the spin polarization along the beam direction as functions of azimuthal angles [8,26] cannot be described by the polarization from the thermal vorticity in hydrodynamic or transport simulations using the Cooper-Frye formula [27,28]. Several models have been proposed to explain the spin polarization

along the beam direction, such as spin polarization from the temperature vorticity [29], the shear-induced polarization [30–34], spin Hall effects [25,35,36], weak magnetic fields induced polarization [37] and corrections from interactions between quarks and background fields [38]. Very recently, the hydrodynamical simulations with the assumption of the isothermal equilibrium [39,40] and the results from blast wave model [41] can give a quantitative description to local spin polarization at very high energy collisions, while, the recent hydrodynamic studies on the spin polarization along the beam direction at p + Pb collisions disagrees with the data measured by CMS experiments [42,43]. Although the azimuthal angle dependence of the spin polarization along the beam direction can be qualitatively understood by incorporating these effects, its quantitative description remains a challenge in general [8,44,45]. We refer readers to recent reviews about spin polarization in heavy ion collisions [46–54].

Meanwhile, the spin alignment of vector mesons in heavy ion collisions was also proposed [3]. The spin alignment of vector mesons is described by the spin density matrix $\rho_{ss'}$, where $s, s' = 0, \pm 1$ represent spin states with the specified spin quantization direction. The component ρ_{00} can be measured through the strong decay of vector mesons [55]. The condition $\rho_{00} > 1/3$ or $\rho_{00} < 1/3$ corresponds to longitudinal and transverse polarization (referred to the polarization vector of the vector meson field) with respect to the spin quantization direction,

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respectively. The first measurement of the global spin alignment of ϕ mesons in Au + Au collisions at $\sqrt{s_{NN}} = 11.5\text{--}200$ GeV was performed by the STAR collaboration [10]. The magnitude of $\rho_{00} - 1/3$ for ϕ mesons at low energies is significantly larger than expected. In contrast, the global spin alignment of K^{*0} mesons was also measured in the same experiment but it was found to be consistent with zero within errors. Recently, the ALICE collaboration also observed the spin alignment of J/Ψ in Pb + Pb collisions at $\sqrt{s_{NN}} = 5.02$ TeV [11].

The spin alignment of ϕ mesons can be described by the relativistic quark coalescence model incorporated into the spin Boltzmann (kinetic) equation for vector mesons, where the quark and antiquark are polarized by the effective ϕ field [56–59]. There are other studies such as the models based on glasma [60,61], self-energy effects [38], and other second-order hydrodynamics effects [62], as well as those employing the light-cone formalism [63]. It has been found that unlike ϕ mesons, rescatterings in the medium lead to a decay in the spin alignment of ρ mesons [64]. The spin alignment of J/Ψ has also been explored within the holographic model [65,66]. The spin alignment of vector mesons can also come from the shear viscous tensor [67,68]. A study of the shear contribution in the quark-gluon plasma [69] suggests that such a contribution is not important as compared to that from the strong force field.

In this work, we investigate the spin alignment of vector mesons using Zubarev’s approach [70–73] in quantum statistical field theory. In this approach, hydrodynamical quantities can be computed by the quantum statistical method in global and local equilibrium. The approach has already been applied to study the spin polarization of Λ hyperons in both global [27,74–76] and local equilibrium [33,34,77].

The structure of the paper is organized as follows. In Sec. II, we introduce the spin density matrix of vector mesons. In Sec. III, we expand the spin density matrix in the gradient expansion. We derive the ρ^{00} in the first and second order in the gradient expansion in Sec. IV. We study the spin alignment induced by the shear tensor under time reversal transformation in Sec. V and summarize in Sec. VI. Throughout this work, we choose the metric $g^{\mu\nu} = \text{diag}\{1, -1, -1, -1\}$ and Levi-Civita tensor $\epsilon^{0123} = 1$. For simplicity, for a four vector $A^\mu = (A^0, \mathbf{A})$, we define $\tilde{A}^\mu = (A^0, -\mathbf{A})$ and $\bar{A}^\mu = (0, \mathbf{A})$.

II. VECTOR FIELDS AND SPIN DENSITY MATRIX

We start from the Lagrangian for charge neutral spin-1 particles,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A \cdot A - A \cdot j, \quad (1)$$

where j^μ represents the source term, and

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (2)$$

is the field strength tensor for the spin-1 particle. Note that the A^μ field is real for charge neutral particles. The equation of motion derived from this Lagrangian, known as the Proca equation, is

$$(\partial^2 + m^2)A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu. \quad (3)$$

Taking another derivative ∂_μ on both sides of the above equation, we obtain

$$\partial_\mu A^\mu = \frac{1}{m^2} \partial_\mu j^\mu. \quad (4)$$

If the source current is conserved, $\partial \cdot j = 0$, the above equation leads to a constraint equation

$$\partial \cdot A = 0. \quad (5)$$

Under this condition, the Proca equation can be reduced to the Klein-Gordon equation for each component of the field,

$$(\partial^2 + m^2)A^\mu = j^\mu. \quad (6)$$

In free space without the source, $j^\mu = 0$, the solution to the Proca equation under the constraint (5) can be expressed as

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \times \sum_{s=\pm 1,0} [a_{\mathbf{p}}^s \epsilon_\mu^s(\mathbf{p}) e^{-ip \cdot x} + a_{\mathbf{p}}^{s\dagger} \epsilon_\mu^{s*}(\mathbf{p}) e^{ip \cdot x}], \quad (7)$$

where $s = \pm 1, 0$ denote three polarization states, and $\epsilon_\mu^s(\mathbf{p})$ is the polarization four-vector corresponding to the spin state s and the on-shell momentum $p^\mu = (E_{\mathbf{p}}, \mathbf{p})$ with $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$, (through it is denoted as a function of the three-momentum $\mathbf{p}$). Here, the creation and annihilation operators satisfy the following commutation rule,

$$[a_{\mathbf{p}}^r, a_{\mathbf{q}}^{s\dagger}] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) \delta_{rs}. \quad (8)$$

Note that the convention in the expansion in (7) is different from Eq. (5) of Ref. [78] by a factor $\sqrt{2E_{\mathbf{p}}}$ so the commutation rule (8) is different from Eq. (6) of Ref. [78] by a factor $2E_{\mathbf{p}}$. The constraint equation (5) becomes,

$$p^\mu \epsilon_\mu^s(\mathbf{p}) = 0, \quad (9)$$

which implies that $\epsilon_\mu^s(\mathbf{p})$ is spacelike and has only three independent components. In the particle’s rest frame, the polarization vector is $\epsilon_s^\mu = (0, \mathbf{e}_s)$ or $\epsilon_s^\mu = (0, -\mathbf{e}_s)$, where $\mathbf{e}_s$ is a unit vector for the spin state s . For instance, if the

quantization direction is along the y axis, we have

$$\begin{aligned}\mathbf{e}_0 &= (0, 1, 0), \\ \mathbf{e}_{+1} &= -\frac{1}{\sqrt{2}}(i, 0, 1), \\ \mathbf{e}_{-1} &= -\frac{1}{\sqrt{2}}(i, 0, -1).\end{aligned}\quad (10)$$

These polarization vectors in the rest frame can be transformed to the laboratory frame by the Lorentz transformation,

$$\epsilon_s^\mu(\mathbf{p}) = \left(\frac{\mathbf{p} \cdot \mathbf{e}_s}{m}, \mathbf{e}_s + \frac{\mathbf{p} \cdot \mathbf{e}_s}{m(E_p + m)} \mathbf{p} \right). \quad (11)$$

One can check that these polarization vectors satisfy orthogonal and completeness relations

$$\epsilon_\mu^s(\mathbf{p}) \epsilon_{s'}^{\mu*}(\mathbf{p}) = -\delta_{ss'}, \quad (12)$$

$$\sum_s \epsilon_s^\mu(\mathbf{p}) \epsilon_s^{\nu*}(\mathbf{p}) = -\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{m^2} \right). \quad (13)$$

We can define the Winger function for the vector fields as

$$W^{\mu\nu}(x, k) = \int d^4y \left\langle A^\nu \left(x - \frac{y}{2} \right) A^\mu \left(x + \frac{y}{2} \right) \right\rangle e^{ik \cdot y}, \quad (14)$$

where $\langle \hat{O} \rangle \equiv \text{Tr}(\hat{\rho} \hat{O})$ with $\hat{\rho}$ being the density operator to be defined shortly. Inserting Eq. (7) for A^μ fields into the above formula, at the leading order in $\hbar$, we obtain

$$\begin{aligned}W^{\mu\nu}(x, k) &= 2\pi\delta(k^2 - m^2) \sum_{r,s} \left\{ \theta(k_0) \epsilon_s^{\nu*}(\mathbf{k}) \epsilon_r^\mu(\mathbf{k}) f_{sr}(x, \mathbf{k}) \right. \\ &\quad \left. + \theta(-k_0) \epsilon_s^\nu(-\mathbf{k}) \epsilon_r^{\mu*}(-\mathbf{k}) [\delta_{rs} + f_{rs}(x, -\mathbf{k})] \right\},\end{aligned}\quad (15)$$

where δ_{rs} stands for the vacuum contribution, and $f_{sr}(x, \mathbf{k})$ is the sr component of the matrix valued spin dependent distribution function (MVSD) [79–82],

$$f_{sr}(x, \mathbf{k}) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{-iq \cdot x} \text{Tr}(\hat{\rho} a_{\mathbf{k}-\mathbf{q}/2}^{s\dagger} a_{\mathbf{k}+\mathbf{q}/2}^r), \quad (16)$$

where $q^\mu = (E_{\mathbf{k}+\mathbf{q}/2} - E_{\mathbf{k}-\mathbf{q}/2}, \mathbf{q})$. Note that the MVSD in (16) is different from that defined in Eq. (12) of Ref. [78] by a factor $2E_{\mathbf{k}}$ and a transpose in spin space. The factor $2E_{\mathbf{k}}$ is due to the convention in the expansion (7) and the corresponding commutation rule (8). The spin density matrix ρ for the vector meson is just the normalized MVSD,

$$\rho_{sr}(x, \mathbf{k}) \equiv \frac{f_{sr}(x, \mathbf{k})}{\text{Tr} f(x, \mathbf{k})}. \quad (17)$$

In general, the MVSD can be defined in a variety of ways. We will present an alternative definition for the MVSD in Sec. IV C.

In this work, we adopt density operators in Zubarev's approach, following Refs. [70,72,73,83]. The local equilibrium density operators have different forms corresponding to different pseudogauges. For example, the Belinfante or canonical forms of density operators are

$$\hat{\rho}_B = \frac{1}{Z_B} \exp \left[- \int_\Sigma d\Sigma_\mu \hat{T}_B^{\mu\nu}(y) \beta_\nu(y) \right], \quad (18)$$

$$\hat{\rho}_C = \frac{1}{Z_C} \exp \left\{ - \int_\Sigma d\Sigma_\mu \left[\hat{T}_C^{\mu\nu}(y) \beta_\nu(y) - \frac{1}{2} \Omega_{\lambda\nu}(y) \hat{S}^{\mu\lambda\nu}(y) \right] \right\}, \quad (19)$$

where $\beta^\mu = u^\mu/T$ with u^μ being the fluid velocity, $\Omega_{\lambda\nu}$ is the spin potential, and Σ is a hypersurface. The subscripts “B” and “C” represent the Belinfante and canonical forms of energy-momentum tensors which are related by pseudo-gauge transformation

$$\begin{aligned}\hat{T}_C^{\mu\nu} &= -F^{\mu\alpha} \partial^\nu A_\alpha + g^{\mu\nu} \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{2} m^2 A_\alpha A^\alpha \right), \\ \hat{T}_B^{\mu\nu} &= \hat{T}_C^{\mu\nu} + \frac{1}{2} \partial_\lambda (\hat{S}^{\lambda\mu\nu} - \hat{S}^{\mu\lambda\nu} - \hat{S}^{\nu\lambda\mu}).\end{aligned}\quad (20)$$

In Eq. (19), the rank-3 spin tensor $\hat{S}^{\mu\alpha\beta}$ is given by

$$\hat{S}^{\mu\alpha\beta} = F^{\mu\beta} A^\alpha - F^{\mu\alpha} A^\beta. \quad (21)$$

As mentioned, we consider real vector fields with vanishing current density in Eqs. (18), (19). If we aim to compute the spin density matrix at the point x , then Σ is the hypersurface passing x [33,84]. The choice of Σ depends on what kind of physics we would like to address, e.g., if we take Σ as the freeze-out hypersurface, $\rho(x, \mathbf{k})$ represents the spin density matrix on that hypersurface.

III. SPIN DENSITY MATRIX FOR VECTOR MESONS IN GRADIENT EXPANSION

In the previous section, we show in Eq. (17) that the spin density matrix for the vector meson is actually the normalized MVSD which can be evaluated by Eq. (16) involving the density operator $\hat{\rho}$. In this section, we will expand $\hat{\rho}$ in terms of creation and annihilation operators and compute the MVSD through Eq. (16).

As shown in Eqs. (18), (19), $\hat{\rho}$ depends on β^μ and $\Omega_{\lambda\nu}$. As a common strategy, we assume that $\beta^\mu(y)$ varies slowly in the region where the correlation length of the system is much larger than the distance $|y - x|$. So $\beta^\mu(y)$ can be expanded in gradients [33],

$$\beta_\mu(y) = \beta_\mu(x) + \partial_\lambda \beta_\mu(x) (y - x)^\lambda + \mathcal{O}(\partial^2). \quad (22)$$

The symmetric and antisymmetric parts of $\partial_\mu \beta_\nu$ are the thermal shear and vorticity tensors, respectively,

$$\begin{aligned}\xi^{\mu\nu} &= \frac{1}{2}(\partial^\mu \beta^\nu + \partial^\nu \beta^\mu), \\ \omega^{\mu\nu} &= -\frac{1}{2}(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu).\end{aligned}\quad (23)$$

It has been widely discussed that the spin potential is proportional to the thermal vorticity tensor in global equilibrium,

$$\Omega^{\mu\nu} = -\frac{1}{2}(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \sim \mathcal{O}(\partial^1). \quad (24)$$

For further discussions on the spin potential, we refer the readers to Ref. [83] in quantum statistical field theory and to Refs. [85–89] in spin hydrodynamics (see, e.g., Ref. [53] for a recent review).

