

# Natural realization of tens-of-GeV dark matter in the general NMSSM

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This study presents a comparative analysis of a light dark matter (DM) scenario in the minimal supersymmetric Standard Model (MSSM), the  $Z_3$ -symmetric next-to-minimal supersymmetric Standard Model, and the general next-to-minimal supersymmetric Standard Model (GNMSSM), incorporating constraints from DM relic density, the LUX-ZEPLIN 2024 experiment, Higgs data, and the Large Hadron Collider (LHC). The results suggest that, among the three frameworks, only GNMSSM can naturally accommodate light DM with a mass below 100 GeV. As such, the viable supersymmetry candidate is primarily Singlino-like. One key advantage of the GNMSSM is the effective decoupling between interactions that establish the relic density and those that control direct detection, allowing the model to satisfy all current experimental bounds simultaneously. We further explore two characteristic mass hierarchies in the GNMSSM parameter space, each exhibiting distinct phenomenological behaviors. The first hierarchy,  $\tilde{S} < \tilde{B} < \tilde{H}$  (Singlino-Bino-Higgsino), involves a relatively light Bino and allows the Higgsino mass parameter,  $\mu_{\text{tot}}$ , to be as low as about 200 GeV, naturally yielding light DM at tens of GeV. The dominant annihilation channels are then  $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow A_s A_s$  in the  $h_1$  scenario and  $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow h_s A_s$  in the  $h_2$  scenario, where  $h_s$  and  $A_s$  denote singlet-dominated  $CP$ -even and  $CP$ -odd Higgs bosons, respectively. The second hierarchy,  $\tilde{S} < \tilde{H} < \tilde{B}$ , corresponds to a heavy Bino. In this case, although the DM phenomenology remains qualitatively similar, LHC constraints require  $\mu_{\text{tot}} \gtrsim 900$  GeV, implying a significant degree of fine-tuning in reproducing the  $Z$ -boson mass.

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## I. INTRODUCTION

Among numerous dark matter (DM) candidates, weakly interacting massive particles (WIMPs) have long been favored by theorists due to their natural ability to produce the observed cosmic abundance through the thermal freeze-out mechanism [1]. According to this mechanism, WIMPs should interact with ordinary matter at a well-defined strength, yielding characteristic cross sections:  $\sim 10^{-45}$  cm<sup>2</sup> for spin-independent (SI) nucleon scattering [2] and  $\sim 10^{-40}$  cm<sup>2</sup> for spin-dependent (SD) interactions [3]. However, direct detection experiments, particularly LUX-ZEPLIN 2024 (LZ 2024) [4], have pushed these upper limits down to  $10^{-48}$  cm<sup>2</sup> and  $10^{-43}$  cm<sup>2</sup>, respectively, posing substantial challenges for conventional WIMP models. To naturally resolve this tension, recent research has increasingly focused on extensions of economical WIMP theories by

constructing DM structures that are nearly decoupled from the Standard Model (SM), leading to so-called “secluded dark matter” models [5].

Cosmological measurements of effective light neutrino numbers and theoretical unitarity constraints place WIMPs within a broad mass range of 3.5 MeV [6] to 100 TeV [7,8]. Within this window, the tens-of-GeV regime deserves special scrutiny for four reasons:

- (1) Many DM theories link DM to electroweak symmetry breaking, providing theoretical motivation for DM at the electroweak scale [9,10].
- (2) Direct detection experiments reach their highest sensitivity in this mass range [4]. Thus, understanding how a WIMP can evade current bounds is invaluable for model building.
- (3) Achieving the correct relic density often requires additional new particles with masses close to that of the WIMP. As such, establishing whether such states can elude Large Hadron Collider (LHC) searches is a key model-building constraint [9–11].
- (4) Most critically, after accounting for current experimental limits, many theories face severe fine-tuning problems when predicting DM in this mass range (see the discussion below). Should future experiments confirm DM in this region, a number of

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theoretical frameworks would face fundamental challenges. Therefore, systematically exploring models that naturally predict DM in this range, and distinguishing them from other theoretical frameworks, constitutes an important research direction at the intersection of particle physics and cosmology.

As one of the most compelling physical models beyond the SM, supersymmetry (SUSY) naturally accommodates WIMP DM [1]. Consequently, studying tens-of-GeV DMs within supersymmetric frameworks has attracted considerable attention [12–15]. This paper systematically compares the minimal supersymmetric Standard Model (MSSM) [16,17] and the next-to-minimal supersymmetric Standard Model with  $Z_3$  symmetry ( $Z_3$ -NMSSM) [18,19], highlighting their characteristics and differences for predicting DM of several tens of GeV. We further demonstrate how the general NMSSM (GNMSSM) can predict a much richer spectrum of DM phenomena and discuss implications for future experiments.

The remainder of this paper is organized as follows. Section II highlights the similarities and differences between the MSSM, the  $Z_3$ -NMSSM, and the GNMSSM, in the context of attempts to realize a DM mass of  $\mathcal{O}(10 \text{ GeV})$ . In this process, the first two models are demonstrated to suffer from severe fine-tuning issues, while the GNMSSM avoids this problem. Section III then focuses on the GNMSSM, examining in detail how it produces a viable tens-of-GeV DM compatible with all current constraints. Conclusions are then summarized in Sec. IV.

## II. SUSY PREDICTIONS OF LIGHT DM

### A. MSSM

To maintain generality, we adopt the *phenomenological MSSM* (pMSSM) [20] as our theoretical framework. Unlike constrained scenarios such as mSUGRA [21] or CMSSM [22], which impose specific universality conditions at a high-energy scale, the pMSSM treats the soft-breaking parameters as free inputs defined directly at the electroweak scale  $Q_{\text{input}} \sim 1 \text{ TeV}$ . This approach avoids overly restrictive universality assumptions while preserving the full predictive power of the low-energy spectrum.

#### 1. Lagrangian and conventions

Following the conventions of Refs. [16,17], the MSSM superpotential is

$$W_{\text{MSSM}} = y_u \hat{Q} \cdot \hat{H}_u \hat{u}^c - y_d \hat{Q} \cdot \hat{H}_d \hat{d}^c - y_e \hat{L} \cdot \hat{H}_d \hat{e}^c + \mu \hat{H}_u \cdot \hat{H}_d, \quad (2.1)$$

where  $\hat{Q}$  and  $\hat{L}$  are the  $SU(2)_L$  quark and lepton doublet superfields;  $\hat{u}^c$ ,  $\hat{d}^c$ ,  $\hat{e}^c$  are the corresponding singlet

superfields;  $\hat{H}_u$  and  $\hat{H}_d$  are the two Higgs doublet superfields with hypercharges  $Y = +\frac{1}{2}$  and  $Y = -\frac{1}{2}$ , respectively;  $y_u$ ,  $y_d$ ,  $y_e$  are  $3 \times 3$  Yukawa coupling matrices in flavor space; and  $\mu$  is the holomorphic Higgsino mass parameter. The  $SU(2)_L$ -invariant product is defined as  $\hat{H}_u \cdot \hat{H}_d = \epsilon_{ab} \hat{H}_u^a \hat{H}_d^b$  with  $\epsilon_{12} = +1$ .

The complete soft supersymmetry-breaking Lagrangian takes the form

$$\begin{aligned} -\mathcal{L}_{\text{soft}}^{\text{MSSM}} = & m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2 + B_\mu (\epsilon_{ab} H_u^a H_d^b + \text{H.c.}) \\ & + m_{\tilde{Q}}^2 |\tilde{Q}|^2 + m_{\tilde{u}}^2 |\tilde{u}^c|^2 + m_{\tilde{d}}^2 |\tilde{d}^c|^2 \\ & + m_{\tilde{L}}^2 |\tilde{L}|^2 + m_{\tilde{e}}^2 |\tilde{e}^c|^2 \\ & + (A_u \tilde{Q} \cdot H_u \tilde{u}^c - A_d \tilde{Q} \cdot H_d \tilde{d}^c - A_e \tilde{L} \cdot H_d \tilde{e}^c + \text{H.c.}) \\ & + \frac{1}{2} (M_1 \tilde{B} \tilde{B} + M_2 \tilde{W}^a \tilde{W}^a + M_3 \tilde{g}^a \tilde{g}^a + \text{H.c.}), \end{aligned} \quad (2.2)$$

where  $m_{H_u}^2$  and  $m_{H_d}^2$  are the Higgs soft masses;  $B_\mu$  is the bilinear soft parameter;  $m_{\tilde{Q}}^2, \dots, m_{\tilde{e}}^2$  are the  $3 \times 3$  sfermion mass-squared matrices;  $A_u, A_d, A_e$  are the  $3 \times 3$  trilinear soft matrices; and  $M_1, M_2, M_3$  are the  $U(1)_Y, SU(2)_L, SU(3)_c$  gaugino masses, respectively. Throughout this paper, the gaugino masses are taken to be real, and all the soft-breaking parameters are assumed to be flavor diagonal. In particular, we denote the  $(3, 3)$  elements in  $A_u$  and  $A_d$ —corresponding to the stop and sbottom trilinear couplings—by  $A_t$  and  $A_b$ , respectively.

Upon electroweak symmetry breaking (EWSB), the neutral components of the two Higgs doublets acquire vacuum expectation values (VEVs):

$$\begin{aligned} \langle H_u^0 \rangle &= v_u, & \langle H_d^0 \rangle &= v_d, \\ v &\equiv \sqrt{v_u^2 + v_d^2} = 174 \text{ GeV}, & \tan \beta &\equiv \frac{v_u}{v_d}. \end{aligned} \quad (2.3)$$

Expanding the neutral Higgs fields around their respective VEVs yields the physical field decomposition:

$$\begin{aligned} H_u^0 &= v_u + \frac{1}{\sqrt{2}} (\phi_u + i\sigma_u), \\ H_d^0 &= v_d + \frac{1}{\sqrt{2}} (\phi_d + i\sigma_d). \end{aligned} \quad (2.4)$$

In the basis  $(\phi_d, \phi_u)$ , the tree-level  $CP$ -even mass-squared matrix takes the form [23]

$$\mathcal{M}_S^2 = \begin{pmatrix} \bar{m}_A^2 \sin^2 \beta + m_Z^2 \cos^2 \beta & -(\bar{m}_A^2 + m_Z^2) \sin \beta \cos \beta \\ -(\bar{m}_A^2 + m_Z^2) \sin \beta \cos \beta & \bar{m}_A^2 \cos^2 \beta + m_Z^2 \sin^2 \beta \end{pmatrix}, \quad (2.5)$$

where  $\bar{m}_A^2 = -2B_\mu/\sin 2\beta$  is the  $CP$ -odd (pseudoscalar) Higgs mass squared and  $m_Z$  is the  $Z$ -boson mass. The two  $CP$ -even mass eigenstates  $h$  (lighter) and  $H$  (heavier) are obtained by diagonalizing  $\mathcal{M}_S^2$  via the rotation angle  $\alpha$ :

$$\begin{pmatrix} H \\ h \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \phi_d \\ \phi_u \end{pmatrix}, \quad (2.6)$$

with eigenvalues

$$m_{h,H}^2 = \frac{1}{2} \left[ \bar{m}_A^2 + m_Z^2 \mp \sqrt{(\bar{m}_A^2 + m_Z^2)^2 - 4\bar{m}_A^2 m_Z^2 \cos^2 2\beta} \right]. \quad (2.7)$$

To facilitate the analysis of Higgs boson properties, it is convenient to introduce the following field combinations:

$$\begin{aligned} H_{\text{SM}} &\equiv \sin \beta \phi_u + \cos \beta \phi_d, & H_{\text{NSM}} &\equiv \cos \beta \phi_u - \sin \beta \phi_d, \\ A_{\text{NSM}} &\equiv \cos \beta \sigma_u - \sin \beta \sigma_d, & G^0 &\equiv \sin \beta \sigma_u + \cos \beta \sigma_d, \end{aligned} \quad (2.8)$$

where  $H_{\text{SM}}$  couples to gauge bosons and fermions with SM strength, while the orthogonal combinations  $H_{\text{NSM}}$  and  $A_{\text{NSM}}$  denote the additional neutral scalar and pseudoscalar fields, respectively, and  $G^0$  is the neutral Goldstone boson. In the decoupling limit  $\bar{m}_A \gg m_Z$ , which is favored by LHC searches for additional Higgs bosons [24,25] and assumed throughout the following discussion, one recovers  $\alpha \rightarrow \beta - \pi/2$  and  $h \rightarrow H_{\text{SM}}$  [23]. Consequently,  $h$  is identified with the SM-like Higgs boson discovered at the LHC [26,27].

Since  $m_h \leq m_Z |\cos 2\beta| < m_Z$  at tree level, reproducing the observed mass  $m_h \simeq 125$  GeV necessitates substantial radiative corrections, dominated by top/stop loops [23]. Including the leading one-loop top/stop contribution [28–32],

$$m_h^2 \simeq m_Z^2 \cos^2 2\beta + \frac{3g_2^2 m_t^4}{8\pi^2 m_W^2} \left[ \ln \frac{M_S^2}{m_t^2} + \frac{A_t^2}{M_S^2} \left( 1 - \frac{A_t^2}{12M_S^2} \right) \right], \quad (2.9)$$

where  $M_S \equiv \sqrt{m_{\tilde{t}_1} m_{\tilde{t}_2}}$  is the geometric mean of the stop masses and  $A_t$  is the stop trilinear coupling. Achieving  $m_h \simeq 125$  GeV generically requires either large stop masses or a large  $|A_t|$  [23].

The MSSM predicts four neutralinos  $\tilde{\chi}_i^0$  ( $i = 1, \dots, 4$ ), which are mass eigenstates formed from the Weyl-spinor basis

$$\psi^0 = (\tilde{B}, \tilde{W}^3, \tilde{H}_d^0, \tilde{H}_u^0)^T, \quad (2.10)$$

corresponding to the Bino  $\tilde{B}$ , the neutral Wino  $\tilde{W}^3$ , and the two neutral Higgsinos  $\tilde{H}_d^0$  and  $\tilde{H}_u^0$ . In this basis, the symmetric neutralino mass matrix reads as [16]

$$\mathcal{M}_{\tilde{\chi}^0} = \begin{pmatrix} M_1 & 0 & -m_Z s_W c_\beta & m_Z s_W s_\beta \\ 0 & M_2 & m_Z c_W c_\beta & -m_Z c_W s_\beta \\ -m_Z s_W c_\beta & m_Z c_W c_\beta & 0 & -\mu \\ m_Z s_W s_\beta & -m_Z c_W s_\beta & -\mu & 0 \end{pmatrix}, \quad (2.11)$$

where  $s_W \equiv \sin \theta_W$  and  $c_W \equiv \cos \theta_W$  are the sine and cosine of the weak mixing angle;  $s_\beta \equiv \sin \beta$  and  $c_\beta \equiv \cos \beta$ ; and  $M_1, M_2$  are the  $U(1)_Y$  and  $SU(2)_L$  gaugino masses, respectively. This matrix is diagonalized by a unitary matrix  $N$ :

$$\begin{aligned} N^* \mathcal{M}_{\tilde{\chi}^0} N^\dagger &= \text{diag}(m_{\tilde{\chi}_1^0}, m_{\tilde{\chi}_2^0}, m_{\tilde{\chi}_3^0}, m_{\tilde{\chi}_4^0}), \\ |m_{\tilde{\chi}_1^0}| &\leq |m_{\tilde{\chi}_2^0}| \leq |m_{\tilde{\chi}_3^0}| \leq |m_{\tilde{\chi}_4^0}|. \end{aligned}$$

The corresponding mass eigenstates are

$$\tilde{\chi}_i^0 = N_{i1} \tilde{B} + N_{i2} \tilde{W}^3 + N_{i3} \tilde{H}_d^0 + N_{i4} \tilde{H}_u^0. \quad (2.12)$$

Following Ref. [33], the mixing matrix elements  $N_{ij}$  can be expressed as simple functions of  $m_{\tilde{\chi}_i^0}$ . The lightest neutralino  $\tilde{\chi}_1^0$  is referred to as Bino-like when  $N_{11}^2$  exceeds both  $N_{12}^2$  and  $N_{13}^2 + N_{14}^2$ , a condition satisfied when  $|M_1| < |M_2|$  and  $|M_1| < |\mu|$ .

## 2. Key features of light DM scenario

In the MSSM, a compelling DM candidate in the tens-of-GeV mass range is the lightest neutralino, which is predominantly Bino-like. This particle couples to ordinary matter primarily through its interactions with the Higgs sector and the  $Z$  boson. The corresponding effective couplings are given by [2]<sup>1</sup>

$$C_{\tilde{B}\tilde{B}H_{\text{SM}}} = g_1^2 \left( \sin 2\beta + \frac{M_1}{\mu} \right) \frac{v}{\mu}, \quad C_{\tilde{B}\tilde{B}H_{\text{NSM}}} = g_1^2 \cos 2\beta \frac{v}{\mu}, \quad (2.13)$$

$$-iC_{\tilde{B}\tilde{B}A_{\text{NSM}}} = g_1^2 \left( 1 + \sin 2\beta \frac{M_1}{\mu} \right) \frac{v}{\mu}, \quad C_{\tilde{B}\tilde{B}Z} = \frac{g_1^3 \cos 2\beta v^2}{4 \sin \theta_W \mu^2}, \quad (2.14)$$

where  $g_1$  denotes the  $U(1)_Y$  gauge coupling;  $H_{\text{SM}}, H_{\text{NSM}}$ , and  $A_{\text{NSM}}$  are the field combinations defined in Eq. (2.8);

<sup>1</sup>For mostly Bino-like DM with small Higgsino admixtures, these couplings yield (to a good approximation) the same numerical results as Eqs. (2.4)–(2.6) in Ref. [34], which were derived using the analytic formulas for  $N_{ij}$  in Ref. [33]. The difference between the two sets of expressions can be traced to higher-dimensional-operator corrections in Ref. [2], arising from expanding the denominator  $[1 - (M_1/\mu)^2]$  in powers of  $M_1^2/\mu^2$ .

and  $|M_1|$  approximates the DM mass to good accuracy. Note that  $\mu$  represents the Higgsino mass, and, as discussed below, it is denoted as  $\mu_{\text{eff}}$  in the  $Z_3$ -NMSSM and  $\mu_{\text{tot}}$  in the GNMSSM.

