

Mass spectra of charged mesons and the quenching of vector meson condensation via exact phase-space diagonalization

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We investigate the dynamics and mass spectra of charged pseudoscalar (π^+) and vector (ρ^+) mesons in a background magnetic field at finite temperature using the two-flavor Nambu–Jona–Lasinio model. By employing a quark propagator that isolates the Schwinger phase from its Landau level expansion, we formulate an analytically controlled phase-space framework utilizing the Wigner–Weyl transform and the Moyal star product. This approach enables the algebraic diagonalization of the Bethe–Salpeter equations for composite states with asymmetric fractional constituent charges. For the pseudoscalar channel, we analytically verify the exact cancellation between the dynamical random phase approximation spatial sum rules and the vacuum gap equation. This identity preserves the generalized Goldstone theorem, causing the π^+ pole mass to track the kinematic zero-point energy drift at order of eB . In the vector channel, our full phase-space evaluation reveals that the Zeeman spin-splitting emerges dynamically from microscopic threshold truncations governed by the chiral Dirac algebra. Notably, we find that the tachyonic instability of the spin-aligned ρ^+ state is quenched. The magnetic catalysis of the chiral condensate drives the continuum threshold ($2M$) upwards, overtaking the Zeeman attraction and preventing vector meson condensation within this mean-field framework. Furthermore, finite-temperature evaluations show a monotonic thermal suppression of the meson masses driven by Pauli blocking, yet all modes remain bound without undergoing Mott dissociation prior to chiral symmetry restoration.

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I. INTRODUCTION

Understanding the phase structure and dynamic properties of strongly interacting matter under extreme conditions is a central endeavor in modern QCD [1]. In recent years, the physics of QCD matter exposed to intense background magnetic fields has attracted significant theoretical and experimental interest [2–4]. Such extreme magnetic environments are physically realized in various systems, ranging from the dense cores of magnetars ($eB \sim 10^{-4}$ – 10^{-2} GeV²) to relativistic heavy-ion collisions at the Relativistic Heavy Ion Collider and the LHC, where transient magnetic fields can reach extraordinary magnitudes on the hadronic scale ($eB \sim 0.1$ – 1.0 GeV²) [5,6]. The presence of these intense magnetic fields explicitly breaks spatial rotational symmetry and modifies the vacuum structure, leading to nonperturbative phenomena such as the chiral magnetic effect [7] and

magnetic catalysis (MC) of chiral symmetry breaking [8–13].

Among these magnetic phenomena, the mass spectrum and structural evolution of charged composite hadrons, specifically the pseudoscalar pion ($\pi^\pm$) and the vector ρ meson ($\rho^\pm$), serve as fundamental probes of the magnetized QCD vacuum. For the $\pi^\pm$ meson, its mass behaves as $m_{\pi^\pm}^2 \approx m_\pi^2 + |eB|$ in the weak-to-moderate field regime [14–26]. However, the dynamics of the charged vector meson ρ^+ present a more controversial theoretical landscape, as its energy level is governed by $m_{\rho^\pm}^2 = m_\rho^2 + (2n + 1 - 2S_z)|eB|$ in the point-particle level [27–29]. The spin-aligned ground state ($n = 0, S_z = 1$) predicts a severe Zeeman attraction $m_{\rho^+}^2 \approx m_\rho^2 - |eB|$. Consequently, at a critical magnetic field $eB_c \approx m_\rho^2$, the macroscopic mass is predicted to plunge into the tachyonic regime, triggering vector meson condensation (VMC) [30–33].

In reality, the charged ρ meson is a composite quark–antiquark ($u\bar{d}$) bound state governed by QCD dynamics at low energies. Recent first-principles lattice-QCD (LQCD) simulations [34–39], together with effective-model studies [40], indicate that the lowest spin-aligned charged-vector mode may initially decrease with the magnetic field but does not reach a tachyonic zero within the explored range.

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These results emphasize that the internal quark structure and its interaction with the QCD vacuum, including magnetic catalysis, are essential for understanding the stability of the vector meson. To describe these composite dynamics, Nambu–Jona-Lasinio (NJL) models [41–49] are widely employed.

Within these models, mesonic bound states are commonly described through the Bethe-Salpeter (BS) equations [50]. Evaluating the constituent-momentum sums and integrals entering the magnetized BS equations requires a consistent treatment of the constituent Landau-level sums, the charge-dependent transverse overlap structure, and the UV regularization. Approximations such as the lowest-Landau-level truncation or simple symmetric UV cutoffs can be useful in suitable regimes, but finite truncations must be implemented carefully to preserve the relevant spatial completeness and projector relations. Magnetic-eigenfunction formulations, most notably the Ritus method [51,52], provide a standard and powerful framework for diagonalizing charged propagators and evaluating observables in a uniform magnetic field. Their formal connections with Schwinger-phase and proper-time descriptions continue to be investigated [53]. For charged composite states whose constituents carry unequal fractional charges, the Ritus construction involves overlaps among magnetic eigenfunctions associated with different charge scales, leading to technically elaborate coordinate-space and Landau-level structures [54]. This complexity is representational rather than physical and motivates complementary ways of developing the same magnetic-translation structure.

The pioneering work of Gorbar and Miransky showed how the Schwinger phases encode magnetic translation symmetry and induce an associated noncommutative algebra (NCA)/Moyal feature [55,56]. Within this hierarchy, the Schwinger phase carries the physical information associated with magnetic translations, while the NCA/Moyal formulation organizes this structure algebraically in phase space. The Ritus and NCA/Moyal formulations should therefore be viewed as complementary representations of the same underlying physics: the Ritus method emphasizes diagonalization in a magnetic eigenfunction basis, whereas the NCA/Moyal formulation makes the charge-dependent transverse geometry and projector algebra explicit in phase space. Building on this perspective, the present work develops a Landau-Wigner phase-space formulation for charged composite states with unequal constituent charges, in which the Laguerre magnetic geometry, the internal and composite Landau-level structure, and the longitudinal bound-state dynamics are neatly separated.

Building on this Landau-Wigner phase-space formulation, we factorize the gauge-dependent Schwinger phases and map the tensor Bethe-Salpeter equations into Wigner-Weyl phase space [57]. Within this NCA/Moyal framework, the Moyal star product enables an algebraic diagonalization of the scattering matrix across the asymmetric

(n_u, n_d) constituent quark Landau levels [57–59]. This projection reduces the spatial convolutions to generalized triple Laguerre overlap integrals, explicitly conserving probability and transverse kinetic energy. Moreover, because the spatial treatment cleanly decouples from the Lorentz spin structure, the method applies uniformly to both pseudoscalar and vector charged mesons. We apply this phase-space diagonalization framework to reevaluate the mass spectra of the magnetized $\rho^\pm$ triplet states, revealing microscopically how chiral dynamics and threshold truncations inherently quench the tachyonic instability.

The paper is organized as follows: in Sec. II, we introduce the magnetized NJL model, formulate the exact gauge-phase factorization, and derive the exact vacuum gap equation. In Sec. III, we apply the Wigner-Weyl projection and the Moyal star product to the pseudoscalar channel, analytically demonstrating the Goldstone protection mechanism and numerically evaluating the π^+ mass. In Sec. IV, we extend the tensor noncommutative framework to the vector channel, exploring the dynamically generated Zeeman splitting, the quenched VMC mechanism, and the finite-temperature mass evolution. Finally, we summarize our findings and conclude in Sec. V. Technical details regarding the analytical proof of the projector algebra and the exact Laguerre overlap sum rules are relegated to the Appendixes.

II. MAGNETIZED QUARK PROPAGATORS AND PHASE-SPACE QUANTIZATION

We use the two-flavor NJL model to investigate the dynamics of charged mesons, the Lagrangian of which in the presence of background magnetic field is given by [41]

$$\begin{aligned} \mathcal{L} = & \bar{\psi}(i\not{D} - \hat{m})\psi + G[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\boldsymbol{\tau}\psi)^2] \\ & - G_V[(\bar{\psi}\boldsymbol{\gamma}\boldsymbol{\tau}\psi)^2 + (\bar{\psi}\boldsymbol{\gamma}\boldsymbol{\gamma}_5\boldsymbol{\tau}\psi)^2], \end{aligned} \quad (1)$$

where $\psi = (u, d)^T$ represents the quark field doublet with $\hat{m} = \text{diag}(m_u, m_d)$ the current quark mass, and $\boldsymbol{\tau}$ are the Pauli matrices in flavor space. The gauge-covariant derivative is defined as $D^\mu = \partial^\mu + i\hat{Q}A^\mu$, which couples the quarks to the external electromagnetic vector potential A^μ via the fractional charge matrix $\hat{Q} = \text{diag}(q_u, q_d) = \text{diag}(\frac{2}{3}e, -\frac{1}{3}e)$. The constants G and G_V denote the effective four-fermion coupling strengths in the scalar-pseudoscalar and vector-axial-vector channels, responsible for generating the π and ρ meson bound states, respectively.

In a constant homogeneous magnetic field $\mathbf{B} = B\hat{z}$, full translational symmetry is explicitly broken. The coordinate-space quark propagator can be factorized into a gauge-dependent Schwinger phase and a translationally invariant, gauge-independent part [3,60–62], as follows:

$$S_f(x, x') = e^{i\Phi_f(x, x')} \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x - x')} \tilde{S}_f(p), \quad (2)$$

where $\Phi_f(x, x') = \frac{1}{2} q_f x_\mu F^{\mu\nu} x'_\nu$ is the Schwinger phase in the symmetric gauge, with $F^{\mu\nu}$ being the electromagnetic tensor. In this exact coincidence limit, the gauge-dependent Schwinger phase $\Phi_f(x, x)$ strictly vanishes, preserving macroscopic gauge invariance. The translationally invariant momentum-space propagator $\tilde{S}_f(p)$ can be expressed as an infinite sum over internal fermionic Landau levels, as follows:

$$\tilde{S}_f(p) = ie^{-\alpha_f} \sum_{n=0}^{\infty} \frac{\mathcal{P}_n(q_f B, p)}{p_{\parallel}^2 - M^2 - 2n|q_f B| + i\epsilon}, \quad (3)$$

where $\alpha_f \equiv \mathbf{p}_{\perp}^2 / |q_f B|$. The numerator $\mathcal{P}_n(q_f B, p)$ encloses the spin dynamics and the dimensionally reduced Dirac structures within each Landau level, as follows:

$$\begin{aligned} \mathcal{P}_n(q_f B, p) = & (\gamma_{\parallel} \cdot p_{\parallel} + M) [\mathcal{P}_{+}^f (-1)^n L_n(2\alpha_f) \\ & + \mathcal{P}_{-}^f (-1)^{n-1} L_{n-1}(2\alpha_f)] \\ & + 2(\boldsymbol{\gamma}_{\perp} \cdot \mathbf{p}_{\perp}) (-1)^{n-1} L_{n-1}^1(2\alpha_f). \end{aligned} \quad (4)$$

Here, $p_{\parallel} = (p_0, 0, 0, p_z)$, $\mathbf{p}_{\perp} = (0, p_x, p_y, 0)$, and the spin projection operators are defined as $\mathcal{P}_{\pm}^f = \frac{1}{2}(1 \pm i s_f \gamma^1 \gamma^2)$ with $s_f = \text{sgn}(q_f B)$. For notational brevity, we define the uncharged projectors as $\mathcal{P}_{\pm} = \frac{1}{2}(1 \pm i \gamma^1 \gamma^2)$. Note that the generalized Laguerre polynomials $L_{-1}^{(a)} \equiv 0$, which effectively annihilates the unphysical spin states in the lowest Landau level ($n = 0$).