In this paper we consider two types of density operators $\hat{Q}_B$ and $\hat{Q}_C$ in Eqs. (18), (19) and make gradient expansion of $\hat{Q}_B$ and $\hat{Q}_C$ to the linear order. Inserting Eqs. (22) and (24) into Eqs. (18) and (19), we obtain

$$\begin{aligned}\hat{Q}_B &= \frac{1}{Z_B} \exp \left[-\beta_\nu(x) \int_\Sigma d\Sigma_\mu \hat{T}_B^{\mu\nu}(y) - \partial_\lambda \beta_\nu(x) \right. \\ &\quad \times \left. \int_\Sigma d\Sigma_\mu \hat{T}_B^{\mu\nu}(y)(y-x)^\lambda + \mathcal{O}(\partial^2) \right],\end{aligned}\quad (25)$$

$$\begin{aligned}\hat{Q}_C &= \frac{1}{Z_C} \exp \left[-\beta_\nu(x) \int_\Sigma d\Sigma_\mu \hat{T}_C^{\mu\nu}(y) - \partial_\lambda \beta_\nu(x) \right. \\ &\quad \times \left. \int_\Sigma d\Sigma_\mu \hat{T}_C^{\mu\nu}(y)(y-x)^\lambda \right. \\ &\quad \left. + \frac{1}{2} \Omega_{\lambda\nu}(x) \int_\Sigma d\Sigma_\mu \hat{S}^{\mu\lambda\nu}(y) + \mathcal{O}(\partial^2) \right].\end{aligned}\quad (26)$$

For convenience, we introduce the momentum operator $\hat{P}^\nu$ and an auxiliary operator $\hat{L}^{\mu\nu}$, both related to $\hat{T}_C^{\mu\nu}$,

$$\hat{P}^\nu = \int_\Sigma d\Sigma_\mu \hat{T}_C^{\mu\nu}(y), \quad (27)$$

$$\hat{L}^{\lambda\nu} = \int_\Sigma d\Sigma_\mu (y-x)^\lambda \hat{T}_C^{\mu\nu}(y). \quad (28)$$

Replacing $\hat{T}_B^{\mu\nu}$ in Eq. (25) by $\hat{T}_C^{\mu\nu}$ in Eq. (20), we can express the density operators as

$$\begin{aligned}\hat{Q}_B &= \frac{1}{Z_B} \exp \left[-\beta_\mu(x) \hat{P}^\mu - \partial_\lambda \beta_\nu(x) \hat{L}^{\lambda\nu}(x) \right. \\ &\quad \left. - \frac{1}{2} \partial_\lambda \beta_\nu(x) \hat{S}_B^{\lambda\nu} + \mathcal{O}(\partial^2) \right],\end{aligned}\quad (29)$$

and

$$\begin{aligned}\hat{Q}_C &= \frac{1}{Z_C} \exp \left[-\beta_\mu(x) \hat{P}^\mu - \partial_\lambda \beta_\nu(x) \hat{L}^{\lambda\nu}(x) \right. \\ &\quad \left. + \frac{1}{2} \Omega_{\lambda\nu}(x) \hat{S}_C^{\lambda\nu} + \mathcal{O}(\partial^2) \right],\end{aligned}\quad (30)$$

where $S_B^{\mu\nu}$ and $S_C^{\mu\nu}$ are operators related to spin tensors

$$\hat{S}_B^{\lambda\nu} = \int_\Sigma d\Sigma_\mu [\hat{S}^{\mu\lambda\nu}(y) + \hat{S}^{\nu\lambda\mu}(y) + \hat{S}^{\lambda\mu\nu}(y)], \quad (31)$$

$$\hat{S}_C^{\lambda\nu} = \int_\Sigma d\Sigma_\mu \hat{S}^{\mu\lambda\nu}(y). \quad (32)$$

In Eqs. (29) and (30) we neglected second order terms in gradient expansion (22) and focus on effects related to $\partial_\mu \beta_\nu$.

Previous studies have shown that the spin polarization vector for Λ and $\bar{\Lambda}$ hyperons in local equilibrium depends on the choice of the hypersurface Σ in Zubarev's approach [33]. Therefore, to study the spin density matrix in local equilibrium, it is essential to specify the hypersurface Σ in density operators (18) and (19). In this paper, we adopt the isochronous hypersurface Σ_0 following Refs. [33,90,91], then the integral on Σ_0 is

$$\int_{\Sigma_0} d\Sigma^\mu = \int d^3y n^\mu, \quad (33)$$

where $n^\mu = (1, 0, 0, 0)$ is the time-directed normal vector for the Σ_0 , so the spin density matrix will depend on n^μ .

Now we deal with the trace in the distribution function (16)

$$\begin{aligned}\langle a_{\mathbf{k}-\mathbf{q}/2}^{s\dagger} a_{\mathbf{k}+\mathbf{q}/2}^r \rangle &= \text{Tr}(\hat{Q}_{B,C} a_{\mathbf{k}-\mathbf{q}/2}^{s\dagger} a_{\mathbf{k}+\mathbf{q}/2}^r) \\ &= \frac{1}{Z_{B,C}} \text{Tr} \{ \exp [\hat{A}(x) + \hat{B}_{B,C}(x)] a_{\mathbf{k}-\mathbf{q}/2}^{s\dagger} a_{\mathbf{k}+\mathbf{q}/2}^r \},\end{aligned}\quad (34)$$

where $Z_{B,C} = \text{Tr} \hat{Q}_{B,C}$, and

$$\hat{A}(x) = -\beta_\mu(x) \hat{P}^\mu, \quad (35)$$

$$\hat{B}_B(x) = -\partial_\lambda \beta_\nu(x) \left(\hat{L}^{\lambda\nu}(x) + \frac{1}{2} \hat{S}_B^{\lambda\nu} \right), \quad (36)$$

$$\hat{B}_C(x) = -\partial_\lambda \beta_\nu(x) \hat{L}^{\lambda\nu}(x) + \frac{1}{2} \Omega_{\lambda\nu}(x) \hat{S}_C^{\lambda\nu}. \quad (37)$$

Hereafter we will suppress indices B and C for simple notations. Using Eq. (8), we obtain the following commutators

$$[\hat{A}, a_{\mathbf{p}}^{s\dagger}] = -\beta_\mu p^\mu a_{\mathbf{p}}^{s\dagger}, \quad (38)$$

$$[\hat{A}, a_{-\mathbf{p}}^s] = \beta_\mu \tilde{p}^\mu a_{-\mathbf{p}}^s, \quad (39)$$

$$[\hat{B}, a_{\mathbf{p}}^{s\dagger}] = \sum_r \Xi_{++}^{sr}(p) a_{\mathbf{p}}^{r\dagger} + \sum_r \Xi_{+-}^{sr}(p) a_{-\mathbf{p}}^r, \quad (40)$$

$$[\hat{B}, a_{-\mathbf{p}}^s] = \sum_r \Xi_{-+}^{sr}(p) a_{\mathbf{p}}^{r\dagger} + \sum_r \Xi_{--}^{sr}(p) a_{-\mathbf{p}}^r, \quad (41)$$

where $\hat{B}$ represents $\hat{B}_B$ or $\hat{B}_C$, and Ξ_{ij}^{sr} ($i, j = \pm$) are given by Eqs. (A21)–(A24) for Belinfante and canonical cases with detailed derivations in Appendix A.

With the help of the Baker-Hausdorff formula

$$\begin{aligned} e^{-X} Y e^X &= Y - [X, Y] + \frac{(-1)^2}{2!} [X, [X, Y]] \\ &\quad + \frac{(-1)^3}{3!} [X, [X, [X, Y]]] + \dots, \end{aligned} \quad (42)$$

and Eqs. (38)–(41), we derive

$$\begin{aligned} e^{-\hat{A}-\hat{B}} a_{\mathbf{p}}^{s\dagger} e^{\hat{A}+\hat{B}} &= \sum_r (e^{\beta \cdot p} \delta_{sr} + D_{++}^{sr}) a_{\mathbf{p}}^{r\dagger} \\ &\quad + \sum_r D_{+-}^{sr} a_{-\mathbf{p}}^r + \mathcal{O}(\partial^3), \end{aligned} \quad (43)$$

$$\begin{aligned} e^{-\hat{A}-\hat{B}} a_{-\mathbf{p}}^s e^{\hat{A}+\hat{B}} &= \sum_r (e^{-\beta \cdot \tilde{p}} \delta_{sr} + D_{--}^{sr}) a_{-\mathbf{p}}^r \\ &\quad + \sum_r D_{-+}^{sr} a_{\mathbf{p}}^{r\dagger} + \mathcal{O}(\partial^3), \end{aligned} \quad (44)$$

where $\hat{B}$ represents $\hat{B}_B$ or $\hat{B}_C$, $\tilde{p}^\mu = (E_{\mathbf{p}}, -\mathbf{p})$, and coefficients D_{ij}^{sr} , with $i, j = \pm$, are given in Eqs. (A25)–(A28). By applying the cyclic property of trace and using Eqs. (43) and (44), we find

$$\begin{aligned} \langle a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \rangle &= \sum_{s'} (e^{\beta \cdot p} \delta_{ss'} + D_{++}^{ss'}) \langle a_{\mathbf{p}}^{s'\dagger} a_{\mathbf{p}}^r \rangle \\ &\quad + \sum_{s'} D_{+-}^{ss'} \langle a_{-\mathbf{p}}^{s'} a_{\mathbf{p}}^r \rangle + \mathcal{O}(\partial^3), \end{aligned} \quad (45)$$

$$\begin{aligned} \langle a_{\mathbf{p}}^r a_{-\mathbf{p}}^s \rangle &= \sum_{s'} (e^{-\beta \cdot \tilde{p}} \delta_{ss'} + D_{--}^{ss'}) \langle a_{-\mathbf{p}}^{s'\dagger} a_{\mathbf{p}}^r \rangle \\ &\quad + \sum_{s'} D_{-+}^{ss'} \langle a_{\mathbf{p}}^{s'\dagger} a_{-\mathbf{p}}^r \rangle + \mathcal{O}(\partial^3), \end{aligned} \quad (46)$$

where $\langle \cdot \rangle$ is defined in (34). Combining Eqs. (45) and (46) with the following relations,

$$\begin{aligned} \langle a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \rangle &= \langle a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^r \rangle + (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \delta_{rs}, \\ \langle a_{-\mathbf{p}}^s a_{\mathbf{p}}^r \rangle &= \langle a_{\mathbf{p}}^r a_{-\mathbf{p}}^s \rangle, \end{aligned} \quad (47)$$

we can solve $\langle a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^r \rangle$ as,

$$\begin{aligned} \langle a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^r \rangle &= (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') [(e^{\beta \cdot p} - 1)I + D_{++} \\ &\quad - D_{+-} \mathcal{A}^{-1} D_{-+}]_{sr}^{-1} + \mathcal{O}(\partial^3), \end{aligned} \quad (48)$$

where $\mathcal{A} = (e^{-\beta \cdot \tilde{p}} - 1)I + D_{--}$ with I being the unity matrix in spin space. Inserting Eq. (48) into Eq. (16), we derive the expression for f_{sr} up to $\mathcal{O}(\partial^2)$,

$$\begin{aligned} f_{sr}(x, \mathbf{k}) &= \int d^3 \mathbf{q} e^{-iq \cdot x} [(e^{\beta \cdot (k-q/2)} - 1)I + D_{++} \\ &\quad - D_{+-} \mathcal{A}^{-1} D_{-+}]_{sr}^{-1} \delta(\mathbf{q}), \end{aligned} \quad (49)$$

where $p^\mu = (k - q/2)^\mu$. Note that D_{++} and D_{--} contain momentum derivatives.

We then consider the gradient expansion for $f_{sr}(x, \mathbf{k})$,

$$f_{sr}(x, \mathbf{k}) = f_{sr}^{(0)}(x, \mathbf{k}) + f_{sr}^{(1)}(x, \mathbf{k}) + f_{sr}^{(2)}(x, \mathbf{k}) + \mathcal{O}(\partial^3), \quad (50)$$

where the superscripts “(i)” ($i = 0, 1, 2, \dots$) denote the power of the gradient. The zeroth order contribution is

$$f_{sr}^{(0)}(x, \mathbf{k}) = \frac{1}{e^{\beta \cdot k} - 1} \delta_{sr} = f_{BE}(x, \mathbf{k}) \delta_{sr}, \quad (51)$$

which automatically yields the Bose-Einstein distribution $f_{BE}(x, \mathbf{k})$ when $s = r$. The first and second order results are,

$$f_{sr}^{(1)}(x, \mathbf{k}) = \int d^3 \mathbf{q} e^{-iq \cdot x} \left\{ -f'_{BE}(x, \mathbf{p}) \Xi_{++} + \frac{1}{2} f''_{BE}(x, \mathbf{p}) [\beta \cdot p, \Xi_{++}] \right\} \delta(\mathbf{q}), \quad (52)$$

$$\begin{aligned} f_{sr}^{(2)}(x, \mathbf{k}) &= \int d^3 \mathbf{q} e^{-iq \cdot x} \left\{ \frac{1}{2} f''_{BE}(x, \mathbf{p}) \Xi_{++}^2 - \frac{1}{6} f'''_{BE}(x, \mathbf{p}) \Xi_{++} [\beta \cdot p, \Xi_{++}] \right. \\ &\quad \left. - \frac{1}{3} f'''_{BE}(x, \mathbf{p}) [\beta \cdot p, \Xi_{++}] \Xi_{++} + \frac{1}{8} f''''_{BE}(x, \mathbf{p}) [\beta \cdot p, \Xi_{++}]^2 \right\}_{sr} \delta(\mathbf{q}) \\ &\quad + \tilde{f}(x, \mathbf{k}) (\Xi_{+-} \Xi_{-+})_{sr}, \end{aligned} \quad (53)$$

where $f_{BE}^{l' \dots l}$ denotes the n th derivative $\partial^n f_{BE} / \partial^n (\beta \cdot k)$ with n being the number of primes, and $\tilde{f}(x, \mathbf{k})$ is defined as

$$\tilde{f}(x, \mathbf{k}) = -\frac{1}{2\beta_0 E_{\mathbf{k}}} \frac{e^{\beta \cdot k}}{(e^{\beta \cdot k} - 1)^2} - \frac{1}{4\beta_0^2 E_{\mathbf{k}}^2} \frac{e^{-\beta \cdot \bar{k}} - e^{\beta \cdot k}}{(e^{\beta \cdot k} - 1)(e^{-\beta \cdot \bar{k}} - 1)}. \quad (54)$$

In general, as mentioned, there will be additional second order terms in the density operators $\hat{q}_{B,C}$. The results presented above do not account for these contributions.