These interactions give rise to both SI and SD scattering off nucleons. The corresponding cross sections can be approximated by [3,34,35]

$$\begin{aligned} \sigma_{\tilde{\chi}_1^0, N}^{\text{SI}} &\simeq 2 \times 10^{-48} \text{ cm}^2 \times \left( \frac{F_u^N + F_d^N}{0.28} \right)^2 \times \left\{ \frac{F_u^N}{F_u^N + F_d^N} \left[ \frac{\cos \alpha}{\sin \beta} \left( \frac{C_{\tilde{\chi}_1^0 \tilde{\chi}_1^0 h}}{0.002} \right) \left( \frac{125 \text{ GeV}}{m_h} \right)^2 + \frac{\sin \alpha}{\sin \beta} \left( \frac{C_{\tilde{\chi}_1^0 \tilde{\chi}_1^0 H}}{0.002} \right) \left( \frac{125 \text{ GeV}}{m_H} \right)^2 \right] \right. \\ &\quad \left. + \frac{F_d^N}{F_u^N + F_d^N} \left[ -\frac{\sin \alpha}{\cos \beta} \left( \frac{C_{\tilde{\chi}_1^0 \tilde{\chi}_1^0 h}}{0.002} \right) \left( \frac{125 \text{ GeV}}{m_h} \right)^2 + \frac{\cos \alpha}{\cos \beta} \left( \frac{C_{\tilde{\chi}_1^0 \tilde{\chi}_1^0 H}}{0.002} \right) \left( \frac{125 \text{ GeV}}{m_H} \right)^2 \right] \right\}^2 \\ &\simeq 2 \times 10^{-48} \text{ cm}^2 \\ &\quad \times \left[ \left( \frac{C_{\tilde{B} \tilde{B} H_{\text{SM}}}}{0.002} \right) \left( \frac{125 \text{ GeV}}{m_h} \right)^2 - \cot 2\beta \left( \frac{C_{\tilde{B} \tilde{B} H_{\text{NSM}}}}{0.002} \right) \left( \frac{125 \text{ GeV}}{m_H} \right)^2 \right]^2, \end{aligned} \quad (2.15)$$

$$\sigma_{\tilde{\chi}_1^0, N}^{\text{SD}} \simeq C_N \times \left( \frac{C_{\tilde{\chi}_1^0 \tilde{\chi}_1^0 Z}}{0.1} \right)^2. \quad (2.16)$$

Here  $C_{\tilde{\chi}_1^0 \tilde{\chi}_1^0 h}$ ,  $C_{\tilde{\chi}_1^0 \tilde{\chi}_1^0 H}$ , and  $C_{\tilde{\chi}_1^0 \tilde{\chi}_1^0 Z}$  denote the DM couplings to the Higgs bosons  $h$  and  $H$  and to the  $Z$  boson, respectively; for Bino-like DM in the decoupling limit, they are well approximated by  $C_{\tilde{B} \tilde{B} H_{\text{SM}}}$ ,  $C_{\tilde{B} \tilde{B} H_{\text{NSM}}}$ , and  $C_{\tilde{B} \tilde{B} Z}$ . The quantities  $F_u^N$  and  $F_d^N$  represent the normalized contributions of up- and down-type quarks to the nucleon mass, respectively. The coefficient  $C_N$  encodes the nucleon spin structure, taking the values  $C_p \simeq 1.8 \times 10^{-40} \text{ cm}^2$  for protons and  $C_n \simeq 1.4 \times 10^{-40} \text{ cm}^2$  for neutrons.

The expressions above expose two important suppression mechanisms for DM-nucleon interactions. First, as the Higgsino mass parameter  $\mu$  increases, the Bino-Higgs coupling is suppressed by a factor of  $1/\mu$ , while the Bino- $Z$  coupling is suppressed even more strongly by  $1/\mu^2$ . Consequently, the SI cross section  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SI}}$  scales as  $1/\mu^2$ , whereas the SD cross section  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SD}}$  is reduced by the steeper factor of  $1/\mu^4$ . Second, when  $\sin 2\beta = -M_1/\mu$ , the coupling  $C_{\tilde{B} \tilde{B} H_{\text{SM}}}$  vanishes identically, leading to a substantial suppression of  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SI}}$  [35].

Before analyzing the DM properties in detail, we summarize the key experimental constraints on the MSSM parameter space:

- (1) To accommodate the observed SM-like Higgs boson mass of approximately 125 GeV, the theory requires  $\tan \beta \gtrsim 4$  [36].
- (2) LHC searches for additional Higgs bosons indicate that, if the heavy  $CP$ -even Higgs boson ( $H$ ) decays exclusively into SM particles, its mass must exceed 350 GeV regardless of  $\tan \beta$  [24].
- (3) Compressed-spectrum scenarios, in which DM coannihilates with electroweakinos to reproduce the observed relic abundance, are constrained by LHC experiments. These constraints imply lower

mass bounds on the electroweakinos of approximately 180 GeV for Higgsinos and 220 GeV for Winos [37].

- (4) LHC searches for electroweakinos via multilepton and fully hadronic final states have established stringent limits within simplified model frameworks: Higgsino masses are excluded below  $\sim 900$  GeV for massless DM, while Wino masses are excluded below 1200 GeV [37,38]. These bounds are further strengthened when light sleptons are present in the decay chain [39].

Combining the theoretical formulation with these experimental constraints, the light DM scenario in the MSSM exhibits the following salient features:

- (1) For large  $\mu$ , the branching ratio of  $H$  decaying into DM pairs is strongly suppressed. Neglecting this subdominant channel,  $m_H$  must exceed 350 GeV. In this regime, the correct relic density can be achieved only through  $Z$ - or  $h$ -funnel annihilation, which requires  $|m_{\tilde{\chi}_1^0}| \simeq m_Z/2$  or  $|m_{\tilde{\chi}_1^0}| \simeq m_h/2$  [34].
- (2) The LZ 2024 constraint on  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SD}}$  implies that, for  $\tan \beta \gtrsim 4$ , the  $\mu$  parameter must exceed 700 GeV, as can be seen from Eqs. (2.14) and (2.16). The SI cross section constraint yields more nuanced boundaries that depend on the sign of  $M_1/\mu$ . Specifically, for  $\bar{m}_A = 1 \text{ TeV}$ ,  $\mu$  must exceed 900 GeV when  $M_1/\mu < 0$  and  $\tan \beta \lesssim 9$ , or when  $M_1/\mu > 0$  and  $\tan \beta \lesssim 25$ ; these  $\tan \beta$  thresholds increase modestly to 10 and 27, respectively, for  $\bar{m}_A = 2 \text{ TeV}$ . Notably, these direct-detection bounds on  $\mu$  are independent of the Higgsino decay modes and therefore complement the LHC constraints.
- (3) The Monte Carlo simulations presented in this work indicate that, for DM annihilating via the  $Z$ - or Higgs-boson resonance, LHC electroweakino

searches require  $\mu \gtrsim 900$  GeV when the kinematic threshold for Higgsino decay into  $H$  is not reached.

The preceding analysis reveals that the MSSM faces significant fine-tuning challenges when accommodating the light DM scenario:

- (1) From the Z-boson mass relation:

$$m_Z^2 = \frac{2(m_{H_d}^2 + \Sigma_d) - 2(m_{H_u}^2 + \Sigma_u) \tan^2 \beta}{\tan^2 \beta - 1} - 2\mu^2, \quad (2.17)$$

where  $\Sigma_d$  and  $\Sigma_u$  are self-energies of corresponding Higgs fields [28–32,40–47], the requirement  $\mu \gtrsim 700$  GeV entails a parameter fine-tuning of approximately  $m_Z^2/(2\mu^2) \simeq 0.8\%$  in reproducing the observed Z-boson mass [48].

- (2) Since a large  $\mu$  severely suppresses the DM couplings to the Z boson and the Higgs bosons, reproducing the correct relic abundance requires the DM mass to satisfy the resonance condition ( $|m_{\tilde{\chi}_1^0}| \simeq m_Z/2$  or  $m_h/2$ ) with high precision, introducing a further fine-tuning problem [49]. Moreover, if  $\mu$  is excessively large, resonant annihilation may prove insufficient to attain the measured relic density, rendering the light DM scenario unviable. We will investigate this issue in a forthcoming work.

It is worth emphasizing that both sources of fine-tuning originate from a single theoretical requirement: the  $\mu$  parameter must be large, which in turn strongly suppresses all DM couplings.

## B. NMSSM

The NMSSM extends the MSSM field content by introducing a single gauge-singlet chiral superfield  $\hat{S}$ , whose scalar component  $S$  acquires a VEV  $v_s \equiv \langle S \rangle$  upon EWSB. This minimal extension dynamically generates an effective  $\mu$  parameter at the electroweak scale through the superpotential coupling  $\lambda \hat{S} \hat{H}_u \cdot \hat{H}_d$ , thereby (partially or fully) resolving the  $\mu$  problem of the MSSM [50]. It also enriches both the Higgs and neutralino spectra in phenomenologically important ways [18]. In what follows, we recapitulate the basics of the NMSSM, adopting the same conventions established in the MSSM discussion above. Since the  $Z_3$ -NMSSM—the most well-studied realization of the NMSSM at the electroweak scale—exhibits significantly different phenomenology from the GNMSSM, we discuss its key features separately.

### 1. Lagrangian and Conventions

The most general renormalizable superpotential of the NMSSM that is consistent with gauge invariance and  $R$ -parity conservation—referred to as the GNMSSM superpotential in the recent literature [51,52]—takes the form [18,53]

$$W_{\text{NMSSM}} = W_{\text{Yukawa}} + \lambda \hat{S} \hat{H}_u \cdot \hat{H}_d + \frac{\kappa}{3} \hat{S}^3 + \mu \hat{H}_u \cdot \hat{H}_d + \frac{1}{2} \mu' \hat{S}^2 + \xi \hat{S}, \quad (2.18)$$

where  $W_{\text{Yukawa}}$  contains the quark and lepton Yukawa couplings identical to those of the MSSM [cf. Eq. (2.1)]. The dimensionless couplings  $\lambda$  and  $\kappa$  govern the singlet-doublet interaction and the singlet cubic self-coupling, respectively. The bilinear mass parameters  $\mu$  and  $\mu'$ , together with the linear tadpole coefficient  $\xi$ , parametrize explicit breaking of the discrete  $Z_3$  symmetry that would otherwise forbid all dimensionful terms in Eq. (2.18). These terms simultaneously resolve the cosmological domain-wall problem associated with the discrete symmetry of the NMSSM and eliminate dangerous higher-order tadpoles [18,53–59], while remaining theoretically well motivated by discrete  $R$  symmetries ( $Z_4^R$  or  $Z_8^R$ ) that naturally generate  $\mu$  and  $\mu'$  at the electroweak scale after SUSY breaking [53,55,58–60]. Since one of the three parameters  $\{\mu, \mu', \xi\}$  can always be eliminated by a constant shift  $\hat{S} \rightarrow \hat{S} + c$  combined with a redefinition of the remaining parameters [53], we set  $\xi = 0$  without loss of generality throughout this study.

The soft SUSY-breaking Lagrangian of the GNMSSM differs from that of the MSSM only in the Higgs sector, and reads as

$$\begin{aligned} -\mathcal{L}_{\text{soft}}^{\text{Higgs}} = & \left[ \lambda A_\lambda S H_u \cdot H_d + \frac{1}{3} \kappa A_\kappa S^3 + m_3^2 H_u \cdot H_d \right. \\ & \left. + \frac{1}{2} m_S'^2 S^2 + \xi' S + \text{H.c.} \right] \\ & + m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2 + m_S^2 |S|^2, \end{aligned} \quad (2.19)$$

where  $m_{H_u}^2$ ,  $m_{H_d}^2$ , and  $m_S^2$  are the soft-breaking mass-squared parameters for the Higgs fields  $H_u$ ,  $H_d$ , and  $S$ , respectively;  $A_\lambda$  and  $A_\kappa$  are the trilinear soft coefficients associated with the superpotential couplings  $\lambda$  and  $\kappa$ ; and  $m_3^2$  and  $m_S'^2$  are the soft counterparts of the bilinear superpotential parameters  $\mu$  and  $\mu'$ , respectively. As in the MSSM, the neutral scalar fields acquire VEVs  $v_u$ ,  $v_d$ , and  $v_s$  after EWSB. Imposing the three stationarity (tadpole) conditions of the scalar potential then trades the soft masses  $m_{H_u}^2$ ,  $m_{H_d}^2$ , and  $m_S^2$  for these VEVs as independent inputs, leaving the physical Higgs spectrum determined by the following set of free parameters [61]:

$$\tan \beta, \lambda, \kappa, v_s, A_\lambda, A_\kappa, \mu, \mu', m_3^2, m_S'^2, \xi'. \quad (2.20)$$

*Higgs sector.* The NMSSM Higgs sector comprises three  $CP$ -even and two  $CP$ -odd physical states after the neutral Goldstone boson  $G^0$  is removed. In the basis  $(H_{\text{NSM}}, H_{\text{SM}}, \text{Re}[S])$ , the  $CP$ -even mass-squared matrix takes the form [18,19]

$$\begin{aligned}
\mathcal{M}_{S,11}^2 &= m_A^2 + (m_Z^2 - \lambda^2 v^2) \sin^2 2\beta, \\
\mathcal{M}_{S,12}^2 &= -\frac{1}{2}(m_Z^2 - \lambda^2 v^2) \sin 4\beta, \\
\mathcal{M}_{S,13}^2 &= -\lambda v(A_\lambda + 2\kappa v_s + \mu') \cos 2\beta, \\
\mathcal{M}_{S,22}^2 &= m_Z^2 \cos^2 2\beta + \lambda^2 v^2 \sin^2 2\beta, \\
\mathcal{M}_{S,23}^2 &= \lambda v[2(\lambda v_s + \mu) - (A_\lambda + 2\kappa v_s + \mu') \sin 2\beta], \\
\mathcal{M}_{S,33}^2 &= \frac{(A_\lambda + \mu') \sin 2\beta}{2v_s} \lambda v^2 + \kappa v_s(A_\kappa + 4\kappa v_s + 3\mu') - \frac{\mu}{v_s} \lambda v^2 - \frac{\xi'}{v_s},
\end{aligned} \tag{2.21}$$

where  $m_A^2 \equiv 2[\lambda v_s(A_\lambda + \kappa v_s + \mu') + m_3^2]/\sin 2\beta$ . Two key features of the NMSSM Higgs sector are immediately apparent from these expressions. First, the diagonal element  $\mathcal{M}_{S,22}^2$  receives an additional positive contribution  $\lambda^2 v^2 \sin^2 2\beta$  relative to the MSSM, which raises the tree-level upper bound on the SM-like Higgs boson mass:

$$m_h^2 \leq m_Z^2 \cos^2 2\beta + \lambda^2 v^2 \sin^2 2\beta, \tag{2.22}$$

thereby reducing the reliance on large radiative corrections from the top-stop sector. Second, the off diagonal element  $\mathcal{M}_{S,23}^2$ , which controls the mixing between  $H_{\text{SM}}$  and  $\text{Re}[S]$ , is proportional to  $\lambda$ , so a small value of  $\lambda$  naturally suppresses the singlet-doublet mixing.

In the  $CP$ -odd sector, the mass-squared matrix is expressed in the basis  $(A_{\text{NSM}} \equiv \cos \beta \text{Im}(H_u^0) + \sin \beta \text{Im}(H_d^0), \text{Im}(S))$ ,<sup>2</sup> and its elements are

$$\begin{aligned}
\mathcal{M}_{P,22}^2 &= \frac{(A_\lambda + 4\kappa v_s + \mu') \sin 2\beta}{2v_s} \lambda v^2 - \kappa v_s(3A_\kappa + \mu') \\
&\quad - \frac{\mu}{v_s} \lambda v^2 - 2m_S^2 - \frac{\xi'}{v_s}, \\
\mathcal{M}_{P,11}^2 &= m_A^2, \quad \mathcal{M}_{P,12}^2 = \lambda v(A_\lambda - 2\kappa v_s - \mu').
\end{aligned} \tag{2.23}$$

The expression for  $\mathcal{M}_{P,11}^2$  confirms that  $m_A$  should be interpreted as the mass of the  $CP$ -odd heavy doublet Higgs field. The expression for  $\mathcal{M}_{P,12}^2$  shows that, as in the  $CP$ -even sector, the singlet-doublet mixing is controlled by  $\lambda$ .

The three  $CP$ -even mass eigenstates  $h_i = \{h, H, h_s\}$  and two  $CP$ -odd mass eigenstates  $a_i = \{A_H, A_s\}$  are obtained by applying unitary rotations  $V$  and  $V_P$  to diagonalize  $\mathcal{M}_S^2$  and  $\mathcal{M}_P^2$ , respectively:

$$h_i = V_{h_i}^{\text{NSM}} H_{\text{NSM}} + V_{h_i}^{\text{SM}} H_{\text{SM}} + V_{h_i}^{\text{S}} \text{Re}[S], \tag{2.24}$$

<sup>2</sup>Note that  $A_{\text{NSM}}$  as defined in the NMSSM differs from its MSSM counterpart; this difference originates from the distinct structures of the respective mass-squared matrices, cf. Eq. (2.24) in Ref. [18] versus Eq. (1.76) in Ref. [23].

$$a_i = V_{P,a_i}^{\text{NSM}} A_{\text{NSM}} + V_{P,a_i}^{\text{S}} \text{Im}[S]. \tag{2.25}$$

Among these states,  $h$  is identified with the scalar boson discovered at the LHC;  $H$  and  $A_H$  are the heavy doublet-dominated Higgs bosons; and  $h_s$  and  $A_s$  are the singlet-dominated states. For convenience, these states are also labeled in ascending mass order as  $h_i$  ( $i = 1, 2, 3$ ) and  $a_i$  ( $i = 1, 2$ ), i.e.,  $m_{h_1} < m_{h_2} < m_{h_3}$  and  $m_{a_1} < m_{a_2}$ . Accordingly,  $h \equiv h_1$  and  $m_{h_s} > m_h$  in the  $h_1$  scenario, while  $h \equiv h_2$  and  $m_h > m_{h_s}$  in the  $h_2$  scenario. The coupling of  $h_i$  to any neutralino pair  $\tilde{\chi} \tilde{\chi}$  ( $\tilde{\chi} = \tilde{B}, \tilde{S}$ ) decomposes, in correspondence with Eq. (2.24), as [52]

$$C_{\tilde{\chi} \tilde{\chi} h_i} = V_{h_i}^{\text{NSM}} C_{\tilde{\chi} \tilde{\chi} H_{\text{NSM}}} + V_{h_i}^{\text{SM}} C_{\tilde{\chi} \tilde{\chi} H_{\text{SM}}} + V_{h_i}^{\text{S}} C_{\tilde{\chi} \tilde{\chi} \text{Re}[S]}. \tag{2.26}$$