Under mean-field approximation, the quark constituent quark mass M is determined by the self-consistent gap equation [4,41],

$$M = m_0 - 2G \langle \bar{\psi} \psi \rangle = m_0 - 2G \sum_{f=u,d} \langle \bar{f} f \rangle, \quad (5)$$

where m_0 is the current quark mass (assuming isospin symmetry $m_u = m_d = m_0$), and the condensate is

$$\langle \bar{f} f \rangle = -i N_c \text{Tr}_D [\lim_{x' \rightarrow x} S_f(x, x')], \quad (6)$$

where Tr_D denotes the trace over the Dirac spinor indices, and the explicit factor N_c arises from the trace over the color space. Substituting the Landau level expansion and evaluating the Dirac trace yields the fundamental vacuum structure. We identify the universal chiral scalar density J_1 , which determines the gap equation. Summing over the flavors, it reads

$$J_1 = \sum_{f=u,d} \frac{1}{2} \left[-i N_c \frac{|q_f B|}{2\pi} \sum_{n_f=0}^{\infty} \beta_{n_f} \int \frac{dp_0 dp_z}{(2\pi)^2} \frac{1}{p_0^2 - p_z^2 - M^2 - 2n_f |q_f B| + i\epsilon} \right], \quad (7)$$

where $\beta_{n_f} = 2 - \delta_{n_f,0}$ explicitly accounts for the spin degeneracy of the Landau levels. This dimensionally reduced integral directly relates to the chiral condensate via $J_1 = \sum_f \langle \bar{f} f \rangle_B / (8M)$.

When constructing the coordinate-space polarization function for the charged pion π^+ , we trace over the closed quark loop, as follows:

$$\Pi_{\pi^+}(x, y) = -i N_c \text{Tr}_D [\gamma_5 S_u(x, y) \gamma_5 S_d(y, x)]. \quad (8)$$

Since $\Phi_d(y, x) = -\Phi_d(x, y)$, the total accumulated Schwinger phase in the loop evaluates to

$$\begin{aligned} \Phi_{\text{loop}}(x, y) = & \Phi_u(x, y) + \Phi_d(y, x) = \frac{1}{2} (q_u - q_d) x_\mu F^{\mu\nu} y_\nu \\ = & \frac{1}{2} q_\pi x_\mu F^{\mu\nu} y_\nu \equiv \Phi_\pi(x, y). \end{aligned} \quad (9)$$

Given the fractional charges $q_u = \frac{2}{3}e$ and $q_d = -\frac{1}{3}e$, this residual phase elegantly matches the macroscopic Schwinger phase of a composite particle carrying the pion's physical net charge $q_\pi = e$.

In the NJL model, mesons appear as collective quark-antiquark excitations generated by the resummation of polarization bubbles. The random phase approximation (RPA) is the leading-order NJL framework for this resummation, and the poles of the resulting scattering amplitude determine the meson masses. In the present formulation, RPA supplies the dynamical bound-state approximation, while the NCA/Moyal framework organizes the magnetic spatial structure.

This bubble-resummation structure and its compact kernel representation are illustrated schematically in Fig. 1. The corresponding scattering T -matrix in the RPA channel satisfies the Bethe-Salpeter integral equation [41,63,64],

$$T(x, y) = 2G \delta^4(x - y) + \int d^4 z K(x, z) T(z, y), \quad (10)$$

where the first term is the local bare contact interaction, and $K(x, z) \equiv 2G \Pi(x, z)$ represents the irreducible one-loop polarization kernel. Because both T and K describe the propagation of the charged pion, they share the identical global Schwinger phase: $T(x, y) = e^{i\Phi_\pi(x, y)} \tilde{T}(x - y)$ and $K(x, y) = e^{i\Phi_\pi(x, y)} \tilde{K}(x - y)$. Particularly, the local bare vertex inherently satisfies this factorization since $2G \delta^4(x - y) = e^{i\Phi_\pi(x, y)} [2G \delta^4(x - y)]$ trivially holds at $x = y$. When performing the spatial convolution over the intermediate

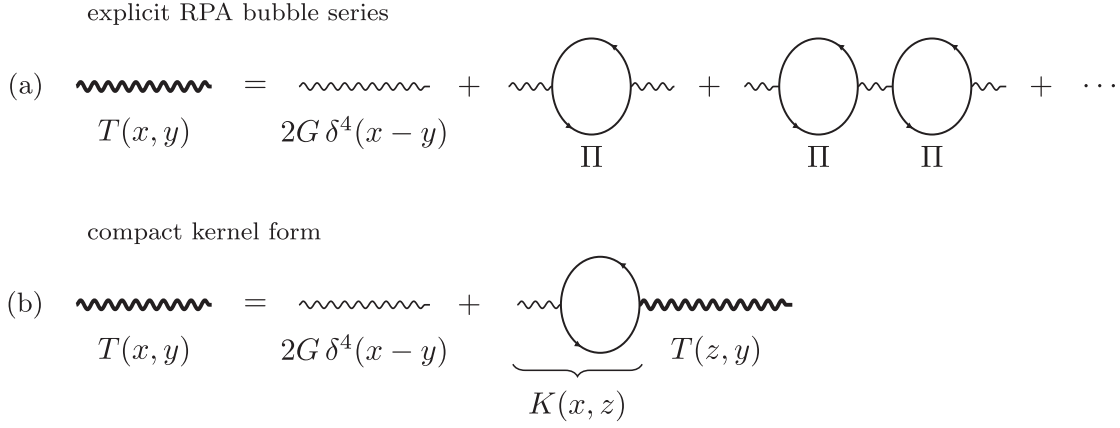


FIG. 1. Diagrammatic representation of the random phase approximation resummation for the quark-antiquark scattering amplitude $T(x, y)$. In panel (a), thin wavy lines denote the local four-fermion interaction $2G\delta^4$, while each fermion loop represents the one-loop polarization function Π ; the thick wavy line denotes the full resummed amplitude. In panel (b), the thin interaction line and the unlabeled fermion loop grouped by the brace define the compact kernel $K(x, z)$, which is connected to the full amplitude $T(z, y)$; the convolution over the intermediate coordinate z is understood. The mesonic collective modes are identified from the poles of the resummed amplitude.

coordinate z , the product of the gauge phases satisfies the geometric identity [60], as follows:

$$\Phi_\pi(x, z) + \Phi_\pi(z, y) = \Phi_\pi(x, y) + \frac{q_\pi}{2}(x - z)_\mu F^{\mu\nu}(z - y)_\nu. \quad (11)$$

By factoring out the global phase $e^{i\Phi_\pi(x, y)}$ from the entire equation, this relation explicitly demonstrates that the coordinate difference inside the integral introduces the additional phase factor $e^{i\frac{q_\pi}{2}(x-z)_\mu F^{\mu\nu}(z-y)_\nu}$.

Upon performing a Fourier transform to momentum space, this phase factor translates into momentum derivatives. The standard convolution integral thereby morphs into the noncommutative Moyal star product [57],

$$\tilde{T}(q) = 2G + \tilde{K}(q) \star \tilde{T}(q), \quad (12)$$

where the bare contact interaction $2G$ drives the lowest-order scattering, and $\tilde{K}(q) \equiv 2G\tilde{\Pi}(q)$ denotes the translationally invariant part of this kernel in momentum space. The star product is formally defined through the infinite-order pseudodifferential operator, as follows:

$$\begin{aligned} f(q) \star g(q) &= f(q) \exp\left(-\frac{i}{2}\tilde{\partial}_{q_\mu}(q_\pi F^{\mu\nu})\tilde{\partial}_{q_\nu}\right)g(q) \\ &= f(q)g(q) - \frac{i}{2}q_\pi F^{\mu\nu}[\partial_{q_\mu}f(q)][\partial_{q_\nu}g(q)] \\ &\quad + \mathcal{O}(F^2). \end{aligned} \quad (13)$$

By expanding the exponential operator, the physical transition from the vacuum to the magnetized medium becomes transparent. The zeroth-order term trivially

recovers the ordinary commutative product, strictly reproducing the standard RPA equation in the B -field free limit. However, the exact retention of the Schwinger phase introduces an infinite tower of higher-order momentum derivatives driven by the background field $F^{\mu\nu}$. These derivative terms entangle the transverse momenta, preventing the standard algebraic factorization of the scattering matrix. To rigorously diagonalize this noncommutative system without compromising the spatial geometry, we must project the equation onto an exact eigenbasis of the star product.

To achieve this, we expand the kernels in terms of the basis functions $\mathcal{P}_n(\alpha_\pi) \equiv 2e^{-\alpha_\pi}(-1)^n L_n(2\alpha_\pi)$, where $\alpha_\pi \equiv q_\perp^2/|q_\pi B|$. These functions act as algebraic projection operators under the Moyal star product, satisfying the noncommutative projector algebra [65], as follows:

$$\mathcal{P}_n(\alpha_\pi) \star \mathcal{P}_{n'}(\alpha_\pi) = \delta_{n, n'} \mathcal{P}_n(\alpha_\pi). \quad (14)$$

This orthogonality can be intuitively understood via the Weyl-Wigner correspondence. The basis functions $\mathcal{P}_n(\alpha_\pi)$ are the exact Wigner transforms of the one-dimensional quantum harmonic oscillator density matrices $\hat{\rho}_n = |n\rangle\langle n|$. Since these Hilbert space projection operators are strictly orthonormal and idempotent ($\hat{\rho}_n \hat{\rho}_{n'} = \delta_{n, n'} \hat{\rho}_n$), their phase-space Wigner transforms naturally inherit this algebraic relation under the star product. A detailed proof using Gaussian generating functions is provided in Appendix A.

Projecting the BS equation onto this mesonic basis collapses the differential star product into decoupled, algebraic scalar equations for each mesonic Landau level n , as follows:

$$\hat{T}_n(q_\parallel) = 2G + 2G\hat{\Pi}_n(q_\parallel)\hat{T}_n(q_\parallel). \quad (15)$$

Therefore, within RPA, the irreducible scattering kernel is driven by the bare contact interaction $2G$ and the projected one-loop polarization tensor $\hat{\Pi}_n(q_{\parallel})$. It leads to the exact solution for the full scattering amplitude:

$$\hat{T}_n(q_{\parallel}) = \frac{2G}{1 - 2G\hat{\Pi}_n(q_{\parallel})}. \quad (16)$$

The pole mass $m_{\pi^{\pm},n}$ of the charged pion in the n th macroscopic Landau level is determined by the divergence of this scattering amplitude, which corresponds strictly to the zeros of the denominator:

$$1 - 2G\hat{\Pi}_n(q_{\parallel}^2 = m_{\pi^{\pm},n}^2) = 0. \quad (17)$$

To extract the dynamical coefficient $\hat{\Pi}_n(q_{\parallel})$ from the unprojected momentum-space tensor $\Pi(q_{\parallel}, \mathbf{q}_{\perp})$, we utilize the integral trace property of the Wigner-Weyl transform:

$$\begin{aligned} \hat{\Pi}_n(q_{\parallel}) &= \frac{1}{2} \int_0^{\infty} d\alpha_{\pi} \Pi(q_{\parallel}, \alpha_{\pi}) \mathcal{P}_n(\alpha_{\pi}) \\ &= \frac{1}{2} \int_0^{\infty} d\alpha_{\pi} \Pi(q_{\parallel}, \alpha_{\pi}) [2e^{-\alpha_{\pi}} (-1)^n L_n(2\alpha_{\pi})]. \end{aligned} \quad (18)$$

This specific projection weight inherently guarantees the correct probability normalization of the mesonic wave function. The physical consistency of this formulation is readily verified in the homogeneous weak-field limit. When the macroscopic transverse momentum vanishes ($\mathbf{q}_{\perp} \rightarrow \mathbf{0}$, hence $\alpha_{\pi} \rightarrow 0$), the unprojected tensor reduces to its translationally invariant form $\Pi(q_{\parallel}, \alpha_{\pi}) \rightarrow \Pi_{\text{homo}}(q_{\parallel})$. Evaluating the projection for the macroscopic ground state ($n = 0$) yields

$$\hat{\Pi}_{n=0}(q_{\parallel}) = \Pi_{\text{homo}}(q_{\parallel}) \int_0^{\infty} d\alpha_{\pi} e^{-\alpha_{\pi}} = \Pi_{\text{homo}}(q_{\parallel}), \quad (19)$$

where the projected amplitude smoothly and analytically recovers the standard vacuum RPA polarization tensor.