After integrating over the $d^3 \mathbf{q}$, Eqs. (52), (53) we obtain

$$f_{sr}^{(1)}(x, \mathbf{k}) = f_{BE}'(x, \mathbf{k}) \mathcal{F}_{sr}, \quad (55)$$

$$f_{sr}^{(2)}(x, \mathbf{k}) = f_2(x, \mathbf{k}) \delta_{sr} + \frac{1}{2} f_{BE}''(x, \mathbf{k}) \sum_{s'} \mathcal{F}_{ss'} \mathcal{F}_{s'r} + \tilde{f}(x, \mathbf{k}) \sum_{s'} \Xi_{+-}^{ss'} \Xi_{-+}^{s'r}, \quad (56)$$

where $f_2(x, \mathbf{k})$ is defined as

$$f_2(x, \mathbf{k}) = (\xi_{\rho\sigma} - \omega_{\rho\sigma})(\xi_{\mu\nu} - \omega_{\mu\nu}) \left[\frac{1}{12} f_{BE}'''(x, \mathbf{k}) k^\sigma \beta_\lambda \left(g^{\rho\nu} - \frac{k^\rho}{E_{\mathbf{k}}} n^\nu \right) \left(g^{\mu\lambda} - \frac{k^\mu}{E_{\mathbf{k}}} n^\lambda \right) + \frac{1}{8} f_{BE}''(x, \mathbf{k}) \left(g^{\rho\nu} - \frac{k^\rho}{E_{\mathbf{k}}} n^\nu \right) \left(g^{\mu\sigma} - \frac{k^\mu}{E_{\mathbf{k}}} n^\sigma \right) + \frac{1}{24} f_{BE}'''(x, \mathbf{k}) k^\sigma k^\nu \beta_0 \left(\frac{\Delta^{\rho\mu}}{E_{\mathbf{k}}} + \frac{\bar{k}^\mu \bar{k}^\rho}{E_{\mathbf{k}}^3} \right) \right], \quad (57)$$

with $\bar{k}^\mu = (0, \mathbf{k})$ and $\Delta^{\mu\nu} = g^{\mu\nu} - g^{\mu 0} g^{\nu 0}$, and the functions $\mathcal{F}_{sr}$ corresponding to Belinfante and canonical cases read

$$\mathcal{F}_{sr}^{(B)} = -i\omega_{\lambda\nu} \left(\frac{m}{E_{\mathbf{k}} + m} n^\nu \Psi_{sr}^\lambda + \frac{1}{2} \Psi_{sr}^{\nu\lambda} \right) + i\xi_{\lambda\nu} \frac{1}{E_{\mathbf{k}}} \frac{m}{E_{\mathbf{k}} + m} k^\nu \Psi_{sr}^\lambda, \quad (58)$$

$$\mathcal{F}_{sr}^{(C)} = \frac{i}{2} \omega_{\lambda\nu} \frac{1}{E_{\mathbf{k}}} \frac{E_{\mathbf{k}} - m}{E_{\mathbf{k}} + m} k^\nu \Psi_{sr}^\lambda - \frac{i}{2} \xi_{\lambda\nu} \frac{1}{E_{\mathbf{k}}} \frac{E_{\mathbf{k}} - m}{E_{\mathbf{k}} + m} k^\nu \Psi_{sr}^\lambda - \frac{i}{2} \Omega_{\lambda\nu} \left(\Psi_{sr}^{\nu\lambda} + \frac{2m}{E_{\mathbf{k}} + m} n^\nu \Psi_{sr}^\lambda + \frac{1}{E_{\mathbf{k}}} \frac{E_{\mathbf{k}} - m}{E_{\mathbf{k}} + m} k^\nu \Psi_{sr}^\lambda \right), \quad (59)$$

where

$$\Psi_{sr}^\mu = \frac{k \cdot \epsilon_s}{m} \epsilon_r^{\mu*} - \frac{k \cdot \epsilon_r^*}{m} \epsilon_s^\mu = \frac{1}{m} (0, -\mathbf{k} \times (\mathbf{e}_s \times \mathbf{e}_r^*)),$$

$$\Psi_{sr}^{\nu\lambda} = \epsilon_r^{\nu*} \epsilon_s^\lambda - \epsilon_r^{\lambda*} \epsilon_s^\nu, \quad (60)$$

and $\epsilon_s^\mu = \epsilon_s^\mu(0) = (0, \mathbf{e}_s)$ is the polarization vector in the particle's rest frame. After a lengthy but straightforward calculation, we obtain expressions of $\sum_{s'} \Xi_{+-}^{ss'} \Xi_{-+}^{s'r}$ for Belinfante and canonical cases,

$$\sum_{s'} \Xi_{(B)++}^{ss'} \Xi_{(B)--}^{s's} = \frac{1}{4} \xi^{\lambda\nu} \xi^{\rho\sigma} (\Xi_{(B)\beta\beta}^{ss})_{\lambda\nu, \rho\sigma}, \quad (61)$$

$$\sum_{s'} \Xi_{(C)++}^{ss'} \Xi_{(C)--}^{s's} = \frac{1}{4} (\xi^{\lambda\nu} - \omega^{\lambda\nu})(\xi^{\rho\sigma} - \omega^{\rho\sigma}) (\Xi_{(C)\beta\beta}^{ss})_{\lambda\nu, \rho\sigma} + \frac{1}{4} \Omega^{\lambda\nu} \Omega^{\rho\sigma} (\Xi_{(C)\Omega\Omega}^{ss})_{\lambda\nu, \rho\sigma} + \frac{1}{4} (\xi^{\lambda\nu} - \omega^{\lambda\nu}) \Omega^{\rho\sigma} [(\Xi_{(C)\beta\Omega}^{ss})_{\lambda\nu, \rho\sigma} + (\Xi_{(C)\beta\Omega}^{ss})_{\rho\sigma, \lambda\nu}^*], \quad (62)$$

where the coefficients Ξ^{ss} are given in Appendix B.

IV. GRADIENT EXPANSION FOR ρ^{00}

A. First order results

To the first order in the gradient expansion, the spin density matrix ρ_{sr} in Eq. (17) is given by

$$\rho_{sr}(x, \mathbf{k}) = \frac{f_{sr}^{(0)} + f_{sr}^{(1)} + \mathcal{O}(\partial^2)}{\text{Tr}(f^{(0)} + f^{(1)} + \mathcal{O}(\partial^2))} = \frac{1}{3} \delta_{sr} + \frac{1}{3f_{BE}} f_{sr}^{(1)} + \mathcal{O}(\partial^2). \quad (63)$$

We can explicitly express the components of ρ_{sr} as

$$\begin{aligned}
\rho_{00}^{(B)}(x, \mathbf{k}) &= \rho_{00}^{(C)}(x, \mathbf{k}) + \mathcal{O}(\partial^2), \\
\rho_{10}^{(B)}(x, \mathbf{k}) &= \rho_{0,-1}^{(B)}(x, \mathbf{k}) \\
&= \frac{f'_{BE}}{3f_{BE}} \left[\frac{1}{E_{\mathbf{k}} + m} \epsilon_{\mu\lambda\rho\sigma} n^\lambda k^\rho \epsilon_{+1}^\sigma \left(\omega^{\mu\nu} n_\nu - \xi^{\mu\nu} \frac{k_\nu}{E_{\mathbf{k}}} \right) + \frac{1}{2} \omega^{\mu\nu} \epsilon_{\mu\nu\rho\sigma} n^\rho \epsilon_{+1}^\sigma \right], \\
\rho_{10}^{(C)}(x, \mathbf{k}) &= \rho_{0,-1}^{(C)}(x, \mathbf{k}) \\
&= -\frac{f'_{BE}}{6f_{BE}} \left[\frac{1}{E_{\mathbf{k}} + m} \epsilon_{\mu\lambda\rho\sigma} n^\lambda k^\rho \epsilon_{+1}^\sigma \left(\omega^{\mu\nu} k_\nu \frac{E_{\mathbf{k}} - m}{mE_{\mathbf{k}}} - \xi^{\mu\nu} k_\nu \frac{E_{\mathbf{k}} - m}{mE_{\mathbf{k}}} - \Omega^{\mu\nu} k_\nu \frac{E_{\mathbf{k}} - m}{mE_{\mathbf{k}}} - 2\Omega^{\mu\nu} n_\nu \right) \right. \\
&\quad \left. - \Omega^{\mu\nu} \epsilon_{\mu\nu\rho\sigma} n^\rho \epsilon_{+1}^\sigma \right] + \mathcal{O}(\partial^2), \\
\rho_{1,-1}^{(B,C)}(x, \mathbf{k}) &= \mathcal{O}(\partial^2).
\end{aligned} \tag{64}$$

In global equilibrium, the Killing condition [83] gives $\xi^{\mu\nu} = 0$ and $\Omega^{\mu\nu} = \omega^{\mu\nu}$. The result in Eq. (64) reproduces that in Ref. [92], and we have $\rho_{sr}^{(B)} = \rho_{sr}^{(C)}$ to the first order in the gradient expansion. In local equilibrium, however, the thermal shear stress tensor $\xi^{\mu\nu}$ is nonvanishing, so we have $\rho_{sr}^{(B)} \neq \rho_{sr}^{(C)}$ except $s = r = 0$ to the first order.

A discussion about the result Eq. (64) is needed. We notice that the unit vector $\mathbf{e}_s$ satisfies the identity $\mathbf{e}_s^* = (-1)^s \mathbf{e}_{-s}$ which leads to

$$\Psi_{sr}^\lambda = (-1)^r \frac{1}{m} (0, -\mathbf{k} \times (\mathbf{e}_s \times \mathbf{e}_{-r})) = -(-1)^{r+s} \Psi_{-r,-s}^\lambda, \tag{65}$$

$$\Psi_{sr}^{\nu\lambda} = (-1)^r (\epsilon_{-r}^\nu \epsilon_s^\lambda - \epsilon_{-r}^\lambda \epsilon_s^\nu) = -(-1)^{r+s} \Psi_{-r,-s}^{\nu\lambda}. \tag{66}$$

Using the above relationship, we obtain

$$\begin{aligned}
\rho_{sr}(x, \mathbf{k}) &= \frac{f_{sr}^{(0)} + f_{sr}^{(1)} + f_{sr}^{(2)} + \mathcal{O}(\partial^3)}{\text{Tr}(f^{(0)} + f^{(1)} + f^{(2)}) + \mathcal{O}(\partial^3)} \\
&= \frac{1}{3} \delta^{sr} + \frac{1}{3f_{BE}} (f_{sr}^{(1)} + f_{sr}^{(2)}) - \frac{1}{9f_{BE}^2} f_{sr}^{(0)} \text{Tr} f^{(2)} + \mathcal{O}(\partial^3).
\end{aligned} \tag{69}$$

We focus on the ρ_{00} component of the spin density matrix. Inserting Eq. (56) into Eq. (69), we obtain

$$\rho_{00}^{(B)}(x, \mathbf{k}) = \frac{1}{3} - \frac{1}{18f_{BE}} f_{BE}'' \Omega_{(B)}^2 - \frac{1}{6f_{BE}} f_{BE}'' (\Omega_{(B)} \cdot \epsilon_0)^2 + \frac{1}{36f_{BE}} \tilde{f} \xi^{\lambda\nu} \xi^{\rho\sigma} \left[3(\Xi_{(B)\beta\beta}^{00})_{\lambda\nu,\rho\sigma} - \sum_s (\Xi_{(B)\beta\beta}^{ss})_{\lambda\nu,\rho\sigma} \right] + \mathcal{O}(\partial^3), \tag{70}$$

for the Belinfante case, and

$$\mathcal{F}_i^{sr} = -(-1)^{r+s} \mathcal{F}_i^{-r-s}, \tag{67}$$

which further leads to

$$f_{sr}^{(1)}(x, \mathbf{k}) = -(-1)^{r+s} f_{-r,-s}^{(1)}(x, \mathbf{k}), \tag{68}$$

and

$$\begin{aligned}
\text{Tr} f^{(1)} &= f_{00}^{(1)} = f_{1,-1}^{(1)} = f_{-1,1}^{(1)} = 0, \\
f_{10}^{(1)} &= f_{0,-1}^{(1)}.
\end{aligned}$$

These properties of $f_{sr}^{(1)}$ follow Eq. (64).

B. Second order results

To the second order in the gradient expansion the spin density matrix reads

$$\begin{aligned} \rho_{00}^{(C)}(x, \mathbf{k}) = & \frac{1}{3} - \frac{1}{18f_{BE}} f_{BE}'' \Omega_{(C)}^2 - \frac{1}{6f_{BE}} f_{BE}'' (\Omega_{(C)} \cdot \epsilon_0)^2 + \frac{1}{36f_{BE}} \tilde{f} \left\{ (\xi^{\lambda\nu} - \omega^{\lambda\nu})(\xi^{\rho\sigma} - \omega^{\rho\sigma}) \right. \\ & \times \left[3(\Xi_{(C)\beta\beta}^{00})_{\lambda\nu,\rho\sigma} - \sum_s (\Xi_{(C)\beta\beta}^{ss})_{\lambda\nu,\rho\sigma} \right] + (\xi^{\lambda\nu} - \omega^{\lambda\nu}) \Omega^{\rho\sigma} \left[3(\Xi_{(C)\beta\Omega}^{00})_{\lambda\nu,\rho\sigma} + 3(\Xi_{(C)\beta\Omega}^{00})_{\rho\sigma,\lambda\nu}^* \right. \\ & \left. \left. - \sum_s (\Xi_{(C)\beta\Omega}^{ss})_{\lambda\nu,\rho\sigma} - \sum_s (\Xi_{(C)\beta\Omega}^{ss})_{\rho\sigma,\lambda\nu}^* \right] + \Omega^{\lambda\nu} \Omega^{\rho\sigma} \left[3(\Xi_{(C)\Omega\Omega}^{00})_{\lambda\nu,\rho\sigma} - \sum_s (\Xi_{(C)\Omega\Omega}^{ss})_{\lambda\nu,\rho\sigma} \right] \right\} + \mathcal{O}(\partial^3), \end{aligned} \quad (71)$$

for the canonical case. Here, we introduced the $\Omega_{(B)}$ and $\Omega_{(C)}$ vectors for both Belinfante and canonical cases

$$\begin{aligned} \Omega_{(B)}^\sigma &= \frac{1}{E_{\mathbf{k}} + m} \epsilon^{\mu\lambda\rho\sigma} n_\lambda k_\rho \left(\omega_{\mu\nu} n^\nu - \xi_{\mu\nu} \frac{k^\nu}{E_{\mathbf{k}}} \right) + \frac{1}{2} \omega_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} n_\rho, \\ \Omega_{(C)}^\sigma &= -\frac{1}{2} \frac{1}{E_{\mathbf{k}} + m} \epsilon^{\mu\lambda\rho\sigma} n_\lambda k_\rho \left(\omega_{\mu\nu} k^\nu \frac{E_{\mathbf{k}} - m}{mE_{\mathbf{k}}} - \xi_{\mu\nu} k^\nu \frac{E_{\mathbf{k}} - m}{mE_{\mathbf{k}}} - \Omega_{\mu\nu} k^\nu \frac{E_{\mathbf{k}} - m}{mE_{\mathbf{k}}} - 2\Omega_{\mu\nu} n^\nu \right) + \frac{1}{2} \Omega_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} n_\rho. \end{aligned} \quad (72)$$

We find that nonvanishing contributions to $\rho_{00} - 1/3$ appear in the second order of gradients. In the Belinfante case, the contributions come from quadratic forms of thermal shear and vorticity tensors, while in the canonical case there are extra terms of $(\partial^\lambda \beta^\nu)(\partial^\rho \beta^\sigma)$, $(\partial^\lambda \beta^\nu) \Omega^{\rho\sigma}$ and $\Omega^{\lambda\nu} \Omega^{\rho\sigma}$.

We can roughly estimate the magnitude of the corrections to $\rho_{00} - \frac{1}{3}$ shown in Eqs. (70), (71). Based on previous hydrodynamic studies of the polarization of Λ hyperons, e.g., in Refs. [25,93], the local polarization induced by thermal vorticity and thermal shear tensor is of the order of 10^{-2} . Consequently, the corrections to $\rho_{00} - \frac{1}{3}$ shown in Eqs. (70), (71) are estimated to be as large as 10^{-4} . On the other hand, determining the exact value and the sign of $\rho_{00} - \frac{1}{3}$ requires systematic hydrodynamic simulations, which will be presented elsewhere.

In the global equilibrium characterized by $\xi^{\mu\nu} = 0$ and $\omega^{\mu\nu} = \Omega^{\mu\nu}$, ρ^{00} converges to the same value for both Belinfante and canonical cases

$$\begin{aligned} \rho_{00}(x, \mathbf{k}) = & \frac{1}{3} - \frac{1}{18f_{BE}} f_{BE}'' \Omega^2 - \frac{1}{6f_{BE}} f_{BE}'' (\Omega \cdot \epsilon_0)^2 \\ & + \mathcal{O}(\partial^3), \end{aligned} \quad (73)$$

where Ω^σ is given as

$$\begin{aligned} \Omega^\sigma &\equiv \Omega_{(B)}^\sigma = \Omega_{(C)}^\sigma \\ &= \frac{1}{E_{\mathbf{k}} + m} \epsilon^{\mu\lambda\rho\sigma} n_\lambda k_\rho \omega_{\mu\nu} n^\nu + \frac{1}{2} \omega_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} n_\rho. \end{aligned} \quad (74)$$

In local equilibrium, the expressions for ρ_{00} are different for Belinfante and canonical cases, even when integrated over space. The difference stems from distinct definitions of the density operator $\hat{q}$ in Eqs. (18) and (19). Notably, in the context of the spin polarization for Λ and $\bar{\Lambda}$ hyperons, $\hat{q}$ is typically chosen to be in the Belinfante form, as utilized in

the modified Cooper-Frye formula [27,28]. However, we note that the results of the canonical and other forms of density operators are very different from those of the Belinfante form [77]. This discrepancy has led to discussions regarding the physical appropriateness of these forms, particularly in how they align with experimental data. The choice of a specific form is often made *a posteriori*, emphasizing the need for further empirical validation [77]. So which form of the density operator has more reflection on the physical reality remains an open question. This underscores the importance of continued experimental and theoretical investigation to resolve this issue.