Since the parameters  $\mu, \mu', m_3^2, m_S^2$ , and  $\xi'$  are not directly accessible to experiment, it is convenient to adopt as theoretical inputs the physical masses of the  $CP$ -odd heavy doublet, the  $CP$ -even and  $CP$ -odd singlet Higgs fields, the Higgsino, and the Singlino, defined respectively as  $m_A$ ,  $m_B \equiv \sqrt{\mathcal{M}_{S,33}^2}$ ,  $m_C \equiv \sqrt{\mathcal{M}_{P,22}^2}$ ,  $\mu_{\text{tot}} \equiv \mu_{\text{eff}} + \mu$  with  $\mu_{\text{eff}} \equiv \lambda v_s$ , and  $m_{\tilde{S}} \equiv \frac{2\kappa}{\lambda} \mu_{\text{eff}} + \mu'$ . In the limit  $\lambda \rightarrow 0$ , the field masses  $m_B$ ,  $m_C$ , and  $m_{\tilde{S}}$  reduce to good approximations of the corresponding physical particle masses. In terms of these inputs, the original parameters are expressed as

$$\begin{aligned}
\mu &= \mu_{\text{tot}} - \lambda v_s, \quad \mu' = m_{\tilde{S}} - 2\kappa v_s, \\
m_3^2 &= m_A^2 \sin \beta \cos \beta - \lambda v_s(\kappa v_s + \mu' + A_\lambda), \\
\xi' &= \left[ \frac{(A_\lambda + \mu') \sin \beta \cos \beta}{v_s} \lambda v^2 + \kappa v_s(A_\kappa + 4\kappa v_s + 3\mu') \right. \\
&\quad \left. - \frac{\mu}{v_s} \lambda v^2 - m_B^2 \right] v_s, \\
m_S^2 &= \frac{1}{2} \left[ m_B^2 - m_C^2 + 2\lambda \kappa \sin 2\beta v^2 - 4\kappa v_s(A_\kappa + \kappa v_s + \mu') \right],
\end{aligned} \tag{2.27}$$

and Eqs. (2.21) and (2.23) reduce to the simplified forms

$$\begin{aligned}\mathcal{M}_{S,11}^2 &= m_A^2 + (m_Z^2 - \lambda^2 v^2) \sin^2 2\beta, & \mathcal{M}_{S,12}^2 &= -(m_Z^2 - \lambda^2 v^2) \sin 2\beta \cos 2\beta, \\ \mathcal{M}_{S,13}^2 &= -\lambda v(A_\lambda + m_{\tilde{S}}) \cos 2\beta, & \mathcal{M}_{S,22}^2 &= m_Z^2 \cos^2 2\beta + \lambda^2 v^2 \sin^2 2\beta, \\ \mathcal{M}_{S,23}^2 &= \lambda v[2\mu_{\text{tot}} - (A_\lambda + m_{\tilde{S}}) \sin 2\beta] \equiv 2\lambda\delta v\mu_{\text{tot}}, & \mathcal{M}_{S,33}^2 &= m_B^2, \\ \mathcal{M}_{P,11}^2 &= m_A^2, & \mathcal{M}_{P,22}^2 &= m_C^2, & \mathcal{M}_{P,12}^2 &= \lambda v(A_\lambda - m_{\tilde{S}}),\end{aligned}\quad (2.28)$$

where  $\delta \equiv 1 - (A_\lambda + m_{\tilde{S}}) \sin 2\beta / (2\mu_{\text{tot}})$  quantifies the degree of cancellation among the different contributions to  $\mathcal{M}_{S,23}^2$  [62] and may replace  $A_\lambda$  as an independent theoretical input. These expressions show that the Higgs mass matrices are fully determined by eight of the eleven parameters:  $\tan\beta$ ,  $\lambda$ ,  $\delta$ ,  $m_A$ ,  $m_B$ ,  $m_C$ ,  $m_{\tilde{S}}$ , and  $\mu_{\text{tot}}$ . In particular, the singlet masses are independent of one another and may all take small values, since they are weakly constrained by current experiments. The remaining three parameters— $\kappa$ ,  $A_\kappa$ , and  $v_s$ —affect only the triple Higgs coupling strengths [61]. In the following analysis, we set  $\xi' = 0$  to simplify the discussion. Under this condition,  $A_\kappa$  is determined by [61]

$$\kappa A_\kappa = m_B^2/v_s + \lambda\mu v^2/v_s^2 - \lambda(A_\lambda + m_{\tilde{S}} - 2\kappa v_s)v^2 \sin 2\beta / (2v_s^2) + 2\kappa^2 v_s - 3\kappa m_{\tilde{S}}.$$

In the limit of large  $m_A$ , the following approximations hold [2]:

$$\begin{aligned}m_{h_s}^2 &\simeq m_B^2 - \frac{\mathcal{M}_{S,13}^4}{m_A^2 - m_B^2}, & m_{A_s}^2 &\simeq m_C^2 - \frac{\mathcal{M}_{P,12}^4}{m_A^2 - m_C^2}, & \frac{V_{P,A_s}^{\text{NSM}}}{V_{P,A_s}^{\text{S}}} &= \frac{\mathcal{M}_{P,12}^2}{m_{A_s}^2 - m_A^2} \simeq 0, \\ \frac{V_h^{\text{S}}}{V_h^{\text{SM}}} &\simeq \frac{\mathcal{M}_{S,23}^2}{m_h^2 - m_B^2}, & V_h^{\text{NSM}} &\sim 0, & V_h^{\text{SM}} &\simeq \sqrt{1 - \left(\frac{V_h^{\text{S}}}{V_h^{\text{SM}}}\right)^2} \sim 1, \\ \frac{V_{h_s}^{\text{SM}}}{V_{h_s}^{\text{S}}} &\simeq \frac{\mathcal{M}_{S,23}^2}{m_{h_s}^2 - m_h^2}, & V_{h_s}^{\text{NSM}} &\sim 0, & V_{h_s}^{\text{S}} &\simeq \sqrt{1 - \left(\frac{V_{h_s}^{\text{SM}}}{V_{h_s}^{\text{S}}}\right)^2} \sim 1.\end{aligned}\quad (2.29)$$

These relations make clear that the singlet fields decouple from the doublet Higgs fields as  $\lambda \rightarrow 0$ , in which limit both  $V_h^{\text{S}}$  and  $V_{h_s}^{\text{SM}}$  vanish.

*Neutralino sector.* The NMSSM extends the four-dimensional neutralino mixing of the MSSM to a  $5 \times 5$  system by including the Singlino  $\tilde{S}$ , the fermionic superpartner of  $\hat{S}$ . In the basis  $\psi^0 = (\tilde{B}, \tilde{W}^0, \tilde{H}_d^0, \tilde{H}_u^0, \tilde{S})^T$ , the symmetric neutralino mass matrix reads as [18]

$$\mathcal{M}_{\tilde{\chi}^0} = \begin{pmatrix} M_1 & 0 & -m_Z s_W c_\beta & m_Z s_W s_\beta & 0 \\ 0 & M_2 & m_Z c_W c_\beta & -m_Z c_W s_\beta & 0 \\ -m_Z s_W c_\beta & m_Z c_W c_\beta & 0 & -\mu_{\text{tot}} & -\lambda v_u \\ m_Z s_W s_\beta & -m_Z c_W s_\beta & -\mu_{\text{tot}} & 0 & -\lambda v_d \\ 0 & 0 & -\lambda v_u & -\lambda v_d & m_{\tilde{S}} \end{pmatrix}, \quad (2.30)$$

where, as introduced before, the total Higgsino mass parameter and the Singlino mass are given by

$$\mu_{\text{tot}} \equiv \lambda v_s + \mu, \quad m_{\tilde{S}} \equiv 2\kappa v_s + \mu'. \quad (2.31)$$

The notation  $\mu_{\text{tot}}$  is used consistently throughout this paper. The five neutralino mass eigenstates  $\tilde{\chi}_i^0$  ( $i = 1, \dots, 5$ , in ascending mass order) are related to the gauge eigenstates by

$$\tilde{\chi}_i^0 = N_{i1}\tilde{B} + N_{i2}\tilde{W}^0 + N_{i3}\tilde{H}_d^0 + N_{i4}\tilde{H}_u^0 + N_{i5}\tilde{S}, \quad (2.32)$$

where the  $5 \times 5$  unitary matrix  $N$  diagonalizes  $\mathcal{M}_{\tilde{\chi}^0}$ . When the lightest neutralino  $\tilde{\chi}_1^0$  is predominantly Singlino-like ( $|N_{15}|^2 \gg |N_{1j}|^2$  for  $j = 1, \dots, 4$ ), its mass is well approximated by  $|m_{\tilde{\chi}_1^0}| \simeq |m_{\tilde{S}}|$ .

## 2. Key features of $Z_3$ -NMSSM

The widely studied  $Z_3$ -NMSSM is recovered from the GNMSSM by imposing a strict  $Z_3$  discrete symmetry, which requires setting all dimensionful parameters in the superpotential (2.18) and in the corresponding soft-break terms (2.19) to zero [18]:

$$\mu=0, \quad \mu'=0, \quad \xi=0, \quad m_3=0, \quad m'_3=0, \quad \xi'=0. \quad (2.33)$$

This parameter choice considerably simplifies the mass matrices. In particular, the Higgsino mass and the Singlino mass reduce to  $\mu_{\text{tot}} = \mu_{\text{eff}} \equiv \lambda v_s$  and  $m_{\tilde{S}} = 2\kappa v_s \equiv 2\kappa\mu_{\text{eff}}/\lambda$ , respectively, revealing that in the constrained theory the Singlino mass is tied to the Higgsino mass by the ratio of the Higgs Yukawa couplings,  $2\kappa/\lambda$ . The entries  $\mathcal{M}_{P,11}^2$ ,  $\mathcal{M}_{S,33}^2$ , and  $\mathcal{M}_{P,22}^2$  reduce to

$$\mathcal{M}_{P,11}^2|_{Z_3} = m_A^2 = \frac{2\lambda v_s(A_\lambda + \kappa v_s)}{\sin 2\beta}, \quad (2.34)$$

$$\mathcal{M}_{S,33}^2|_{Z_3} = m_B^2 = \frac{A_\lambda \sin 2\beta}{2v_s} \lambda v^2 + \kappa v_s (A_\kappa + 4\kappa v_s), \quad (2.35)$$

$$\mathcal{M}_{P,22}^2|_{Z_3} = m_C^2 = \frac{(A_\lambda + 4\kappa v_s) \sin 2\beta}{2v_s} \lambda v^2 - 3\kappa v_s A_\kappa. \quad (2.36)$$

The structure of these matrices shows that all observables in the Higgs and neutralino sectors are controlled by only six parameters:  $\lambda$ ,  $\kappa$ ,  $\tan \beta$ ,  $v_s$ ,  $A_\lambda$ , and  $A_\kappa$ , of which the latter three can be traded for  $\mu_{\text{eff}}$ ,  $m_A$ , and  $m_C$  as theoretical inputs. Notably,  $m_B$ ,  $m_{\tilde{S}}$ , and  $\delta \equiv 1 - (A_\lambda + m_{\tilde{S}}) \sin 2\beta / (2\mu_{\text{tot}})$  are no longer independent quantities, in contrast to the GNMSSM.

In the  $Z_3$ -NMSSM, a DM candidate in the tens-of-GeV mass range can be either a Bino-dominated or a Singlino-dominated lightest neutralino. Beyond the couplings listed in Eqs. (2.13) and (2.14), the following interactions also govern DM phenomenology [2]:

$$\begin{aligned} C_{\tilde{B}\tilde{B}\text{Re}[S]} &= iC_{\tilde{B}\tilde{B}\text{Im}[S]} = -\frac{\lambda g_1^2 \sin 2\beta}{2} \frac{v^2}{\mu_{\text{eff}}^2}, \\ C_{\tilde{B}\tilde{S}H_{\text{SM}}} &= -2\sqrt{2}\lambda g_1 \cos 2\beta \frac{v}{\mu_{\text{eff}}}, \quad iC_{\tilde{B}\tilde{S}A_{\text{NSM}}} = -\sqrt{2}\lambda g_1 \cos 2\beta \frac{(M_1 + m_{\tilde{S}})v}{\mu_{\text{eff}}^2}, \\ C_{\tilde{B}\tilde{S}H_{\text{NSM}}} &= \sqrt{2}\lambda g_1 \left[ 2\sin 2\beta + \frac{(M_1 - m_{\tilde{S}})}{\mu_{\text{eff}}} \right] \frac{v}{\mu_{\text{eff}}}, \\ C_{\tilde{B}\tilde{S}\text{Re}[S]} &= iC_{\tilde{B}\tilde{S}\text{Im}[S]} = \sqrt{2}\lambda^2 g_1 \cos 2\beta \frac{v^2}{\mu_{\text{eff}}^2}, \quad C_{\tilde{B}\tilde{S}Z} = \frac{\sqrt{2}\lambda g_1^2 \sin 2\beta}{\sin \theta_W} \frac{v^2}{\mu_{\text{eff}}^2}, \\ C_{\tilde{S}\tilde{S}H_{\text{SM}}} &= -2\lambda^2 \sin 2\beta \frac{v}{\mu_{\text{eff}}} + 2\lambda^2 \frac{m_{\tilde{S}}v}{\mu_{\text{eff}}^2}, \quad C_{\tilde{S}\tilde{S}H_{\text{NSM}}} = -2\lambda^2 \cos 2\beta \frac{v}{\mu_{\text{eff}}}, \\ C_{\tilde{S}\tilde{S}\text{Re}[S]} &= iC_{\tilde{S}\tilde{S}\text{Im}[S]} = \lambda^3 \sin 2\beta \frac{v^2}{\mu_{\text{eff}}^2} - 2\kappa \left( 1 - \lambda^2 \frac{v^2}{\mu_{\text{eff}}^2} \right), \\ iC_{\tilde{S}\tilde{S}A_{\text{NSM}}} &= 2\lambda^2 \left[ 1 - \sin 2\beta \frac{m_{\tilde{S}}}{\mu_{\text{eff}}} \right] \frac{v}{\mu_{\text{eff}}}, \quad C_{\tilde{S}\tilde{S}Z} = -\frac{\lambda^2 g_1 \cos 2\beta}{2 \sin \theta_W} \frac{v^2}{\mu_{\text{eff}}^2}, \end{aligned} \quad (2.37)$$

where, as emphasized before,  $\mu_{\text{eff}}$  represents the Higgsino mass, and it should be replaced by  $\mu_{\text{tot}}$  when these expressions are applied to the GNMSSM. In contrast to the interactions in Eqs. (2.13) and (2.14), these couplings depend not only on  $\mu_{\text{eff}}$  but also critically on the Yukawa coupling coefficients  $\lambda$  and  $\kappa$ .

In analyzing DM phenomenology within this framework, three fundamental constraints must be taken into account:

### (1) Higgs data constraints.

In the NMSSM, mixing between the singlet  $CP$ -even Higgs field and the SM-like Higgs field modifies the properties of the observed Higgs boson,

an effect that is most pronounced when diagonalizing the  $CP$ -even Higgs mass matrix.

A global fit of Higgs properties to the LHC data, performed with the latest HIGGSTOOLS package [63], has established that  $h$  is predominantly composed of the field  $\text{Re}[H_u^0]$ , while  $H$  has its largest component from  $\text{Re}[H_d^0]$ .

The fit further restricts  $|\mathcal{M}_{S,23}^2/(m_h^2 - \mathcal{M}_{S,33}^2)| \lesssim 0.4$  [64], or equivalently  $\lambda \delta \mu_{\text{eff}} \lesssim |\mathcal{M}_{S,33}^2 - m_h^2|/(900 \text{ GeV})$ . Quantitatively, fixing  $\mu_{\text{eff}} = 700 \text{ GeV}$  and  $\delta \simeq 1$  requires  $\lambda \lesssim 0.009$  for  $\mathcal{M}_{S,33} = 100 \text{ GeV}$  and  $\lambda \lesssim 0.039$  for  $\mathcal{M}_{S,33} = 200 \text{ GeV}$ . In addition,

perturbativity up to the grand unified theory (GUT) scale imposes  $\lambda \lesssim 0.7$  [18].

(2) *Singlino mass constraint.*

For the Singlino to play a significant role in DM physics, its mass  $m_{\tilde{S}} = 2\kappa\mu_{\text{eff}}/\lambda$  must lie in the tens-of-GeV range, which implies

$$\kappa \lesssim \frac{\lambda}{20} \times \frac{m_{\tilde{S}}}{100 \text{ GeV}} \times \frac{1 \text{ TeV}}{\mu_{\text{eff}}}. \quad (2.38)$$

Consequently, in light DM scenarios  $\kappa$  must be at least 1 order of magnitude smaller than  $\lambda$  when  $\mu_{\text{eff}} \gtrsim 700 \text{ GeV}$ , or smaller than  $\lambda/4$  when  $\mu_{\text{eff}} \gtrsim 200 \text{ GeV}$ .

(3) *DM-nucleon scattering.*

In the NMSSM, the SI DM-nucleon cross section can be approximated by [2]

$$\begin{aligned} \sigma_{\tilde{\chi},N}^{\text{SI}} \simeq & 2 \times 10^{-48} \text{ cm}^2 \times \left[ \left( \frac{125 \text{ GeV}}{m_h} \right)^2 \left( \frac{C_{\tilde{\chi}\tilde{\chi}h}}{0.002} \right) \right. \\ & + \left( \frac{125 \text{ GeV}}{m_H} \right)^2 \left( \frac{C_{\tilde{\chi}\tilde{\chi}H}}{0.002} \right) + \left( \frac{125 \text{ GeV}}{m_{h_s}} \right)^2 \\ & \left. \times \left( \frac{C_{\tilde{\chi}\tilde{\chi}h_s}}{0.002} \right) \right]^2, \end{aligned} \quad (2.39)$$

where the three terms correspond to  $t$ -channel exchanges of the SM-like Higgs  $h$ , the heavier doublet-like Higgs  $H$ , and the singlet-like Higgs  $h_s$ , respectively. The SD cross section retains the same form as in Eq. (2.16). For Singlino-like DM specifically, more detailed expressions for both cross sections have been derived in Refs. [51,61,65]. These results show that the leading contribution to the SI cross section scales as  $\sigma_{\tilde{\chi},N}^{\text{SI}} \propto \lambda^4 v^2 / \mu_{\text{eff}}^2$  for a sufficiently heavy  $h_s$ , or as  $\propto \lambda^2 \kappa^2$  for a moderately light  $h_s$ , while the SD cross section satisfies  $\sigma_{\tilde{\chi},N}^{\text{SD}} \propto \lambda^4 v^4 / \mu_{\text{eff}}^4$ . Both cross sections thus exhibit a strong dependence on  $\lambda$ . The LZ 2024 results constrain this coupling to  $\lambda \lesssim 0.05$  [61], placing stringent limits on the viability of Singlino-like DM in the NMSSM.