The primary computational task is therefore to evaluate the unprojected $\Pi(q_{\parallel}, \mathbf{q}_{\perp})$ from the trace of two magnetized quark propagators and perform the projection integral in Eq. (18). Because $|q_u B| \neq |q_d B|$, the substitution will generate asymmetric products of $L_{n_u}(|q_u B|)$ and $L_{n_d}(|q_d B|)$. Integrating these internal functions against the external mesonic state $L_n(|q_{\pi} B|)$ produces complex generalized triple Laguerre overlap integrals, which we derive in the subsequent sections.

III. DYNAMICS OF THE CHARGED PION IN A MAGNETIC FIELD

We evaluate the unprojected polarization tensor for the charged pion π^+ by tracing over the u and $\bar{d}$ quark propagators. Let the internal loop momentum be p , and

the external mesonic momentum be $q = (q_0, \mathbf{q}_{\perp}, q_z)$. It is required to maintain a nonvanishing transverse momentum $\mathbf{q}_{\perp} \neq \mathbf{0}$, as it governs the spatial Moyal star product and the ensuing Laguerre projection. To simplify the longitudinal kinematics without loss of physical insight regarding mass generation, we evaluate the tensor in the longitudinal rest frame by setting $q_z = 0$. The loop momentum routing is defined such that the u -quark carries momentum $k = p + q$ and the d -quark carries p .

To maintain algebraic consistency with the universal gap equation background J_1 defined in Sec. II, we evaluate the properly scaled unprojected polarization tensor by factoring out the global Dirac scalar factor of 4, as follows:

$$\Pi_{\pi^+}^{\text{scaled}}(q) = -iN_c \int \frac{d^4 p}{(2\pi)^4} \frac{1}{4} \text{Tr}_D[\gamma_5 \tilde{S}_u(k) \gamma_5 \tilde{S}_d(p)]. \quad (20)$$

Because $\mathcal{P}_{\pm}^d = \mathcal{P}_{\mp}^u$, the algebraic multiplication of the longitudinal structures generates crossing terms, yielding an inherent parity phase of $(-1)^{n_u+n_d}$ that aligns with the fundamental spatial basis, where n_u and n_d are the internal Landau level indices. Under the chiral flip ($\gamma_5 \gamma_i \gamma_5 = -\gamma_i$), the transverse trace yields $\text{Tr}_D[-\gamma_i \gamma_j] = 4\delta_{ij}$. Combining all the parity signs, the scaled trace evaluates to an additive combination, as follows:

$$\begin{aligned} &\frac{1}{4} \text{Tr}[\gamma_5 \mathcal{P}_{n_u}^u(k) \gamma_5 \mathcal{P}_{n_d}^d(p)] \\ &= (-1)^{n_u+n_d} \left[\frac{1}{2} S_L \cdot \mathcal{F}_{n_u, n_d}^{LL}(\mathbf{k}_{\perp}, \mathbf{p}_{\perp}) + \mathcal{F}_{n_u, n_d}^{TT}(\mathbf{k}_{\perp}, \mathbf{p}_{\perp}) \right], \end{aligned} \quad (21)$$

where the longitudinal kinematic invariant $S_L = k_{\parallel} \cdot p_{\parallel} - M^2$. The corresponding transverse spatial weighting functions are

$$\mathcal{F}_{n_u, n_d}^{LL} = L_{n_u}(2\alpha_u) L_{n_d-1}(2\alpha_d) + L_{n_u-1}(2\alpha_u) L_{n_d}(2\alpha_d), \quad (22)$$

$$\mathcal{F}_{n_u, n_d}^{TT} = 4(\mathbf{k}_{\perp} \cdot \mathbf{p}_{\perp}) L_{n_u-1}^1(2\alpha_u) L_{n_d-1}^1(2\alpha_d), \quad (23)$$

where $\alpha_u = \mathbf{k}_{\perp}^2 / |q_u B|$ and $\alpha_d = \mathbf{p}_{\perp}^2 / |q_d B|$. Note that the generalized Laguerre polynomial definition inherently enforces $L_{-1}^1(2\alpha) \equiv 0$. This ensures that unphysical spin configurations in the lowest Landau level are analytically annihilated, maintaining accurate state counting and numerical stability.

The spatial overlap factors for the macroscopic mesonic ground state ($n = 0$, where $L_0(2\alpha_{\pi}) = 1$) are obtained by integrating the Dirac trace weights over the transverse phase space. The canonical type I (scalar) and type II (vector) overlap integrals are

$$\mathcal{J}_{n_u, n_d, 0}^{(1)} = \int d^2\mathbf{k}_\perp d^2\mathbf{p}_\perp e^{-\alpha_u - \alpha_d - \alpha_\pi} (-1)^{n_u + n_d} L_{n_u}(2\alpha_u) L_{n_d}(2\alpha_d), \quad (24)$$

$$\mathcal{J}_{n_u, n_d, 0}^{(2)} = \int d^2\mathbf{k}_\perp d^2\mathbf{p}_\perp e^{-\alpha_u - \alpha_d - \alpha_\pi} (\mathbf{k}_\perp \cdot \mathbf{p}_\perp) (-1)^{n_u + n_d} L_{n_u-1}^1(2\alpha_u) L_{n_d-1}^1(2\alpha_d), \quad (25)$$

By mapping the internal phase space via $\mathbf{k}_\perp = \mathbf{p}_\perp + \mathbf{q}_\perp$, the differential volume is conserved ($d^2\mathbf{q}_\perp d^2\mathbf{p}_\perp = d^2\mathbf{k}_\perp d^2\mathbf{p}_\perp$). As detailed in Appendix B, we systematically evaluate these integrals by mapping them into parameter space using generating functions to extract finite numerical recurrence relations. In the weak magnetic field regime ($|q_\pi B| \rightarrow 0$), the analytical Wigner projection measure asymptotically collapses into a momentum-space Dirac δ function: $\lim_{|q_\pi B| \rightarrow 0} [\frac{1}{\pi|q_\pi B|} d^2\mathbf{q}_\perp] e^{-\mathbf{q}_\perp^2/|q_\pi B|} = \delta^{(2)}(\mathbf{q}_\perp) d^2\mathbf{q}_\perp$. This δ -function limit recovers the continuous vacuum polarization function without altering the normalization of the transverse phase-space measure.

Applying the Wigner-Weyl Laguerre projection to the lowest macroscopic ground state ($n = 0$) yields the dynamical coefficient

$$\hat{\Pi}_{n=0}(q_0^2) = \frac{4iN_c}{\pi|q_\pi B|} \sum_{n_u, n_d=0}^{\infty} \int \frac{d\omega dp_z}{(2\pi)^2} \frac{-\frac{1}{2}S_L(\mathcal{J}_{n_u, n_d-1, 0}^{(1)} + \mathcal{J}_{n_u-1, n_d, 0}^{(1)} + \mathcal{J}_{n_u, n_d, 0}^{(2)})}{D_u(\omega + q_0)D_d(\omega)}. \quad (26)$$

Here, the inverse quark propagators governing the longitudinal energy poles are $D_u(\omega + q_0) = (\omega + q_0)^2 - p_z^2 - M^2 - 2n_u|q_u B|$, $D_d(\omega) = \omega^2 - p_z^2 - M^2 - 2n_d|q_d B|$ and $\omega = p_0$ for convenience.

We isolate the vacuum condensate background from the dynamic mesonic excitation via a partial fraction decomposition on the longitudinal invariant, as follows:

$$-\frac{S_L}{D_u D_d} = -\left[\frac{1}{2D_d} + \frac{1}{2D_u}\right] + \frac{q_0^2/2 - n_u|q_u B| - n_d|q_d B|}{D_u D_d}. \quad (27)$$

This decomposition uncouples the polarization tensor into a static background and a dynamic kernel: $\hat{\Pi}_{n=0}(q_0^2) = J_1 + J_m(q_0^2)$.

The single-denominator terms constitute the vacuum condensate background. Because $D_d(\omega)$ depends exclusively on n_d , the infinite summation over n_u acts solely on the spatial overlap factor. Utilizing the completeness relation of the type I overlap integral derived in Appendix B, we have

$$\sum_{n_u=0}^{\infty} \mathcal{J}_{n_u, n_d, 0}^{(1)} = \frac{\pi^2}{2} |q_\pi B| |q_d B|. \quad (28)$$

Substituting this geometric identity back, the overall prefactor evaluates analytically to the charge-dependent factor $iN_c \frac{|q_d B|}{2\pi}$. Accounting for the (u, d) -flavor spaces and combining all the factors, the two single-denominator terms exactly recover the vacuum gap equation prefactor $-iN_c \frac{|q_f B|}{4\pi}$, aligning with the J_1 background.

The remainder forms the dynamic two-particle scattering kernel $J_m(q_0^2)$, which governs the charged pion mass generation, as follows:

$$J_m(q_0^2) = -\frac{4iN_c}{\pi|q_\pi B|} \sum_{n_u, n_d=0}^{\infty} \int \frac{d\omega dp_z}{(2\pi)^2} \frac{\mathcal{N}_{\pi^+}}{D_u(\omega + q_0)D_d(\omega)}, \quad (29)$$

where the scalar dynamic mass scale is $R_{\pi^+}^2 = n_u|q_u B| + n_d|q_d B| - q_0^2/2$, and the consolidated transverse spatial weight is

$$\mathcal{N}_{\pi^+} = \frac{1}{2} R_{\pi^+}^2 (\mathcal{J}_{n_u, n_d-1, 0}^{(1)} + \mathcal{J}_{n_u-1, n_d, 0}^{(1)} - \mathcal{J}_{n_u, n_d, 0}^{(2)}). \quad (30)$$

Applying the Matsubara summation formalism ($\omega \rightarrow i\omega_m = i(2m+1)\pi T$), the kernel splits into vacuum and medium components ($J_m = J_m^{\text{vac}} + J_m^{\text{med}}$), as follows:

$$J_m^{\text{vac}}(q_0^2) = -\frac{4N_c}{\pi|q_\pi B|} \sum_{n_u, n_d=0}^{\infty} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{\mathcal{N}_{\pi^+}}{2E_u E_d} \times \left[\frac{E_u + E_d}{q_0^2 - (E_u + E_d)^2} \right], \quad (31)$$

$$J_m^{\text{med}}(q_0^2, T) = \frac{4N_c}{\pi|q_\pi B|} \sum_{n_u, n_d=0}^{\infty} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{\mathcal{N}_{\pi^+}}{2E_u E_d} \left[\frac{E_u + E_d}{q_0^2 - (E_u + E_d)^2} (n_F(E_u) + n_F(E_d)) \right. \\ \left. - \frac{E_u - E_d}{q_0^2 - (E_u - E_d)^2} (n_F(E_u) - n_F(E_d)) \right]. \quad (32)$$

To manage UV divergences, we apply a 3D soft cutoff regularization with a smooth damping function $f_\Lambda(x) = [1 + (x/\Lambda^2)^5]^{-1}$. The transverse regularization scale is designated to depend exclusively on the Landau level of a *single* reference constituent quark. By leaving the conjugate constituent's phase space unconstrained by the UV cutoff, the infinite summation over its discrete energy levels cleanly factors out, ensuring the analytical measure annihilation proceeds without truncation artifacts.