C. Another choice for MVSD

There is an alternative definition for the MVSD f_{sr} . From the definition of the Wigner function in Eq. (15), we can decompose the Wigner function into timelike and spacelike parts in momentum [81],

$$W^{\mu\nu}(x, k) = W_+^{\mu\nu}(x, k) + W_-^{\mu\nu}(x, k) + W_S^{\mu\nu}(x, k), \quad (75)$$

where $W_S^{\mu\nu}(x, k)$ is the spacelike part and $W_\pm^{\mu\nu}(x, k)$ are the timelike parts (upper/lower sign corresponds to the positive/negative energy branch respectively), defined as

$$W_+^{\mu\nu}(x, k) = W^{\mu\nu}(x, k) \theta(k_0) \theta(k^2), \quad (76)$$

$$W_-^{\mu\nu}(x, k) = W^{\mu\nu}(x, k) \theta(-k_0) \theta(k^2), \quad (77)$$

$$W_S^{\mu\nu}(x, k) = W^{\mu\nu}(x, k) \theta(-k^2). \quad (78)$$

Different from Eq. (16), one can also introduce an alternative MVSD (also see Refs. [79–82] for the fermion's MVSD),

$$\begin{aligned}
f_{sr}(x, k) &= \frac{1}{2\pi} \epsilon_\mu^{r*}(\mathbf{k}) \epsilon_\nu^s(\mathbf{k}) W_+^{\mu\nu}(x, k) \\
&= \frac{1}{(2\pi)^3} \int \frac{d^3 q}{\sqrt{4E_{\mathbf{k}+\mathbf{q}/2} E_{\mathbf{k}-\mathbf{q}/2}}} \sum_{s', r'} e^{-iq \cdot x} \delta\left(k_0 - \frac{E_{\mathbf{k}-\mathbf{q}/2} + E_{\mathbf{k}+\mathbf{q}/2}}{2}\right) \langle a_{\mathbf{k}-\mathbf{q}/2}^{s'\dagger} a_{\mathbf{k}+\mathbf{q}/2}^{r'} \rangle \\
&\quad \times \epsilon_\mu^{r*}(\mathbf{k}) \epsilon_\nu^s(\mathbf{k}) \epsilon_{s'}^{\nu*}\left(\mathbf{k} - \frac{\mathbf{q}}{2}\right) \epsilon_{r'}^\mu\left(\mathbf{k} + \frac{\mathbf{q}}{2}\right).
\end{aligned} \tag{79}$$

We can make a gradient expansion following the same procedure as in Eq. (50), we obtain

$$\begin{aligned}
f_{sr}^{(0)}(x, k) &= \delta(k^2 - m^2) \theta(k_0) f_{BE}(x, \mathbf{k}) \delta_{sr}, \\
f_{sr}^{(1)}(x, k) &= \delta(k^2 - m^2) \theta(k_0) f'_{BE}(x, \mathbf{k}) \mathcal{F}_{sr}, \\
f_{sr}^{(2)}(x, k) &= \delta(k^2 - m^2) \theta(k_0) \left[f_2^{(+)}(x, \mathbf{k}) \delta_{sr} + \frac{1}{2} f''_{BE}(x, \mathbf{k}) \sum_{s'} \mathcal{F}_{ss'} \mathcal{F}_{s'r} + \tilde{f}(x, \mathbf{k}) \sum_{s'} \Xi_{+-}^{ss'} \Xi_{-+}^{s'r} \right. \\
&\quad \left. - \frac{1}{4} f''_{BE}(x, \mathbf{k}) (\xi_{\lambda\nu} - \omega_{\lambda\nu}) (\xi_{\rho\sigma} - \omega_{\rho\sigma}) \frac{k^\nu k^\sigma}{m^2} (\bar{\partial}_k^\lambda k^\alpha) (\bar{\partial}_k^\rho k^\beta) \epsilon_\alpha^{r*}(\mathbf{k}) \epsilon_\beta^s(\mathbf{k}) \right] \\
&\quad - \frac{1}{16} \delta'(k_0 - E_{\mathbf{k}}) f''_{BE}(x, \mathbf{k}) (\xi_{\lambda\nu} - \omega_{\lambda\nu}) (\xi_{\rho\sigma} - \omega_{\rho\sigma}) k^\sigma k^\nu \left(\frac{\Delta^{\rho\lambda}}{E_{\mathbf{k}}^2} + \frac{\bar{k}^\rho \bar{k}^\lambda}{E_{\mathbf{k}}^4} \right) \delta_{sr},
\end{aligned} \tag{81}$$

with $\delta'(x) = \partial\delta(x)/\partial x$ and

$$f_2^{(+)}(x, \mathbf{k}) = f_2(x, \mathbf{k}) - \frac{1}{4} f''_{BE}(\xi_{\lambda\nu} - \omega_{\lambda\nu}) (\xi_{\rho\sigma} - \omega_{\rho\sigma}) k^\sigma k^\nu \left(\frac{1}{2} \frac{\Delta^{\lambda\rho}}{E_{\mathbf{k}}^2} + \frac{\bar{k}^\lambda \bar{k}^\rho}{E_{\mathbf{k}}^4} \right), \tag{82}$$

$$\mathcal{F}_{sr}^{(B)} = -i\omega_{\lambda\nu} \epsilon_r^{\nu*}(\mathbf{k}) \epsilon_s^\lambda(\mathbf{k}) + i\xi_{\lambda\nu} k^\nu \frac{1}{E_{\mathbf{k}}} [\epsilon_r^{0*}(\mathbf{k}) \epsilon_s^\lambda(\mathbf{k}) - \epsilon_s^0(\mathbf{k}) \epsilon_r^{\lambda*}(\mathbf{k})], \tag{83}$$

$$\mathcal{F}_{sr}^{(C)} = -i\Omega_{\lambda\nu} \epsilon_r^{\nu*}(\mathbf{k}) \epsilon_s^\lambda(\mathbf{k}) - i(\omega_{\lambda\nu} - \Omega_{\lambda\nu} - \xi_{\lambda\nu}) k^\nu \frac{1}{2E_{\mathbf{k}}} [\epsilon_r^{0*}(\mathbf{k}) \epsilon_s^\lambda(\mathbf{k}) - \epsilon_s^0(\mathbf{k}) \epsilon_r^{\lambda*}(\mathbf{k})]. \tag{84}$$

In the comparison with Eqs. (58) and (59), $\mathcal{F}_{sr}^{(B)}$ and $\mathcal{F}_{sr}^{(C)}$ in Eqs. (83) and (84) have an additional term

$$\Delta \mathcal{F}_{sr}^{(B)} = \Delta \mathcal{F}_{sr}^{(C)} = (\partial_\mu \beta_\nu) k^\nu [i \bar{\partial}_k^\mu \epsilon_\beta^{r*}(\mathbf{k})] \epsilon_s^\beta(\mathbf{k}), \tag{85}$$

where $\bar{\partial}_k^\mu \equiv (0, -\nabla_{\mathbf{k}})$. In global equilibrium, we observe that $\mathcal{F}_{sr}^{(B,C)}$ in Eqs. (83) and (84) do not depend on n^μ , while $f_{sr}^{(2)}$ in Eq. (81), $\mathcal{F}_{sr}^{(B,C)}$ in Eqs. (58), (59) and $f_{sr}^{(1,2)}$ in Eqs. (55) and (56) do depend on n^μ . According to Eq. (85), the extra term proportional to $\bar{\partial}_k^\mu \epsilon_\beta^{r*}(\mathbf{k})$ cancels the terms related to n^μ in $\mathcal{F}_{sr}^{(B,C)}$ in Eqs. (58), (59).

From Eqs. (17) and (69) we can obtain the spin density matrix as the normalized MVSD in Eqs. (81),

$$\begin{aligned}
\rho_{00}^{(B)}(x, \mathbf{k}) &= \frac{1}{3} - \frac{1}{18 f_{BE}} f''_{BE} \Omega_{(B)}^2 - \frac{1}{6 f_{BE}} f''_{BE} (\Omega_{(B)} \cdot \epsilon_0)^2 - \frac{1}{36} f''_{BE} (\xi_{\lambda\nu} - \omega_{\lambda\nu}) (\xi_{\rho\sigma} - \omega_{\rho\sigma}) \Xi_{(+)}^{\lambda\nu, \rho\sigma} \\
&\quad + \frac{1}{36 f_{BE}} \tilde{f} \xi^{\lambda\nu} \xi^{\rho\sigma} \left[3 (\Xi_{(B)\beta\beta}^{ss})_{\lambda\nu, \rho\sigma} - \sum_s (\Xi_{(B)\beta\beta}^{ss})_{\lambda\nu, \rho\sigma} \right] + \mathcal{O}(\partial^3),
\end{aligned} \tag{86}$$

for the Belinfante case, and

$$\begin{aligned}
 \rho_{00}^{(C)}(x, \mathbf{k}) = & \frac{1}{3} - \frac{1}{18f_{BE}} f_{BE}'' \Omega_{(C)}^2 - \frac{1}{6f_{BE}} f_{BE}'' (\Omega_{(C)} \cdot \epsilon_0)^2 - \frac{1}{36} f_{BE}'' (\xi_{\lambda\nu} - \omega_{\lambda\nu}) (\xi_{\rho\sigma} - \omega_{\rho\sigma}) \Xi_{(+)}^{\lambda\nu, \rho\sigma} \\
 & + \frac{1}{36f_{BE}} \tilde{f} \left\{ (\xi^{\lambda\nu} - \omega^{\lambda\nu}) (\xi^{\rho\sigma} - \omega^{\rho\sigma}) \left[3(\Xi_{(C)\beta\beta}^{00})_{\lambda\nu, \rho\sigma} - \sum_s (\Xi_{(C)\beta\beta}^{ss})_{\lambda\nu, \rho\sigma} \right] \right. \\
 & + (\xi^{\lambda\nu} - \omega^{\lambda\nu}) \Omega^{\rho\sigma} \left[3(\Xi_{(C)\beta\Omega}^{00})_{\lambda\nu, \rho\sigma} + 3(\Xi_{(C)\beta\Omega}^{00})_{\rho\sigma, \lambda\nu}^* - \sum_s (\Xi_{(C)\beta\Omega}^{ss})_{\lambda\nu, \rho\sigma} - \sum_s (\Xi_{(C)\beta\Omega}^{ss})_{\rho\sigma, \lambda\nu}^* \right] \\
 & \left. + \Omega^{\lambda\nu} \Omega^{\rho\sigma} \left[3(\Xi_{(C)\Omega\Omega}^{00})_{\lambda\nu, \rho\sigma} - \sum_s (\Xi_{(C)\Omega\Omega}^{ss})_{\lambda\nu, \rho\sigma} \right] \right\} + \mathcal{O}(\partial^3), \quad (87)
 \end{aligned}$$

for the canonical case, where we defined

$$\Omega_{(B)}^\sigma = \frac{1}{m} \epsilon^{\mu\lambda\rho\sigma} n_\lambda k_\rho \left(\omega_{\mu\nu} n^\nu \frac{m}{E_{\mathbf{k}} + m} + \omega_{\mu\nu} \frac{k^\nu}{E_{\mathbf{k}} + m} - \xi_{\mu\nu} \frac{k^\nu}{E_{\mathbf{k}}} \right) + \frac{1}{2} \omega_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} n_\rho, \quad (88)$$

$$\Omega_{(C)}^\sigma = -\frac{1}{2m} \epsilon^{\mu\lambda\rho\sigma} n_\lambda k_\rho \left(\xi_{\mu\nu} \frac{k^\nu}{E_{\mathbf{k}}} - \omega_{\mu\nu} \frac{k^\nu}{E_{\mathbf{k}}} - \Omega_{\mu\nu} \frac{k^\nu}{E_{\mathbf{k}}} \frac{E_{\mathbf{k}} - m}{E_{\mathbf{k}} + m} - \frac{2m}{E_{\mathbf{k}} + m} \Omega^{\mu\nu} n_\nu \right) + \frac{1}{2} \Omega_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} n_\rho, \quad (89)$$

$$\begin{aligned}
 \Xi_{(+)}^{\lambda\nu, \rho\sigma} = & 3 \frac{k^\nu k^\sigma}{m^2} \left[\epsilon_0^{\lambda*} \epsilon_0^\rho + \frac{m}{E_{\mathbf{k}}(E_{\mathbf{k}} + m)} \left(\frac{k \cdot \epsilon_0^*}{m} \epsilon_0^\rho \bar{k}^\lambda + \frac{k \cdot \epsilon_0}{m} \epsilon_0^{\lambda*} \bar{k}^\rho \right) + \frac{m^2}{E_{\mathbf{k}}^2(E_{\mathbf{k}} + m)^2} \frac{k \cdot \epsilon_0^*}{m} \frac{k \cdot \epsilon_0}{m} \bar{k}^\lambda \bar{k}^\rho \right] \\
 & + \frac{k^\nu k^\sigma}{m^2} \left(\Delta^{\lambda\rho} + \frac{\bar{k}^\lambda \bar{k}^\rho}{E_{\mathbf{k}}^2} \right). \quad (90)
 \end{aligned}$$

Here, we simply drop $\delta(k^2 - m^2)$ in the numerator and denominator in the definition of the spin density matrix (69) and take $k_0 = E_{\mathbf{k}}$.

We can define the momentum dependent spin density matrix as,

$$\begin{aligned}
 \rho_{sr}(\mathbf{k}) &= \frac{\int d^3x f_{sr}(x, k)}{\int d^3x \text{Tr} f(x, k)} = \frac{\delta(k^2 - m^2) \text{Tr}(\hat{a}_{\mathbf{k}}^{s\dagger} a_{\mathbf{k}}^r)}{\sum_{s'} \delta(k^2 - m^2) \text{Tr}(\hat{a}_{\mathbf{k}}^{s'\dagger} a_{\mathbf{k}}^{s'})} \\
 &= \frac{\text{Tr}(\hat{a}_{\mathbf{k}}^{s\dagger} a_{\mathbf{k}}^r)}{\sum_{s'} \text{Tr}(\hat{a}_{\mathbf{k}}^{s'\dagger} a_{\mathbf{k}}^{s'})}, \quad (91)
 \end{aligned}$$

where in the third equality, $\delta(k^2 - m^2)$ functions cancel in the numerator and denominator and $k_0 = E_{\mathbf{k}}$ is taken. Interestingly, although $\rho_{00}^{(B,C)}(x, \mathbf{k})$ in Eqs. (86) and (87) differ from $\rho_{00}^{(B,C)}(x, \mathbf{k})$ in Eqs. (70), (71), the momentum dependent spin density matrix is the same for both choices of MVSDs.

V. TIME REVERSAL PROPERTY OF SPIN ALIGNMENT INDUCED BY SHEAR STRESS TENSOR

In this section, we discuss the time reversal property of the spin alignment induced by the shear stress tensor, which is related to the question if the spin alignment induced by the shear stress tensor dissipative.