When DM is predominantly Bino-like, the LZ 2024 constraint on the SD cross section requires  $\mu_{\text{eff}} \gtrsim 700 \text{ GeV}$  for  $\tan\beta \gtrsim 4$ , as in the MSSM. Beyond the canonical  $Z$ - and  $h$ -funnel annihilation channels, the correct relic abundance can also be achieved through Bino–Singlino coannihilation when the two masses are nearly degenerate [2], or via resonant annihilation through  $h_s$  and  $A_s$  [49,66,67]. The relevant interactions— $C_{\tilde{B}\tilde{B}\text{Re}[S]}$ ,  $C_{\tilde{B}\tilde{B}\text{Im}[S]}$ ,  $C_{\tilde{B}\tilde{S}H_{\text{SM}}}$ ,  $C_{\tilde{B}\tilde{S}\text{Re}[S]}$ ,  $C_{\tilde{B}\tilde{S}\text{Im}[S]}$ ,  $C_{\tilde{B}\tilde{S}Z}$ ,  $C_{\tilde{S}\tilde{S}H_{\text{SM}}}$ ,  $C_{\tilde{S}\tilde{S}\text{Re}[S]}$ ,  $C_{\tilde{S}\tilde{S}\text{Im}[S]}$ , and  $C_{\tilde{S}\tilde{S}Z}$ —are all suppressed by at least one power of  $1/\mu_{\text{eff}}$  and typically also by  $\lambda$ . This suppression necessitates fine-tuning to reproduce the observed relic

density. The situation is further complicated by the finite width of the mediator: if the couplings are sufficiently weak, resonant annihilation alone may prove insufficient to attain the measured DM density.

For Singlino-dominated DM, the LZ 2024 constraints on both the SI and SD cross sections require  $\lambda \lesssim 0.05$ , although no definitive lower bound on  $\mu_{\text{eff}}$  can be established [61]. LHC constraints on compressed-spectrum scenarios independently require  $\mu_{\text{eff}} \gtrsim 180 \text{ GeV}$  [37], while electroweakino searches via multilepton channels impose even stronger limits in other scenarios [37–39]. Under these conditions, all interactions in Eq. (2.37) are strongly suppressed by  $\lambda$  or  $\kappa$ , where  $\kappa \lesssim \lambda/4$  according to Eq. (2.38) and the LHC bounds on  $\mu_{\text{eff}}$ . Consequently, annihilation processes such as  $\tilde{S}\tilde{S} \rightarrow h_s A_s, h_s h_s, A_s A_s$  cannot generate the observed relic density due to the extremely small value of  $\kappa$  [49]. Resonant annihilation via  $h_s$  or  $A_s$  is also subject to substantial fine-tuning, since the coupling strengths  $C_{\tilde{S}\tilde{S}\text{Re}[S]}$  and  $C_{\tilde{S}\tilde{S}\text{Im}[S]}$  are necessarily small [61].

The conclusion from the above analysis is clear: because every DM coupling is strongly suppressed, the  $Z_3$ -NMSSM requires severe fine-tuning to accommodate a viable light DM candidate.

### 3. Key features of GNMSSM

In the GNMSSM, the Bino and Singlino interactions are identical to those of the  $Z_3$ -NMSSM when  $\mu$  in Eqs. (2.13) and (2.14) and  $\mu_{\text{eff}}$  in Eq. (2.37) are interpreted as the Higgsino mass, denoted by  $\mu_{\text{tot}}$  in the GNMSSM; however, the enlarged superpotential decouples several key mass parameters [52,61]. Specifically, the Higgsino mass  $\mu_{\text{tot}}$  and the Singlino mass  $m_{\tilde{S}}$  defined in Eq. (2.31) can be adjusted independently. This feature represents a critical departure from the  $Z_3$ -NMSSM, in which the Singlino mass is tied to the Higgsino mass by the ratio  $2\kappa/\lambda$ , and  $\kappa$  must satisfy  $\kappa < \lambda/2$  for viable Singlino-like DM scenarios, or  $\kappa < \lambda/4$  if the DM is further required to be lighter than 100 GeV. By contrast, the GNMSSM permits  $|\kappa|$  to be much larger than  $\lambda$  in this scenario [51]. This freedom profoundly alters the Singlino DM phenomenology while leaving the Bino-like DM properties essentially unchanged.

A Singlino-like DM  $\tilde{\chi}_1^0$  with mass  $|m_{\tilde{\chi}_1^0}| \simeq |m_{\tilde{S}}|$  can therefore annihilate through “secluded-sector” channels controlled by  $\kappa$ , e.g.,  $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow h_s A_s, h_s h_s, A_s A_s$ . Consequently, the thermally averaged annihilation cross section exhibits a characteristic  $\kappa^4$  dependence [61]. The direct-detection cross section, by contrast, is proportional to  $\lambda^4$  or  $\lambda^2 \kappa^2$ —as in the  $Z_3$ -NMSSM—and remains naturally suppressed by the smallness of  $\lambda$  [61]. This separation of roles provides a substantially broader DM parameter space that can simultaneously accommodate the observed relic density and current direct-detection bounds, in particular without requiring resonance enhancements in the annihilation [51,61]. Furthermore, the smaller  $\lambda$  values

characteristic of the viable parameter space enhance the stability of the quartic self-coupling, thereby mitigating potential issues with charge- or color-breaking vacua that can arise in certain regions of the  $Z_3$ -NMSSM parameter space [68,69].

From the collider perspective, the LHC phenomenology of the GNMSSM presents both challenges and opportunities. The scalar and neutralino spectrum is enriched, since the singlet scalar masses  $m_{h_s}$  and  $m_{A_s}$  and the Singlino mass  $m_{\tilde{g}}$  can all be set independently, giving rise to diverse and potentially distinctive collider signatures [52]. Supersymmetric particle decay chains can be longer and more intricate, with significant momentum transferred to particles in the secluded sector [51]. While this may reduce the sensitivity of conventional searches, the Higgsino mass can be as low as about 200 GeV, simultaneously opening new avenues for novel detection strategies [70].

In summary, the GNMSSM constitutes a well-motivated extension of the  $Z_3$ -NMSSM, offering greater theoretical consistency and phenomenological flexibility. By introducing  $\mu$ ,  $\mu'$ , and optionally  $\xi$  terms that break the  $Z_3$  symmetry, the model allows  $\lambda$  and  $\kappa$  to play distinct roles in DM physics:  $\lambda$  can remain small to satisfy direct-detection constraints, while  $\kappa$  can take larger values to dominate the

DM self-annihilation. These properties make the GNMSSM framework particularly attractive for simultaneously addressing the challenges of SUSY naturalness, DM phenomenology, and Higgs physics in a coherent and less fine-tuned manner. As demonstrated in the following section, the GNMSSM naturally predicts a viable DM candidate in the tens-of-GeV mass range, since several key DM couplings remain appreciable even after all current experimental constraints have been imposed.

### III. COMPREHENSIVE STUDY OF GNMSSM

This section presents a comprehensive numerical framework for studying light DM in the GNMSSM. The model was built using the SARAH-4.15.3 [71–74] package. The particle spectra and flavor observables were calculated by the SPheno-4.0.5 [75,76] and FLAVORKIT [77] codes, respectively. DM observables were computed with the micrOMEGAs-5.0.4 [78–83] code.

#### A. Research strategy

A joint likelihood function  $\mathcal{L}$  was first formulated to guide the sampling process and accurately delineate the GNMSSM parameter space as follows:

$$\begin{aligned}\mathcal{L} &\equiv \mathcal{L}_{\Omega h^2} \times \mathcal{L}_{\text{LZ}} \times \mathcal{L}_{\text{Const}}, \\ \mathcal{L}_{\Omega h^2} &= \exp \left[ -\frac{1}{2} \left( \frac{\Omega h^2 - 0.120}{0.012} \right)^2 \right], \quad \mathcal{L}_{\text{LZ}} = \exp \left[ -\frac{1}{2} \left( \frac{\sigma_{\text{eff}}^{\text{SI}}}{\delta_{\sigma}} \right)^2 \right], \\ \mathcal{L}_{\text{Const}} &= \begin{cases} 1 & \text{if satisfying all experimental constraints} \\ \exp(-100) & \text{otherwise.} \end{cases}\end{aligned}\quad (3.1)$$

In this construction, the relic abundance  $\Omega h^2$  is assumed to follow a Gaussian distribution  $\mathcal{L}_{\Omega h^2}$ , with its central value derived from the Planck experiment [84] and a theoretical uncertainty of 10%. Given the smallness of the SI scattering cross section, and consequently that  $\sigma_p^{\text{SI}}$  may significantly deviate from  $\sigma_n^{\text{SI}}$ , the effective SI cross section in the Gaussian likelihood term  $\mathcal{L}_{\text{LZ}}$  is defined as  $\sigma_{\text{eff}}^{\text{SI}} = 0.169\sigma_p^{\text{SI}} + 0.347\sigma_n^{\text{SI}} + 0.484\sqrt{\sigma_p^{\text{SI}} \times \sigma_n^{\text{SI}}}$  [85] for comparison with the results from LZ 2022 [86]. The variance square is then given by  $\delta_{\sigma}^2 \equiv \text{UL}_{\sigma}^2 / 1.64^2 + (0.2\sigma)^2$ , where  $\text{UL}_{\sigma}$  refers to the 90% CL upper limit on the SI scattering cross section, and  $0.2\sigma$  accounts for uncertainties [87]. The  $\mathcal{L}_{\text{Const}}$  term encompasses the following experimental constraints:

- (1) *Higgs data fit*: The properties of  $h$  must align with the LHC Higgs data at 95% CL. This alignment is enforced by the HIGGSIGNAL 2.6.2 code [88–91] with the requirement that the  $p$  value be greater than 0.05, where a 3 GeV combined theoretical and experimental uncertainty for  $m_h$  is also considered.

In addition, HIGGSTOOLS-1.2 [63], which incorporates 129 Higgs observables, was used to update constraints from the Higgs data by imposing the condition  $\Delta\chi^2 \equiv \chi_{\text{Higgs}}^2 - \chi_{\text{SM},125}^2 \lesssim 6.18$ , where  $\chi_{\text{SM},125}^2 \simeq 176.3$  corresponds to a pure SM Higgs boson in the GNMSSM [64].

- (2) *Extra Higgs searches*: The HIGGSBOUNDS 5.10.2 code [92–96] was employed to conduct comprehensive searches for additional Higgs bosons beyond the SM at the large electron-positron collider (LEP), Tevatron and LHC. This search was further complemented by HIGGSTOOLS-1.2 [63] with enhanced search capabilities.
- (3) *Indirect DM searches*: The Fermi-LAT Collaboration has accumulated years of dwarf galaxy observations, yielding constraints on DM annihilation cross sections.<sup>3</sup> The likelihood function from Refs. [97,98] was applied to evaluate these bounds. The latest MADHAT

<sup>3</sup>See [https://www-glast.stanford.edu/pub\\_data/1048](https://www-glast.stanford.edu/pub_data/1048).

TABLE I. An exploration of the parameter space for the  $h_1$  scenario. All inputs were assumed to exhibit a flat prior distribution, reflecting their physical meaning in the small  $\lambda$  case as shown in Ref. [52]. The parameter  $v_s$  was defined as  $\langle S \rangle$ , and  $m_B$  and  $m_C$  are the mass parameters for the  $CP$ -even and -odd singlet Higgs fields, respectively. The soft trilinear coefficients for the third-generation squarks were set equal ( $A_t = A_b$ ) for simplicity and allowed to vary due to their significant radiative corrections to the SM-like Higgs boson mass. We fixed  $m_A = 2$  TeV,  $M_1 = 150$  GeV,  $M_3 = 3$  TeV, and set all other unmentioned dimensional parameters to 2 TeV. All parameters were defined at the renormalization scale  $Q_{\text{input}} = 1$  TeV. The  $h_2$  scenario shared the same parameter space as in the  $h_1$  scenario, except that  $5 \text{ GeV} \leq m_B \leq 130 \text{ GeV}$ .

| Parameter        | Prior | Range          | Parameter                     | Prior | Range    |
|------------------|-------|----------------|-------------------------------|-------|----------|
| $\lambda$        | Flat  | $10^{-3}$ –0.2 | $\mu_{\text{tot}}/\text{GeV}$ | Flat  | 200–1000 |
| $\kappa$         | Flat  | −0.7–0.7       | $m_B/\text{GeV}$              | Flat  | 95–250   |
| $\delta$         | Flat  | −1.0–1.0       | $m_C/\text{GeV}$              | Flat  | 5–250    |
| $\tan \beta$     | Flat  | 1–60           | $m_{\tilde{S}}/\text{GeV}$    | Flat  | −102–102 |
| $v_s/\text{GeV}$ | Flat  | $1$ – $10^3$   | $A_t/\text{TeV}$              | Flat  | −5.0–5.0 |

package [99,100], which integrates 14 years of public Fermi-LAT data from 54 dwarf spheroidal galaxies, was also employed to enhance the robustness of this analysis.

- (4) *B-physics observables*: The branching ratios of  $B_s \rightarrow \mu^+ \mu^-$  and  $B \rightarrow X_s \gamma$  should agree with experimental measurements at the  $2\sigma$  level [101].

Equation (3.1) was used to implement the MultiNest algorithm with  $n_{\text{live}} = 8000$  [102] and conduct comprehensive scans over the parameter space shown in Table I. Samples obtained in this way were then surveyed to see their compatibility with LHC-based electroweakino search results. This was done by performing a series of Monte Carlo simulations for the following production channels:

$$\begin{aligned}
 pp &\rightarrow \tilde{\chi}_\alpha^0 \tilde{\chi}_\beta^0, & \alpha, \beta &= 2, 3, 4, 5; & \alpha &\leq \beta, \\
 pp &\rightarrow \tilde{\chi}_i^\pm \tilde{\chi}_j^\mp, & i, j &= 1, 2, \\
 pp &\rightarrow \tilde{\chi}_\alpha^0 \tilde{\chi}_i^\pm, & \alpha &= 2, 3, 4, 5; & i &= 1, 2.
 \end{aligned} \quad (3.2)$$

In this process, corresponding cross sections were first calculated to next-to-leading order using the program PROSPINO2 [103], with LHC fast simulations conducted by the SMOBELS-3.0.0 [104] tool. Specifically, we selected all analyses included in the `smodels-database-release-3.0.0` and combined signal regions when available, to provide a preliminary assessment. Given the exclusion capabilities of SMOBELS-3.0.0 are limited by the database and strict working prerequisites, we further examined surviving samples by simulating with the Monte Carlo method the analyses listed in Table II. In more detail, we generated  $10^5$  events for these processes using the program MADGRAPH\_aMC@NLO [105,106], and implemented parton showers and hadronization using PYTHIA8 [107]. The resulting event files were then fed into the CHECKMATE-2.0.37 [108–110] package, which integrates the modular DELPHES-3.5 [111] framework for detector simulations. The

$R$  value, defined as  $R \equiv \max\{S_i/S_{i,\text{obs}}^{95}\}$ , was subsequently calculated for all involved analyses, where  $S_i$  represents a simulated event number for the  $i$ -th signal region and  $S_{i,\text{obs}}^{95}$  is the corresponding 95% CL upper limit. A value of  $R > 1$  typically implies the sample is experimentally excluded if the involved uncertainties are neglected [112], while  $R < 1$  indicates consistency with experimental data.

## B. Numerical results

The included parameter space scan produced 23732 samples for the  $h_1$  scenario and 21724 samples for the  $h_2$  scenario. It is notable that the LZ experiment recently published its 2024 results [4], which prompted us to refine these samples using new constraints that significantly reduced sample sizes to 8983 and 6155, respectively. The previously mentioned LHC constraints further narrowed the surviving samples to 6379 for the  $h_1$  scenario and 4874 for the  $h_2$  scenario.

To characterize light DM properties comprehensively, Table III classifies samples satisfying both the LZ 2024 constraints and the LHC threshold criteria  $R < 1$  by their dominant annihilation channels. The analysis identifies  $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow A_s A_s$  as the predominant channel in the  $h_1$  scenario. This  $p$ -wave process proceeds under the threshold conditions  $|m_{\tilde{\chi}_1^0}| \gtrsim m_{A_s}$  and  $m_{h_s} \gtrsim 2|m_{\tilde{\chi}_1^0}| - m_{A_s}$ . At freeze-out temperature, the thermally averaged cross section  $\langle \sigma v \rangle_{x_F}^{A_s A_s}$  can be approximated by [10]

$$\begin{aligned}
 \langle \sigma v \rangle_{x_F}^{A_s A_s} \simeq & \frac{v_F^2}{128\pi m_{\tilde{\chi}_1^0}^2} \sqrt{1 - \frac{m_{A_s}^2}{m_{\tilde{\chi}_1^0}^2}} 8\kappa^4 \left\{ \frac{1}{3} \frac{\left[ \left( \frac{m_{A_s}}{m_{\tilde{\chi}_1^0}} \right)^2 - 1 \right]^2}{\left[ \frac{1}{2} \left( \frac{m_{A_s}}{m_{\tilde{\chi}_1^0}} \right)^2 - 1 \right]^4} \right. \\
 & \left. + \frac{\left( \frac{A_s - m_{\tilde{S}}}{m_{\tilde{\chi}_1^0}} \right)^2}{\left[ \left( \frac{m_{h_s}}{m_{\tilde{\chi}_1^0}} \right)^2 - 4 \right]^2 + \left( \frac{m_{h_s}}{m_{\tilde{\chi}_1^0}} \right)^2 \left( \frac{\Gamma_{h_s}}{m_{\tilde{\chi}_1^0}} \right)^2} \right\}, \quad (3.3)
 \end{aligned}$$

where  $v_F \simeq 0.3$  represents the DM velocity at freeze-out. The first and second terms in the braces arise from  $t$ - and

TABLE II. LHC analyses for electroweakino pair production included in this study, categorized by final-state signatures.

