

Black string in the standard model

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The Swampland cobordism conjecture [arXiv:1909.10355.] predicts various new objects in a theory with dynamical gravity. Applying this idea to the Standard Model of particle physics, a string object is predicted. We numerically constructed such an object as a black string solution.

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I. INTRODUCTION

The Swampland cobordism conjecture [1] states that the cobordism class of quantum gravity is trivial. In particular, the conjecture implies that all of the string compactifications must be null cobordant. This leads to new non-supersymmetric branes in string theory. For instance, the reflection 7-brane [2] in type IIB and various objects in heterotic string theory are predicted [3–6].

In addition to string theory, the Swampland conjectures apply to the standard model (SM) as we believe that it is in the landscape. A natural cobordism group associated with the SM is Ω_n^{Spin} , which is the bordism group of n -dimensional manifolds with the spin structure. When Ω_n^{Spin} is nontrivial, it implies the existence of codimension $n + 1$ new branes in the SM. In this respect, it is interesting that Ω_1^{Spin} is nontrivial, $\Omega_1^{\text{Spin}} = \mathbb{Z}_2$. The nontrivial element is generated by the circle S^1 with the periodic boundary condition of the fermions. This implies that the SM has a stringlike object with the $\mathbb{Z}_2$ charge flux. In this paper, we establish the existence of the object by numerically constructing such a stringlike object as a black string solution. The black string is supported by the Casimir energy of the SM particles and is an intrinsically quantum object. We also compute the tension and the Hawking temperature of the black string. Notice that the black string in the SM was considered in Ref. [7] with a different motivation, but the explicit construction was not established there.

Let us clarify the difference between our work and the previous studies that apply the cobordism conjecture to string theory. In contrast to such a top-down approach, our work takes a bottom-up perspective. The absolute stability of the $\mathbb{Z}_2$ -charged black string we will construct depends on the UV completion. This is because the relevant dimension and the tangential structure of the spacetime may differ in the UV completion (denoted by Ω_{UV} , see Fig. 1). For instance, if the UV completion is given by 4d theory (e.g., asymptotic safety scenario of the gravity [8]), then the relevant group would be $\Omega_{\text{UV}} = \Omega_1^{\text{Spin}}$, in which case the black string is stable. On the other hand, if the SM is UV completed by ten- or eleven-dimensional string theory, Ω_{UV} is not equal to Ω_1^{Spin} in general, and the black string can be stable only when it corresponds to a nontrivial element of Ω_{UV} . In this way, the stability of the black string depends on the UV completion. However, even in the unstable case, we expect the black string to be a metastable configuration since its decay process should involve the UV degrees of freedom (for instance, the metric of internal space changes to trivialize the defect).

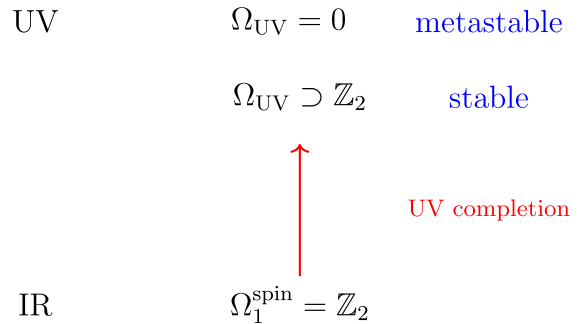


FIG. 1. The absolute stability of the black string in the SM depends on the UV completion.

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This paper is organized as follows. In Sec. II, we review the black string supported by the Casimir energy in the SM. In Sec. III, we construct the black string solution numerically. The tension and Hawking temperature of the black string are also computed. We conclude in Sec. IV.

II. SETUP

In this section, we introduce the Casimir energy as a key ingredient in our study.