If a transport coefficient in hydrodynamics is even under the time reversal transformation (T even), the transport

effect is called nondissipative, while it is called dissipative if the corresponding transport coefficient is T odd [94,95]. For example, the chiral magnetic effect is described by $\mathbf{j} = \xi_B \mathbf{B}$ with ξ_B being the transport coefficient and is nondissipative as it does not lead to entropy production [96,97]. This can be seen by the time reversal transformation of the current and magnetic field, $\mathbf{j} \rightarrow -\mathbf{j}$ and $\mathbf{B} \rightarrow -\mathbf{B}$, so the coefficient ξ_B is T even.

We consider the spin alignment induced by the shear stress tensor in this work. From Eq. (79), we have

$$\rho_{sr}(x, \mathbf{k}) \sim \epsilon_\mu^{r*}(\mathbf{k}) \epsilon_\nu^s(\mathbf{k}) W^{\mu\nu}. \quad (92)$$

Given $W^{\mu\nu}$ being a symmetric rank-2 tensor, it can be associated with other symmetric tensors. In this context, we consider the following terms

$$\begin{aligned}
 \epsilon_\mu^{r*}(\mathbf{k}) \epsilon_\nu^s(\mathbf{k}) W^{\mu\nu} &\sim a \xi^{\mu\nu} \epsilon_\mu^{r*}(\mathbf{k}) \epsilon_\nu^s(\mathbf{k}) + b (\xi^{\mu\lambda} \xi_\lambda^\nu + \xi^{\mu\lambda} \xi_\lambda^\nu) \\
 &\times \epsilon_\mu^{r*}(\mathbf{k}) \epsilon_\nu^s(\mathbf{k}) + \mathcal{O}(\xi_{\mu\nu}^3), \quad (93)
 \end{aligned}$$

where a and b are interpreted as transport coefficients. Under the time reversal transformation, the polarization vector transforms as

$$\epsilon_0^s(\mathbf{k}) \rightarrow (-1)^{s+1} \epsilon_0^{-s}(-\mathbf{k}), \quad \epsilon_i^s(\mathbf{k}) \rightarrow (-1)^s \epsilon_i^{-s}(-\mathbf{k}), \quad (94)$$

which leads to the time reversal transformation of the spin density matrix ρ_{sr} as

TABLE I. Time reversal transformation (TRT) for components of the shear stress tensor and spin density matrix.

Terms	∂_0	β_0	ξ_{00}	ξ_{0i}	ξ_{ij}	$(-1)^{s+r}\rho_{sr}(t, \mathbf{x}, \mathbf{k})$
TRT	$-\partial_0$	$+\beta_0$	$-\xi_{00}$	$+\xi_{0i}$	$-\xi_{ij}$	$\rho_{-r,-s}(-t, \mathbf{x}, -\mathbf{k})$
Terms	∂_i	β_i	$(-1)^{s+r}\epsilon_0^{r*}(\mathbf{k})\epsilon_0^s(\mathbf{k})$	$(-1)^{s+r}\epsilon_0^{r*}(\mathbf{k})\epsilon_i^s(\mathbf{k})$	$(-1)^{s+r}\epsilon_i^{r*}(\mathbf{k})\epsilon_j^s(\mathbf{k})$	
TRT	$+\partial_i$	$-\beta_i$	$+\epsilon_0^{-r*}(-\mathbf{k})\epsilon_0^{-s}(-\mathbf{k})$	$-\epsilon_0^{-r*}(-\mathbf{k})\epsilon_i^{-s}(-\mathbf{k})$	$+\epsilon_i^{-r*}(-\mathbf{k})\epsilon_j^{-s}(-\mathbf{k})$	

$$\rho_{sr}(t, \mathbf{x}, \mathbf{k}) \rightarrow (-1)^{s+r}\rho_{-r,-s}(-t, \mathbf{x}, -\mathbf{k}). \quad (95)$$

Additional related quantities under the time reversal transformation are detailed in Table I. From this analysis, we find that the transport coefficient a is T odd while b is T even for ρ_{00} . Therefore, following this analysis, it implies that the spin alignment induced by the shear stress tensor in the first order of the gradient expansion is dissipative.

The density operator $\hat{Q}_R$ for a real system in local equilibrium differs from $\hat{Q}$ introduced in Eqs. (18) and (19). From $\hat{Q}_R$ and $\hat{Q}$ we can obtain statistical average values of the energy momentum tensor $T_R^{\mu\nu} \equiv \text{Tr}(\hat{Q}_R \hat{T}^{\mu\nu})$ and $T^{\mu\nu} \equiv \text{Tr}(\hat{Q} \hat{T}^{\mu\nu})$ which are related by [72,84],

$$n_\mu T_R^{\mu\nu} = n_\mu T^{\mu\nu}, \quad (96)$$

where n_μ is the unit vector perpendicular to the hypersurface Σ chosen in the density operator $\hat{Q}$. It is important to note that $T_R^{\mu\nu}$ is not necessarily equal to $T^{\mu\nu}$, which is computed from the density operators in Eqs. (18) and (19). Following the Refs. [72,84], one can demonstrate that the entropy production rate is given by

$$\partial_\mu s^\mu = (T_R^{\mu\nu} - T^{\mu\nu})\partial_\mu \beta_\nu. \quad (97)$$

When $\hat{Q} = \hat{Q}_R$, we have $T_R^{\mu\nu} = T^{\mu\nu}$, resulting in conserved entropy. Consequently, any thermal quantity $\text{Tr}(\hat{Q} \hat{O})$ computed using the density operator $\hat{Q}$ in Eqs. (18) and (19) is nondissipative, while the real ensemble averaged quantity $\text{Tr}(\hat{Q}_R \hat{O})$ can be dissipative which can be computed using the Kubo formula [71,72],

$$\text{Tr}[\hat{Q}_R \hat{O}(x)] - \text{Tr}[\hat{Q} \hat{O}(x)] = \partial_\mu \beta_\nu(x) \lim_{k \rightarrow 0} \text{Im} T \int_{t_0}^t d^4 x' \int_{t_0}^{t'} d\theta \langle [\hat{T}^{\mu\nu}(\theta, \mathbf{x}'), \hat{O}(x)] \rangle_{\beta(x)} e^{-ik \cdot (x' - x)}. \quad (98)$$

For the spin alignment induced by the shear stress tensor computed using the above formula, we refer the readers to Ref. [69]. We also refer to the very recent analysis of the nature of dissipation in shear-induced polarization within the framework of quantum kinetic theory [98].

VI. SUMMARY

We have computed the spin density matrix element ρ_{00} up to the second order of gradient expansion at local equilibrium using Zubarev's approach. The spin density matrix is obtained through the matrix valued spin dependent distribution function f_{sr} . It is defined as the ensemble average on the density operators $\hat{Q}$, which depend on the choice of the energy-momentum and spin tensors or pseudogauge. We choose two forms of the energy-momentum and spin tensors: Belinfante and canonical ones. Then we obtain f_{sr} up to the second order by making the gradient expansion for $\hat{Q}$.

The results for ρ_{00} to the first order are given in Eqs. (64). We find $\rho_{00} = 1/3$, meaning that there is no contribution from the thermal vorticity and shear stress tensor to the first order. The off-diagonal elements of the spin density matrix are nonzero.

The results for ρ_{00} in the second order are presented in Eqs. (70) and (71) with nonvanishing contributions from both the thermal vorticity and shear stress tensors. The exact expression for ρ_{00} depends on the pseudogauge, so ρ_{00} are different in the Belinfante and canonical cases. In global equilibrium with Killing conditions, ρ_{00} in both cases converge to a unique expression as shown in Eq. (73). The choice of the pseudogauge is *a posteriori* and should be validated by experimental data.

The definition of f_{sr} is also not unique. We discussed an alternative choice of f_{sr} in Eq. (79). The expressions for f_{sr} up to the second order of gradient expansion are given in Eq. (81). Although the spin density matrices are different from the previous one, they give the same integrated result over the spatial coordinate, as shown in Eq. (91).

We finally discussed the time reversal property of the spin alignment induced by the shear stress tensor, which is related to the question of if the spin alignment induced by the shear stress tensor is dissipative. The effective transport coefficient for the spin alignment induced by the thermal shear stress tensor is T odd in the first order, implying that the first order effect is dissipative. For the ensemble defined by the density operators in Eqs. (25) and (26), the contribution to the spin alignment from the thermal shear

stress tensor only appears in the second order, since such density operators can only give nondissipative effects.

Note added. Recently, we were informed of Ref. [99] which works on a similar topic and appeared on arXiv on the same day.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DERIVATIONS OF EQS. (38)–(44)

From Eqs. (7), (21), (20) and (27), we obtain

$$\begin{aligned} \hat{P}^\nu = & \frac{1}{2} \int d^3y n_\mu \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \sum_{s,r} \left[\Phi_{(1)++}^{sr;\mu\nu}(p, q) a_{\mathbf{p}}^{s\dagger} a_{\mathbf{q}}^r e^{i(p-q)\cdot y} + \Phi_{(1)-+}^{sr;\mu\nu}(p, q) a_{\mathbf{p}}^s a_{\mathbf{q}}^{r\dagger} e^{-i(p-q)\cdot y} \right. \\ & \left. - \Phi_{(1)--}^{sr;\mu\nu}(p, q) a_{\mathbf{p}}^s a_{\mathbf{q}}^r e^{-i(p+q)\cdot y} - \Phi_{(1)++}^{sr;\mu\nu}(p, q) a_{\mathbf{p}}^{s\dagger} a_{\mathbf{q}}^{r\dagger} e^{i(p+q)\cdot y} \right], \end{aligned} \quad (\text{A1})$$

where the auxiliary functions $\Phi_{(1)}$ are

$$\begin{aligned} \Phi_{(1)++}^{sr;\mu\nu}(p, q) &= [\Phi_{(1)-+}^{sr;\mu\nu}(p, q)]^* \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{q}}}} \{ [(p \cdot q - m^2) g^{\mu\nu} - 2p^\mu q^\nu] [\epsilon_s^*(\mathbf{p}) \cdot \epsilon_r(\mathbf{q})] - g^{\mu\nu} [q \cdot \epsilon_s^*(\mathbf{p})] [p \cdot \epsilon_r(\mathbf{q})] + 2q^\nu \epsilon_s^{\mu*}(\mathbf{p}) [p \cdot \epsilon_r(\mathbf{q})] \}, \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \Phi_{(1)--}^{sr;\mu\nu}(p, q) &= [\Phi_{(1)++}^{sr;\mu\nu}(p, q)]^* \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{q}}}} \{ [(p \cdot q + m^2) g^{\mu\nu} - 2p^\mu q^\nu] [\epsilon_s(\mathbf{p}) \cdot \epsilon_r(\mathbf{q})] - g^{\mu\nu} [q \cdot \epsilon_s(\mathbf{p})] [p \cdot \epsilon_r(\mathbf{q})] + 2q^\nu \epsilon_s^\mu(\mathbf{p}) [p \cdot \epsilon_r(\mathbf{q})] \}. \end{aligned} \quad (\text{A3})$$

After integrating over y and q in Eq. (A1) and replacing p with q as the integral variable, we obtain

$$\begin{aligned} \hat{P}^\nu = & \frac{1}{2} n_\mu \int \frac{d^3q}{(2\pi)^3} \sum_{s,r} \left[\Phi_{(1)++}^{sr;\mu\nu}(q, q) a_{\mathbf{q}}^{s\dagger} a_{\mathbf{q}}^r + \Phi_{(1)-+}^{sr;\mu\nu}(q, q) a_{\mathbf{q}}^s a_{\mathbf{q}}^{r\dagger} \right. \\ & \left. - \Phi_{(1)--}^{sr;\mu\nu}(q, \tilde{q}) a_{\mathbf{q}}^s a_{-\mathbf{q}}^r e^{-2iE_{\mathbf{q}}t_y} - \Phi_{(1)++}^{sr;\mu\nu}(q, \tilde{q}) a_{\mathbf{q}}^{s\dagger} a_{-\mathbf{q}}^{r\dagger} e^{2iE_{\mathbf{q}}t_y} \right], \end{aligned} \quad (\text{A4})$$

where $\tilde{q} = (E_{\mathbf{q}}, -\mathbf{q})$. The commutators are

$$\begin{aligned} [\hat{P}^\nu, a_{\mathbf{p}}^{s\dagger}] &= \frac{1}{2} n_\mu \sum_r \left[\Phi_{(1)++}^{rs;\mu\nu}(p, p) + \Phi_{(1)-+}^{sr;\mu\nu}(p, p) \right] a_{\mathbf{p}}^{r\dagger} - \frac{1}{2} n_\mu \sum_r \left[\Phi_{(1)--}^{rs;\mu\nu}(\tilde{p}, p) + \Phi_{(1)-+}^{sr;\mu\nu}(p, \tilde{p}) \right] a_{-\mathbf{p}}^r e^{-2iE_{\mathbf{p}}t_y} \\ &= p^\nu a_{\mathbf{p}}^{s\dagger}, \\ [\hat{P}^\nu, a_{-\mathbf{p}}^s] &= -\frac{1}{2} n_\mu \sum_r \left[\Phi_{(1)++}^{rs;\mu\nu}(\tilde{p}, \tilde{p}) + \Phi_{(1)-+}^{sr;\mu\nu}(\tilde{p}, \tilde{p}) \right] a_{-\mathbf{p}}^r + \frac{1}{2} n_\mu \sum_r \left[\Phi_{(1)++}^{rs;\mu\nu}(p, \tilde{p}) + \Phi_{(1)-+}^{sr;\mu\nu}(\tilde{p}, p) \right] a_{\mathbf{p}}^{r\dagger} e^{2iE_{\mathbf{p}}t_y} \\ &= -\tilde{p}^\nu a_{-\mathbf{p}}^s, \end{aligned} \quad (\text{A5})$$

where we have used following identities

$$\begin{aligned}
n_\mu \left[\Phi_{(1)++}^{rs;\mu\nu}(p, p) + \Phi_{(1)--}^{sr;\mu\nu}(p, p) \right] &= 2p^\nu \delta_{sr}, \\
n_\mu \left[\Phi_{(1)++}^{rs;\mu\nu}(\tilde{p}, p) + \Phi_{(1)--}^{sr;\mu\nu}(p, \tilde{p}) \right] &= n_\mu \left[\Phi_{(1)++}^{rs;\mu\nu}(p, \tilde{p}) + \Phi_{(1)++}^{sr;\mu\nu}(\tilde{p}, p) \right] = 0.
\end{aligned} \tag{A6}$$

Analogous to the above formula, we obtain expressions for other operators. From Eqs. (27), (28) and (A1) we obtain

$$\begin{aligned}
\hat{L}^{\lambda\nu} &= -x^\lambda \hat{P}^\nu + \frac{1}{2} \int d^3y y^\lambda n_\mu \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \sum_{s,r} \left[\Phi_{(1)++}^{sr;\mu\nu}(p, q) a_{\mathbf{p}}^{s\dagger} a_{\mathbf{q}}^r e^{i(p-q)\cdot y} + \Phi_{(1)++}^{sr;\mu\nu}(p, q) a_{\mathbf{p}}^s a_{\mathbf{q}}^{r\dagger} e^{-i(p-q)\cdot y} \right. \\
&\quad \left. - \Phi_{(1)++}^{sr;\mu\nu}(p, q) a_{\mathbf{p}}^s a_{\mathbf{q}}^r e^{-i(p+q)\cdot y} - \Phi_{(1)++}^{sr;\mu\nu}(p, q) a_{\mathbf{p}}^{s\dagger} a_{\mathbf{q}}^{r\dagger} e^{i(p+q)\cdot y} \right].
\end{aligned} \tag{A7}$$