| Scenario                                                                                                                                                                                                                                                                                                                                               | Final state                                                                   | Analysis                                                                                                                                                                                                                                                                                                                                                |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\tilde{\chi}_1^+ \tilde{\chi}_1^- \rightarrow 2\tilde{\ell}\nu(\ell\bar{\nu})$                                                                                                                                                                                                                                                                        | $2\ell + E_T^{\text{miss}}$                                                   | ATLAS-CONF-2016-096 (13.3 fb <sup>-1</sup> ) [113]<br>ATLAS-1908-08215 (139 fb <sup>-1</sup> ) [114]                                                                                                                                                                                                                                                    |
| $\tilde{\chi}_1^+ \tilde{\chi}_1^- \rightarrow WW\tilde{\chi}_1^0\tilde{\chi}_1^0$                                                                                                                                                                                                                                                                     | $2\ell + E_T^{\text{miss}}$                                                   | ATLAS-1908-08215 (139 fb <sup>-1</sup> ) [114]                                                                                                                                                                                                                                                                                                          |
| $\tilde{\chi}_2^0 \tilde{\chi}_1^\pm \rightarrow \ell\tilde{\ell}\tilde{\ell}\nu(\ell\bar{\nu}\nu\bar{\nu})$                                                                                                                                                                                                                                           | $3\ell + E_T^{\text{miss}}$                                                   | ATLAS-CONF-2016-096 (13.3 fb <sup>-1</sup> ) [113]<br>ATLAS-1803-02762 (36.1 fb <sup>-1</sup> ) [115]<br>CMS-SUS-16-039 (35.9 fb <sup>-1</sup> ) [116]                                                                                                                                                                                                  |
| $\tilde{\chi}_2^0 \tilde{\chi}_1^\pm \rightarrow WZ\tilde{\chi}_1^0\tilde{\chi}_1^0$                                                                                                                                                                                                                                                                   | $3\ell + E_T^{\text{miss}}$                                                   | ATLAS-2106-01676 (139 fb <sup>-1</sup> ) [37]<br>ATLAS-CONF-2019-020 (139 fb <sup>-1</sup> ) [117]                                                                                                                                                                                                                                                      |
| $\tilde{\chi}_1^0 \tilde{\chi}_2^0 \rightarrow Z\tilde{\chi}_1^0\tilde{\chi}_1^0$<br>$\tilde{\chi}_1^0 \tilde{\chi}_1^\pm \rightarrow W\tilde{\chi}_1^0\tilde{\chi}_1^0$<br>$\tilde{\chi}_2^0 \tilde{\chi}_1^\pm \rightarrow WZ\tilde{\chi}_1^0\tilde{\chi}_1^0$<br>$\tilde{\chi}_1^+ \tilde{\chi}_1^- \rightarrow WW\tilde{\chi}_1^0\tilde{\chi}_1^0$ | $n\ell(n \geq 4) + E_T^{\text{miss}}$                                         | ATLAS-2103-11684 (139 fb <sup>-1</sup> ) [118]                                                                                                                                                                                                                                                                                                          |
| $\tilde{\chi}_2^0 \tilde{\chi}_1^\pm \rightarrow \tau\bar{\tau}\tilde{\tau}\nu$                                                                                                                                                                                                                                                                        | $3\tau + E_T^{\text{miss}}$                                                   | CMS-SUS-16-039 (35.9 fb <sup>-1</sup> ) [116]                                                                                                                                                                                                                                                                                                           |
| $\tilde{\chi}_2^0 \tilde{\chi}_1^\pm \rightarrow \ell\bar{\ell}\tilde{\ell}\nu$                                                                                                                                                                                                                                                                        | $2\ell + 1\tau + E_T^{\text{miss}}$                                           | CMS-SUS-16-039 (35.9 fb <sup>-1</sup> ) [116]                                                                                                                                                                                                                                                                                                           |
| $\tilde{\chi}_2^0 \tilde{\chi}_1^\pm \rightarrow WZ\tilde{\chi}_1^0\tilde{\chi}_1^0$<br>$\tilde{\chi}_1^+ \tilde{\chi}_1^- \rightarrow WW\tilde{\chi}_1^0\tilde{\chi}_1^0$                                                                                                                                                                             | $2j(\text{large}) + E_T^{\text{miss}}$                                        | ATLAS-2108-07586 (139 fb <sup>-1</sup> ) [38]                                                                                                                                                                                                                                                                                                           |
| $\tilde{\chi}_2^0 \tilde{\chi}_3^0 \rightarrow ZZ(Zh)\tilde{\chi}_1^0\tilde{\chi}_1^0$<br>$\tilde{\chi}_2^0 \tilde{\chi}_1^\pm \rightarrow WZ(W\bar{h})\tilde{\chi}_1^0\tilde{\chi}_1^0$                                                                                                                                                               | $j(\text{large}) + b(\text{large}) + E_T^{\text{miss}}$                       | ATLAS-2108-07586 (139 fb <sup>-1</sup> ) [38]                                                                                                                                                                                                                                                                                                           |
| $\tilde{\chi}_2^0 \tilde{\chi}_1^\pm \rightarrow WZ\tilde{\chi}_1^0\tilde{\chi}_1^0$                                                                                                                                                                                                                                                                   | $2\ell(\text{soft}) + \text{jets} + E_T^{\text{miss}}$                        | CMS-PAS-SUS-16-025 (12.9 fb <sup>-1</sup> ) [119]                                                                                                                                                                                                                                                                                                       |
| $\tilde{\chi}_2^0 \tilde{\chi}_1^\pm \rightarrow WZ\tilde{\chi}_1^0\tilde{\chi}_1^0$                                                                                                                                                                                                                                                                   | $n\ell(n \geq 2) + nj(n \geq 0) + E_T^{\text{miss}}$                          | ATLAS-1712-08119 (36.1 fb <sup>-1</sup> ) [120]<br>ATLAS-1803-02762 (36.1 fb <sup>-1</sup> ) [115]<br>ATLAS-1911-12606 (139 fb <sup>-1</sup> ) [121]<br>ATLAS-2106-01676 (139 fb <sup>-1</sup> ) [37]<br>CMS-SUS-16-039 (35.9 fb <sup>-1</sup> ) [116]<br>CMS-SUS-16-048 (35.9 fb <sup>-1</sup> ) [122]<br>CMS-SUS-20-001 (137 fb <sup>-1</sup> ) [123] |
| $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow ZZ(Zh)(hh)\tilde{G}\tilde{G}$                                                                                                                                                                                                                                                                           | $n\ell(n \geq 2) + nj(n \geq 0) + E_T^{\text{miss}}$                          | CMS-SUS-16-039 (35.9 fb <sup>-1</sup> ) [116]<br>CMS-SUS-20-001 (137 fb <sup>-1</sup> ) [123]                                                                                                                                                                                                                                                           |
| $\tilde{\chi}_2^0 \tilde{\chi}_1^\pm \rightarrow Wh\tilde{\chi}_1^0\tilde{\chi}_1^0$                                                                                                                                                                                                                                                                   | $n\ell(n \geq 1) + nb(n \geq 0) + nj(n \geq 0) + E_T^{\text{miss}}$           | ATLAS-1909-09226 (139 fb <sup>-1</sup> ) [124]<br>CMS-SUS-16-039 (35.9 fb <sup>-1</sup> ) [116]                                                                                                                                                                                                                                                         |
| $\tilde{\chi}_2^0 \tilde{\chi}_1^\pm \rightarrow Wh\tilde{\chi}_1^0\tilde{\chi}_1^0$                                                                                                                                                                                                                                                                   | $2\gamma + nl(n \geq 1) + E_T^{\text{miss}}$                                  | ATLAS-2004-10894 (139 fb <sup>-1</sup> ) [125]                                                                                                                                                                                                                                                                                                          |
| $\tilde{\chi}_2^0 \tilde{\chi}_1^\pm \rightarrow Wh\tilde{\chi}_1^0\tilde{\chi}_1^0$                                                                                                                                                                                                                                                                   | $2\gamma + nj(n \geq 2) + E_T^{\text{miss}}$                                  | ATLAS-2004-10894 (139 fb <sup>-1</sup> ) [125]                                                                                                                                                                                                                                                                                                          |
| $\tilde{\chi}_2^0 \tilde{\chi}_1^\mp \rightarrow Wh(WZ)\tilde{\chi}_1^0\tilde{\chi}_1^0$<br>$\tilde{\chi}_1^+ \tilde{\chi}_1^- \rightarrow WW\tilde{\chi}_1^0\tilde{\chi}_1^0$                                                                                                                                                                         | $2\gamma + n\ell(n \geq 0) + nb(n \geq 0) + nj(n \geq 0) + E_T^{\text{miss}}$ | ATLAS-1802-03158 (36.1 fb <sup>-1</sup> ) [126]                                                                                                                                                                                                                                                                                                         |

$s$ -channel Feynman diagrams, respectively. This formula reveals several key features:

- (1) *Noninterference property*: The  $s$ - and  $t$ -channel contributions do not interfere. Both scale as  $\kappa^4/m_{\tilde{\chi}_1^0}^2$ , modulated by a common phase-space factor  $\sqrt{1-R_1}$  with  $R_1 \equiv (m_{A_s}/m_{\tilde{\chi}_1^0})^2$ .

- (2)  *$t$ -channel behavior*: For fixed  $m_{\tilde{\chi}_1^0}$ , the  $t$ -channel contribution depends solely on  $\kappa$  and the squared mass ratio  $R_1$ . When normalized to its value at  $R_1 = 0$ , it equals approximately 0.87, 0.68, 0.42, and 0.14 for  $R_1 = 0.2, 0.4, 0.6$ , and  $0.8$ , respectively—decreasing monotonically as  $R_1$  grows. Thus, its overall

TABLE III. Primary annihilation channels and their proportions for light DM in both  $h_1$  and  $h_2$  scenarios, with  $M_1 = 150$  GeV and  $f = u, d, c, s, b, \tau$ . The first row of each scenario takes into account all samples that meet LZ 2024 constraints. The second row corresponds to those further satisfying the LHC constraints, namely predicting an  $R$  value less than unity.

| Scenario | Sample size | $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow h_s h_s$ | $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow A_s A_s$ | $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow h_s A_s$ | $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow f \bar{f}$ |
|----------|-------------|---------------------------------------------------------|---------------------------------------------------------|---------------------------------------------------------|-----------------------------------------------------------|
| $h_1$    | 8983        | 0                                                       | 98.3%                                                   | 1.0%                                                    | 0.6%                                                      |
|          | 6379        | 0                                                       | 98.3%                                                   | 1.2%                                                    | 0.4%                                                      |
| $h_2$    | 6155        | 0                                                       | 9.8%                                                    | 90.0%                                                   | 0                                                         |
|          | 4874        | 0                                                       | 9.1%                                                    | 90.8%                                                   | 0                                                         |

contribution is more sensitive to  $\kappa$  than to the squared mass ratio  $R_1$ .

- (3) *s-channel complexity*: In contrast to the  $t$  channel, the  $s$ -channel contribution depends on additional parameters: the mass of  $h_s$ , its width  $\Gamma_{h_s}$ , and the combination  $D \equiv (A_\kappa - m_{\tilde{S}})/m_{\tilde{\chi}_1^0}$  arising from the  $h_s A_s A_s$  coupling. Among the samples from our scan,  $\Gamma_{h_s}$  remains very small and  $h_s$  rarely approaches resonance during annihilation. Consequently, the width term can be safely neglected.
- (4) *Magnitude comparison*: While the  $t$ -channel amplitude in braces never exceeds  $1/3$ , the  $s$ -channel component can be several orders of magnitude larger when  $|D|$  is substantial and  $m_{h_s}$  approaches  $2|m_{\tilde{\chi}_1^0}|$ . Consequently, the  $s$ -channel annihilation often governs the relic density.

Figure 1 projects the samples dominated by  $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow A_s A_s$  annihilation onto the  $|m_{\tilde{\chi}_1^0}|-|D|$  and  $|m_{\tilde{\chi}_1^0}|-|\kappa|$  planes. Compared to the full parameter space defined in Table I, accommodating the observed relic density while satisfying the LZ 2024 constraints requires  $|\kappa| \lesssim 0.3$  and

$|m_{\tilde{\chi}_1^0}| \gtrsim 10$  GeV. Further analysis reveals distinct mass-dependent behaviors across three regimes:

- (1) *Light DM regime* (10–30 GeV): As shown in the left panel of Fig. 1, the  $s$ -channel contribution remains subdominant due to  $m_{h_s}^2 \gg m_{\tilde{\chi}_1^0}^2$ , given that  $m_{h_s}$  exceeds 125 GeV. It gradually strengthens and becomes comparable with the  $t$  channel as  $|m_{\tilde{\chi}_1^0}|$  approaches 30 GeV. By contrast, despite substantial  $R_1$  values (numerically  $0.6 \lesssim R_1 \lesssim 0.8$ ) that suppress the  $t$ -channel amplitude, sufficiently large  $\kappa$  ensures  $t$ -channel dominance in determining the relic abundance, as illustrated in the right panel of Fig. 1.
- (2) *Intermediate masses* (30–60 GeV): The  $s$ -channel contribution increases progressively and begins to surpass the  $t$  channel, which diminishes due to its inverse proportionality to  $m_{\tilde{\chi}_1^0}^2$ .
- (3) *Heavy DM regime* (> 60 GeV): A dramatic transition occurs wherein the  $s$  channel typically dominates and intensifies further with increasing mass. This dominance arises through two mechanisms. First,

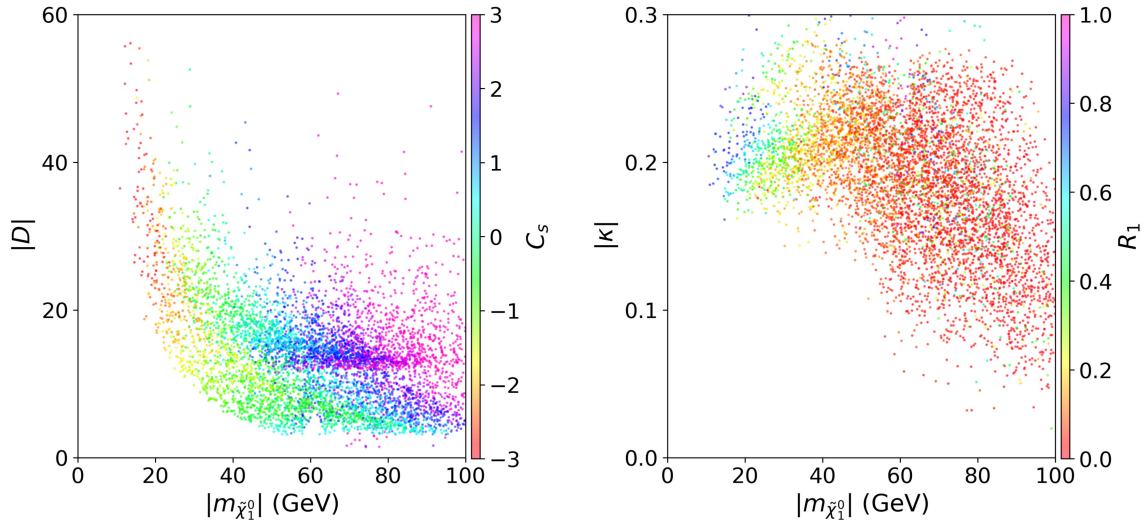


FIG. 1. Scatter plots for samples dominated by  $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow A_s A_s$  annihilation and satisfying all experimental constraints in the  $h_1$  scenario. Left panel:  $|D| \equiv |(A_\kappa - m_{\tilde{S}})/m_{\tilde{\chi}_1^0}|$  versus  $|m_{\tilde{\chi}_1^0}|$ , with color scale  $C_s$  defined by  $2\{\ln |(A_\kappa - m_{\tilde{S}})/m_{\tilde{\chi}_1^0}| - \ln |(m_{h_s}/m_{\tilde{\chi}_1^0})^2 - 4|\}$ , indicating relative  $s$ -channel strength in Eq. (3.3). Right panel:  $|\kappa|$  versus  $|m_{\tilde{\chi}_1^0}|$ , colored by the squared mass ratio  $R_1 \equiv (m_{A_s}/m_{\tilde{\chi}_1^0})^2$ .

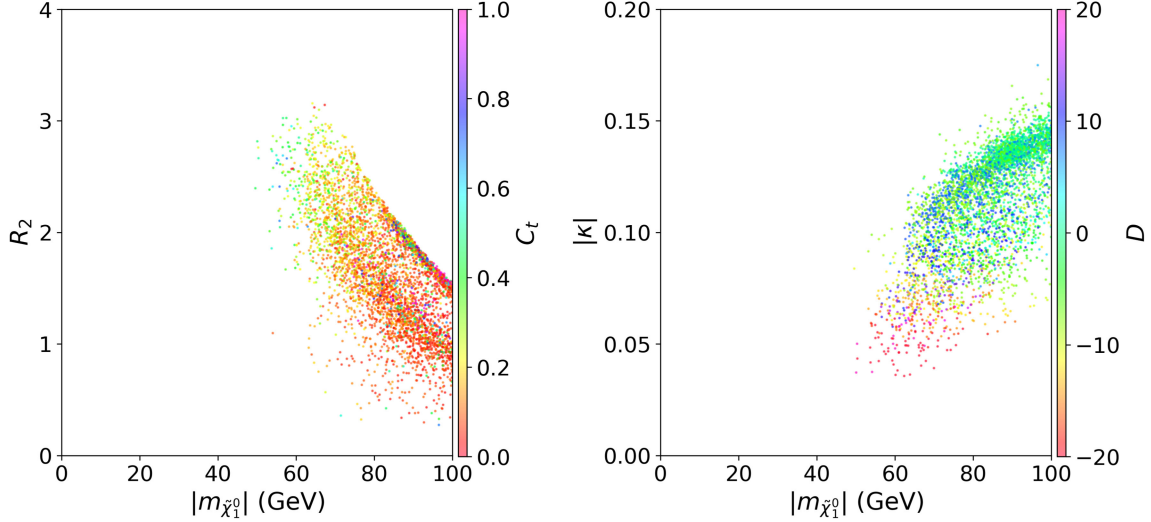


FIG. 2. Scatter plots for samples dominated by  $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow h_s A_s$  annihilation and satisfying all experimental constraints in the  $h_2$  scenario. Left panel:  $R_2 \equiv (m_{h_s}/m_{\tilde{\chi}_1^0})^2$  versus  $|m_{\tilde{\chi}_1^0}|$ , with a color bar  $C_t$  defined by  $8m_{A_s}^2/(4m_{\tilde{\chi}_1^0}^2 - m_{h_s}^2)$ , which measures the partial  $t$ -channel contributions of Eq. (3.4). Right panel:  $|\kappa|$  versus  $|m_{\tilde{\chi}_1^0}|$ , with a color bar representing the  $s$ -channel contribution  $D \equiv (A_\kappa - m_{\tilde{S}})/m_{\tilde{\chi}_1^0}$  in Eq. (3.4).

most samples satisfy  $0.6 \lesssim m_{h_s}/(2|m_{\tilde{\chi}_1^0}|) \lesssim 1.4$ , where near-resonance effects enhance the  $s$  channel. Second, for samples far from resonance,  $|A_\kappa/m_{\tilde{\chi}_1^0}|$  becomes sufficiently large to ensure  $s$ -channel dominance. These mechanisms allow  $\kappa$  as low as 0.04 to produce the measured abundance at  $|m_{\tilde{\chi}_1^0}| \simeq 100$  GeV.

In the  $h_2$  scenario, DM achieves the observed relic abundance primarily through the  $s$ -wave annihilation channel  $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow h_s A_s$ , as listed in Table III. This process requires the kinematic condition  $2|m_{\tilde{\chi}_1^0}| \gtrsim m_{h_s} + m_{A_s}$ , suggesting that all the singlet-dominated particles are likely to be simultaneously light. Compared with the  $h_1$  scenario with a massive  $h_s$ , the  $h_2$  case is more tightly constrained by current Higgs coupling measurements and by LHC searches for exotic decays of the SM-like Higgs boson [127]. The underlying reason is that, according to the approximation in Eq. (2.29),  $h$  typically contains a more substantial singlet admixture in this scenario. Consequently, its couplings to singlet-dominated states become stronger when  $\kappa$  is sizable, leading to potentially large branching fractions for exotic decays such as  $h \rightarrow \tilde{\chi}_1^0 \tilde{\chi}_1^0$ ,  $h_s h_s$ , and  $A_s A_s$ .