A. Vacuum solutions

As in Sec. I, we look for the black string solution. However, as is well known, there is no event horizon whose topology is other than spherical in $d = 4$ general relativity under the assumption of the dominant energy condition (see Sec. 9.3 in [9]). This can be seen by using general vacuum solutions of the Einstein equation with cylindrical symmetry [10]:

$$ds^2 = -\alpha(\Sigma z)^{\frac{4\sigma}{\Sigma}} dt^2 + \beta(\Sigma z)^{\frac{-4\sigma(1-2\sigma)}{\Sigma}} dx^2 + dz^2 + \frac{1}{a}(\Sigma z)^{\frac{2(1-2\sigma)}{\Sigma}} d\phi^2, \quad (1)$$

where $\Sigma = 4\sigma^2 - 2\sigma + 1$, σ takes real arbitrary value,¹ and z , x , and ϕ parametrize the radial direction, the longitudinal direction, and the angle around the x axes, respectively (see Fig. 2). The coefficients α , β are arbitrary positive numbers corresponding to rescaling of t , x .

The curvature singularity is located at $z = 0$, and there is no horizon that hides the singularity. In order to have a black string solution without naked singularity, we consider the quantum effect.

B. Black string horizon supported by quantum effect

The key observation to construct the black string is that the fields going around the string feel the nontrivial boundary condition [7]. Consequently, the Casimir energy is generated as a quantum effect. The Casimir energy associated to the S^1 surrounding the string is (see, e.g., [11])

$$V_{\text{Cas}} = - \sum_{\text{particle}} (-1)^{2s_p} n_p \frac{m_p^4}{2\pi^2} \times \sum_{n=1}^{\infty} \frac{\cos(2\pi n \theta_p)}{(2\pi R m_p n)^2} K_2(2\pi R m_p n). \quad (2)$$

The sum is over particles in the SM, s_p is its spin, m_p is its mass, and n_p is the real degrees of freedom. R is the radial distance from the string, which is along with the vertical axis x of the cylindrical coordinate (see Fig. 2). The parameter θ_p corresponds to the boundary condition when the particle moves around the string. The periodic and antiperiodic boundary conditions correspond to $\theta_p = 0$ and $\theta_p = 1/2$,

¹Especially, $\sigma = 0$ corresponds to the flat metric.

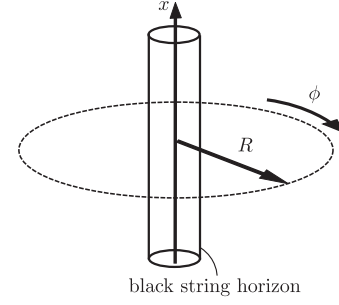


FIG. 2. Our coordinate system.

respectively. We consider the periodic boundary condition both for bosons and fermions.²

As the Casimir energy violates the dominant and null energy conditions, the no-go theorem of the black string is circumvented.

III. BLACK STRING SOLUTION

We look for black string solutions in the presence of the Casimir energy. In this paper, we assume that neutrinos are Majorana and the normal hierarchy, whose lightest mass is 0 (the heaviest neutrino mass is $m_3 = 0.05$ eV [12]). See Sec. I of Supplemental Material [13] for the plot of the Casimir energy in this setup. In this section, we first introduce the metric ansatz and boundary conditions, and then solve the Einstein equation numerically.

A. Metric ansatz and boundary conditions

Extremal case. It is natural to assume the boost invariance in the direction in which this black string extends (x axis), as in the case of extremal black p branes [14]. Therefore, we employ the following metric ansatz:

$$ds^2 = A(z)^2 (-dt^2 + dx^2) + \frac{M_P^2}{m_3^4} dz^2 + R(z)^2 d\phi^2, \quad (3)$$

where $M_P z / m_3^2$ is a proper distance of radial direction. The scale M_P / m_3^2 is introduced in such a way that z is dimensionless and the equation of motion (EOM) is simplified. Note that $z \rightarrow +\infty$ and $z \rightarrow -\infty$ correspond to asymptotic infinity and the horizon, respectively.

As for the boundary conditions, we require the metric to approach to the flat for $z \rightarrow \infty$ and to the $\text{AdS}_3 \times S^1$ for $z \rightarrow -\infty$ (see [7] and Sec. I of Supplemental Material [13] for details):

$$A(z) \xrightarrow{z \rightarrow -\infty} e^{\frac{M_P}{m_3^2} \frac{z}{l_{\text{AdS}}}}, \quad R(z) \xrightarrow{z \rightarrow -\infty} R_0^{\text{ext}}, \quad (4)$$

$$A(z) \xrightarrow{z \rightarrow \infty} \text{const}, \quad R(z) \xrightarrow{z \rightarrow \infty} \text{const} \times \frac{M_P}{m_3^2} z. \quad (5)$$

²This corresponds to the nontrivial element of Ω_1^{Spin} .