From Eqs. (21), (31) and (32), we obtain

$$\begin{aligned}
\hat{S}_B^{\lambda\nu} &= -2 \int d^3y n_\mu \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \sum_{s,r} \left[\Phi_{(2)++}^{sr;\mu\lambda\nu}(p, q) a_{\mathbf{p}}^{s\dagger} a_{\mathbf{q}}^r e^{i(p-q)\cdot y} + \Phi_{(2)++}^{sr;\mu\lambda\nu}(p, q) a_{\mathbf{p}}^s a_{\mathbf{q}}^{r\dagger} e^{-i(p-q)\cdot y} \right. \\
&\quad \left. - \Phi_{(2)++}^{sr;\mu\lambda\nu}(p, q) a_{\mathbf{p}}^s a_{\mathbf{q}}^r e^{-i(p+q)\cdot y} - \Phi_{(2)++}^{sr;\mu\lambda\nu}(p, q) a_{\mathbf{p}}^{s\dagger} a_{\mathbf{q}}^{r\dagger} e^{i(p+q)\cdot y} \right],
\end{aligned} \tag{A8}$$

$$\begin{aligned}
\hat{S}_C^{\lambda\nu} &= - \int d^3y n_\mu \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \sum_{s,r} \left\{ \left[\Phi_{(2)++}^{sr;\mu\lambda\nu}(p, q) - \Phi_{(2)++}^{sr;\mu\nu\lambda}(p, q) \right] a_{\mathbf{p}}^{s\dagger} a_{\mathbf{q}}^r e^{i(p-q)\cdot y} \right. \\
&\quad + \left[\Phi_{(2)++}^{sr;\mu\lambda\nu}(p, q) - \Phi_{(2)++}^{sr;\mu\nu\lambda}(p, q) \right] a_{\mathbf{p}}^s a_{\mathbf{q}}^{r\dagger} e^{-i(p-q)\cdot y} - \left[\Phi_{(2)++}^{sr;\mu\lambda\nu}(p, q) - \Phi_{(2)++}^{sr;\mu\nu\lambda}(p, q) \right] a_{\mathbf{p}}^s a_{\mathbf{q}}^r e^{-i(p+q)\cdot y} \\
&\quad \left. - \left[\Phi_{(2)++}^{sr;\mu\lambda\nu}(p, q) - \Phi_{(2)++}^{sr;\mu\nu\lambda}(p, q) \right] a_{\mathbf{p}}^{s\dagger} a_{\mathbf{q}}^{r\dagger} e^{i(p+q)\cdot y} \right\},
\end{aligned} \tag{A9}$$

where auxiliary functions $\Phi_{(2)}$ are

$$\begin{aligned}
\Phi_{(2)++}^{sr;\mu\lambda\nu}(p, q) &= \Phi_{(2)++}^{sr;\mu\lambda\nu*}(p, q) \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{q}}}} i[p^\mu \epsilon_s^{\lambda*}(\mathbf{p}) - p^\lambda \epsilon_s^{\mu*}(\mathbf{p})] \epsilon_r^\nu(\mathbf{q}),
\end{aligned} \tag{A10}$$

$$\begin{aligned}
\Phi_{(2)++}^{sr;\mu\lambda\nu}(p, q) &= \Phi_{(2)++}^{sr;\mu\lambda\nu*}(p, q) \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{q}}}} i[p^\mu \epsilon_s^\lambda(\mathbf{p}) - p^\lambda \epsilon_s^\mu(\mathbf{p})] \epsilon_r^\nu(\mathbf{q}).
\end{aligned} \tag{A11}$$

Now we integrate over y and q in Eqs. (A7)–(A9) and replace p with q as the integral variable, we obtain

$$\begin{aligned}
\hat{L}^{\lambda\nu} &= -x^\lambda \hat{P}^\nu + \frac{1}{2} n_\mu \int \frac{d^3q}{(2\pi)^3} \sum_{s,r} \left\{ \left[t_y \frac{q^\lambda}{E_q} \Phi_{(1)++}^{sr;\mu\nu}(q, q) + i g_i^{\lambda} \partial_{q(1)}^i \Phi_{(1)++}^{sr;\mu\nu}(q, q) \right] a_{\mathbf{q}}^{s\dagger} a_{\mathbf{q}}^r + i g_i^{\lambda} \Phi_{(1)++}^{sr;\mu\nu}(q, q) (\partial_q^i a_{\mathbf{q}}^{s\dagger}) a_{\mathbf{q}}^r \right. \\
&\quad + \left[t_y \frac{q^\lambda}{E_q} \Phi_{(1)++}^{sr;\mu\nu}(q, q) - i g_i^{\lambda} \partial_{q(1)}^i \Phi_{(1)++}^{sr;\mu\nu}(q, q) \right] a_{\mathbf{q}}^s a_{\mathbf{q}}^{r\dagger} - i g_i^{\lambda} \Phi_{(1)++}^{sr;\mu\nu}(q, q) (\partial_q^i a_{\mathbf{q}}^s) a_{\mathbf{q}}^{r\dagger} \\
&\quad + \left[t_y \frac{q^\lambda}{E_q} \Phi_{(1)++}^{sr;\mu\nu}(q, \tilde{q}) - i g_i^{\lambda} \partial_{q(1)}^i \Phi_{(1)++}^{sr;\mu\nu}(q, \tilde{q}) \right] e^{-2iE_q t_y} a_{\mathbf{q}}^s a_{-\mathbf{q}}^r - i g_i^{\lambda} \Phi_{(1)++}^{sr;\mu\nu}(q, \tilde{q}) e^{-2iE_q t_y} (\partial_q^i a_{\mathbf{q}}^s) a_{-\mathbf{q}}^r \\
&\quad \left. + \left[t_y \frac{q^\lambda}{E_q} \Phi_{(1)++}^{sr;\mu\nu}(q, \tilde{q}) + i g_i^{\lambda} \partial_{q(1)}^i \Phi_{(1)++}^{sr;\mu\nu}(q, \tilde{q}) \right] e^{2iE_q t_y} a_{\mathbf{q}}^{s\dagger} a_{-\mathbf{q}}^{r\dagger} + i g_i^{\lambda} \Phi_{(1)++}^{sr;\mu\nu}(q, \tilde{q}) e^{2iE_q t_y} (\partial_q^i a_{\mathbf{q}}^{s\dagger}) a_{-\mathbf{q}}^{r\dagger} \right\},
\end{aligned} \tag{A12}$$

$$\hat{S}_B^{\lambda\nu} = -2n_\mu \int \frac{d^3q}{(2\pi)^3} \sum_{s,r} \left[\Phi_{(2)++}^{sr;\mu\lambda\nu}(q, q) a_{\mathbf{q}}^{s\dagger} a_{\mathbf{q}}^r + \Phi_{(2)-+}^{sr;\mu\lambda\nu}(q, q) a_{\mathbf{q}}^s a_{\mathbf{q}}^{r\dagger} \right. \\ \left. - \Phi_{(2)--}^{sr;\mu\lambda\nu}(q, \tilde{q}) a_{\mathbf{q}}^s a_{-\mathbf{q}}^r e^{-2iE_{\mathbf{q}}t_y} - \Phi_{(2)++}^{sr;\mu\lambda\nu}(q, \tilde{q}) a_{\mathbf{q}}^{s\dagger} a_{-\mathbf{q}}^{r\dagger} e^{2iE_{\mathbf{q}}t_y} \right], \quad (\text{A13})$$

$$\hat{S}_C^{\lambda\nu} = -n_\mu \int \frac{d^3q}{(2\pi)^3} \sum_{s,r} \left\{ \left[\Phi_{(2)++}^{sr;\mu\lambda\nu}(q, q) - \Phi_{(2)-+}^{sr;\mu\lambda\nu}(q, q) \right] a_{\mathbf{q}}^{s\dagger} a_{\mathbf{q}}^r + \left[\Phi_{(2)-+}^{sr;\mu\lambda\nu}(q, q) - \Phi_{(2)++}^{sr;\mu\lambda\nu}(q, q) \right] a_{\mathbf{q}}^s a_{\mathbf{q}}^{r\dagger} \right. \\ \left. - \left[\Phi_{(2)--}^{sr;\mu\lambda\nu}(q, \tilde{q}) - \Phi_{(2)++}^{sr;\mu\lambda\nu}(q, \tilde{q}) \right] a_{\mathbf{q}}^s a_{-\mathbf{q}}^r e^{-2iE_{\mathbf{q}}t_y} - \left[\Phi_{(2)++}^{sr;\mu\lambda\nu}(q, \tilde{q}) - \Phi_{(2)--}^{sr;\mu\lambda\nu}(q, \tilde{q}) \right] a_{\mathbf{q}}^{s\dagger} a_{-\mathbf{q}}^{r\dagger} e^{2iE_{\mathbf{q}}t_y} \right\}, \quad (\text{A14})$$

where $\partial_{(1)}$ acts only on the first variable of $\Phi(1, 2)$ in Eq. (A12). We can also define $\partial_{(2)}$, which acts only on the second variable of $\Phi(1, 2)$. The commutators between OAM and spin tensor operators are

$$[\hat{L}^{\lambda\nu}, a_{\mathbf{p}}^{s\dagger}] = -x^\lambda p^\nu a_{\mathbf{p}}^{s\dagger} + \frac{1}{2} \left\{ \left[in_\mu g_i^\lambda \left(\partial_{p(1)}^i \Phi_{(1)++}^{rs;\mu\nu}(p, p) + \partial_{p(2)}^i \Phi_{(1)-+}^{sr;\mu\nu}(p, p) \right) + 2t_y \frac{1}{E_p} p^\lambda p^\nu \delta_{sr} + 2i\delta_{sr} p^\nu g_i^\lambda \partial_p^i \right] a_{\mathbf{p}}^{r\dagger} \right. \\ \left. + in_\mu g_i^\lambda \left[\partial_{p(1)}^i \Phi_{(1)++}^{rs;\mu\nu}(\tilde{p}, p) + \partial_{p(2)}^i \Phi_{(1)++}^{sr;\mu\nu}(p, \tilde{p}) \right] e^{-2iE_p t_y} a_{-\mathbf{p}}^r \right\}, \quad (\text{A15})$$

$$[\hat{L}^{\lambda\nu}, a_{-\mathbf{p}}^s] = x^\lambda \tilde{p}^\nu a_{-\mathbf{p}}^s + \frac{1}{2} \left\{ \left[-in_\mu g_i^\lambda \left(\partial_{p(1)}^i \Phi_{(1)-+}^{rs;\mu\nu}(\tilde{p}, \tilde{p}) + \partial_{p(2)}^i \Phi_{(1)-+}^{sr;\mu\nu}(\tilde{p}, \tilde{p}) \right) - 2t_y \frac{\tilde{p}^\lambda \tilde{p}^\nu}{E_p} \delta_{sr} - 2i\delta_{sr} \tilde{p}^\nu g_i^\lambda \partial_p^i \right] a_{-\mathbf{p}}^{r\dagger} \right. \\ \left. - in_\mu g_i^\lambda \left[\partial_{p(1)}^i \Phi_{(1)++}^{rs;\mu\nu}(p, \tilde{p}) + \partial_{p(2)}^i \Phi_{(1)++}^{sr;\mu\nu}(\tilde{p}, p) \right] e^{2iE_p t_y} a_{\mathbf{p}}^{r\dagger} \right\}, \quad (\text{A16})$$

$$[\hat{S}_B^{\lambda\nu}, a_{\mathbf{p}}^{s\dagger}] = -2n_\mu \left\{ \left[\Phi_{(2)++}^{rs;\mu\lambda\nu}(p, p) + \Phi_{(2)-+}^{sr;\mu\lambda\nu}(p, p) \right] a_{\mathbf{p}}^{r\dagger} - \left[\Phi_{(2)++}^{rs;\mu\lambda\nu}(\tilde{p}, p) + \Phi_{(2)--}^{sr;\mu\lambda\nu}(p, \tilde{p}) \right] e^{-2iE_p t_y} a_{-\mathbf{p}}^r \right\}, \quad (\text{A17})$$

$$[\hat{S}_B^{\lambda\nu}, a_{-\mathbf{p}}^s] = -2n_\mu \left\{ -\left[\Phi_{(2)++}^{sr;\mu\lambda\nu}(\tilde{p}, \tilde{p}) + \Phi_{(2)-+}^{rs;\mu\lambda\nu}(\tilde{p}, \tilde{p}) \right] a_{-\mathbf{p}}^r + \left[\Phi_{(2)++}^{rs;\mu\lambda\nu}(p, \tilde{p}) + \Phi_{(2)++}^{sr;\mu\lambda\nu}(\tilde{p}, p) \right] e^{2iE_p t_y} a_{\mathbf{p}}^{r\dagger} \right\}, \quad (\text{A18})$$

$$[\hat{S}_C^{\lambda\nu}, a_{\mathbf{p}}^{s\dagger}] = -n_\mu \left\{ \left[\Phi_{(2)++}^{rs;\mu\lambda\nu}(p, p) - \Phi_{(2)-+}^{rs;\mu\lambda\nu}(p, p) \right] a_{\mathbf{p}}^{r\dagger} + \left[\Phi_{(2)-+}^{sr;\mu\lambda\nu}(p, p) - \Phi_{(2)-+}^{sr;\mu\lambda\nu}(p, p) \right] a_{\mathbf{p}}^{r\dagger} \right. \\ \left. - \left[\Phi_{(2)++}^{rs;\mu\lambda\nu}(\tilde{p}, p) - \Phi_{(2)++}^{rs;\mu\lambda\nu}(\tilde{p}, p) \right] e^{-2iE_p t_y} a_{-\mathbf{p}}^r - \left[\Phi_{(2)++}^{sr;\mu\lambda\nu}(p, \tilde{p}) - \Phi_{(2)++}^{sr;\mu\lambda\nu}(p, \tilde{p}) \right] e^{-2iE_p t_y} a_{-\mathbf{p}}^r \right\}, \quad (\text{A19})$$

$$[\hat{S}_C^{\lambda\nu}, a_{-\mathbf{p}}^s] = -n_\mu \left\{ -\left[\Phi_{(2)++}^{sr;\mu\lambda\nu}(\tilde{p}, \tilde{p}) - \Phi_{(2)-+}^{sr;\mu\lambda\nu}(\tilde{p}, \tilde{p}) \right] a_{-\mathbf{p}}^r - \left[\Phi_{(2)-+}^{rs;\mu\lambda\nu}(\tilde{p}, \tilde{p}) - \Phi_{(2)-+}^{rs;\mu\lambda\nu}(\tilde{p}, \tilde{p}) \right] a_{-\mathbf{p}}^r \right. \\ \left. + \left[\Phi_{(2)++}^{sr;\mu\lambda\nu}(\tilde{p}, p) - \Phi_{(2)++}^{sr;\mu\lambda\nu}(\tilde{p}, p) \right] e^{2iE_p t_y} a_{\mathbf{p}}^{r\dagger} + \left[\Phi_{(2)++}^{rs;\mu\lambda\nu}(p, \tilde{p}) - \Phi_{(2)++}^{rs;\mu\lambda\nu}(p, \tilde{p}) \right] e^{2iE_p t_y} a_{\mathbf{p}}^{r\dagger} \right\}. \quad (\text{A20})$$