The combined constraints from Higgs physics and the measured relic density yield a preferred particle spectrum characterized by  $m_{h_s} > |m_{\tilde{\chi}_1^0}| \gtrsim 60$  GeV and  $m_{A_s}^2/m_{\tilde{\chi}_1^0}^2 < 1/4$  for most samples, with an absolute upper limit of  $m_{A_s}^2/m_{\tilde{\chi}_1^0}^2 < 1/2$  across all surviving samples. These features are illustrated in the left panel of Fig. 2, where samples dominated by the  $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow h_s A_s$  annihilation channel are projected onto  $R_2 \equiv m_{h_s}^2/m_{\tilde{\chi}_1^0}^2$  versus the  $|m_{\tilde{\chi}_1^0}|$  plane.

The color scale  $C_t \equiv 8m_{A_s}^2/(4m_{\tilde{\chi}_1^0}^2 - m_{h_s}^2)$  quantifies the partial  $t$ -channel contribution as introduced below. The right panel displays the same samples in the  $|\kappa|$  versus  $|m_{\tilde{\chi}_1^0}|$  plane, where the color variable  $D \equiv (A_\kappa - m_{\tilde{S}})/m_{\tilde{\chi}_1^0}$  represents the combination governing the  $s$ -channel contribution.

The thermally averaged cross section at freeze-out for this channel,  $\langle \sigma v \rangle_{x_F}^{h_s A_s}$ , is expressed as [2,61,128]

$$\langle \sigma v \rangle_{x_F}^{h_s A_s} \simeq \frac{1}{256\pi m_{\tilde{\chi}_1^0}^2} \left\{ \left[ 4 - \frac{(m_{h_s} + m_{A_s})^2}{m_{\tilde{\chi}_1^0}^2} \right] \times \left[ 4 - \frac{(m_{h_s} - m_{A_s})^2}{m_{\tilde{\chi}_1^0}^2} \right] \right\}^{1/2} |\mathcal{A}_s + \mathcal{A}_t|^2,$$

where the  $s$ - and  $t$ -channel amplitudes,  $\mathcal{A}_s$  and  $\mathcal{A}_t$ , depend primarily on the spectrum and couplings of the singlet-dominated fields. For small  $\lambda$  and  $m_{A_s}^2 \ll 4m_{\tilde{\chi}_1^0}^2$ , this expression simplifies to

$$\langle \sigma v \rangle_{x_F}^{h_s A_s} \simeq \frac{1}{256\pi m_{\tilde{\chi}_1^0}^2} \left[ 4 - \left( \frac{m_{h_s}}{m_{\tilde{\chi}_1^0}} \right)^2 \right] |\mathcal{A}_s + \mathcal{A}_t|^2, \quad (3.4)$$

with the amplitudes approximated as

$$\mathcal{A}_s \simeq \kappa^2 D, \quad \mathcal{A}_t \simeq \kappa^2 (4 + C_t). \quad (3.5)$$

The color coding in Fig. 2 reveals that the amplitude  $\mathcal{A}_s$  varies substantially from  $-20\kappa^2$  to  $20\kappa^2$ , while  $\mathcal{A}_t$  remains relatively stable between  $4\kappa^2$  and  $5\kappa^2$ . The sign of  $\mathcal{A}_s$  determines whether it interferes constructively or

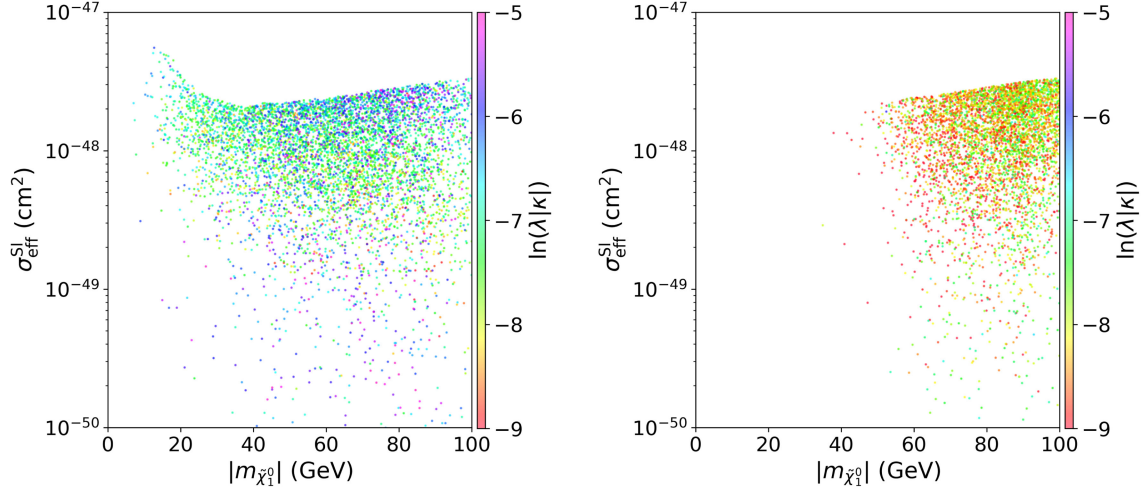


FIG. 3. Scatter plots of  $\sigma_{\text{eff}}^{\text{SI}}$  versus  $|m_{\tilde{\chi}_1^0}|$  for the samples in the  $h_1$  scenario (left panel) and the  $h_2$  scenario (right panel). The color bar is defined as  $\ln(\lambda|\kappa|)$ , describing the coupling strength involved in the scattering.

destructively with  $\mathcal{A}_t$ , thereby significantly affecting the required magnitude of  $\kappa$  for achieving the measured relic density. Specifically, when  $D \simeq 20$ , the  $s$ -channel contribution dominates and constructive interference allows  $|\kappa|$  as low as 0.03 for  $|m_{\tilde{\chi}_1^0}| \simeq 60$  GeV to acquire the density. Conversely, in the  $|D| \lesssim 10$  regime—which characterizes the majority of surviving samples—the two amplitudes become comparable. When destructive interference occurs, larger  $|\kappa|$  values up to 0.16 are necessary at  $|m_{\tilde{\chi}_1^0}| \simeq 100$  GeV to compensate for the reduced total amplitude. These relationships are evident in the right panel of Fig. 2.

We clarify that in the  $h_2$  scenario, the singlet-like Higgs mass satisfies  $m_{h_s} \lesssim 125$  GeV, and this upper limit is consistently achieved when  $|m_{\tilde{\chi}_1^0}| \gtrsim 70$  GeV for the obtained samples. Consequently, the maximum reach of  $R_2$  decreases monotonically with increasing  $|m_{\tilde{\chi}_1^0}|$ , as seen in the left panel of Fig. 2. Besides, since  $\langle \sigma v \rangle_{\tilde{\chi}_1^0 \tilde{\chi}_1^0}^{h_s A_s}$  is proportional to  $\kappa^4/m_{\tilde{\chi}_1^0}^2$ , larger  $|\kappa|$  values are required to maintain the measured relic abundance as  $|m_{\tilde{\chi}_1^0}|$  increases, a trend clearly visible in the right panel. Additionally, as exhibited by the correlation between the two panels, for fixed  $|m_{\tilde{\chi}_1^0}| \simeq 60$  GeV, an increase in  $R_2$ —arising from enhanced  $m_{h_s}$  and consequently reduced annihilation phase space—requires a correspondingly larger  $\kappa^2 D$  to preserve the measured relic density. This reflects the interplay of the parameters.

Next, we turn to DM-nucleon scattering. For  $m_{h_s} \lesssim 250$  GeV and  $m_H \simeq 2$  TeV, the spin-independent cross section  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SI}}$  is dominated by a  $t$ -channel exchange of  $h$  and  $h_s$ , with the leading contribution scaling as  $\lambda^2 \kappa^2$  due to a relatively light  $h_s$  [61]. Crucially, the two exchange amplitudes can interfere destructively, producing a notable cancellation that intricately interplays among samples with different  $\lambda\kappa$  values. Furthermore, with  $M_1 = 150$  GeV in

our analysis, the DM retains a Bino admixture up to the 0.1% level. Despite being small, this component yields a non-negligible contribution to  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SI}}$  because the Bino-Bino-SM Higgs coupling  $C_{\tilde{B}\tilde{B}H_{\text{SM}}}$  is much larger than the Singlino-Singlino-SM Higgs coupling  $C_{\tilde{S}\tilde{S}H_{\text{SM}}}$  when the Higgsino mass lies around several hundred GeV. Additionally, the small  $\tilde{H}_d$  component in  $\tilde{\chi}_1^0$  may also be non-negligible in the scattering since  $\tilde{S} - \tilde{H}_d - H_{\text{SM}}$  coupling  $C_{\tilde{S}\tilde{H}_d H_{\text{SM}}}$  is determined by  $\lambda$ . These effects, not previously emphasized in the literature, can substantially enhance the cancellations among different contributions to  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SI}}$ . These observations have important implications for satisfying LZ 2024 constraints: strong destructive interference permits larger  $\lambda|\kappa|$  values, while weak cancellation favors smaller  $\lambda|\kappa|$ .

This phenomenon is numerically verified in Fig. 3, where the left and right panels display results for the  $h_1$  and  $h_2$  scenarios, respectively. The figure reveals that the cancellation in the  $h_2$  scenario is significantly weaker than in the  $h_1$  scenario, thereby favoring smaller  $\lambda|\kappa|$ . Nevertheless, despite this difference in cancellation strengths, both scenarios can suppress  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SI}}$  to values as low as  $10^{-50}$  cm<sup>2</sup>. Notably, the required smallness of  $\lambda$  naturally aligns with constraints from Higgs data fitting, as elaborated in Sec. II B 2.

Finally, we examine the LHC constraints on supersymmetry. For this purpose, we project samples obtained from parameter scans and refined by LZ 2024 constraints onto  $|m_{\tilde{\chi}_1^0}|$ - $\mu_{\text{tot}}$  and  $|\kappa|$ - $\lambda$  planes, as shown in Fig. 4. Several key observations emerge:

- (1) When the Bino-like neutralino serves as the next-to-lightest supersymmetric particle (NLSP), the decay cascades are longer and the final states have higher multiplicities (often including a light  $A_s$ ), which

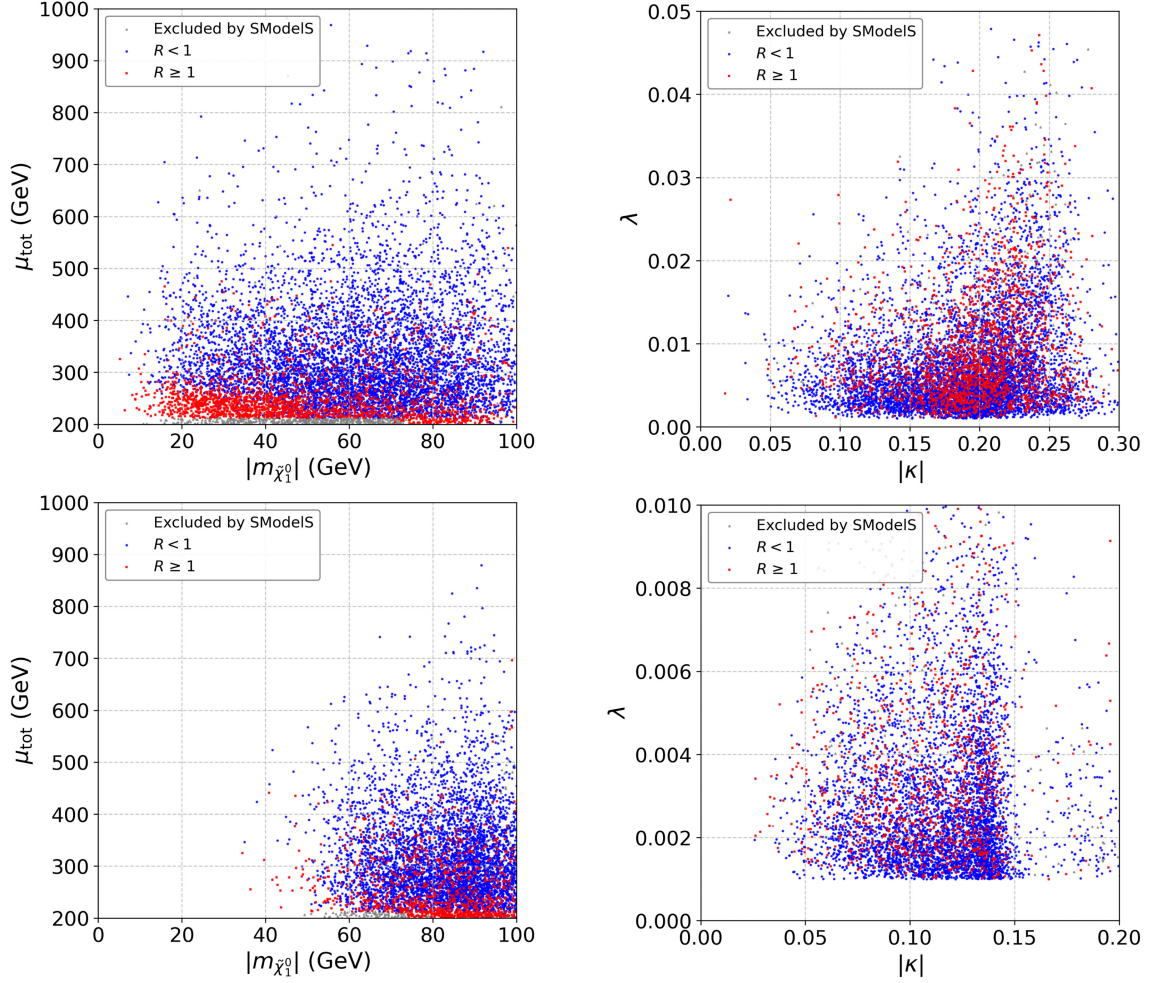


FIG. 4. LHC constraints on the parameters  $|m_{\tilde{\chi}_1^0}|$ ,  $\mu_{\text{tot}}$ ,  $|\kappa|$ , and  $\lambda$ . Samples include those obtained by the scans and refined by the LZ 2024 constraints. They are classified by exclusion levels, as depicted in the legend. Gray markers are excluded by SModelS-3.0.0, blue markers ( $R < 1$ ) are consistent with all experiments, and red markers ( $R \geq 1$ ) are excluded by LHC data. The upper panels correspond to the  $h_1$  scenario, while the lower panels refer to the  $h_2$  scenario.

degrades search sensitivity and relaxes the LHC bounds. In this hierarchy, i.e.,  $\tilde{S} < \tilde{B} < \tilde{H}$ , samples with  $\mu_{\text{tot}}$  as low as  $\sim 200$  GeV remain viable, alleviating the fine-tuning in predicting Z-boson mass and preserving theoretical naturalness.

- (2) Samples excluded by our recast are primarily characterized by final states containing three or more leptons, which fall into two categories [116]: (i) No  $\tau$  leptons, at least two opposite-sign same-flavor (OSSF) lepton pairs (electrons or muons), and missing transverse momentum  $p_T^{\text{miss}} \in (100, 150)$  GeV or  $p_T^{\text{miss}} \geq 200$  GeV and (ii) Two  $\tau$  leptons, fewer than two OSSF pairs, and  $p_T^{\text{miss}} \geq 150$  GeV. Note that the substantial missing transverse momentum in these signatures constitutes a characteristic feature of supersymmetric particle production.
- (3) The LHC and LZ 2024 constraints exhibit mutual consistency, as surviving samples display stable

distributions in  $\lambda$  and  $|\kappa|$  parameter space. Specifically, in the  $h_1$  scenario,  $\lambda$  was bounded above by  $\lambda \lesssim 0.048$ , while  $|\kappa|$  could extend up to 0.3. By contrast, the  $h_2$  scenario imposes more stringent constraints:  $\lambda \leq 0.01$  and  $|\kappa| \in (0.03, 0.2)$ , with most samples concentrated near  $|\kappa| \simeq 0.15$ .

The characteristic features of surviving samples are exemplified by two benchmark points detailed in Table IV. Both have low  $M_1$  and  $\mu_{\text{tot}}$ , and consequently the DM acquires small but non-negligible Bino and Higgsino admixtures, especially in the  $h_1$  case. These admixtures strengthen the destructive interference among the contributions to  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SI}}$ , thereby suppressing the effective cross section  $\sigma_{\text{eff}}^{\text{SI}}$  below the LZ 2024 limit. In addition, the sparticle spectra yield multistep decay chains with several competing modes (often involving a light  $A_s$ ), which weakens current LHC sensitivities and allows these benchmarks to evade existing searches.