Substituting the chiral symmetry breaking condition $(1 - 2GJ_1 = m_0/M)$ into the RPA pole equation manifests Goldstone's theorem: for a finite current quark mass m_0 , the generalized mass equation reduces to a form dependent only on J_m , as follows:

$$\frac{m_0}{M} - 2G[J_m^{\text{vac, cutoff}}(m_{\pi^\pm}^2) + J_m^{\text{med}}(m_{\pi^\pm}^2, T)] = 0. \quad (33)$$

We first examine the spatial dynamics generated by the asymmetric constituent charges. In Fig. 2, we present the unprojected spatial overlap integrals in the (n_u, n_d) phase space at $|eB| = 0.3 \text{ GeV}^2$. For a neutral meson, identical charge scalings enforce a Kronecker- δ orthogonality ($n_u = n_d$). In contrast, the absolute constituent charges of the charged pion satisfy a 2:1 ratio ($|q_u| = 2|q_d|$). As shown in the left and middle panels, the Moyal star product dynamically generates a localized strong-coupling ridge along the kinematic trajectory $n_d \approx 2n_u$, corresponding to the geometric matching of their transverse spatial dispersions. The right panel shows the dimensionally normalized

type II vector overlap, confirming its function as a transverse kinetic energy extraction operator that projects out the scale $\frac{1}{2}\mathcal{C}_{\text{norm}} \cdot (n_f|q_f B|)$.

We numerically evaluate the physical mass of the charged pion π^+ . The NJL parameters are fixed at $m_0 = 0.0052 \text{ GeV}$, $\Lambda = 0.6213 \text{ GeV}$, and $G = 5.1881 \text{ GeV}^{-2}$, targeting $m_\pi \approx 138 \text{ MeV}$ and $\langle \bar{u}u \rangle^{1/3} \approx -250 \text{ MeV}$ in the vacuum limit [66–69]. In Fig. 3, we present the constituent quark mass M and the pion mass m_{π^+} as functions of the magnetic field at $T = 0 \text{ MeV}$ and $T = 80 \text{ MeV}$. The macroscopic mass evolution of the π^+ rigorously tracks the kinematic zero-point energy drift, $m_{\pi^\pm}^2 \approx m_\pi(0)^2 + |eB|$. Because the dynamically generated vacuum condensate background J_1 is canceled exactly by the scalar interaction term via the gap equation, the π^+ meson remains protected by the generalized Goldstone theorem. The minor numerical nonsmoothness near the weak-field limit ($|eB| < 0.05 \text{ GeV}^2$) is a computational artifact arising from evaluating the soft-cutoff regularizer in the increasingly dense discrete continuum.

Figure 4 illustrates the thermal mass shift $\Delta m_\pi^+ = m_\pi^+(T, eB) - m_\pi^+(0, eB)$. The monotonic suppression of the pion mass as temperature rises is dynamically generated by the Fermi-Dirac distributions within the medium kernel J_m^{med} . Thermal scattering and Pauli blocking effectively weaken the quark-antiquark binding interaction, pushing the pole mass downward prior to the critical temperature of chiral symmetry restoration.

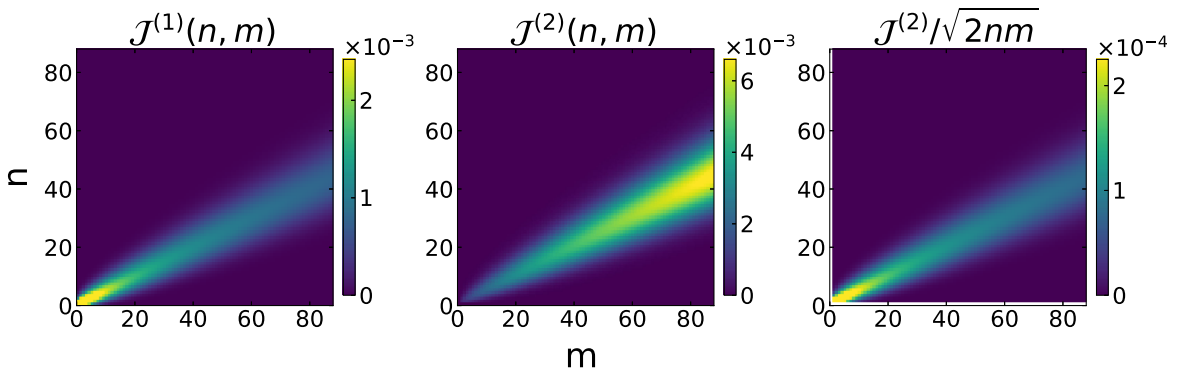


FIG. 2. Numerical evaluation of the generalized triple Laguerre overlap integrals in the internal constituent phase space (n_u, n_d) for the macroscopic ground state at $|eB| = 0.3 \text{ GeV}^2$. Left: type I scalar overlap $\mathcal{J}^{(1)}(n_u, n_d, 0)$. Middle: type II vector overlap $\mathcal{J}^{(2)}(n_u, n_d, 0)$. Right: normalized type II overlap $\mathcal{J}^{(2)}/\sqrt{2nm}$, shown only for $n, m \geq 1$. Color bar tick values are multiplied by 10^{-3} , 10^{-3} , and 10^{-4} in the left, middle, and right panels, respectively. A broad coupling ridge emerges along the trajectory corresponding to the 1:2 charge ratio, demonstrating the dynamic phase-space matching enforced by the Moyal star product for charged constituents.

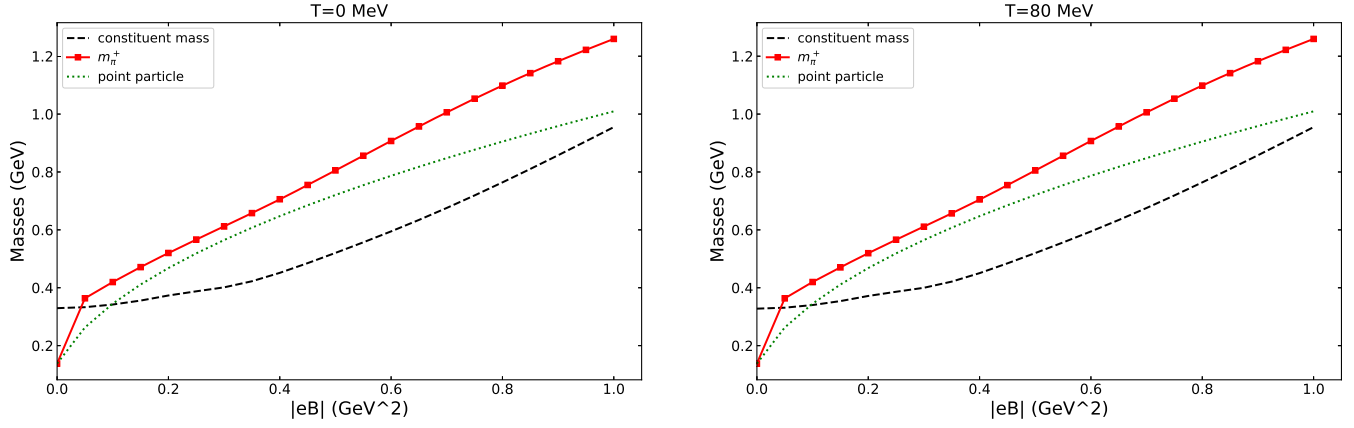


FIG. 3. Constituent quark mass M (black dashed line) and the charged pion mass m_{π^+} (red line) as a function of the magnetic field $|eB|$ at $T = 0$ MeV (left) and $T = 80$ MeV (right). The green dotted line represents the point-particle kinematic limit $\sqrt{m_{\pi}(0)^2 + |eB|}$. The composite pion mass tracks this zero-point drift due to the gap equation cancellation.

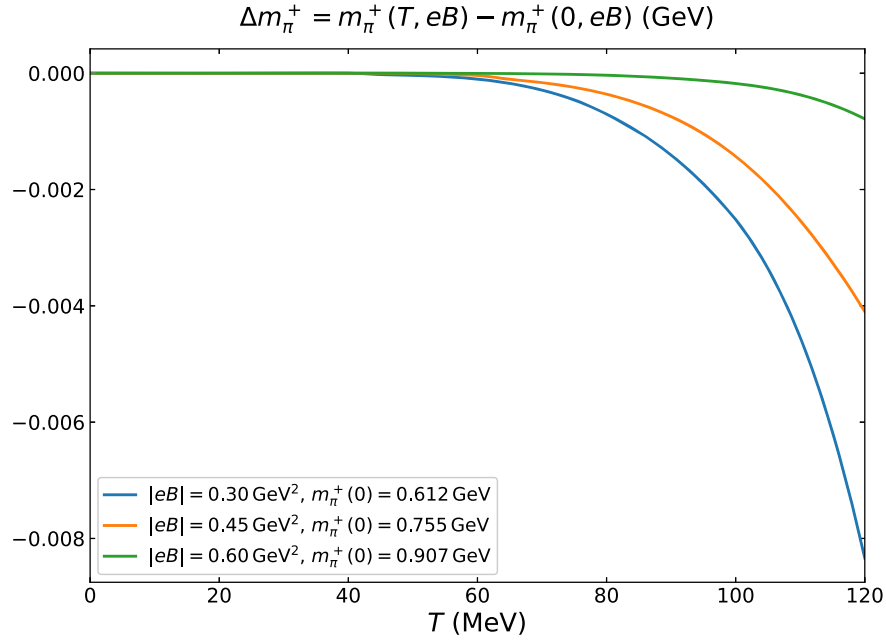


FIG. 4. Temperature-induced mass shift $\Delta m_{\pi^+} = m_{\pi^+}(T, eB) - m_{\pi^+}(0, eB)$ of the charged pion at fixed magnetic fields $|eB| = 0.30, 0.45, \text{ and } 0.60 \text{ GeV}^2$. The downward trend is driven by Pauli blocking and thermal scattering within the finite-temperature Matsubara summations.

IV. DYNAMICS OF THE CHARGED ρ MESON IN A MAGNETIC FIELD

The algebraic diagonalization achieved for the pseudo-scalar π^+ meson via the Moyal star product provides a universal framework for charged composite states. We now extend this formalism to the charged vector meson ρ^+ , which introduces Lorentz tensor structures and spin-magnetic couplings.

The ρ^+ meson shares the same valence quark content ($u\bar{d}$) and net electric charge ($q_{\rho} = e$) as the π^+ meson. The

irreducible quark-antiquark polarization kernel for the vector channel is given by

$$\begin{aligned} K^{\mu\nu}(x, y) &= 2G_V \Pi^{\mu\nu}(x, y) \\ &= 2iG_V N_c \text{Tr}_D [\gamma^\mu S_u(x, y) \gamma^\nu S_d(y, x)], \end{aligned} \quad (34)$$

where G_V is the vector four-fermion coupling constant. This kernel accumulates the identical macroscopic Schwinger phase: $K^{\mu\nu}(x, y) = e^{i\Phi_{\rho}(x, y)} \tilde{K}^{\mu\nu}(x - y)$. Factoring out this phase and performing the Fourier transform, the convolution

translates into the tensor star product, as follows:

$$\tilde{T}^{\mu\nu}(q) = 2G_V + \tilde{K}^{\mu\alpha}(q) \star \tilde{T}_\alpha^\nu(q). \quad (35)$$

To diagonalize this equation, the projection basis must separate the spatial phase space and the Lorentz spin structure. The background magnetic field $\mathbf{B} = B\hat{z}$ breaks the $SO(1,3)$ Lorentz symmetry down to $SO(1,1)_\parallel \times SO(2)_\perp$, splitting the physical spin-1 states into three polarization modes governed by the spin projection s_z .