Here l_{AdS} is the AdS_3 radius near the horizon, and R_0^{ext} is the horizon size at which radion potential (28) has a minimum. The coefficient of z of $R(z)$ as $z \rightarrow \infty$ determines a deficit angle of the black string.

Nonextremal case. Motivated by the vacuum solution (1), we consider the following metric ansatz:

$$ds^2 = -A(z)^2 dt^2 + B(z)^2 dx^2 + \frac{M_P^2}{m_3^4} dz^2 + R(z)^2 d\phi^2. \quad (6)$$

This ansatz no longer has Lorentz invariance on the (tx) plane. For $z \rightarrow \infty$, Casimir energy is suppressed, and black string solutions should approach Eq. (1). This provides the boundary conditions at infinity. For instance, we see $R(z) \rightarrow (1/\sqrt{a}) \cdot (\Sigma z)^{\frac{(1-2\sigma)}{4\sigma^2-2\sigma+1}}$. The coefficient a of $R(z)$ is physical and determines the deficit angle.

On the other hand, for small z , we assume that the horizon is located at finite z .³ Without loss of generality, we put the horizon at $z = 0$. We impose

$$A(0) = 0, \quad B'(0) = 0, \quad R(0) = R_0, \quad R'(0) = 0 \quad (7)$$

at the horizon to avoid the curvature singularity at $z = 0$ (see Sec. II of Supplemental Material [13]). Here we have introduced a parameter R_0 to parametrize the solutions. As varying R_0 , one obtains a family of the solutions.

B. Einstein equations

We derive the Einstein equation from the action

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} M_P^2 \mathcal{R} - V_{\text{Cas}} + \dots \right). \quad (8)$$

The variation of the action by metric gives

$$-\mathcal{R}_{\mu\nu} + \frac{g_{\mu\nu}}{2} \mathcal{R} = g_{\mu\nu} \frac{V_{\text{Cas}}}{M_P^2} + \delta_{\mu 3} \delta_{\nu 3} \frac{R^3}{M_P^2} \frac{\partial V_{\text{Cas}}}{\partial R}. \quad (9)$$

This is the Einstein equation to be solved. The right-hand side is the energy-momentum tensor of the Casimir energy.

Extremal case. The Einstein equation (9) with the ansatz (3) is

$$R'' + R' \frac{A'}{A} + R \frac{A''}{A} = -R \tilde{V}_{\text{Cas}}, \quad (10)$$

$$\frac{A'}{A} \left(2R' + R \frac{A'}{A} \right) = -R \tilde{V}_{\text{Cas}}, \quad (11)$$

$$\frac{A'^2}{A^2} + 2 \frac{A''}{A} = -\tilde{V}_{\text{Cas}} - R \partial_R \tilde{V}_{\text{Cas}}, \quad (12)$$

³This is again the analogy of the nonextremal RN black hole case, where its throat region has a finite length of proper distance.

where the prime is the derivative with respect to z , and $\tilde{V}_{\text{Cas}} := V_{\text{Cas}}/m_3^4$. There are three equations despite only two unknown functions $A(z)$ and $R(z)$. This is because the first order equation (11) is the Hamiltonian constraint.

Equation (11) can be solved for A'/A :

$$\frac{A'}{A} = -\frac{R'}{R} + \sqrt{\frac{R'^2}{R^2} - \tilde{V}_{\text{Cas}}}, \quad (13)$$

where we have chosen the branch to be consistent with the boundary condition (5). By using (10), (12), and (13) we obtain

$$R'' + \gamma R' = -R \left(\tilde{V}_{\text{Cas}} - \frac{1}{2} R \partial_R \tilde{V}_{\text{Cas}} \right), \quad (14)$$

$$\gamma = -2 \left(\frac{R'}{R} - \sqrt{\frac{R'^2}{R^2} - \tilde{V}_{\text{Cas}}} \right). \quad (15)$$

Equation (14) has an interpretation as a classical mechanical EOM with the friction γ and the potential U :

$$\partial_R U = R \left(\tilde{V}_{\text{Cas}} - \frac{1}{2} R \partial_R \tilde{V}_{\text{Cas}} \right), \quad (16)$$

with z being the “time variable.” This potential has a local maximum at the point $R = R_0^{\text{ext}}$, where $\partial_R U = 0$ is satisfied. The solution we are looking for is the one with $R|_{z=-\infty} = R_0^{\text{ext}}$ and $R|_{z=+\infty} = +\infty$.