Using Eqs. (36), (37), the commutation coefficients of Eqs. (40), (41) become,

$$\Xi_{(B)++}^{sr}(p) = -\Xi_{(B)--}^{sr*}(p \leftrightarrow \tilde{p}) \\ = x^\lambda p^\nu (\partial_\lambda \beta_\nu) \delta_{sr} - (\partial_\lambda \beta_\nu) \left[\frac{1}{2} in_\mu g_i^\lambda \partial_p^i \left(\partial_{p(1)}^i \Phi_{(1)++}^{rs;\mu\nu}(p, p) + \partial_{p(2)}^i \Phi_{(1)-+}^{sr;\mu\nu}(p, p) \right) \right. \\ \left. + t_y \frac{1}{E_p} p^\lambda p^\nu \delta_{sr} + i\delta_{sr} p^\nu g_i^\lambda \partial_p^i - n_\mu \Phi_{(2)++}^{rs;\mu\lambda\nu}(p, p) - n_\mu \Phi_{(2)-+}^{sr;\mu\lambda\nu}(p, p) \right], \quad (\text{A21})$$

$$\Xi_{(B)+-}^{sr}(p) = -\Xi_{(B)-+}^{sr*}(p \leftrightarrow \tilde{p}) \\ = -(\partial_\lambda \beta_\nu) e^{-2iE_p t_y} n_\mu \left\{ \frac{1}{2} i g_i^\lambda \left[\partial_{p(1)}^i \Phi_{(1)++}^{rs;\mu\nu}(\tilde{p}, p) + \partial_{p(2)}^i \Phi_{(1)++}^{sr;\mu\nu}(p, \tilde{p}) \right] + \Phi_{(2)++}^{rs;\mu\lambda\nu}(\tilde{p}, p) + \Phi_{(2)++}^{sr;\mu\lambda\nu}(p, \tilde{p}) \right\}, \quad (\text{A22})$$

$$\begin{aligned}
\Xi_{(C)++}^{sr}(p) &= -\Xi_{(C)--}^{sr*}(p \leftrightarrow \tilde{p}) \\
&= x^\lambda p^\nu (\partial_\lambda \beta_\nu) \delta_{sr} - (\partial_\lambda \beta_\nu) \left\{ \frac{1}{2} i n_\mu g_i^\lambda \partial_p^i \left[\partial_{p(1)}^i \Phi_{(1)++}^{rs;\mu\nu}(p, p) + \partial_{p(2)}^i \Phi_{(1)-+}^{sr;\mu\nu}(p, p) \right] + t_y \frac{1}{E_p} p^\lambda p^\nu \delta_{sr} + i \delta_{sr} p^\nu g_i^\lambda \partial_p^i \right\} \\
&\quad - \frac{1}{2} \Omega_{\lambda\nu} n_\mu \left[\Phi_{(2)++}^{rs;\mu\lambda\nu}(p, p) - \Phi_{(2)++}^{rs;\mu\nu\lambda}(p, p) + \Phi_{(2)-+}^{sr;\mu\lambda\nu}(p, p) - \Phi_{(2)-+}^{sr;\mu\nu\lambda}(p, p) \right], \tag{A23}
\end{aligned}$$

$$\begin{aligned}
\Xi_{(C)+-}^{sr}(p) &= -\Xi_{(C)-+}^{sr*}(p \leftrightarrow \tilde{p}) \\
&= -i \frac{1}{2} (\partial_\lambda \beta_\nu) e^{-2iE_p t_y} n_\mu g_i^\lambda \left[\partial_{p(1)}^i \Phi_{(1)--}^{rs;\mu\nu}(\tilde{p}, p) + \partial_{p(2)}^i \Phi_{(1)--}^{sr;\mu\nu}(p, \tilde{p}) \right] + \frac{1}{2} \Omega_{\lambda\nu} e^{-2iE_p t_y} n_\mu \\
&\quad \times \left[\Phi_{(2)--}^{rs;\mu\lambda\nu}(\tilde{p}, p) - \Phi_{(2)--}^{rs;\mu\nu\lambda}(\tilde{p}, p) + \Phi_{(2)--}^{sr;\mu\lambda\nu}(p, \tilde{p}) - \Phi_{(2)--}^{sr;\mu\nu\lambda}(p, \tilde{p}) \right]. \tag{A24}
\end{aligned}$$

We use Eqs. (38)–(41) and (42) to evaluate Eqs. (43) and (44), the coefficients are obtained as

$$\begin{aligned}
D_{++}^{sr} &= -e^{\beta \cdot p} \int_0^1 d\lambda e^{-\lambda \beta \cdot p} \Xi_{++}^{sr} e^{\lambda \beta \cdot p} + \sum_{s'} e^{\beta \cdot p} \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 e^{-\lambda_1 \beta \cdot p} \Xi_{++}^{ss'} e^{\lambda_1 \beta \cdot p} e^{-\lambda_2 \beta \cdot p} \Xi_{++}^{s'r} e^{\lambda_2 \beta \cdot p} \\
&\quad + \sum_{s'} e^{\beta \cdot p} \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 e^{-\lambda_1 \beta \cdot p} \Xi_{++}^{ss'} e^{-\lambda_1 \beta \cdot \tilde{p}} e^{\lambda_2 \beta \cdot \tilde{p}} \Xi_{++}^{s'r} e^{\lambda_2 \beta \cdot p}, \tag{A25}
\end{aligned}$$

$$\begin{aligned}
D_{+-}^{sr} &= -e^{\beta \cdot p} \int_0^1 d\lambda e^{-\lambda \beta \cdot p} \Xi_{+-}^{sr} e^{-\lambda \beta \cdot \tilde{p}} + \sum_{s'} e^{\beta \cdot p} \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 e^{-\lambda_1 \beta \cdot p} \Xi_{+-}^{ss'} e^{\lambda_1 \beta \cdot p} e^{-\lambda_2 \beta \cdot p} \Xi_{+-}^{s'r} e^{-\lambda_2 \beta \cdot \tilde{p}} \\
&\quad + \sum_{s'} e^{\beta \cdot p} \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 e^{-\lambda_1 \beta \cdot p} \Xi_{+-}^{ss'} e^{-\lambda_1 \beta \cdot \tilde{p}} e^{\lambda_2 \beta \cdot \tilde{p}} \Xi_{+-}^{s'r} e^{-\lambda_2 \beta \cdot \tilde{p}}, \tag{A26}
\end{aligned}$$

$$\begin{aligned}
D_{--}^{sr} &= -e^{-\beta \cdot \tilde{p}} \int_0^1 d\lambda e^{\lambda \beta \cdot \tilde{p}} \Xi_{--}^{sr} e^{-\lambda \beta \cdot \tilde{p}} + \sum_{s'} e^{-\beta \cdot \tilde{p}} \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 e^{\lambda_1 \beta \cdot \tilde{p}} \Xi_{--}^{ss'} e^{-\lambda_1 \beta \cdot \tilde{p}} e^{\lambda_2 \beta \cdot \tilde{p}} \Xi_{--}^{s'r} e^{-\lambda_2 \beta \cdot \tilde{p}} \\
&\quad + \sum_{s'} e^{-\beta \cdot \tilde{p}} \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 e^{\lambda_1 \beta \cdot \tilde{p}} \Xi_{--}^{ss'} e^{\lambda_1 \beta \cdot p} e^{-\lambda_2 \beta \cdot p} \Xi_{--}^{s'r} e^{-\lambda_2 \beta \cdot \tilde{p}}, \tag{A27}
\end{aligned}$$

$$\begin{aligned}
D_{-+}^{sr} &= -e^{-\beta \cdot \tilde{p}} \int_0^1 d\lambda e^{\lambda \beta \cdot \tilde{p}} \Xi_{-+}^{sr} e^{\lambda \beta \cdot p} + \sum_{s'} e^{-\beta \cdot \tilde{p}} \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 e^{\lambda_1 \beta \cdot \tilde{p}} \Xi_{-+}^{ss'} e^{-\lambda_1 \beta \cdot \tilde{p}} e^{\lambda_2 \beta \cdot \tilde{p}} \Xi_{-+}^{s'r} e^{\lambda_2 \beta \cdot p} \\
&\quad + \sum_{s'} e^{-\beta \cdot \tilde{p}} \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 e^{\lambda_1 \beta \cdot \tilde{p}} \Xi_{-+}^{ss'} e^{\lambda_1 \beta \cdot p} e^{-\lambda_2 \beta \cdot p} \Xi_{-+}^{s'r} e^{\lambda_2 \beta \cdot p}. \tag{A28}
\end{aligned}$$

APPENDIX B: COEFFICIENTS FOR EQS. (62), (61)

Here, we list the coefficients in Eqs. (61) and (62),

$$\begin{aligned}
(\Xi_{(B)\beta\beta}^{ss})_{\lambda\nu,\rho\sigma} &= -\tilde{\Delta}_{\lambda\nu} \tilde{\Delta}_{\rho\sigma} - \tilde{\Delta}_{\lambda\nu} (\epsilon_\rho^{s*} \epsilon_\sigma^s + \epsilon_\sigma^{s*} \epsilon_\rho^s) - \tilde{\Delta}_{\rho\sigma} (\epsilon_\lambda^{s*} \epsilon_\nu^s + \epsilon_\nu^{s*} \epsilon_\lambda^s) + (\epsilon_\rho^{s*} \epsilon_\nu^s \tilde{\Delta}_{\lambda\sigma} + \epsilon_\sigma^{s*} \epsilon_\nu^s \tilde{\Delta}_{\lambda\rho} + \epsilon_\rho^{s*} \epsilon_\lambda^s \tilde{\Delta}_{\nu\sigma} + \epsilon_\sigma^{s*} \epsilon_\lambda^s \tilde{\Delta}_{\nu\rho}) \\
&\quad - \frac{m}{E_{\mathbf{k}}(E_{\mathbf{k}} + m)} \tilde{\Delta}_{\lambda\nu} \left[\frac{k \cdot \epsilon_s^*}{m} (\epsilon_\rho^s \bar{k}_\sigma + \epsilon_\sigma^s \bar{k}_\rho) + \frac{k \cdot \epsilon_s}{m} (\epsilon_\rho^{s*} \bar{k}_\sigma + \epsilon_\sigma^{s*} \bar{k}_\rho) \right] \\
&\quad - \frac{m}{E_{\mathbf{k}}(E_{\mathbf{k}} + m)} \tilde{\Delta}_{\rho\sigma} \left[\frac{k \cdot \epsilon_s^*}{m} (\epsilon_\lambda^s \bar{k}_\nu + \epsilon_\nu^s \bar{k}_\lambda) + \frac{k \cdot \epsilon_s}{m} (\epsilon_\lambda^{s*} \bar{k}_\nu + \epsilon_\nu^{s*} \bar{k}_\lambda) \right] \\
&\quad + \frac{m}{E_{\mathbf{k}}(E_{\mathbf{k}} + m)} \frac{k \cdot \epsilon_s^*}{m} (\tilde{\Delta}_{\lambda\rho} \bar{k}_\sigma \epsilon_\nu^s + \tilde{\Delta}_{\nu\rho} \bar{k}_\sigma \epsilon_\lambda^s + \tilde{\Delta}_{\lambda\sigma} \bar{k}_\rho \epsilon_\nu^s + \tilde{\Delta}_{\nu\sigma} \bar{k}_\rho \epsilon_\lambda^s) \\
&\quad + \frac{m}{E_{\mathbf{k}}(E_{\mathbf{k}} + m)} \frac{k \cdot \epsilon_s}{m} (\tilde{\Delta}_{\rho\lambda} \bar{k}_\nu \epsilon_\sigma^{s*} + \tilde{\Delta}_{\lambda\sigma} \bar{k}_\nu \epsilon_\rho^{s*} + \tilde{\Delta}_{\nu\sigma} \bar{k}_\lambda \epsilon_\rho^{s*} + \tilde{\Delta}_{\rho\lambda} \bar{k}_\lambda \epsilon_\sigma^{s*}) \\
&\quad + \frac{m^2}{E_{\mathbf{k}}^2(E_{\mathbf{k}} + m)^2} \frac{k \cdot \epsilon_s^*}{m} \frac{k \cdot \epsilon_s}{m} (\tilde{\Delta}_{\lambda\rho} \bar{k}_\nu \bar{k}_\sigma + \tilde{\Delta}_{\lambda\sigma} \bar{k}_\nu \bar{k}_\rho + \tilde{\Delta}_{\nu\rho} \bar{k}_\lambda \bar{k}_\sigma + \tilde{\Delta}_{\nu\sigma} \bar{k}_\lambda \bar{k}_\rho - 2\tilde{\Delta}_{\lambda\nu} \bar{k}_\rho \bar{k}_\sigma - 2\tilde{\Delta}_{\rho\sigma} \bar{k}_\lambda \bar{k}_\nu), \tag{B1}
\end{aligned}$$

$$\begin{aligned}
(\Xi_{(C)\beta\beta}^{ss})_{\lambda\nu,\rho\sigma} = & -\tilde{\Delta}_{\lambda\nu}\tilde{\Delta}_{\rho\sigma}\delta_{ss} + \frac{\tilde{k}_\nu\tilde{k}_\sigma\bar{k}^2}{E_{\mathbf{k}}^2 m^2} \epsilon_\lambda^{s*} \epsilon_\rho^s - \frac{k_\sigma k \cdot \epsilon_s^*}{E_{\mathbf{k}} m} \epsilon_\rho^s \tilde{\Delta}_{\lambda\nu} + \frac{\tilde{k}_\sigma k \cdot \epsilon_s}{E_{\mathbf{k}} m} \epsilon_\rho^{s*} \left[\left(\Delta_{\lambda\nu} - \frac{\bar{k}_\lambda \bar{k}_\nu}{E_{\mathbf{k}}^2} \right) - \frac{\bar{k}_\lambda \tilde{k}_\nu}{m E_{\mathbf{k}}} \right] \\
& - \frac{k_\nu k \cdot \epsilon_s}{E_{\mathbf{k}} m} \epsilon_\lambda^{s*} \tilde{\Delta}_{\rho\sigma} + \frac{\tilde{k}_\nu k \cdot \epsilon_s^*}{E_{\mathbf{k}} m} \epsilon_\lambda^s \left[\left(\Delta_{\rho\sigma} - \frac{\bar{k}_\rho \bar{k}_\sigma}{E_{\mathbf{k}}^2} \right) - \frac{\bar{k}_\rho \tilde{k}_\sigma}{m E_{\mathbf{k}}} \right] - (m\Delta_{\lambda\nu} - u_\nu \bar{k}_\lambda) \frac{2\bar{k}_\rho \tilde{k}_\sigma}{E_{\mathbf{k}}^2 (E_{\mathbf{k}} + m)} \frac{k \cdot \epsilon_s^* k \cdot \epsilon_s}{m m} \\
& - (m\Delta_{\rho\sigma} - u_\sigma \bar{k}_\rho) \frac{2\bar{k}_\lambda \tilde{k}_\nu}{E_{\mathbf{k}}^2 (E_{\mathbf{k}} + m)} \frac{k \cdot \epsilon_s^* k \cdot \epsilon_s}{m m} + \frac{k_\nu k_\sigma}{E_{\mathbf{k}}^2} \Delta_{\lambda\rho} \frac{k \cdot \epsilon_s^* k \cdot \epsilon_s}{m m}, \tag{B2}
\end{aligned}$$

$$\begin{aligned}
(\Xi_{(C)\beta\Omega}^{ss})_{\lambda\nu,\rho\sigma} = & \frac{\tilde{k}_\nu\tilde{k}_\sigma\bar{k}^2}{E_{\mathbf{k}}^2 m^2} \epsilon_\rho^{s*} \epsilon_\lambda^s - \frac{\tilde{k}_\nu k_\sigma \bar{k}_\rho k \cdot \epsilon_s^*}{E_{\mathbf{k}}^2 m m} \epsilon_\lambda^s - \tilde{\Delta}_{\lambda\nu} \frac{k_\sigma k \cdot \epsilon_s^*}{E_{\mathbf{k}} m} \epsilon_\rho^s + \frac{\tilde{k}_\sigma k \cdot \epsilon_s}{E_{\mathbf{k}} m} \epsilon_\rho^{s*} \left(\Delta_{\lambda\nu} - \frac{\bar{k}_\lambda \bar{k}_\nu}{E_{\mathbf{k}}^2} \right) - \frac{\bar{k}_\lambda \tilde{k}_\nu \tilde{k}_\sigma k \cdot \epsilon_s}{m E_{\mathbf{k}}^2 m} \epsilon_\rho^{s*} \\
& + \Delta_{\lambda\rho} \frac{k_\nu k_\sigma k \cdot \epsilon_s^* k \cdot \epsilon_s}{E_{\mathbf{k}}^2 m m} + \bar{k}_\rho k_\sigma \bar{k}_\lambda \bar{k}_\nu \frac{2}{E_{\mathbf{k}}^3 (E_{\mathbf{k}} + m)} \frac{k \cdot \epsilon_s^* k \cdot \epsilon_s}{m m}, \tag{B3}
\end{aligned}$$

$$(\Xi_{(C)\Omega\Omega}^{ss})_{\lambda\nu,\rho\sigma} = \frac{\tilde{k}_\nu\tilde{k}_\sigma\bar{k}^2}{E_{\mathbf{k}}^2 m^2} \epsilon_\rho^{s*} \epsilon_\lambda^s - \frac{\tilde{k}_\nu k_\sigma \bar{k}_\rho k \cdot \epsilon_s^*}{E_{\mathbf{k}}^2 m m} \epsilon_\lambda^s - \frac{k_\nu \tilde{k}_\sigma \bar{k}_\lambda k \cdot \epsilon_s}{E_{\mathbf{k}}^2 m m} \epsilon_\rho^{s*} + \frac{k_\nu k_\sigma k \cdot \epsilon_s^* k \cdot \epsilon_s}{E_{\mathbf{k}}^2 m m} \Delta_{\lambda\rho}, \tag{B4}$$

where we defined

$$\tilde{\Delta}_{\mu\nu} = \Delta_{\mu\nu} + \frac{\bar{k}_\mu \bar{k}_\nu}{E_{\mathbf{k}}^2}. \tag{B5}$$

