

Komar superpotentials in extended theories of gravity

H. Arthur Weldon^{*}

Department of Physics and Astronomy, *West Virginia University*, Morgantown, West Virginia 26506, USA



(Received 7 August 2026; accepted 9 September 2026; published 7 October 2026)

This paper investigates Lagrangians with arbitrary dependence on the Riemann tensor and any number of derivatives. Noether's second theorem is applied separately to $\mathcal{L}_G$ and $\mathcal{L}_M$, where the matter Lagrangian depends on a scalar field. For $\mathcal{L}_G$ diffeomorphism invariance requires that the generalized Einstein tensor satisfy $\nabla_\alpha E^{\alpha\beta} = 0$ and that there is a current satisfying $0 = \partial_\alpha(\sqrt{g}J_{G2}^\alpha)$. The main result is an explicit formula for the superpotential $\Phi^{[\alpha\mu]}$ that solves $\partial_\mu \Phi^{[\alpha\mu]} = \sqrt{g}J_{G2}^\alpha$. Some applications to $f(R)$ gravity and various quadratic gravity theories are discussed. There are analogous results, including superpotentials, for the matter sector whenever the couplings to the matter fields contain derivatives of the metric.

DOI: [10.1103/wzzq-1xfz](https://doi.org/10.1103/wzzq-1xfz)

I. INTRODUCTION

The consequences of diffeomorphism invariance for any gravitational action are expressed in Noether's second theorem [1–10]. Invariance under an infinitesimal coordinate transformation

$$x^\lambda \rightarrow x^\lambda + \xi^\lambda(x^0, x^1, x^2, x^3) \quad (1.1)$$

where ξ^λ has arbitrary spacetime dependence leads to an identity of the form

$$0 = \mathcal{E} + \partial_\alpha \mathcal{J}^\alpha \quad (1.2)$$

where both terms are linear in ξ and its derivatives. This identity, due to Noether, expresses general diffeomorphism invariance and holds for any values of the metric and the matter fields.

Historically this principle has been applied to the total Lagrangian for gravitation and matter with $\mathcal{L}_G = \sqrt{g}R/16\pi G$, where $g = -\det(g_{\mu\nu}) > 0$. The results are often presented as follows. First, if the field equations for the metric are satisfied then $\mathcal{E} = 0$. Second, the current has the form

$$\mathcal{J}^\alpha = \mathcal{J}_1^\alpha + \mathcal{J}_2^\alpha \quad (1.3)$$

where $\mathcal{J}_1^\alpha$ vanishes when $\mathcal{E}$ vanishes, i.e., when the field equations are satisfied. Third, $\partial_\alpha \mathcal{J}_2^\alpha = 0$ for any metric

without regard for the field equations. And finally, the conserved current can be obtained from a superpotential $\mathcal{J}_2^\alpha = \partial_\mu \Phi^{[\alpha\mu]}$ with

$$\Phi^{[\alpha\mu]} = \frac{\sqrt{g}}{16\pi G} (\nabla^\mu \xi^\alpha - \nabla^\alpha \xi^\mu). \quad (1.4)$$

These results apparently led Einstein and many others to define the energy-momentum of the gravitational field by pseudotensors [11–14].

In 1959 Komar used various physical arguments, not Noether's theorem, to argue that the volume integral

$$Q_\xi = \int_V d^3x \partial_j \Phi^{[0j]} = \int_{\partial V} dS_j \Phi^{[0j]} \quad (1.5)$$

is a sensible accounting of the energy, momentum, and angular momentum of the gravitational field for appropriate choices of ξ [15–18]. Thus $\Phi^{[\alpha\mu]}$ is called the Komar superpotential even though it was previously known from Noether's second theorem.

There are now extended theories of gravity [19–22] motivated by cosmological problems (dark energy and inflation, dark matter and large scale structure) and also by quantum gravity. One class includes Lagrangians of the form $\sqrt{g}f(R)$, where R is the Ricci scalar [23–27]. Another category are Lagrangians that are linear combinations of the scalars R^2 , $R_{\alpha\beta}R^{\alpha\beta}$, and $R_{\rho\sigma\tau\omega}R^{\rho\sigma\tau\omega}$ [28–35]. A special type of the quadratic form is Gauss-Bonnet gravity where the quadratic term is a total derivative [36–38].