We define the orthonormal polarization vectors for the three states $\lambda \in \{+, -, \parallel\}$ [27,70,71] as follows:

$$\epsilon_+^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0) \quad (s_z = +1, \text{Right-handed transverse}), \quad (36)$$

$$\epsilon_-^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0) \quad (s_z = -1, \text{Left-handed transverse}), \quad (37)$$

$$\epsilon_\parallel^\mu = (0, 0, 0, 1) \quad (s_z = 0, \text{Longitudinal}). \quad (38)$$

These construct the spin projection tensors $P_\lambda^{\mu\nu} = (\epsilon_\lambda^\mu)^* \epsilon_\lambda^\nu$. The complete eigenbasis for the vector meson is the direct product of the macroscopic Laguerre functions $\mathcal{P}_n(\alpha_\rho)$ with $\alpha_\rho \equiv \mathbf{q}_\perp^2/|q_\rho B|$ and the spin projectors $P_\lambda^{\mu\nu}$. Because the spatial star product commutes with constant Lorentz tensors, the projection algebra factorizes as follows:

$$[P_\lambda^{\mu\alpha} \mathcal{P}_n(\alpha_\rho)] \star [P_{\lambda'\alpha}^\nu \mathcal{P}_{n'}(\alpha_\rho)] = \delta_{n,n'} \delta_{\lambda,\lambda'} [P_\lambda^{\mu\nu} \mathcal{P}_n(\alpha_\rho)]. \quad (39)$$

Projecting the tensor BS equation onto this basis decouples it into three independent scalar equations. The projected dynamical coefficient for each Landau level n and spin state λ is

$$\hat{\Pi}_{n,\lambda}(q_\parallel) = \int_0^\infty d\alpha_\rho [(\epsilon_\lambda^\mu)^* \Pi_{\mu\nu}(q_\parallel, \alpha_\rho) \epsilon_\lambda^\nu] \times [e^{-\alpha_\rho} (-1)^n L_n(2\alpha_\rho)]. \quad (40)$$

Consequently, the physical masses $m_{\rho^+,n}(\lambda)$ of the charged vector meson are determined by three generalized RPA pole equations:

$$1 - 2G_V \hat{\Pi}_{n,\lambda}(q_\parallel^2 = m_{\rho^+,n}^2(\lambda)) = 0 \quad \text{for } \lambda \in \{+, -, \parallel\}. \quad (41)$$

The unprojected tensor for a specific physical state λ is strictly defined as

$$\Pi_{\rho^+}(\lambda) = iN_c \int \frac{d^4 p}{(2\pi)^4} \text{Tr}_D[(\gamma \cdot \epsilon_\lambda^*) \tilde{S}_u(k) (\gamma \cdot \epsilon_\lambda) \tilde{S}_d(p)]. \quad (42)$$

For the longitudinal mode ($\lambda = \parallel$), the vertex γ^3 commutes with the transverse spin projectors $\mathcal{P}_\pm$. Similar to the pseudoscalar channel, this forces a chiral mismatch between the oppositely charged u and $\bar{d}$ quarks, generating misaligned Laguerre pairings. For the transverse modes ($\lambda = \pm$), the chiral vertices ($\gamma^\pm = (\gamma^1 \pm i\gamma^2)/2$) dynamically flip the spin projectors ($\gamma^\pm \mathcal{P}_\mp = \mathcal{P}_\pm \gamma^\pm$). This compensates for the underlying charge asymmetry of the $u\bar{d}$ pair, resulting in strictly aligned Laguerre configurations with uniform parity $(-1)^{n_u+n_d}$.

These chiral vertices also alter the transverse spatial dynamics. By decomposing the transverse momentum operator, $\boldsymbol{\gamma}_\perp \cdot \mathbf{k}_\perp = \frac{1}{2}(\gamma^+ k^- + \gamma^- k^+)$, the internal transverse Dirac chain for the right-handed mode ($\lambda = +$) reads

$$\begin{aligned} & \gamma^-(\boldsymbol{\gamma}_\perp \cdot \mathbf{k}_\perp) \gamma^+(\boldsymbol{\gamma}_\perp \cdot \mathbf{p}_\perp) \\ &= \frac{1}{4} \gamma^-(\gamma^+ k^- + \gamma^- k^+) \gamma^+(\gamma^+ p^- + \gamma^- p^+) \equiv 0. \end{aligned} \quad (43)$$

An identical algebraic annihilation occurs for the left-handed mode ($\lambda = -$). Physically, this demonstrates that fully polarized transverse vector modes ($|s_z| = 1$) cannot couple to the orbital transverse momentum fluctuations of the internal quark pair, as their chiral spin channels are maximally saturated. Consequently, the transverse momentum trace rigorously vanishes, suppressing any type II spatial overlap for these two modes. Finally, the scaled unprojected traces evaluate to

$$\begin{aligned} \frac{1}{4} \text{Tr}_{\text{Dirac}}[\lambda = \parallel] &= (-1)^{n_u+n_d-1} \left[\frac{1}{2} S_L^\parallel \cdot \mathcal{F}_{n_u,n_d}^{LL,\parallel}(\mathbf{k}_\perp, \mathbf{p}_\perp) \right. \\ &\quad \left. + \mathcal{F}_{n_u,n_d}^{TT,\parallel}(\mathbf{k}_\perp, \mathbf{p}_\perp) \right], \end{aligned} \quad (44)$$

$$\frac{1}{4} \text{Tr}_{\text{Dirac}}[\lambda = \pm] = (-1)^{n_u+n_d} [S_L^\pm \cdot \mathcal{F}_{n_u,n_d}^{LL,\pm}(\mathbf{k}_\perp, \mathbf{p}_\perp)]. \quad (45)$$

The longitudinal invariants are $S_L^\parallel = k_0 p_0 + k_3 p_3 - M^2$ and $S_L^\pm = k_\parallel \cdot p_\parallel - M^2$. The spatial weighting functions separate according to the chiral algebra, as follows:

$$\mathcal{F}_{n_u,n_d}^{LL,\parallel} = L_{n_u}(2\alpha_u) L_{n_d-1}(2\alpha_d) + L_{n_u-1}(2\alpha_u) L_{n_d}(2\alpha_d), \quad (46)$$

$$\mathcal{F}_{n_u,n_d}^{TT,\parallel} = 4(\mathbf{k}_\perp \cdot \mathbf{p}_\perp) L_{n_u-1}^1(2\alpha_u) L_{n_d-1}^1(2\alpha_d), \quad (47)$$

$$\begin{aligned}\mathcal{F}_{n_u, n_d}^{LL,+} &= L_{n_u}(2\alpha_u)L_{n_d}(2\alpha_d), \\ \mathcal{F}_{n_u, n_d}^{LL,-} &= L_{n_u-1}(2\alpha_u)L_{n_d-1}(2\alpha_d).\end{aligned}\quad (48)$$

Following the Laguerre projection onto the mesonic ground state ($n = 0$, $q_3 = 0$), the spatial integrations strictly yield the canonical overlaps. For $\lambda = \parallel$, we obtain

$$\mathcal{J}_{n_u, n_d, 0}^{(1), \parallel} = \mathcal{J}_{n_u, n_d-1, 0}^{(1)} + \mathcal{J}_{n_u-1, n_d, 0}^{(1)}, \quad \mathcal{J}_{n_u, n_d, 0}^{(2), \parallel} = \mathcal{J}_{n_u, n_d, 0}^{(2)}.\quad (49)$$

For $\lambda = \pm$, the integration isolates only the scalar overlaps, as follows:

$$\mathcal{J}_{n_u, n_d, 0}^{(1), +} = \mathcal{J}_{n_u, n_d, 0}^{(1)}, \quad \mathcal{J}_{n_u, n_d, 0}^{(1), -} = \mathcal{J}_{n_u-1, n_d-1, 0}^{(1)}.\quad (50)$$

By performing the algebraic partial fraction decomposition, we separate the static vacuum background from the dynamic scattering kernel. And the projected tensors decouple as $\hat{\Pi}_{\pm}(q_0^2) = J_1 + J_{m, \pm}(q_0^2)$ and $\hat{\Pi}_{\parallel}(q_0^2) = J_1 + J_{m, \parallel}(q_0^2)$.

Applying the Matsubara summation, the dynamic kernels separate into vacuum and medium components as follows:

$$J_{m, \lambda}^{\text{vac}}(q_0^2) = -\frac{4N_c}{\pi|q_\rho B|} \sum_{n_u, n_d=0}^{\infty} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{\mathcal{N}_\lambda}{2E_u E_d} \left[\frac{E_u + E_d}{q_0^2 - (E_u + E_d)^2} \right],\quad (51)$$

$$J_{m, \lambda}^{\text{med}}(q_0^2, T) = \frac{4N_c}{\pi|q_\rho B|} \sum_{n_u, n_d=0}^{\infty} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{\mathcal{N}_\lambda}{2E_u E_d} \left[\frac{E_u + E_d}{q_0^2 - (E_u + E_d)^2} (n_F(E_u) + n_F(E_d)) - \frac{E_u - E_d}{q_0^2 - (E_u - E_d)^2} (n_F(E_u) - n_F(E_d)) \right],\quad (52)$$

where the consolidated transverse spatial weights are

$$\mathcal{N}_{\parallel} = \frac{1}{2} R_{\parallel}^2(q_0^2, p_z) (\mathcal{J}_{n_u, n_d-1, 0}^{(1)} + \mathcal{J}_{n_u-1, n_d, 0}^{(1)}) - \mathcal{J}_{n_u, n_d, 0}^{(2)},\quad (53)$$

$$\mathcal{N}_{\pm} = R_{\pm}^2(q_0^2) \mathcal{J}_{n_u, n_d, 0}^{(1), \pm},\quad (54)$$

and the dynamic mass scales dictating the thresholds are $R_{\pm}^2(q_0^2) = n_u|q_u B| + n_d|q_d B| - q_0^2/2$ and $R_{\parallel}^2(q_0^2, p_z) = R_{\pm}^2(q_0^2) - 2p_z^2$. The RPA equations governing the masses $m_{\rho^{\pm}}(\lambda)$ are thus

$$1 - 2G_V[J_1 + J_{m, \lambda}(q_0^2 = m_{\rho^{\pm}, \lambda}^2)] = 0.\quad (55)$$

A fundamental feature of these equations is that the Zeeman splitting emerges dynamically from microscopic *threshold truncations* enforced by the chiral Dirac algebra, rather than phenomenological preassignments. The right-handed state ($\lambda = +$) isolates the aligned pair $L_{n_u}L_{n_d}$, allowing it to occupy the absolute ground state ($n_u = 0, n_d = 0$). Accessing the lowest unshifted kinematic threshold ($2M$), it experiences maximal binding without zero-point magnetic penalties, driving its mass downward ($m^2 - eB$). Conversely, the left-handed state ($\lambda = -$) isolates $L_{n_u-1}L_{n_d-1}$, enforcing a strict truncation ($n_u \geq 1, n_d \geq 1$). The injection of zero-point magnetic energy suppresses the interaction kernel, driving the mass steeply upward ($m^2 + eB$). The longitudinal state ($\lambda = \parallel$) sits structurally between them, requiring exactly one constituent quark to be magnetically excited.

To regularize the vacuum integrals, we implement the asymmetric 3D soft cutoff established in the pseudoscalar channel, applying the form factor f_Λ based exclusively on the longitudinal momentum and Landau level of a single reference constituent quark. Otherwise, a symmetric regularization would entangle the internal phase space and break the noncommutative projector algebra, artificially suppressing the type II extraction.

To numerically evaluate the RPA equations, the parameters are recalibrated for the vector channel. We use the 3D soft-cutoff parameters: $m_0 = 0.005$ GeV, $\Lambda = 0.582$ GeV, and $G = 2.388/\Lambda^2 \approx 7.0500$ GeV⁻². The generated constituent quark mass is $M \approx 0.457$ GeV. Targeting $m_\rho = 0.770$ GeV requires $G_V = 5.9432$ GeV⁻² ($G_V/G \approx 0.843$) [54,70,72,73]. Figure 5 illustrates the zero-field thermal calibration, confirming the degenerate triplet states and demonstrating the stability of the baseline prior to symmetry breaking.