There are three integration constants in the solution of (13) and (14). One of them is fixed by the condition above. The other two are unphysical parameters corresponding to the rescaling of t , x , and the shift of z .

Nonextremal case. With the metric ansatz (6), the Einstein equation (9) becomes

$$\frac{R''}{R} + \frac{R'}{R} \frac{B'}{B} + \frac{B''}{B} = -\tilde{V}_{\text{Cas}}, \quad (17)$$

$$\frac{R''}{R} + \frac{R'}{R} \frac{A'}{A} + \frac{A''}{A} = -\tilde{V}_{\text{Cas}}, \quad (18)$$

$$\frac{A' B'}{A B} + \frac{B' R'}{B R} + \frac{R' A'}{R A} = -\tilde{V}_{\text{Cas}}, \quad (19)$$

$$\frac{A' B'}{A B} + \frac{A''}{A} + \frac{B''}{B} = -\tilde{V}_{\text{Cas}} - R \partial_R \tilde{V}_{\text{Cas}}. \quad (20)$$

Among them, the first order equation (19) is again a constraint equation. Equation (19) can be solved for A'/A :

$$\frac{A'}{A} = \frac{-\tilde{V}_{\text{Cas}} B R - B' R'}{(B R)'} \quad (21)$$

By using (18), (20), and (21), we obtain

$$\frac{R''}{R} + \frac{B'^2 R'}{B(BR)'} + \frac{BR'}{(BR)'} \tilde{V}_{\text{Cas}} - \frac{R}{2} \partial_R \tilde{V}_{\text{Cas}} = 0. \quad (22)$$

By solving Eqs. (17) and (22), we obtain B and R . Then, the solution for A is obtained by substituting B and R into (21). Notice that there are five integration constants. Four of them are fixed by the boundary condition (7), leaving one unphysical parameter corresponding to the rescaling of x . Thus, the only physical parameter is the horizon size R_0 .

C. Numerical results

Extremal case. We utilize two different methods independently, the shooting method and the modified gradient flow method [15] to solve (13) and (14) with the boundary conditions (4) and (5), where the two methods show good agreement in the solutions. The solutions are shown in Fig. 3, in which R interpolates the S^1 compacted region where R is almost constant and the flat region where R is proportional to z .

The important physical parameter is $R'(\infty)$. For our solution,

$$R'(\infty) = \mathcal{O}(10^{-2}) m_3^{-1}. \quad (23)$$

This indicates that the opening angle is $\theta_o = 2\pi R'(\infty) \times (m_3^2/M_P) \sim 2\pi \times 10^{-2} \times (m_3/M_P) \ll 1$ and the deficit angle is $\theta = 2\pi - \theta_o = \mathcal{O}(1)$.

The solution for A also interpolates the AdS_3 region where A exponentially decays with AdS radius $l_{\text{AdS}} \sim 6.2 \times 10^{-3} \times (M_P/m_3^2)$ and the flat region where A is almost constant. Since A goes to 0 as $z \rightarrow -\infty$, $z = -\infty$ is the coordinate singularity and corresponds to the horizon.