TABLE IV. Detailed information for two benchmark points aligning with all currently available experimental results, with  $M_1 = 150$  GeV.

| Benchmark point BP1 ( $h_1$ scenario)                                                                                                       |             |                                   |                                     | Benchmark point BP2 ( $h_2$ scenario)                                                                                                       |             |                                   |                                     |
|---------------------------------------------------------------------------------------------------------------------------------------------|-------------|-----------------------------------|-------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------|-------------|-----------------------------------|-------------------------------------|
| $\lambda$                                                                                                                                   | 0.008       | $m_{h_s}$                         | 148.6 GeV                           | $\lambda$                                                                                                                                   | 0.002       | $m_{h_s}$                         | 100.4 GeV                           |
| $\kappa$                                                                                                                                    | -0.194      | $m_h$                             | 125.4 GeV                           | $\kappa$                                                                                                                                    | 0.110       | $m_h$                             | 125.5 GeV                           |
| $\delta$                                                                                                                                    | 0.375       | $m_H$                             | 1877.6 GeV                          | $\delta$                                                                                                                                    | 0.721       | $m_H$                             | 1840.6 GeV                          |
| $\tan\beta$                                                                                                                                 | 22.57       | $m_{A_s}$                         | 11.7 GeV                            | $\tan\beta$                                                                                                                                 | 23.88       | $m_{A_s}$                         | 10.7 GeV                            |
| $\mu$                                                                                                                                       | 224.9 GeV   | $m_{\tilde{\chi}_1^0}$            | -52.4 GeV                           | $\mu$                                                                                                                                       | 299.0 GeV   | $m_{\tilde{\chi}_1^0}$            | -73.3 GeV                           |
| $\mu_{\text{eff}}$                                                                                                                          | 3.2 GeV     | $m_{\tilde{\chi}_2^0}$            | 141.7 GeV                           | $\mu_{\text{eff}}$                                                                                                                          | 1.0 GeV     | $m_{\tilde{\chi}_2^0}$            | 146.3 GeV                           |
| $\mu_{\text{tot}}$                                                                                                                          | 228.1 GeV   | $m_{\tilde{\chi}_3^0}$            | -239.0 GeV                          | $\mu_{\text{tot}}$                                                                                                                          | 300.0 GeV   | $m_{\tilde{\chi}_3^0}$            | -312.1 GeV                          |
| $A_\lambda$                                                                                                                                 | 3275.5 GeV  | $m_{\tilde{\chi}_4^0}$            | 244.6 GeV                           | $A_\lambda$                                                                                                                                 | 2078.0 GeV  | $m_{\tilde{\chi}_4^0}$            | 313.2 GeV                           |
| $A_\kappa$                                                                                                                                  | -288.8 GeV  | $m_{\tilde{\chi}_5^0}$            | 2020.9 GeV                          | $A_\kappa$                                                                                                                                  | 521.3 GeV   | $m_{\tilde{\chi}_5^0}$            | 2021.0 GeV                          |
| $A_t$                                                                                                                                       | -3846.0 GeV | $m_{\tilde{\chi}_1^\pm}$          | 235.0 GeV                           | $A_t$                                                                                                                                       | -3546.2 GeV | $m_{\tilde{\chi}_1^\pm}$          | 308.4 GeV                           |
| $M_1$                                                                                                                                       | 150.0 GeV   | $m_{\tilde{\chi}_2^\pm}$          | 2021.2 GeV                          | $M_1$                                                                                                                                       | 150.0 GeV   | $m_{\tilde{\chi}_2^\pm}$          | 2021.2 GeV                          |
| $m_B$                                                                                                                                       | 149.9 GeV   | $\Omega h^2$                      | 0.122                               | $m_B$                                                                                                                                       | 101.5 GeV   | $\Omega h^2$                      | 0.130                               |
| $m_C$                                                                                                                                       | 16.7 GeV    | $\sigma_{\text{eff}}^{\text{SI}}$ | $2.00 \times 10^{-49} \text{ cm}^2$ | $m_C$                                                                                                                                       | 19.9 GeV    | $\sigma_{\text{eff}}^{\text{SI}}$ | $1.22 \times 10^{-48} \text{ cm}^2$ |
| $m_{\tilde{S}}$                                                                                                                             | -52.5 GeV   | $\sigma_n^{\text{SD}}$            | $4.00 \times 10^{-47} \text{ cm}^2$ | $m_{\tilde{S}}$                                                                                                                             | -73.4 GeV   | $\sigma_n^{\text{SD}}$            | $5.13 \times 10^{-50} \text{ cm}^2$ |
| $V_{h_s}^S, V_{h_s}^{\text{SM}}, V_{h_s}^S, V_{h_s}^{\text{SM}}$                                                                            |             |                                   |                                     | $V_{h_s}^S, V_{h_s}^{\text{SM}}, V_{h_s}^S, V_{h_s}^{\text{SM}}$                                                                            |             |                                   |                                     |
| -0.999, -0.004, 0.004, -0.999                                                                                                               |             |                                   |                                     | 0.999, -0.025, 0.025, 0.999                                                                                                                 |             |                                   |                                     |
| $N_{11}, N_{12}, N_{13}, N_{14}, N_{15}$                                                                                                    |             |                                   |                                     | $N_{11}, N_{12}, N_{13}, N_{14}, N_{15}$                                                                                                    |             |                                   |                                     |
| -0.001, 0.000, 0.001, 0.006, -0.999                                                                                                         |             |                                   |                                     | 0.000, 0.000, 0.000, 0.001, -0.999                                                                                                          |             |                                   |                                     |
| $N_{21}, N_{22}, N_{23}, N_{24}, N_{25}$                                                                                                    |             |                                   |                                     | $N_{21}, N_{22}, N_{23}, N_{24}, N_{25}$                                                                                                    |             |                                   |                                     |
| 0.939, -0.008, 0.290, -0.183, -0.002                                                                                                        |             |                                   |                                     | 0.978, -0.004, 0.186, -0.094, 0.000                                                                                                         |             |                                   |                                     |
| $N_{31}, N_{32}, N_{33}, N_{34}, N_{35}$                                                                                                    |             |                                   |                                     | $N_{31}, N_{32}, N_{33}, N_{34}, N_{35}$                                                                                                    |             |                                   |                                     |
| -0.077, 0.024, 0.699, 0.711, 0.005                                                                                                          |             |                                   |                                     | -0.065, 0.023, 0.701, 0.710, 0.001                                                                                                          |             |                                   |                                     |
| $N_{41}, N_{42}, N_{43}, N_{44}, N_{45}$                                                                                                    |             |                                   |                                     | $N_{41}, N_{42}, N_{43}, N_{44}, N_{45}$                                                                                                    |             |                                   |                                     |
| -0.334, -0.031, 0.654, -0.678, -0.003                                                                                                       |             |                                   |                                     | -0.198, -0.033, 0.688, -0.697, 0.000                                                                                                        |             |                                   |                                     |
| $N_{51}, N_{52}, N_{53}, N_{54}, N_{55}$                                                                                                    |             |                                   |                                     | $N_{51}, N_{52}, N_{53}, N_{54}, N_{55}$                                                                                                    |             |                                   |                                     |
| 0.001, -0.999, -0.006, 0.040, 0.000                                                                                                         |             |                                   |                                     | 0.001, -0.999, -0.008, 0.040, 0.000                                                                                                         |             |                                   |                                     |
| Annihilations                                                                                                                               |             |                                   |                                     | Annihilations                                                                                                                               |             |                                   |                                     |
| Fractions [%]                                                                                                                               |             |                                   |                                     | Fractions [%]                                                                                                                               |             |                                   |                                     |
| $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow A_s A_s$                                                                                     |             |                                   |                                     | $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow h_s A_s / A_s A_s$                                                                           |             |                                   |                                     |
| 99.8                                                                                                                                        |             |                                   |                                     | 70.0/30.0                                                                                                                                   |             |                                   |                                     |
| Decays                                                                                                                                      |             |                                   |                                     | Decays                                                                                                                                      |             |                                   |                                     |
| Branching ratios [%]                                                                                                                        |             |                                   |                                     | Branching ratios [%]                                                                                                                        |             |                                   |                                     |
| $h_s \rightarrow A_s A_s / \tilde{\chi}_1^0 \tilde{\chi}_1^0$                                                                               |             |                                   |                                     | $h_s \rightarrow A_s A_s$                                                                                                                   |             |                                   |                                     |
| 77.8/22.2                                                                                                                                   |             |                                   |                                     | 100.0                                                                                                                                       |             |                                   |                                     |
| $h \rightarrow A_s A_s / \tilde{\chi}_1^0 \tilde{\chi}_1^0 / \dots$                                                                         |             |                                   |                                     | $h \rightarrow A_s A_s / \dots$                                                                                                             |             |                                   |                                     |
| 10.0/1.0/...                                                                                                                                |             |                                   |                                     | 9.6/...                                                                                                                                     |             |                                   |                                     |
| $H \rightarrow b\bar{b} / \tau\bar{\tau} / \dots$                                                                                           |             |                                   |                                     | $H \rightarrow b\bar{b} / \tau\bar{\tau} / \dots$                                                                                           |             |                                   |                                     |
| 70.1/13.1/...                                                                                                                               |             |                                   |                                     | 71.8/13.4/...                                                                                                                               |             |                                   |                                     |
| $A_s \rightarrow b\bar{b} / \tau\bar{\tau}$                                                                                                 |             |                                   |                                     | $A_s \rightarrow b\bar{b} / \tau\bar{\tau}$                                                                                                 |             |                                   |                                     |
| 88.8/9.9                                                                                                                                    |             |                                   |                                     | 88.4/10.2                                                                                                                                   |             |                                   |                                     |
| $\tilde{\chi}_2^0 \rightarrow \tilde{\chi}_1^0 A_s$                                                                                         |             |                                   |                                     | $\tilde{\chi}_2^0 \rightarrow \tilde{\chi}_1^0 A_s$                                                                                         |             |                                   |                                     |
| 100.0                                                                                                                                       |             |                                   |                                     | 100.0                                                                                                                                       |             |                                   |                                     |
| $\tilde{\chi}_3^0 \rightarrow \tilde{\chi}_2^0 Z$                                                                                           |             |                                   |                                     | $\tilde{\chi}_3^0 \rightarrow \tilde{\chi}_2^0 h / \tilde{\chi}_2^0 Z$                                                                      |             |                                   |                                     |
| 98.1                                                                                                                                        |             |                                   |                                     | 4.8/95.2                                                                                                                                    |             |                                   |                                     |
| $\tilde{\chi}_4^0 \rightarrow \tilde{\chi}_2^0 Z$                                                                                           |             |                                   |                                     | $\tilde{\chi}_4^0 \rightarrow \tilde{\chi}_2^0 h / \tilde{\chi}_2^0 Z$                                                                      |             |                                   |                                     |
| 98.5                                                                                                                                        |             |                                   |                                     | 81.2/18.8                                                                                                                                   |             |                                   |                                     |
| $\tilde{\chi}_5^0 \rightarrow \tilde{\chi}_3^0 h / \tilde{\chi}_3^0 Z / \tilde{\chi}_4^0 h / \tilde{\chi}_4^0 Z / \tilde{\chi}_1^\pm W^\mp$ |             |                                   |                                     | $\tilde{\chi}_5^0 \rightarrow \tilde{\chi}_3^0 h / \tilde{\chi}_3^0 Z / \tilde{\chi}_4^0 h / \tilde{\chi}_4^0 Z / \tilde{\chi}_1^\pm W^\mp$ |             |                                   |                                     |
| 8.2/15.5/14.7/7.6/47.6                                                                                                                      |             |                                   |                                     | 8.4/15.5/13.7/7.3/47.7                                                                                                                      |             |                                   |                                     |
| $\tilde{\chi}_1^\pm \rightarrow \tilde{\chi}_2^0 W^\pm$                                                                                     |             |                                   |                                     | $\tilde{\chi}_1^\pm \rightarrow \tilde{\chi}_1^0 W^\pm$                                                                                     |             |                                   |                                     |
| 99.5                                                                                                                                        |             |                                   |                                     | 100.0                                                                                                                                       |             |                                   |                                     |
| $\tilde{\chi}_2^\pm \rightarrow \tilde{\chi}_3^0 W^\pm / \tilde{\chi}_4^0 W^\pm / \tilde{\chi}_1^\pm h / \tilde{\chi}_1^\pm Z$              |             |                                   |                                     | $\tilde{\chi}_2^\pm \rightarrow \tilde{\chi}_3^0 W^\pm / \tilde{\chi}_4^0 W^\pm / \tilde{\chi}_1^\pm h / \tilde{\chi}_1^\pm Z$              |             |                                   |                                     |
| 23.8/22.2/23.8/23.7                                                                                                                         |             |                                   |                                     | 23.8/23.3/23.9/23.6                                                                                                                         |             |                                   |                                     |
| $R$ value                                                                                                                                   |             |                                   |                                     | $R$ value                                                                                                                                   |             |                                   |                                     |
| 0.398                                                                                                                                       |             |                                   |                                     | 0.295                                                                                                                                       |             |                                   |                                     |

### C. Subtleties of LHC restrictions

The results summarized in Table III might suggest that the LHC constraints are largely inconsequential for light DM scenarios. This impression is misleading: depending on the sparticle mass spectrum, LHC searches can impose very strong limits on such scenarios. To illustrate this point, we now consider an alternative configuration in which the Bino is sufficiently heavy such that the sparticle spectrum exhibits the hierarchy  $m_{\tilde{S}} \ll m_{\tilde{H}} \ll M_1$  ( $\tilde{S} < \tilde{H} < \tilde{B}$ ). Compared to the previously studied configuration, this hierarchy substantially simplifies the Higgsino decay patterns, which typically strengthens the sensitivity of LHC constraints. Specifically, we repeated the analysis strategy outlined previously, fixing  $M_1 = 2$  TeV. Comprehensive

exploratory scans of the GNMSSM parameter space and careful examination of the acquired samples indicate that all samples with Higgsino masses below approximately 900 GeV are excluded by LHC experiments. This issue is discussed extensively below.

When characterizing light DM scenarios in the heavy Bino case, we followed the same procedure as described in Sec. III A, with the sole modification being that the Higgsino mass was scanned in the range 850 GeV to 1200 GeV. This scan initially produced 24842 samples for the  $h_1$  scenario and 24902 samples for the  $h_2$  scenario. After applying LZ 2024 constraints, these numbers were substantially reduced to 2885 and 139, respectively. The LHC simulation further narrowed the surviving sets to 1217

TABLE V. A series of results similar to those of Table III, but differing in two regards. 1)  $M_1 = 2$  TeV. 2) In the second row of each scenario, the first numbers correspond to those further satisfying the LHC constraints (i.e., the typical  $R < 1$  results), while the values inside the parentheses apply to the loose case ( $R \leq 1.5$ ).

| Scenario | Sample size | $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow h_s h_s$ | $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow A_s A_s$ | $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow h_s A_s$ | $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow f \bar{f}$ |
|----------|-------------|---------------------------------------------------------|---------------------------------------------------------|---------------------------------------------------------|-----------------------------------------------------------|
| $h_1$    | 2885        | 0                                                       | 97.6%                                                   | 1.7%                                                    | 0.6%                                                      |
|          | 1217 (2117) | 0 (0)                                                   | 98.4% (97.8%)                                           | 1.2% (1.7%)                                             | 0.4% (0.5%)                                               |
| $h_2$    | 139         | 3.6%                                                    | 8.6%                                                    | 86.3%                                                   | 1.4%                                                      |
|          | 15 (123)    | 0 (4.1%)                                                | 0 (8.9%)                                                | 100.0% (85.4%)                                          | 0 (1.6%)                                                  |

for the  $h_1$  scenario and only 15 for the  $h_2$  scenario by the nominal requirement  $R < 1$ . For completeness, we also adopt a loose LHC criterion  $R \leq 1.5$  to account for implementation uncertainties in the recasting (see discussion below). Following this analysis strategy, we documented the updated sampling results in Table V, where the values inside the parentheses apply to the loose case.

Compared with the  $M_1 = 150$  GeV case (Table III), significantly fewer samples satisfy the LZ 2024 constraints when  $M_1 = 2$  TeV. The primary reason is that for large  $M_1$  and  $\mu_{\text{tot}}$ , the DM becomes an almost pure Singlino with negligible Bino and Higgsino admixtures. As discussed earlier, small Bino and  $\tilde{H}_d$  components are crucial for facilitating destructive interference among different contributions to  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SI}}$ . Specifically, their markedly stronger coupling to the SM-like Higgs boson, relative to that of the Singlino, can generate sizable scattering amplitudes that interfere destructively with the pure-Singlino amplitude. When these admixtures are driven to (nearly) zero by  $M_1 = 2$  TeV and  $\mu_{\text{tot}} \gtrsim 900$  GeV, this interference leverage is lost, and the cancellation is greatly weakened, thereby impeding effective suppression of the scattering cross

section. Consequently, most samples fail to satisfy the LZ 2024 constraints and are thus excluded even when  $\lambda$  is as low as 0.01. Given that this effect is particularly pronounced in the  $h_2$  scenario, resulting in its severely limited viable sample size, we henceforth focus on the  $h_1$  scenario.

In the heavy-Bino case, the dominant annihilation channel remains  $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow A_s A_s$ , with parameter-space distributions qualitatively similar to those at  $M_1 = 150$  GeV but with much lower sample density. Accordingly, the physical interpretations established in the  $M_1 = 150$  GeV analysis remain fully applicable, and we refrain from reiterating the detailed arguments here. One salient feature, however, warrants emphasis: the weakened cancellation among different contributions to  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SI}}$  causes  $\sigma_{\text{eff}}^{\text{SI}}$  to be substantially elevated. As a direct consequence, surviving samples tend to cluster near the LZ 2024 exclusion boundary, and configurations with very small  $\sigma_{\text{eff}}^{\text{SI}}$  are considerably rarer compared to the  $M_1 = 150$  GeV case shown in Fig. 3.

On the collider side, the adjustable sparticle spectrum consists of  $\tilde{\chi}_{2,3}^0$  and charginos  $\tilde{\chi}_1^\pm$ , with their masses

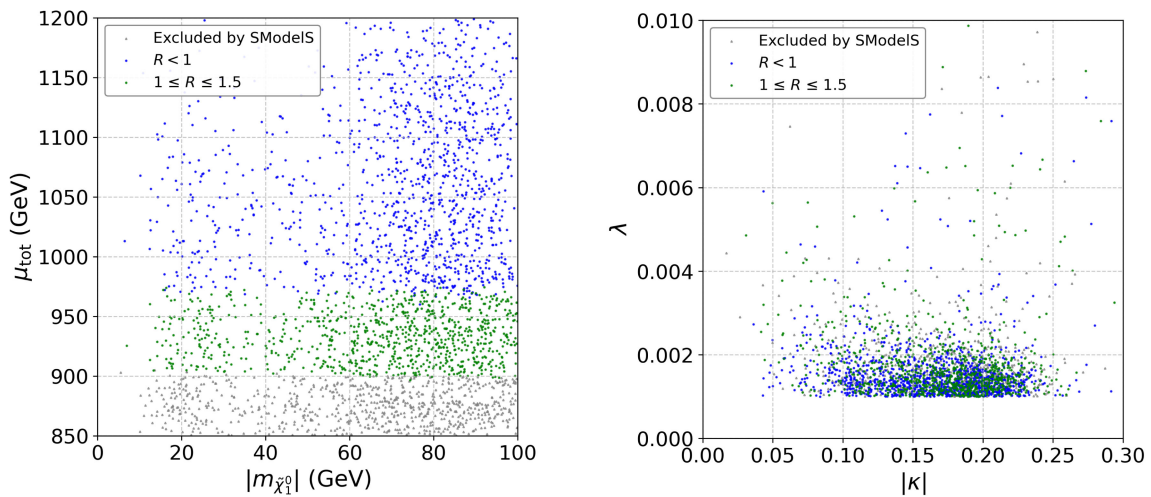


FIG. 5. Similar to Fig. 4, but for  $M_1 = 2$  TeV in the  $h_1$  scenario. One difference from Fig. 4 is that samples with  $R \geq 1$  are further categorized: green markers ( $1 \leq R \leq 1.5$ ) represent those satisfying a loose LHC exclusion criterion, suggesting potential compatibility with LHC data once theoretical and experimental uncertainties are properly accounted for, including those from sparticle production cross section calculations and Monte Carlo simulations.