In Fig. 6, we present the numerical solutions for the charged ρ triplet states as functions of $|eB|$ at $T = 0$ MeV and $T = 80$ MeV. The dynamic Zeeman splitting is unambiguously observed ($m_{\rho^-} > m_{\rho^0} > m_{\rho^+}$). A critical physical behavior emerges regarding the right-handed ($\lambda = +$) state. While early point-particle models predict a tachyonic instability ($m \rightarrow 0$) leading to vector meson condensation [31,32], our full phase-space evaluation shows an initial dip followed by an arrest and reversal. This is driven by the dynamic competition with magnetic catalysis. The magnetic field rapidly increases the constituent quark mass M (black dashed line), pushing the kinematic threshold ($2M$) upwards. This surge in the

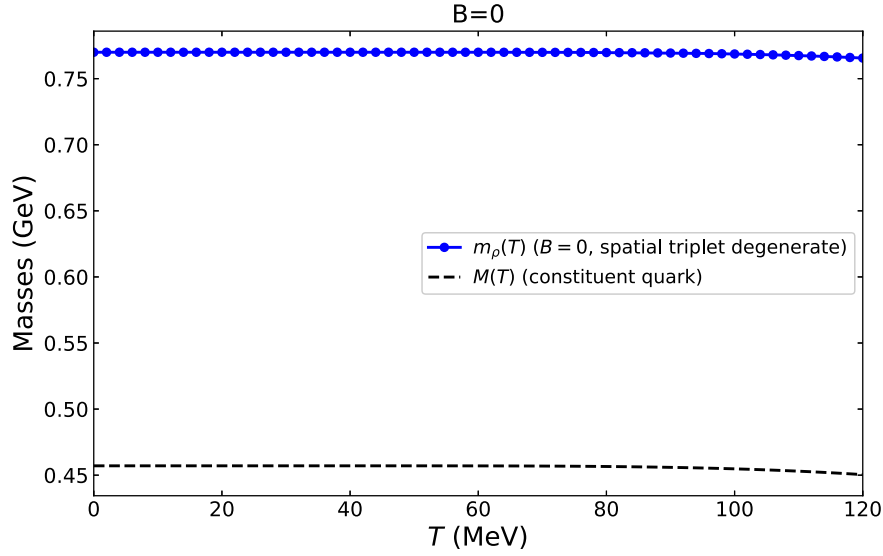


FIG. 5. Phenomenological vacuum calibration for the vector channel. The constituent quark mass M (black dashed line) and the unmagnetized ρ meson mass (blue line) are shown as functions of temperature T at $|eB| = 0$. The spatial triplet modes are strictly degenerate, fitting to $m_\rho = 0.770$ GeV at $T = 0$.

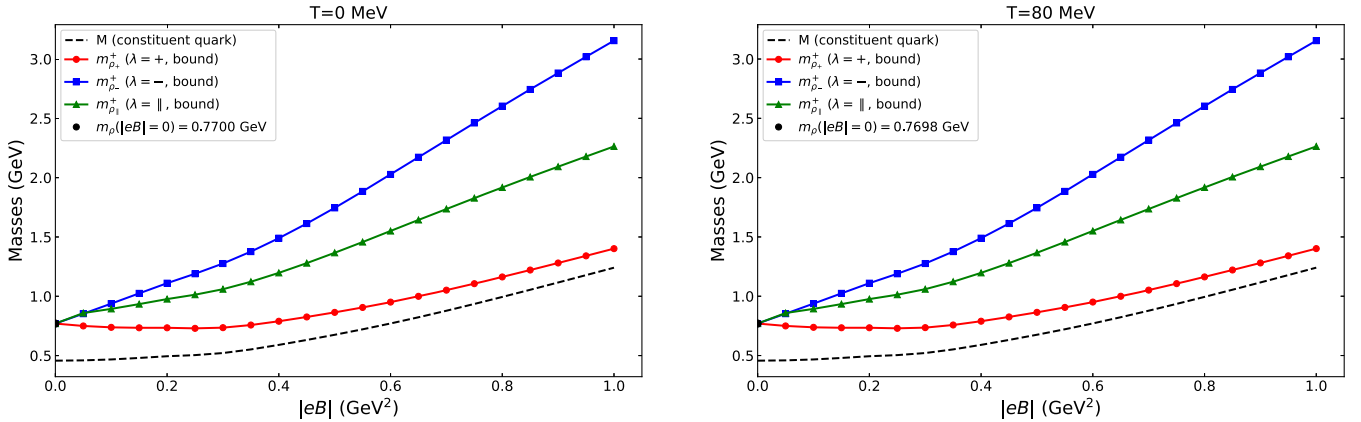


FIG. 6. Pole masses of the charged ρ meson triplet states ($\lambda = +, -, ||$) as a function of the magnetic field $|eB|$ at $T = 0$ MeV (left) and $T = 80$ MeV (right). The black dashed line indicates the constituent quark mass M . The anticipated VMC for the ρ^+ state is dynamically quenched as the MC of the constituent mass overtakes the Zeeman attraction.

threshold fundamentally overtakes the magnetic binding enhancement, quenching the tachyonic instability within this regularized framework.

Taken together, the resolved $\lambda = +, -, ||$ channels make the stabilization mechanism explicit: the spin-aligned channel exhibits the strongest Zeeman attraction, while magnetic catalysis raises the constituent mass and the corresponding two-quark continuum thresholds. In the numerical solution, the latter effect overcomes the initial downward trend and drives the $\lambda = +$ pole upward before a tachyonic instability is reached.

It is necessary to discuss these findings in the context of recent LQCD evaluations [34–36,38,39]. While our result qualitatively aligns with early nonperturbative calculations

indicating quenched VMC, some recent high-precision lattice data suggest a stronger downward trajectory for the ρ^+ mass at moderate magnetic fields. This quantitative discrepancy highlights the limitations of the current mean-field approximation. Specifically, effects such as inverse magnetic catalysis [10,11,74], where the running coupling constant decreases with eB [69], or the explicit incorporation of anomalous magnetic moments [23,43,47,75] for the constituent quarks could weaken the threshold surge and enhance the Zeeman binding, potentially bringing the model closer to recent lattice predictions. These nontrivial magnetic refinements are deferred to subsequent studies.

Finally, we investigate the thermal evolution of the spin-split states. Figure 7 displays the temperature-induced mass

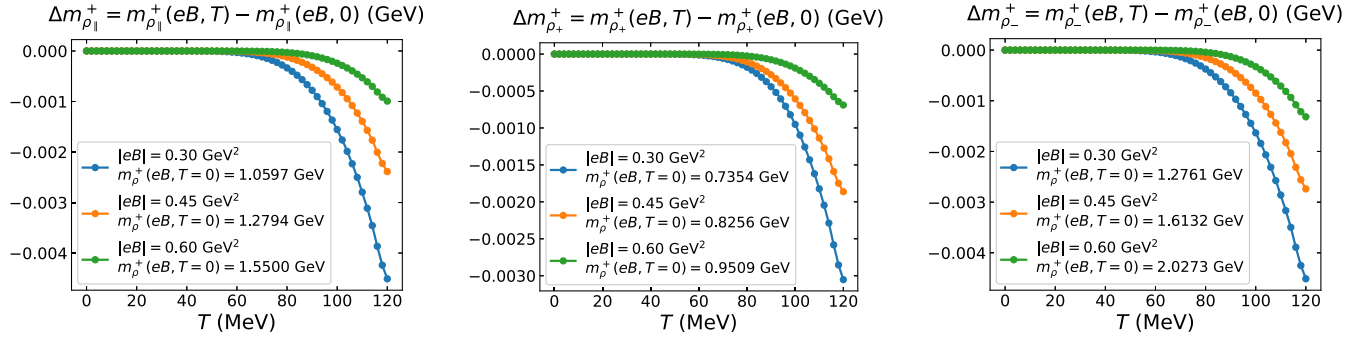


FIG. 7. Temperature-induced mass shifts $\Delta m_{\rho_\lambda}^+ = m_{\rho_\lambda}^+(eB, T) - m_{\rho_\lambda}^+(eB, 0)$ for the longitudinal ($\lambda = \parallel$, left), right-handed ($\lambda = +$, middle), and left-handed ($\lambda = -$, right) polarization states at fixed magnetic fields. The monotonic thermal suppression originates from the Pauli-blocking modification of the thermal quark-antiquark scattering kernel. All modes maintain bound-state solutions without undergoing Mott dissociation within the evaluated temperature range.

shifts $\Delta m_\rho = m_\rho(eB, T) - m_\rho(eB, 0)$. Across the computed domain ($T \leq 120$ MeV), the mass shifts are negative and exhibit monotonic suppression. The monotonic thermal suppression originates from the Fermi-Dirac occupation factors in the quark-antiquark polarization kernel, which reweight the intermediate states and shift the RPA pole toward lower energy as the temperature increases. Importantly, all triplet states remain robustly bound; no Mott dissociation is observed, indicating that the magnetic confinement mechanism is highly resilient against moderate thermal fluctuations prior to the chiral phase transition.

V. SUMMARY AND CONCLUSIONS

While mathematically equivalent to the standard Schwinger proper-time integral, our approach offers a distinct physical advantage by explicitly expanding the translationally invariant propagator in the Landau level representation. The traditional proper-time integral effectively integrates out the intermediate discrete states, thereby obscuring the underlying energy spectrum. In contrast, our explicit summation transparently exposes the internal microscopic kinematics and the state-by-state dynamics of the constituent u and $\bar{d}$ quarks, which otherwise remain heavily entangled in formalisms relying on mixed coordinate-momentum representations. To resolve the asymmetric nonlocal interactions induced by the fractional constituent charges, we formulated an analytically controlled NCA/Moyal phase-space framework via the Wigner-Weyl transform and the Moyal star product. This method achieves an algebraic diagonalization of the tensor Bethe-Salpeter integral equations, translating complex spatial convolutions into generalized triple Laguerre overlap integrals. Especially, this explicit phase-space mapping exposes the spatial projector structure underlying Landau-level completeness and magnetic translation symmetry. By identifying these analytical sum rules, we constructed an improved, asymmetric UV truncation scheme that respects the spatial factors. This geometrically consistent

regularization circumvents the probability nonconservation introduced by naïve cutoffs, supporting mathematically robust and physically transparent bound-state solutions within the evaluated magnetic regimes.

Applying this formalism to the pseudoscalar channel, we analytically verified the cancellation between the spatial sum rules of the RPA kernel and the vacuum gap equation. This identity preserves the generalized Goldstone theorem under magnetic fields. Consequently, the pole mass of the charged pion tracks the kinematic zero-point energy drift of $|eB|$. This behavior reflects the generalized Goldstone consistency established through the projector sum rules and the gap-equation cancellation. The asymmetric soft-cutoff regularization, in turn, supports a consistent numerical realization of the same structure.

In the vector channel, our phase-space evaluation shows that the Zeeman spin-splitting of the ρ^+ meson is generated by threshold truncations dictated by the chiral Dirac algebra, rather than being introduced phenomenologically. While the longitudinal ($\lambda = \parallel$) and left-handed ($\lambda = -$) modes acquire zero-point magnetic kinetic energy penalties, the right-handed ($\lambda = +$) state accesses the ground state ($n_u = 0, n_d = 0$), leading to an initial Zeeman mass reduction. This mechanism addresses limitations in models that treat the Zeeman shift and constituent mass generation independently. We found that the tachyonic instability of the ρ^+ state is quenched by chiral symmetry breaking. Magnetic catalysis drives the constituent quark mass M upwards. Because the mesonic pole is bounded by the continuum threshold ($2M$), the increase of this threshold overtakes the Zeeman attraction, preventing the mass squared from becoming negative and thus avoiding vector meson condensation within this mean-field framework [75,76].

At finite temperatures, the meson masses exhibit monotonic thermal suppression across all polarization states. This is driven by thermal scattering and Pauli blocking within the Matsubara medium kernel, which reduces the effective quark-antiquark binding. Despite this suppression, all mesonic modes remain bound. The absence of

Mott dissociation within the explored temperature range ($T \leq 120$ MeV) indicates that the magnetic binding remains stable against thermal fluctuations prior to chiral restoration.

In conclusion, the Moyal star product framework provides a consistent approach for exploring composite states in magnetic fields [77,78]. Future extensions could include the introduction of a finite baryon chemical potential (μ_B) [74,79–83] or a chiral chemical potential (μ_5) [11,80]. The interplay between finite density and the magnetic field could induce inverse magnetic catalysis, potentially suppressing the $2M$ threshold increase and retriggering the tachyonic instability [74,80,84]. Additionally, adapting this phase-space projection to real-time formalisms will allow for the calculation of spectral functions [85–87] and transport coefficients of magnetized mesonic matter, relevant for the phenomenological interpretation of heavy-ion collisions [71,88–90].