Let us calculate the Hawking temperature. We start from the metric (6) and change the radial coordinate from z to R :

$$ds^2 = -A(z)^2 dt^2 + \frac{M_P^2}{m_3^4} \frac{dR^2}{R^2} + B(z)^2 dx^2 + R^2 d\phi^2, \quad (24)$$

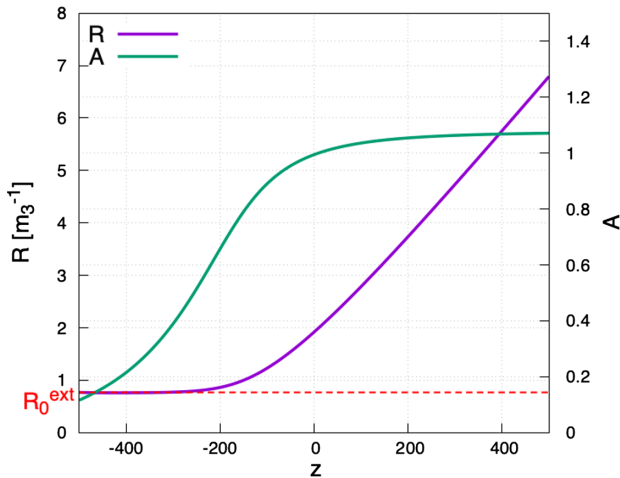


FIG. 3. The extremal solution for R and A as a function of z . The left axis is for R , and the right axis is for A .

where $R' \equiv \frac{dR}{dz}$. It is sufficient to consider only the (tR) sector of the metric. We concentrate on the metric near the horizon, and the metric components are expanded as

$$A^2 \sim \tilde{\alpha}(R - R_0), \quad R'^2 \sim \tilde{\beta}(R - R_0),$$

$$\tilde{\alpha} := \frac{\partial A^2}{\partial R} \Big|_{\text{horizon}}, \quad \tilde{\beta} := \frac{\partial R'^2}{\partial R} \Big|_{\text{horizon}}. \quad (25)$$

Then, the (tR) sector of the metric is

$$ds^2 = -\tilde{\alpha} \xi dt^2 + \frac{M_P^2}{m_3^4} \frac{d\xi^2}{\tilde{\beta} \xi} + \dots, \quad \xi := R - R_0. \quad (26)$$

Furthermore, we introduce the new coordinate ζ as

$$\frac{M_P^2}{m_3^4} \frac{d\xi^2}{\tilde{\beta} \xi} = d\zeta^2, \quad (27)$$

and the metric becomes

$$ds^2 = -\frac{\tilde{\alpha} \tilde{\beta}}{4} \left(\frac{m_3^2}{M_P} \right)^2 \zeta^2 dt^2 + d\zeta^2 + \dots \quad (28)$$

When the metric is Wick rotated ($t \rightarrow -i\tau$), the metric has a conical singularity, in general. In order to avoid it, the periodicity of τ has to be

$$(\text{periodicity of } \tau) = \frac{4\pi}{\sqrt{\tilde{\alpha} \tilde{\beta}}} \left(\frac{M_P}{m_3^2} \right). \quad (29)$$

This corresponds to the inverse temperature in the context of the quantum statistical physics, and therefore the Hawking temperature is given by

$$T_H = \frac{\sqrt{\tilde{\alpha} \tilde{\beta}}}{4\pi} \left(\frac{m_3^2}{M_P} \right). \quad (30)$$

For the extremal black string, the Taylor expansion of $A(R)^2$ starts from $(R - R_0^{\text{ext}})^2$. This means that $\tilde{\alpha} = 0$ and $T_H = 0$, as expected.

Nonextremal case. We simply integrate the differential equations (17), (21), and (22) starting from the horizon $z = 0$. The numerical solutions for several different values of the horizon size R_0 are plotted in Figs. 4–6. The solutions approach the vacuum ones (1) (red dashed line) for large z , as it should be. Here, σ and a are determined by fitting to the numerical solutions.

In Fig. 7, we plot σ and $1/\sqrt{a}$ as functions of R_0 . We observe that when R_0 approaches R_0^{ext} , σ approaches 0 as expected. Similarly, when $R_0 = R_0^{\text{ext}}$, the value of a is consistent with our previous extremal result, which characterizes the deficit angle.