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- [1] Z.-T. Liang and X.-N. Wang, *Phys. Rev. Lett.* **94**, 102301 (2005); **96**, 039901(E) (2006).
[2] S. A. Voloshin, *arXiv:nucl-th/0410089*.
[3] Z.-T. Liang and X.-N. Wang, *Phys. Lett. B* **629**, 20 (2005).
[4] J.-H. Gao, S.-W. Chen, W.-T. Deng, Z.-T. Liang, Q. Wang, and X.-N. Wang, *Phys. Rev. C* **77**, 044902 (2008).
[5] F. Becattini, F. Piccinini, and J. Rizzo, *Phys. Rev. C* **77**, 024906 (2008).
[6] B. I. Abelev *et al.* (STAR Collaboration), *Phys. Rev. C* **77**, 061902 (2008).
[7] L. Adamczyk *et al.* (STAR Collaboration), *Nature (London)* **548**, 62 (2017).
[8] J. Adam *et al.* (STAR Collaboration), *Phys. Rev. Lett.* **123**, 132301 (2019).
[9] J. Adam *et al.* (STAR Collaboration), *Phys. Rev. Lett.* **126**, 162301 (2021).
[10] M. S. Abdallah *et al.* (STAR Collaboration), *Nature (London)* **614**, 244 (2023).
[11] S. Acharya *et al.* (ALICE Collaboration), *Phys. Rev. Lett.* **131**, 042303 (2023).
[12] S. Acharya *et al.* (ALICE Collaboration), *Phys. Rev. Lett.* **125**, 012301 (2020).
[13] R. Abou Yassine *et al.* (HADES Collaboration), *Phys. Lett. B* **835**, 137506 (2022).
[14] I. Karpenko and F. Becattini, *Eur. Phys. J. C* **77**, 213 (2017).
[15] Y. Xie, D. Wang, and L. P. Csernai, *Phys. Rev. C* **95**, 031901 (2017).
[16] H. Li, L.-G. Pang, Q. Wang, and X.-L. Xia, *Phys. Rev. C* **96**, 054908 (2017).
[17] Y. Sun and C. M. Ko, *Phys. Rev. C* **96**, 024906 (2017).
[18] S. Shi, K. Li, and J. Liao, *Phys. Lett. B* **788**, 409 (2019).
[19] D.-X. Wei, W.-T. Deng, and X.-G. Huang, *Phys. Rev. C* **99**, 014905 (2019).
[20] X.-L. Xia, H. Li, Z.-B. Tang, and Q. Wang, *Phys. Rev. C* **98**, 024905 (2018).
[21] O. Vitiuk, L. V. Bravina, and E. E. Zabrodin, *Phys. Lett. B* **803**, 135298 (2020).
[22] B. Fu, K. Xu, X.-G. Huang, and H. Song, *Phys. Rev. C* **103**, 024903 (2021).
[23] S. Ryu, V. Jovic, and C. Shen, *Phys. Rev. C* **104**, 054908 (2021).
[24] A. Lei, D. Wang, D.-M. Zhou, B.-H. Sa, and L. P. Csernai, *Phys. Rev. C* **104**, 054903 (2021).
[25] X.-Y. Wu, C. Yi, G.-Y. Qin, and S. Pu, *Phys. Rev. C* **105**, 064909 (2022).
[26] T. Niida (STAR Collaboration), *Nucl. Phys. A* **982**, 511 (2019).
[27] F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, *Ann. Phys. (Amsterdam)* **338**, 32 (2013).
[28] R.-h. Fang, L.-g. Pang, Q. Wang, and X.-n. Wang, *Phys. Rev. C* **94**, 024904 (2016).
[29] H.-Z. Wu, L.-G. Pang, X.-G. Huang, and Q. Wang, *Phys. Rev. Res.* **1**, 033058 (2019).
[30] S. Y. F. Liu and Y. Yin, *J. High Energy Phys.* **07** (2021) 188.
[31] Y.-C. Liu and X.-G. Huang, *Sci. China Phys. Mech. Astron.* **65**, 272011 (2022).
[32] B. Fu, S. Y. F. Liu, L. Pang, H. Song, and Y. Yin, *Phys. Rev. Lett.* **127**, 142301 (2021).

- [33] F. Becattini, M. Buzzegoli, and A. Palermo, *Phys. Lett. B* **820**, 136519 (2021).
- [34] F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko, and A. Palermo, *Phys. Rev. Lett.* **127**, 272302 (2021).
- [35] S. Y. F. Liu and Y. Yin, *Phys. Rev. D* **104**, 054043 (2021).
- [36] B. Fu, L. Pang, H. Song, and Y. Yin, arXiv:2201.12970.
- [37] J.-A. Sun and L. Yan, *Chin. Phys. C* **49**, 071001 (2025).
- [38] S. Fang, S. Pu, and D.-L. Yang, *Phys. Rev. D* **109**, 034034 (2024).
- [39] A. Palermo, E. Grossi, I. Karpenko, and F. Becattini, *Eur. Phys. J. C* **84**, 920 (2024).
- [40] C. Yi, S. Fang, D.-L. Wang, and S. Pu, Local spin polarization of Λ hyperons and its interaction corrections, in *31st International Conference on Ultra-relativistic Nucleus-Nucleus Collisions* (2025), arXiv:2509.00377.
- [41] A. Arslan, W.-B. Dong, G.-L. Ma, S. Pu, and Q. Wang, *Phys. Rev. C* **111**, 044911 (2025).
- [42] C. Yi, X.-Y. Wu, J. Zhu, S. Pu, and G.-Y. Qin, *Phys. Rev. C* **111**, 044901 (2025).
- [43] C. Yi, X.-Y. Wu, J. Zhu, S. Pu, and G.-Y. Qin, Hydrodynamic effects on spin polarization along the beam direction in Au + Au and p + Pb collisions, in *31st International Conference on Ultra-relativistic Nucleus-Nucleus Collisions* (2025), arXiv:2509.00380.
- [44] S. Acharya *et al.* (ALICE Collaboration), *Phys. Rev. Lett.* **128**, 172005 (2022).
- [45] M. Abdulhamid *et al.* (STAR Collaboration), *Phys. Rev. Lett.* **131**, 202301 (2023).
- [46] Q. Wang, *Nucl. Phys. A* **967**, 225 (2017).
- [47] W. Florkowski, A. Kumar, and R. Ryblewski, *Prog. Part. Nucl. Phys.* **108**, 103709 (2019).
- [48] Z.-T. Liang, M. A. Lisa, and X.-N. Wang, *Nucl. Phys. News* **30**, 10 (2020).
- [49] F. Becattini and M. A. Lisa, *Annu. Rev. Nucl. Part. Sci.* **70**, 395 (2020).
- [50] X.-G. Huang, J. Liao, Q. Wang, and X.-L. Xia, *Lect. Notes Phys.* **987**, 281 (2021).
- [51] J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, *Lect. Notes Phys.* **987**, 195 (2021).
- [52] Y. Hidaka, S. Pu, Q. Wang, and D.-L. Yang, *Prog. Part. Nucl. Phys.* **127**, 103989 (2022).
- [53] F. Becattini, M. Buzzegoli, T. Niida, S. Pu, A.-H. Tang, and Q. Wang, *Int. J. Mod. Phys. E* **33**, 2430006 (2024).
- [54] T. Niida and S. A. Voloshin, *Int. J. Mod. Phys. E* **33**, 2430010 (2024).
- [55] P. Faccioli, C. Lourenco, J. Seixas, and H. K. Wohri, *Eur. Phys. J. C* **69**, 657 (2010).
- [56] X.-L. Sheng, L. Oliva, and Q. Wang, *Phys. Rev. D* **101**, 096005 (2020).
- [57] X.-L. Sheng, Q. Wang, and X.-N. Wang, *Phys. Rev. D* **102**, 056013 (2020).
- [58] X.-L. Sheng, L. Oliva, Z.-T. Liang, Q. Wang, and X.-N. Wang, *Phys. Rev. Lett.* **131**, 042304 (2023).
- [59] X.-L. Sheng, S. Pu, and Q. Wang, *Phys. Rev. C* **108**, 054902 (2023).
- [60] A. Kumar, B. Müller, and D.-L. Yang, *Phys. Rev. D* **107**, 076025 (2023).
- [61] A. Kumar, B. Müller, and D.-L. Yang, *Phys. Rev. D* **108**, 016020 (2023).
- [62] A. Kumar, D.-L. Yang, and P. Gubler, *Phys. Rev. D* **109**, 054038 (2024).
- [63] B. Fu, F. Gao, Y. Liu, and H. Song, *Phys. Lett. B* **855**, 138821 (2024).
- [64] Y.-L. Yin, W.-B. Dong, J.-Y. Pang, S. Pu, and Q. Wang, *Phys. Rev. C* **110**, 024905 (2024).
- [65] Y.-Q. Zhao, X.-L. Sheng, S.-W. Li, and D. Hou, *J. High Energy Phys.* **08** (2024) 070.
- [66] X.-L. Sheng, Y.-Q. Zhao, S.-W. Li, F. Becattini, and D. Hou, *Phys. Rev. D* **110**, 056047 (2024).
- [67] F. Li and S. Y. F. Liu, arXiv:2206.11890.
- [68] D. Wagner, N. Weickgenannt, and E. Speranza, *Phys. Rev. D* **108**, 116017 (2023).
- [69] W.-B. Dong, Y.-L. Yin, X.-L. Sheng, S.-Z. Yang, and Q. Wang, *Phys. Rev. D* **109**, 056025 (2024).
- [70] D. N. Zubarev, A. V. Prozorkevich, and S. A. Smolyanskii, *Theor. Math. Phys.* **40**, 821 (1979).
- [71] D. Zubarev, *J. Sov. Math.* **16**, 1509 (1981).
- [72] F. Becattini, M. Buzzegoli, and E. Grossi, *Particles* **2**, 197 (2019).
- [73] F. Becattini, G. Inghirami, V. Rolando, A. Beraudo, L. Del Zanna, A. De Pace, M. Nardi, G. Pagliara, and V. Chandra, *Eur. Phys. J. C* **75**, 406 (2015); **78**, 354(E) (2018).
- [74] F. Becattini, I. Karpenko, M. Lisa, I. Upsal, and S. Voloshin, *Phys. Rev. C* **95**, 054902 (2017).
- [75] A. Palermo, M. Buzzegoli, and F. Becattini, *J. High Energy Phys.* **10** (2021) 077.
- [76] A. Palermo and F. Becattini, *Eur. Phys. J. Plus* **138**, 547 (2023).
- [77] M. Buzzegoli, *Phys. Rev. C* **105**, 044907 (2022).
- [78] X.-L. Sheng, L. Oliva, Z.-T. Liang, Q. Wang, and X.-N. Wang, *Phys. Rev. D* **109**, 036004 (2024).
- [79] S. R. De Groot, W. A. van Leeuwen, and Ch. G. van Weert, *Relativistic Kinetic Theory: Principles and Applications* (North-Holland Publishing Company, Amsterdam, New York, and Oxford, 1980).
- [80] F. Becattini, J. Liao, and M. Lisa, *Lect. Notes Phys.* **987**, 1 (2021).
- [81] F. Becattini, *Lect. Notes Phys.* **987**, 15 (2021).
- [82] X.-L. Sheng, N. Weickgenannt, E. Speranza, D. H. Rischke, and Q. Wang, *Phys. Rev. D* **104**, 016029 (2021).
- [83] F. Becattini, W. Florkowski, and E. Speranza, *Phys. Lett. B* **789**, 419 (2019).
- [84] C. G. Van Weert, *Ann. Phys. (N.Y.)* **140**, 133 (1982).
- [85] K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, *Phys. Lett. B* **795**, 100 (2019).
- [86] K. Fukushima and S. Pu, *Phys. Lett. B* **817**, 136346 (2021).
- [87] D.-L. Wang, S. Fang, and S. Pu, *Phys. Rev. D* **104**, 114043 (2021).
- [88] D.-L. Wang, X.-Q. Xie, S. Fang, and S. Pu, *Phys. Rev. D* **105**, 114050 (2022).
- [89] X.-Q. Xie, D.-L. Wang, C. Yang, and S. Pu, *Phys. Rev. D* **108**, 094031 (2023).
- [90] F. Becattini, L. Csernai, and D. J. Wang, *Phys. Rev. C* **88**, 034905 (2013); **93**, 069901(E) (2016).
- [91] M. Buzzegoli, E. Grossi, and F. Becattini, *J. High Energy Phys.* **10** (2017) 091; **07** (2018) 119(E).
- [92] J.-H. Gao and S.-Z. Yang, *Chin. Phys. C* **48**, 053114 (2024).
- [93] C. Yi, S. Pu, and D.-L. Yang, *Phys. Rev. C* **104**, 064901 (2021).

-
- [94] D. E. Kharzeev and H.-U. Yee, [Phys. Rev. D **84**, 045025 \(2011\)](#).
- [95] J.-W. Chen, T. Ishii, S. Pu, and N. Yamamoto, [Phys. Rev. D **93**, 125023 \(2016\)](#).
- [96] D. T. Son and P. Surowka, [Phys. Rev. Lett. **103**, 191601 \(2009\)](#).
- [97] S. Pu, J.-h. Gao, and Q. Wang, [Phys. Rev. D **83**, 094017 \(2011\)](#).
- [98] J.-R. Wang, S. Fang, D.-L. Yang, and S. Pu, [arXiv:2507.15238](#).
- [99] Z.-H. Zhang, X.-G. Huang, F. Becattini, and X.-L. Sheng, [J. High Energy Phys. **07** \(2025\) 224](#).