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: ANALYTICAL PROOF OF THE NONCOMMUTATIVE PROJECTOR ALGEBRA

While the idempotent projector algebra $\mathcal{P}_n(\alpha_\pi) \star \mathcal{P}_{n'}(\alpha_\pi) = \delta_{n,n'} \mathcal{P}_n(\alpha_\pi)$ can be indirectly inferred via the Weyl-Wigner correspondence to the Hilbert space quantum harmonic oscillator, we present here a purely analytical proof operating strictly within the phase-space geometry using generating functions [57,59,91].

For the Moyal star product between two arbitrary functions f and g in the transverse momentum space, characterized by the noncommutative parameter $\theta = |q_\pi B|$, the infinite-order pseudodifferential operator strictly translates into its well-known nonlocal integral representation via the Fourier transform. This representation reads [92]

$$(f \star g)(\mathbf{q}_\perp) = \frac{1}{\pi^2 \theta^2} \int d^2 \mathbf{u} \int d^2 \mathbf{v} f(\mathbf{q}_\perp + \mathbf{u}) g(\mathbf{q}_\perp + \mathbf{v}) \times \exp\left(\frac{2i}{\theta}(\mathbf{u} \times \mathbf{v})\right). \quad (\text{A1})$$

To evaluate the star product of two concentric Gaussian distributions, we substitute the isotropic functions $f(\mathbf{q}_\perp) = \exp(-c_1 \mathbf{q}_\perp^2 / \theta) \equiv e^{-c_1 \alpha_\pi}$ and $g(\mathbf{q}_\perp) = \exp(-c_2 \mathbf{q}_\perp^2 / \theta) \equiv e^{-c_2 \alpha_\pi}$ into the integral representation. The algebraic

interplay between the Gaussian exponents and the non-commutative cross-product phase allows for exact analytical integration over the intermediate variables $\mathbf{u}$ and $\mathbf{v}$. Exploiting the closure of multidimensional Gaussian integrals, the convolution rigorously reduces to the fundamental Gaussian star product lemma [91]

$$e^{-c_1 \alpha_\pi} \star e^{-c_2 \alpha_\pi} = \frac{1}{1 + c_1 c_2} \exp\left(-\frac{c_1 + c_2}{1 + c_1 c_2} \alpha_\pi\right), \quad (\text{A2})$$

where c_1 and c_2 are complex constants ensuring convergence.

Recall the explicit definition of the macroscopic mesonic Wigner basis functions [65]:

$$\mathcal{P}_n(\alpha_\pi) = 2e^{-\alpha_\pi} (-1)^n L_n(2\alpha_\pi). \quad (\text{A3})$$

To evaluate the star product across all Landau levels simultaneously, we construct the generating function $\mathcal{G}(t, \alpha_\pi)$ for this basis using a formal parameter t . Utilizing the standard generalized Laguerre generating series [93],

$$\sum_{n=0}^{\infty} L_n(x) z^n = (1 - z)^{-1} \exp\left[-\frac{xz}{1 - z}\right] \quad \text{for } |z| < 1, \quad (\text{A4})$$

and substituting $z = -t$ alongside $x = 2\alpha_\pi$, the generating function organizes into a pure Gaussian form, as follows:

$$\begin{aligned} \mathcal{G}(t, \alpha_\pi) &\equiv \sum_{n=0}^{\infty} t^n \mathcal{P}_n(\alpha_\pi) = 2e^{-\alpha_\pi} \sum_{n=0}^{\infty} (-t)^n L_n(2\alpha_\pi) \\ &= \frac{2}{1 + t} \exp\left(-\alpha_\pi \frac{1 - t}{1 + t}\right). \end{aligned} \quad (\text{A5})$$

We now evaluate the star product of two independent generating functions, $\mathcal{G}(t_1, \alpha_\pi) \star \mathcal{G}(t_2, \alpha_\pi)$. By defining the rational parameters $c_1 = \frac{1-t_1}{1+t_1}$ and $c_2 = \frac{1-t_2}{1+t_2}$, we directly apply the Gaussian lemma in Eq. (A2). The governing coefficients analytically evaluate to

$$1 + c_1 c_2 = \frac{2(1 + t_1 t_2)}{(1 + t_1)(1 + t_2)}, \quad (\text{A6})$$

$$c_1 + c_2 = \frac{2(1 - t_1 t_2)}{(1 + t_1)(1 + t_2)}. \quad (\text{A7})$$

The combined exponent simplifies into an argument dependent exclusively on the product $t_1 t_2$, as follows:

$$\frac{c_1 + c_2}{1 + c_1 c_2} = \frac{1 - t_1 t_2}{1 + t_1 t_2}. \quad (\text{A8})$$

Reassembling the prefactors with the exponential kernel yields the strict algebraic identity

$$\begin{aligned}\mathcal{G}(t_1, \alpha_\pi) \star \mathcal{G}(t_2, \alpha_\pi) &= \left(\frac{2}{1+t_1} e^{-c_1 \alpha_\pi} \right) \star \left(\frac{2}{1+t_2} e^{-c_2 \alpha_\pi} \right) \\ &= \frac{4}{(1+t_1)(1+t_2)} \left[\frac{(1+t_1)(1+t_2)}{2(1+t_1 t_2)} \right] \exp \left(-\alpha_\pi \frac{1-t_1 t_2}{1+t_1 t_2} \right) \\ &= \frac{2}{1+t_1 t_2} \exp \left(-\alpha_\pi \frac{1-t_1 t_2}{1+t_1 t_2} \right) \equiv \mathcal{G}(t_1 t_2, \alpha_\pi).\end{aligned}\quad (\text{A9})$$

This identity, $\mathcal{G}(t_1, \alpha_\pi) \star \mathcal{G}(t_2, \alpha_\pi) = \mathcal{G}(t_1 t_2, \alpha_\pi)$, demonstrates that the star product maps the phase-space convolution into a simple multiplicative algebra in the parameter space. Expanding both sides via their respective Taylor series, we obtain:

$$\text{lhs} = \left(\sum_{n=0}^{\infty} t_1^n \mathcal{P}_n(\alpha_\pi) \right) \star \left(\sum_{m=0}^{\infty} t_2^m \mathcal{P}_m(\alpha_\pi) \right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} t_1^n t_2^m (\mathcal{P}_n(\alpha_\pi) \star \mathcal{P}_m(\alpha_\pi)), \quad (\text{A10})$$

$$\text{rhs} = \sum_{k=0}^{\infty} (t_1 t_2)^k \mathcal{P}_k(\alpha_\pi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} t_1^n t_2^m \delta_{n,m} \mathcal{P}_n(\alpha_\pi). \quad (\text{A11})$$

Equating the multilinear coefficients of $t_1^n t_2^m$ completes the proof, as follows:

$$\mathcal{P}_n(\alpha_\pi) \star \mathcal{P}_m(\alpha_\pi) = \delta_{n,m} \mathcal{P}_n(\alpha_\pi). \quad (\text{A12})$$

This establishes the idempotent Wigner projection algebra purely through exact analytical integration, cleanly circumventing the infinite-order pseudodifferential operator via the integral representation.

APPENDIX B: TRIPLE LAGUERRE OVERLAP INTEGRALS AND RECURRENCE RELATIONS

Direct integration of Laguerre polynomials in Cartesian coordinates becomes computationally demanding for high Landau levels. To systematically evaluate these integrals,

we map the problem into a parameter space using generating functions, which naturally absorb the alternating phases $(-1)^n$ inherent in the magnetized propagators.

1. Type I (scalar) overlap integrals

For the standard Laguerre polynomials $L_n(x)$ present in the longitudinal trace, we utilize the generating function in Eq. (A4). To perform the transverse phase-space integration, we define the composite 2D momentum vector $\mathbf{v} = (\mathbf{k}_\perp, \mathbf{p}_\perp)^T$. Substituting the explicit definitions of α_u , α_d , and α_π into the generating function product, the exponent organizes into a quadratic form: $-\frac{1}{2} \mathbf{v}^T \mathbf{A}(\vec{t}) \mathbf{v}$, where $\vec{t} \equiv (t_1, t_2, t_3)$ denotes a set of dimensionless auxiliary variables introduced via the generating functions of the macroscopic Laguerre polynomials to decouple the transverse Gaussian integrals. The symmetric 2×2 phase-space matrix $\mathbf{A}(\vec{t})$ explicitly encodes the charge asymmetry, as follows:

$$\mathbf{A}(t_1, t_2, t_3) = 2 \begin{pmatrix} \frac{1}{|q_u B|(1+t_1)} + \frac{1}{|q_\pi B|(1+t_3)} & -\frac{1}{|q_\pi B|(1+t_3)} \\ -\frac{1}{|q_\pi B|(1+t_3)} & \frac{1}{|q_d B|(1+t_2)} + \frac{1}{|q_\pi B|(1+t_3)} \end{pmatrix}. \quad (\text{B1})$$

Because the Gaussian integral evaluates to $[\det \mathbf{A}]^{-1}$ for four real variables, the master generating function $\mathcal{G}^{(1)}(t_1, t_2, t_3)$ for the type I integrals $\mathcal{J}_{n_u, n_d, n}^{(1)}$ evaluates to a rational algebraic fraction, as follows

$$\mathcal{G}^{(1)}(t_1, t_2, t_3) = \frac{\pi^2 |q_u B| |q_d B|}{(1+t_1)(1+t_2)(1+t_3) \det \mathbf{A}(t_1, t_2, t_3)}. \quad (\text{B2})$$

The off-diagonal cross-coupling terms inherent in the spatial coordinates square out and cancel during the determinant evaluation over the 2D symmetric phase space. Consequently, the determinant depends exclusively on rational powers of the charge ratios $r_u = |q_u/q_\pi|$ and $r_d = |q_d/q_\pi|$. With $r_u + r_d = 1$, it takes the exact form

$$\det \mathbf{A}(\vec{t}) = \Delta_0 + \Delta_1 t_1 + \Delta_2 t_2 + \Delta_3 t_3 + \Delta_{12} t_1 t_2 + \Delta_{13} t_1 t_3 + \Delta_{23} t_2 t_3 + \Delta_{123} t_1 t_2 t_3, \quad (\text{B3})$$

where the rational coefficients are

$$\begin{aligned}\Delta_0 &= 4, & \Delta_1 &= -4(r_u - r_d), & \Delta_2 &= 4(r_u - r_d), & \Delta_3 &= 0, \\ \Delta_{12} &= -4(r_u - r_d)^2, & \Delta_{13} &= 4, & \Delta_{23} &= 4, & \Delta_{123} &= 0.\end{aligned}\quad (\text{B4})$$

To extract a robust numerical recurrence relation, we define the master polynomial $\mathcal{P}^{(1)}(\vec{t}) \equiv (1 + t_1)(1 + t_2)(1 + t_3) \det \mathbf{A}(\vec{t})$. The generating function satisfies the identity

$$\mathcal{P}^{(1)}(t_1, t_2, t_3) \mathcal{G}^{(1)}(t_1, t_2, t_3) = \pi^2 |q_u B| |q_d B|. \quad (\text{B5})$$

By expanding $\mathcal{P}^{(1)}(\vec{t}) = \sum_{i,j,k} p_{i,j,k} t_1^i t_2^j t_3^k$ and matching the power series coefficients for any combined index state (n_u, n_d, n) strictly greater than the origin $(0, 0, 0)$, the constant term cancels out, dictating the fundamental constant-coefficient ladder recurrence, as follows:

$$\sum_{i,j,k} p_{i,j,k} \mathcal{J}_{n_u-i, n_d-j, n-k}^{(1)} = 0. \quad (\text{B6})$$

The resulting linear algebraic system involves no index-weighted prefactors, helping maintain numerical consistency at high Landau levels.

2. Completeness and probability conservation

The physical consistency of the spatial projection framework is verified by evaluating the completeness limit for the macroscopic ground state $(n = 0)$. Summing over all possible internal states of one constituent quark must analytically recover a constant density of states.