In the nonextremal case, the black string should have a nonzero temperature. The Hawking temperature as a

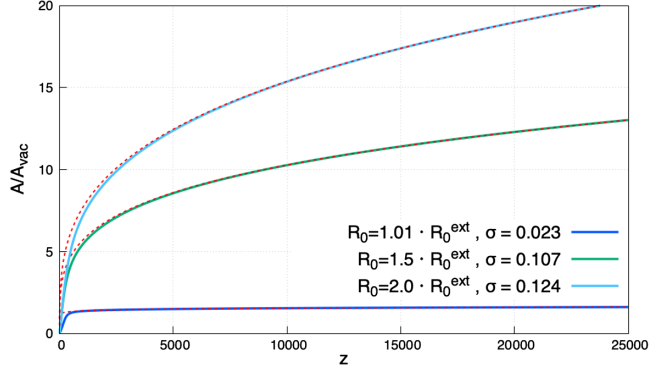


FIG. 4. The nonextremal solution for A as a function of z , where $A_{\text{vac}} := (M_P/m_3^2)^{\frac{2\sigma}{\Sigma}}$. The normalization of A is chosen in such a way that $\alpha = 1$ is realized for large z (1).

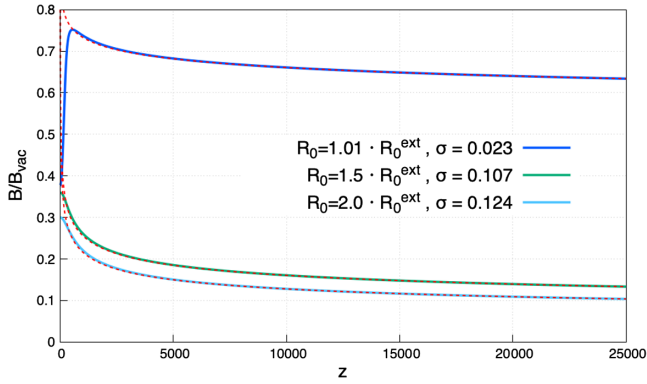


FIG. 5. The nonextremal solution for B as a function of z , where $B_{\text{vac}} := (M_P/m_3^2)^{\frac{-2\sigma(1-2\sigma)}{\Sigma}}$. The normalization of B is chosen in such a way that $\beta = 1$ is realized for large z (1).

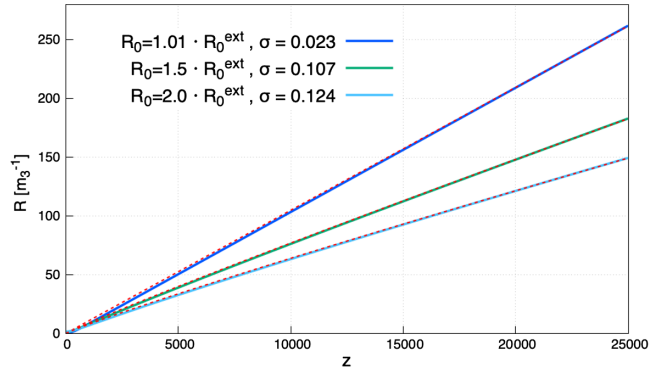


FIG. 6. The nonextremal solution for R as a function of z .

function of R_0 is plotted in Fig. 8. In this calculation, A is normalized so that $A = 1$ at a large z , instead of the way used in Sec. III ($\alpha = \beta = 1$). The behavior is very similar to that of the RN black hole.

It is valuable to see the behavior of the Hawking temperature when the string does not have $\mathbb{Z}_2$ charge.

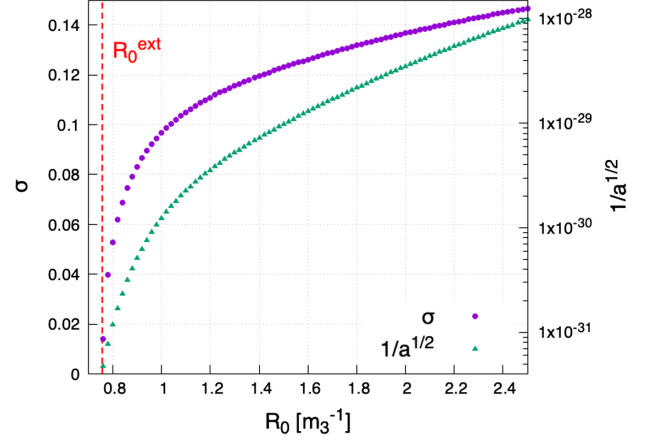


FIG. 7. The parameter σ and $1/\sqrt{a}$ as a function of the horizon size R_0 .