TABLE VI. A list of results similar to Table IV, but with  $M_1 = 2$  TeV.

| Benchmark point BP3 ( $h_1$ scenario)                                                                                          |                                      |                                   |                                     | Benchmark point BP4 ( $h_2$ scenario)                                                                                          |                                      |                                   |                                     |
|--------------------------------------------------------------------------------------------------------------------------------|--------------------------------------|-----------------------------------|-------------------------------------|--------------------------------------------------------------------------------------------------------------------------------|--------------------------------------|-----------------------------------|-------------------------------------|
| $\lambda$                                                                                                                      | 0.001                                | $m_{h_s}$                         | 192.1 GeV                           | $\lambda$                                                                                                                      | 0.001                                | $m_{h_s}$                         | 94.4 GeV                            |
| $\kappa$                                                                                                                       | 0.186                                | $m_h$                             | 125.3 GeV                           | $\kappa$                                                                                                                       | -0.076                               | $m_h$                             | 125.2 GeV                           |
| $\delta$                                                                                                                       | 0.923                                | $m_H$                             | 2550.0 GeV                          | $\delta$                                                                                                                       | 0.721                                | $m_H$                             | 1677.7 GeV                          |
| $\tan \beta$                                                                                                                   | 29.51                                | $m_{A_s}$                         | 10.0 GeV                            | $\tan \beta$                                                                                                                   | 17.89                                | $m_{A_s}$                         | 15.7 GeV                            |
| $\mu$                                                                                                                          | 968.1 GeV                            | $m_{\tilde{\chi}_1^0}$            | 35.0 GeV                            | $\mu$                                                                                                                          | 967.6 GeV                            | $m_{\tilde{\chi}_1^0}$            | -85.4 GeV                           |
| $\mu_{\text{eff}}$                                                                                                             | 0.349 GeV                            | $m_{\tilde{\chi}_2^0}$            | 981.8 GeV                           | $\mu_{\text{eff}}$                                                                                                             | 0.154 GeV                            | $m_{\tilde{\chi}_2^0}$            | 979.1 GeV                           |
| $\mu_{\text{tot}}$                                                                                                             | 968.5 GeV                            | $m_{\tilde{\chi}_3^0}$            | -986.7 GeV                          | $\mu_{\text{tot}}$                                                                                                             | 967.8 GeV                            | $m_{\tilde{\chi}_3^0}$            | -985.2 GeV                          |
| $A_\lambda$                                                                                                                    | 2182.2 GeV                           | $m_{\tilde{\chi}_4^0}$            | 2006.6 GeV                          | $A_\lambda$                                                                                                                    | 4923.6 GeV                           | $m_{\tilde{\chi}_4^0}$            | 2006.8 GeV                          |
| $A_\kappa$                                                                                                                     | 645.3 GeV                            | $m_{\tilde{\chi}_5^0}$            | 2021.2 GeV                          | $A_\kappa$                                                                                                                     | -640.3 GeV                           | $m_{\tilde{\chi}_5^0}$            | 2021.3 GeV                          |
| $A_t$                                                                                                                          | 2694.7 GeV                           | $m_{\tilde{\chi}_1^\pm}$          | 984.0 GeV                           | $A_t$                                                                                                                          | -3493.0 GeV                          | $m_{\tilde{\chi}_1^\pm}$          | 981.8 GeV                           |
| $M_1$                                                                                                                          | 2000.0 GeV                           | $m_{\tilde{\chi}_2^\pm}$          | 2021.0 GeV                          | $M_1$                                                                                                                          | 2000.0 GeV                           | $m_{\tilde{\chi}_2^\pm}$          | 2021.2 GeV                          |
| $m_B$                                                                                                                          | 194.6 GeV                            | $\Omega h^2$                      | 0.136                               | $m_B$                                                                                                                          | 96.4 GeV                             | $\Omega h^2$                      | 0.127                               |
| $m_C$                                                                                                                          | 25.5 GeV                             | $\sigma_{\text{eff}}^{\text{SI}}$ | $1.90 \times 10^{-48} \text{ cm}^2$ | $m_C$                                                                                                                          | 22.7 GeV                             | $\sigma_{\text{eff}}^{\text{SI}}$ | $2.61 \times 10^{-48} \text{ cm}^2$ |
| $m_{\tilde{S}}$                                                                                                                | 35.1 GeV                             | $\sigma_n^{\text{SD}}$            | $3.72 \times 10^{-53} \text{ cm}^2$ | $m_{\tilde{S}}$                                                                                                                | -85.4 GeV                            | $\sigma_n^{\text{SD}}$            | $3.93 \times 10^{-53} \text{ cm}^2$ |
| $V_{h_s}^S, V_{h_s}^{\text{SM}}, V_h^S, V_h^{\text{SM}}$                                                                       | -0.999, -0.015, 0.015, -0.999        |                                   |                                     | $V_{h_s}^S, V_{h_s}^{\text{SM}}, V_h^S, V_h^{\text{SM}}$                                                                       | 0.999, -0.036, 0.036, 0.998          |                                   |                                     |
| $N_{11}, N_{12}, N_{13}, N_{14}, N_{15}$                                                                                       | 0.000, 0.000, 0.000, 0.000, 1.000    |                                   |                                     | $N_{11}, N_{12}, N_{13}, N_{14}, N_{15}$                                                                                       | 0.000, 0.000, 0.000, 0.000, 0.999    |                                   |                                     |
| $N_{21}, N_{22}, N_{23}, N_{24}, N_{25}$                                                                                       | -0.031, 0.054, -0.707, 0.704, 0.000  |                                   |                                     | $N_{21}, N_{22}, N_{23}, N_{24}, N_{25}$                                                                                       | -0.032, -0.056, -0.707, 0.704, 0.000 |                                   |                                     |
| $N_{31}, N_{32}, N_{33}, N_{34}, N_{35}$                                                                                       | 0.010, -0.018, -0.707, -0.707, 0.000 |                                   |                                     | $N_{31}, N_{32}, N_{33}, N_{34}, N_{35}$                                                                                       | 0.010, -0.017, -0.706, -0.708, 0.000 |                                   |                                     |
| $N_{41}, N_{42}, N_{43}, N_{44}, N_{45}$                                                                                       | -0.474, -0.880, -0.016, 0.031, 0.000 |                                   |                                     | $N_{41}, N_{42}, N_{43}, N_{44}, N_{45}$                                                                                       | -0.476, -0.879, -0.016, 0.031, 0.000 |                                   |                                     |
| $N_{51}, N_{52}, N_{53}, N_{54}, N_{55}$                                                                                       | 0.880, -0.472, -0.025, 0.050, 0.000  |                                   |                                     | $N_{51}, N_{52}, N_{53}, N_{54}, N_{55}$                                                                                       | 0.879, -0.474, -0.027, 0.051, 0.000  |                                   |                                     |
| Annihilations                                                                                                                  | Fractions [%]                        |                                   |                                     | Annihilations                                                                                                                  | Fractions [%]                        |                                   |                                     |
| $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow A_s A_s$                                                                        | 100.0                                |                                   |                                     | $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow h_s A_s / A_s A_s$                                                              | 98.7/1.2                             |                                   |                                     |
| Decays                                                                                                                         | Branching ratios [%]                 |                                   |                                     | Decays                                                                                                                         | Branching ratios [%]                 |                                   |                                     |
| $h_s \rightarrow A_s A_s / \tilde{\chi}_1^0 \tilde{\chi}_1^0$                                                                  | 86.1/13.9                            |                                   |                                     | $h_s \rightarrow A_s A_s$                                                                                                      | 100.0                                |                                   |                                     |
| $h \rightarrow A_s A_s / \dots$                                                                                                | 10.4/...                             |                                   |                                     | $h \rightarrow A_s A_s / \dots$                                                                                                | 8.4/...                              |                                   |                                     |
| $H \rightarrow b\bar{b} / \tau\bar{\tau}$                                                                                      | 79.6/19.5                            |                                   |                                     | $H \rightarrow b\bar{b} / \tau\bar{\tau}$                                                                                      | 81.1/15.7                            |                                   |                                     |
| $A_s \rightarrow b\bar{b} / \tau\bar{\tau}$                                                                                    | 87.9/10.6                            |                                   |                                     | $A_s \rightarrow b\bar{b} / \tau\bar{\tau}$                                                                                    | 89.5/9.6                             |                                   |                                     |
| $\tilde{\chi}_2^0 \rightarrow \tilde{\chi}_1^0 h / \tilde{\chi}_1^0 Z$                                                         | 51.7/47.8                            |                                   |                                     | $\tilde{\chi}_2^0 \rightarrow \tilde{\chi}_1^0 h / \tilde{\chi}_1^0 Z$                                                         | 37.8/62.0                            |                                   |                                     |
| $\tilde{\chi}_3^0 \rightarrow \tilde{\chi}_1^0 h / \tilde{\chi}_1^0 Z$                                                         | 49.3/50.4                            |                                   |                                     | $\tilde{\chi}_3^0 \rightarrow \tilde{\chi}_1^0 h / \tilde{\chi}_1^0 Z$                                                         | 63.5/36.5                            |                                   |                                     |
| $\tilde{\chi}_4^0 \rightarrow \tilde{\chi}_2^0 h / \tilde{\chi}_3^0 Z / \tilde{\chi}_1^\pm W^\mp$                              | 10.2/9.4/75.7                        |                                   |                                     | $\tilde{\chi}_4^0 \rightarrow \tilde{\chi}_2^0 h / \tilde{\chi}_3^0 Z / \tilde{\chi}_1^\pm W^\mp$                              | 9.9/9.4/76.1                         |                                   |                                     |
| $\tilde{\chi}_5^0 \rightarrow \tilde{\chi}_2^0 h / \tilde{\chi}_3^0 Z / \tilde{\chi}_3^0 h / \tilde{\chi}_3^0 Z$               | 44.2/4.6/4.6/42.4                    |                                   |                                     | $\tilde{\chi}_5^0 \rightarrow \tilde{\chi}_2^0 h / \tilde{\chi}_3^0 Z / \tilde{\chi}_3^0 h / \tilde{\chi}_3^0 Z$               | 43.0/4.1/4.1/42.4                    |                                   |                                     |
| $\tilde{\chi}_1^\pm \rightarrow \tilde{\chi}_1^0 W^\pm$                                                                        | 100.0                                |                                   |                                     | $\tilde{\chi}_1^\pm \rightarrow \tilde{\chi}_1^0 W^\pm$                                                                        | 100.0                                |                                   |                                     |
| $\tilde{\chi}_2^\pm \rightarrow \tilde{\chi}_2^0 W^\pm / \tilde{\chi}_3^0 W^\pm / \tilde{\chi}_1^\pm h / \tilde{\chi}_1^\pm Z$ | 23.3/23.4/24.6/23.4                  |                                   |                                     | $\tilde{\chi}_2^\pm \rightarrow \tilde{\chi}_2^0 W^\pm / \tilde{\chi}_3^0 W^\pm / \tilde{\chi}_1^\pm h / \tilde{\chi}_1^\pm Z$ | 23.4/23.7/24.1/23.7                  |                                   |                                     |
| $R$ value                                                                                                                      | 0.963                                |                                   |                                     | $R$ value                                                                                                                      | 0.958                                |                                   |                                     |

approximated by  $|m_{\tilde{\chi}_{2,3}^0}| \simeq m_{\tilde{\chi}_1^\pm} \simeq \mu_{\text{tot}}$ , when  $M_1 = 2$  TeV. Within the simplified mass hierarchy, the dominant decay channels for these particles are  $\tilde{\chi}_{2,3}^0, \tilde{\chi}_1^\pm \rightarrow \tilde{\chi}_1^0 + W/Z/h$ , as exemplified by benchmark point BP3 in Table VI. These signatures closely resemble those targeted by the ATLAS search for charginos and neutralinos decaying to fully hadronic final states in the MSSM, as documented in Ref. [38] (ATLAS-2108-07586). Critically, in light DM scenarios, this analysis sets a lower mass limit of approximately  $\sim 900$  GeV on the NLSP mass, as illustrated in the left panel of Fig. 5. The right panel further shows that surviving points concentrate in narrow bands,  $\lambda \in (0.001, 0.01)$  and  $|\kappa| \in (0.05, 0.25)$ .

An important technical note concerns the  $R \leq 1.5$  exclusion criterion, which was adopted to account for

implementation uncertainties encountered during the recasting of the ATLAS-2108-07586 analysis into the CHECKMATE-2.0.37 framework [109,110]. Consequently, interpreting samples yielding  $R \leq 1.5$  as phenomenologically consistent with current LHC experiments is justified in this context. The resulting samples with  $1 \leq R \leq 1.5$  are depicted by the green-shaded region in Fig. 5.

Table VI presents two benchmark points that provide further insights. Compared to Table IV, the large  $\mu_{\text{tot}}$  value produces an almost pure Singlino DM, which precludes effective cancellation among different contributions to  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SI}}$ , leading to surviving samples with moderately large  $\sigma_{\text{eff}}^{\text{SI}}$ . Moreover, the simpler decay topologies in the  $\tilde{S} < \tilde{H} < \tilde{B}$  hierarchy yield more detectable signatures at the LHC, resulting in notably more restrictive experimental limits.

#### IV. CONCLUSION

Light supersymmetric DM with a mass of several tens of GeV warrants particular attention, as such light particles are challenging to realize within simple supersymmetric frameworks such as the MSSM or the  $Z_3$ -NMSSM. Through a comprehensive theoretical and numerical analysis incorporating relic density, LZ 2024, Higgs data, and LHC constraints, this study establishes the GNMSSM as a robust and phenomenologically viable framework capable of naturally accommodating light supersymmetric DM. The central virtue of the GNMSSM is the effective decoupling between interactions governing thermal relic abundance and those controlling direct detection, which is enabled by allowing  $\mu$  and  $\mu'$  parameters to break the  $Z_3$  symmetry of the  $Z_3$ -NMSSM and thereby disentangle  $\lambda$  and  $\kappa$ . This feature broadens the viable parameter space and yields distinctive signatures that can help discriminate supersymmetric frameworks in future experiments. Our systematic parameter space exploration across two configurations— $M_1 = 150$  GeV and  $M_1 = 2$  TeV—yields the following key insights:

- (1) *Diverse self-annihilation mechanisms:* In the  $h_1$  scenario, DM achieves the observed relic density primarily through the  $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow A_s A_s$  channel, exhibiting pronounced mass-dependent behavior across three distinct regimes. In the low-mass region ( $\lesssim 30$  GeV), the  $t$  channel dominates the annihilation process. The intermediate-mass region (30–60 GeV) features a delicate interplay between  $t$ - and  $s$ -channel contributions. In the high-mass region ( $\gtrsim 60$  GeV),  $s$ -channel processes typically prevail through both resonant and nonresonant effects. In contrast, the  $h_2$  scenario yields sufficient samples only for the  $M_1 = 150$  GeV configuration, where DM achieves the correct relic density primarily via  $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \rightarrow h_s A_s$ , with comparable  $s$ - and  $t$ -channel contributions in most parameter regions.
- (2) *Correlation between direct detection and naturalness:* For a heavy doublet-like Higgs  $H$ , the compatibility of  $\sigma_{\text{eff}}^{\text{SI}}$  with LZ 2024 constraints critically depends on the strength of destructive interference among various contributions to  $\sigma_{\tilde{\chi}_1^0, N}^{\text{SI}}$ . After incorporating constraints from LHC supersymmetry searches, this cancellation mechanism is substantially stronger at  $M_1 = 150$  GeV than at  $M_1 = 2$  TeV. Specifically, the former configuration can suppress  $\sigma_{\text{eff}}^{\text{SI}}$  down to  $10^{-50}$  cm<sup>2</sup> while maintaining a larger viable parameter space, thereby more readily satisfying experimental constraints. Crucially, the  $M_1 = 150$  GeV configuration permits  $\mu_{\text{tot}}$  values as low as  $\sim 200$  GeV, revealing an intrinsic consistency between direct detection constraints and theoretical naturalness.
- (3) *NLSP-dependent LHC constraints:* The identity of NLSP profoundly influences LHC constraints.

When  $M_1 = 150$  GeV, the NLSP is a Bino-dominated  $\tilde{\chi}_2^0$ . The resulting diverse decay modes and prolonged cascade decay chains degrade LHC search sensitivities, thereby allowing  $\mu_{\text{tot}}$  values as low as  $\sim 200$  GeV without contradicting current experimental bounds. Consequently, the model preserves extensive parameter space for future LHC exploration: in the  $h_1$  scenario,  $|m_{\tilde{\chi}_1^0}| \in (10, 100)$  GeV,  $\lambda \in (0, 0.05)$ , and  $|\kappa| \in (0.05, 0.3)$ , while in the  $h_2$  scenario,  $|m_{\tilde{\chi}_1^0}| \in (50, 100)$  GeV,  $\lambda \in (0, 0.01)$ , and  $|\kappa| \in (0.03, 0.2)$ . Both scenarios feature  $m_{\tilde{\chi}_2^0} \simeq 150$  GeV. Conversely, when  $M_1 = 2$  TeV, the NLSP becomes Higgsino dominated, leading to simplified decay topologies where Higgsino-like particles decay directly to the DM. LHC experiments then impose a stringent bound of  $\mu_{\text{tot}} \gtrsim 900$  GeV for such simplified signatures, substantially restricting the viable parameter space.

In summary, the GNMSSM circumvents the  $Z_3$ -NMSSM constraint  $2|\kappa| < \lambda$  in predicting Singlino-dominated DM, thereby enabling effective decoupling between DM annihilation mechanisms and DM-nucleon scattering processes. Our analysis identifies two distinct mass hierarchies within the GNMSSM:  $\tilde{S} < \tilde{B} < \tilde{H}$  and  $\tilde{S} < \tilde{H} < \tilde{B}$ , both capable of yielding phenomenologically viable DM candidates in the tens-of-GeV range. The  $\tilde{S} < \tilde{B} < \tilde{H}$  hierarchy emerges as the preferred configuration, allowing  $\mu_{\text{tot}}$  values as low as  $\sim 200$  GeV and naturally accommodating Singlino-dominated light DM with proper DM relic abundance. Moreover, the presence of accompanying light particles such as  $A_s$  with characteristic LHC signatures provides additional handles for distinguishing between different supersymmetric models. By contrast, although the alternative  $\tilde{S} < \tilde{H} < \tilde{B}$  hierarchy is theoretically simpler, it requires  $\mu_{\text{tot}} \gtrsim 900$  GeV based on LHC constraints, resulting in approximately 0.5% fine-tuning in electroweak symmetry breaking.

By systematically exploring the previously uncharted  $\tilde{S} < \tilde{B} < \tilde{H}$  parameter space, this work advances our understanding of supersymmetric DM in challenging mass regimes and provides concrete guidance for future experimental searches across multiple frontiers: direct detection experiments approaching  $10^{-50}$  cm<sup>2</sup> sensitivity, collider searches for electroweakinos with complex decay chains involving light singlet states, and indirect detection through secluded-sector annihilation products. We anticipate that forthcoming experiments will provide crucial tests of the mass hierarchy identified in this study, ultimately helping to establish the correct theoretical framework for describing DM.

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## DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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