Recall the integral definition for the canonical type I overlap in the mesonic ground state $(L_0(2\alpha_\pi) = 1)$, which is

$$\begin{aligned}\mathcal{J}_{n_u, n_d, 0}^{(1)} &= \int d^2 \mathbf{k}_\perp d^2 \mathbf{p}_\perp e^{-\alpha_u - \alpha_d - \alpha_\pi} (-1)^{n_u + n_d} \\ &\times L_{n_u}(2\alpha_u) L_{n_d}(2\alpha_d).\end{aligned}\quad (\text{B7})$$

Summing this expression over the u -quark Landau levels moves the series inside the integration, as follows:

$$\begin{aligned}\sum_{n_u=0}^{\infty} \mathcal{J}_{n_u, n_d, 0}^{(1)} &= \int d^2 \mathbf{k}_\perp d^2 \mathbf{p}_\perp e^{-\alpha_u - \alpha_d - \alpha_\pi} (-1)^{n_d} L_{n_d}(2\alpha_d) \\ &\times \left[\sum_{n_u=0}^{\infty} (-1)^{n_u} L_{n_u}(2\alpha_u) \right].\end{aligned}\quad (\text{B8})$$

The evaluation of this infinite sum relies on a sequence of standard analytic reductions. First, applying the completeness relation $\sum_{n_u=0}^{\infty} (-1)^{n_u} L_{n_u}(2\alpha_u) = \frac{1}{2} e^{\alpha_u}$ entirely eliminates the u -quark polynomial structure. With the

kinematic relation $\alpha_\pi = (\mathbf{k}_\perp - \mathbf{p}_\perp)^2 / |q_\pi B|$, the integration over the internal momentum $d^2 \mathbf{k}_\perp$ factors into a decoupled 2D Gaussian, yielding a geometric factor of $\pi |q_\pi B|$. Finally, the integration over the d -quark phase space $(d^2 \mathbf{p}_\perp = \pi |q_d B| d\alpha_d)$ is resolved via the standard Laguerre orthogonality relation $\int_0^\infty e^{-\alpha_d} L_{n_d}(2\alpha_d) d\alpha_d = (-1)^{n_d} [93]$. This strictly neutralizes the global alternating phase $(-1)^{n_d}$.

Combining these integrations, the infinite sum smoothly collapses to a purely geometric constant, as follows:

$$\sum_{n_u=0}^{\infty} \mathcal{J}_{n_u, n_d, 0}^{(1)} = \frac{\pi^2}{2} |q_d B| |q_\pi B| \equiv \mathcal{C}_{\text{norm}}. \quad (\text{B9})$$

This mathematically proves that $\mathcal{C}_{\text{norm}}$ is a universal constant independent of the initial state n_d , ensuring proper normalization without leakage across the asymmetric charge geometries. Notably, this summation identity is inherently symmetric under the flavor exchange $u \leftrightarrow d$, as an equivalent convergence is obtained by summing over the n_d Landau levels for a fixed n_u , thereby maintaining the internal consistency of the Wigner-Weyl projection framework.

This completeness identity also determines how the UV regularization is implemented in the pseudoscalar channel. After the partial-fraction decomposition, the single-denominator condensate terms contain a conjugate Landau-level sum acting only on the transverse overlap functions. Since the sum in Eq. (B9) collapses exactly to $\mathcal{C}_{\text{norm}}$, it must remain untruncated in the analytical completeness reduction. We therefore apply the soft cutoff to one reference constituent while retaining the conjugate sum, allowing the condensate background to recover the gap-equation structure analytically. A symmetric cutoff on both constituent Landau-level indices would truncate this completeness identity and introduce a regulator-dependent mismatch between the gap and pole equations. The asymmetric prescription is therefore an algebra-preserving finite regularization rather than a new renormalization scheme.

The type II vector overlap integral $\mathcal{J}^{(2)}$ encloses the localized extraction of transverse kinetic energy. For the macroscopic ground state, it reads

$$\begin{aligned}\mathcal{J}_{n_u, n_d, 0}^{(2)} &= \int d^2 \mathbf{k}_\perp d^2 \mathbf{p}_\perp e^{-\alpha_u - \alpha_d - \alpha_\pi} (\mathbf{k}_\perp \cdot \mathbf{p}_\perp) (-1)^{n_u + n_d} \\ &\times L_{n_u-1}^1(2\alpha_u) L_{n_d-1}^1(2\alpha_d).\end{aligned}\quad (\text{B10})$$

For the lowest Landau level $(n_u = 0)$, the associated Laguerre polynomial $L_{-1}^1 \equiv 0$, rendering the state physically trivial. The infinite summation must strictly

commence from $n_u = 1$, as follows:

$$\sum_{n_u=1}^{\infty} \mathcal{J}_{n_u, n_d, 0}^{(2)} = \int d^2 \mathbf{k}_{\perp} d^2 \mathbf{p}_{\perp} e^{-\alpha_u - \alpha_d - \alpha_{\pi}} (\mathbf{k}_{\perp} \cdot \mathbf{p}_{\perp}) (-1)^{n_d} L_{n_d-1}^1(2\alpha_d) \left[\sum_{n_u=1}^{\infty} (-1)^{n_u} L_{n_u-1}^1(2\alpha_u) \right]. \quad (\text{B11})$$

The bracketed term collapses to $-\frac{1}{4}e^{\alpha_u}$, annihilating the u -quark integration measure. Applying the kinematic shift $\mathbf{k}_{\perp} = \mathbf{q}_{\perp} + \mathbf{p}_{\perp}$, the inner product expands to $(\mathbf{q}_{\perp} \cdot \mathbf{p}_{\perp} + \mathbf{p}_{\perp}^2)$. The odd-parity cross term $(\mathbf{q}_{\perp} \cdot \mathbf{p}_{\perp})$ strictly vanishes upon integration over the symmetric 2D domain. The residual term exclusively isolates the momentum squared of the spectator quark: $\mathbf{p}_{\perp}^2 (\pi |q_{\pi} B|)$.

Substituting this topological reduction back leaves an integration solely over the d -quark configurations:

$$\sum_{n_u=1}^{\infty} \mathcal{J}_{n_u, n_d, 0}^{(2)} = \left(-\frac{\pi}{4} |q_{\pi} B| \right) \int d^2 \mathbf{p}_{\perp} e^{-\alpha_d} \mathbf{p}_{\perp}^2 (-1)^{n_d} L_{n_d-1}^1(2\alpha_d). \quad (\text{B12})$$

By invoking the exact differential identity $x L_{n-1}^1(x) = n L_{n-1}(x) - n L_n(x)$ [93], the residual integral over $x = 2\alpha_d$ resolves to $-n_d(-1)^{n_d}$. This neutralizes both the global alternating phase and the initial negative coefficient, yielding

$$\sum_{n_u=1}^{\infty} \mathcal{J}_{n_u, n_d, 0}^{(2)} = \frac{\pi^2}{4} |q_{\pi} B| |q_d B| \cdot (n_d |q_d B|) \equiv \frac{1}{2} C_{\text{norm}} \cdot (n_d |q_d B|). \quad (\text{B13})$$

Unlike the type I scalar projection that yields a universal geometric constant, the type II sum rule operates as a transverse kinetic energy projector. It explicitly extracts the intrinsic transverse scale $(n_d |q_d B|)$ of the spectator d -quark, with the same mechanism applying to the u -quark.

3. Type II (vector) overlaps and auxiliary recurrence

Type II integrals $\mathcal{J}^{(2)}$ involve the bilinear momentum factor $(\mathbf{k}_{\perp} \cdot \mathbf{p}_{\perp})$. We express this inner product as a symmetric quadratic form over $\mathbf{v}$, as follows:

$$\mathbf{k}_{\perp} \cdot \mathbf{p}_{\perp} = \frac{1}{2} \mathbf{v}^T \mathbf{M} \mathbf{v}, \quad \text{where } \mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (\text{B14})$$

According to the Gaussian moment generating theorem, inserting this quadratic form evaluates to $\frac{1}{2} \text{Tr}(\mathbf{M} \mathbf{A}^{-1})$ multiplied by the bare scalar integral. Because $\mathbf{A}^{-1} = \text{adj}(\mathbf{A}) / \det \mathbf{A}$, the raw generating function modifies to

$$\mathcal{G}^{(2)}(\vec{t}) = \frac{\pi^2 |q_u B| |q_d B|}{(1+t_1)^2 (1+t_2)^2 (1+t_3)} \frac{\frac{1}{2} \text{Tr}[\text{Madj}(\mathbf{A})]}{[\det \mathbf{A}(\vec{t})]^2}. \quad (\text{B15})$$

By substituting $\lambda_i = \frac{1-t_i}{1+t_i}$ into $\mathbf{A}$, the off-diagonal coupling yields $\frac{1}{2} \text{Tr}(\text{Madj} \mathbf{A}) = \frac{2}{|q_{\pi} B|} \frac{1-t_3}{1+t_3}$. Concurrently, because the λ_3^2 terms cancel out in $\det \mathbf{A}$, its denominator is strictly $(1+t_1)(1+t_2)(1+t_3)$. The squared determinant inversion thus perfectly annihilates the external measure, as follows:

$$\begin{aligned} \mathcal{G}^{(2)}(\vec{t}) &= \frac{\pi^2 |q_u B| |q_d B|}{(1+t_1)^2 (1+t_2)^2 (1+t_3)} \times \left(\frac{2(1-t_3)}{|q_{\pi} B| (1+t_3)} \right) \\ &\quad \times \frac{(1+t_1)^2 (1+t_2)^2 (1+t_3)^2}{[\mathcal{P}^{(1)}(\vec{t})]^2} \\ &= \frac{2\pi^2 |q_u B| |q_d B| / |q_{\pi} B|}{[\mathcal{P}^{(1)}(\vec{t})]^2} (1-t_3). \end{aligned} \quad (\text{B16})$$

Here, $\mathcal{P}^{(1)}(\vec{t})$ is the cleared determinant polynomial from the type I framework. The vector coupling polynomial rigorous collapses into a simple first-order binomial dependent exclusively on the external mesonic variable t_3 , as follows:

$$P_{\text{vec}}(\vec{t}) = 1 - t_3. \quad (\text{B17})$$

To evaluate $\mathcal{J}^{(2)}$ efficiently, we introduce a set of auxiliary scalar integrals $\mathcal{I}_{m_u, m_d, n}$, which are defined exclusively by the squared denominator $[\mathcal{P}^{(1)}(\vec{t})]^{-2}$ of the generating function. The algebraic structure of this denominator dictates that these auxiliary integrals satisfy a finite homogeneous recurrence relation. By expanding the characteristic polynomial as $\mathcal{P}_{\text{aux}}(\vec{t}) \equiv [\mathcal{P}^{(1)}(\vec{t})]^2 = \sum q_{i,j,k} t_1^i t_2^j t_3^k$, the recurrence is explicitly given by

$$\sum_{i,j,k} q_{i,j,k} \mathcal{I}_{m_u-i, m_d-j, n-k} = 0. \quad (\text{B18})$$

The physical type II overlaps are reconstructed by applying P_{vec} onto the auxiliary lattice. Within the parameter space, multiplying the series by t_3 corresponds to a backwards discrete index shift on the mesonic Landau level n . Introducing the shift operator $\hat{S}_3 \mathcal{I}_{m_u, m_d, n} = \mathcal{I}_{m_u, m_d, n-1}$,

the reconstruction forms a direct difference equation:

$$\mathcal{J}_{m_u+1, m_d+1, n}^{(2)} \propto P_{\text{vec}}(\hat{S}_3) \mathcal{I}_{m_u, m_d, n} = \mathcal{I}_{m_u, m_d, n} - \mathcal{I}_{m_u, m_d, n-1}. \quad (\text{B19})$$

This algebraic decoupling mechanism supports stable double-precision evaluation and circumvents the numerical instability intrinsic to high-dimensional Laguerre integrations.

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