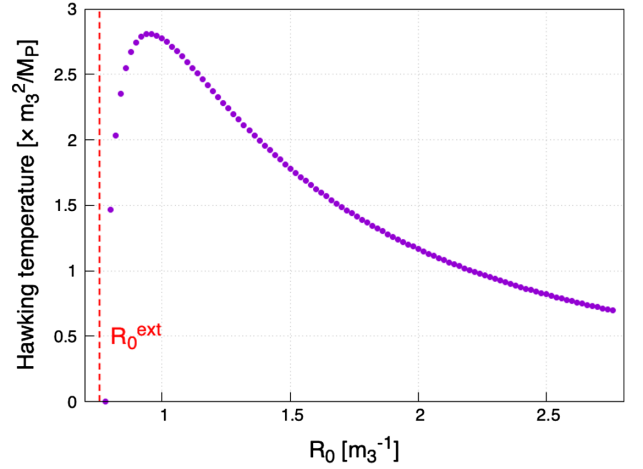


FIG. 8. The Hawking temperature as a function of the horizon size R_0 .

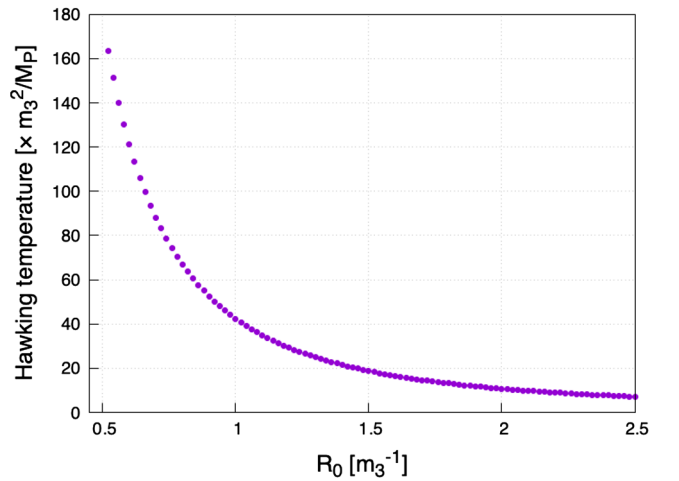


FIG. 9. The Hawking temperature as a function of the horizon size R_0 in the case where the black string does not have $\mathbb{Z}_2$ charge.

One can construct the black string solutions without $\mathbb{Z}_2$ charge by replacing the periodic boundary conditions for neutrinos in the Casimir energy with usual antiperiodic ones. In this case, there is no extremal solution as expected. Hence, the Hawking temperature behaves as the function of horizon size as in the case of the Schwarzschild black hole, as shown in Fig. 9. The black string may evaporate until the event horizon becomes Planckian in size.

IV. DISCUSSION AND CONCLUSION

In this paper, we have constructed the black string solution. There are several future directions to be explored. For instance, the stability of the solution is important. The perturbative analysis based on the gauge invariant variable would be necessary to this end. We have assumed the spin structure as a tangential structure of the spacetime. However, if the $B - L$ gauge field is introduced, it is more natural to consider the Spin^c structure. In this case, the bordism group is $\Omega_1^{\text{Spin}^c} = 0$. This implies the black string solution has an instability associated with the $B - L$ gauge field.

Another interesting direction is to construct the black solutions in string theories. The same technology we have used in this paper could be applied to the string theory branes.⁴ We will report these issues in a future

⁴See Ref. [16] for the solutions of heterotic black 0 and 4-branes, and Ref. [14] for the 6-brane.

publication. Furthermore, in the case of supersymmetric 7-branes [17], it involves an orbifold identification, which is viewed as an additional contribution to the deficit angle. It is interesting to study a similar effect in our context.

It is also worth considering phenomenological and cosmological applications. Particularly, it is quite nontrivial if the black string that we considered above can be produced in the Universe. Since the size of the black string is very thick (even comparable to the Hubble horizon size for the temperature around m_3), their dynamics after the production should be different from conventional cosmic strings [18].

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DATA AVAILABILITY

No data were created or analyzed in this study.

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