In extended theories of gravity it is known that Noether's second theorem [5,6,8,9] leads to conserved currents that depend on an arbitrary vector ξ . The main task of this paper is to find explicit formulas for the corresponding superpotentials.

^{*}Contact author: hweldon@mail.wvu.edu

A. Outline and summary

Sections II, III, and IV investigate a general class of extended theories with a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}_G(g_{**}, R_{****}) + \mathcal{L}_M(g_{**}, R_{****}, \phi, \partial_\mu \phi) \quad (1.6)$$

where matter is represented by the spin zero field ϕ . Both terms can depend on scalars formed by contractions of the metric with Riemann tensors. Lagrangians of this type include $f(R)$ theories and quadratic theories and allows more complicated terms such as $R^{\rho\tau}R^{\sigma\omega}R_{\rho\sigma\tau\omega}$, $R^\rho_\lambda R^{\lambda\sigma\tau\omega}R_{\rho\sigma\tau\omega}$, or $R_{\alpha\beta\rho\sigma}E^{\rho\sigma\tau\omega}R_{\tau\omega\mu\nu}g^{\alpha\mu}R^{\beta\nu}$ and also couplings to matter such as $R\phi^2$ or $R^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)$.

It is customary to consider the diffeomorphism invariance of the total Lagrangian, which leads to $\mathcal{E}$ in (1.2) being a linear combination of the field equations, but it is more productive to treat the diffeomorphism invariance of $\mathcal{L}_G$ and of $\mathcal{L}_M$ separately.

In Sec. II diffeomorphism invariance is expressed in terms of the Lie derivative. For $\mathcal{L}_G$ of the type (1.6) Sec. III shows that the generalized Einstein tensor

$$\sqrt{g}E^{\alpha\beta} = \frac{\partial\mathcal{L}_G}{\partial g_{\alpha\beta}} - \partial_\mu \frac{\partial\mathcal{L}_G}{\partial(\partial_\mu g_{\alpha\beta})} + \partial_\mu \partial_\nu \frac{\partial\mathcal{L}_G}{\partial(\partial_\mu \partial_\nu g_{\alpha\beta})} \quad (1.7)$$

satisfies

$$\nabla_\alpha E^{\alpha\beta} = 0 \quad (1.8)$$

for any metric, without regard to the field equations. The Noether identity (1.2) is

$$0 = 2\sqrt{g}E^{\alpha\beta}\nabla_\alpha \xi_\beta + \partial_\alpha \{-2\sqrt{g}E^{\alpha\beta}\xi_\beta + \mathcal{J}_{G2}^\alpha\} \quad (1.9)$$

and because of (1.8) this simplifies to

$$\partial_\alpha \mathcal{J}_{G2}^\alpha = 0. \quad (1.10)$$

This current is expected to be expressible as the derivative of a superpotential defined by $\partial_\mu \Phi^{[\alpha\mu]} = \mathcal{J}_{G2}^\alpha$. Section III displays the formula for the superpotential, which is proven in Appendix B.1. In terms of the tensor M

$$\sqrt{g}M^{\alpha\mu\nu\beta} = \frac{\partial\mathcal{L}_G}{\partial R_{\alpha\mu\nu\beta}} \quad (1.11)$$

the superpotential is

$$\Phi_G^{[\alpha\mu]} = 4\sqrt{g}(\nabla_\nu M^{\alpha\mu\nu\beta})\xi_\beta - 2\sqrt{g}M^{\alpha\mu\nu\beta}\nabla_\nu \xi_\beta, \quad (1.12)$$

(which was found by other means in [39]). As a by-product the generalized Einstein tensor is expressed as

$$E^{\alpha\beta} = 2\nabla_\mu \nabla_\nu M^{\alpha\{\mu\nu\}\beta} - \frac{1}{2}M^{\alpha\mu\nu\phi}R_{\mu\nu\phi}^\beta - \frac{1}{2}M^{\beta\mu\nu\phi}R_{\mu\nu\phi}^\alpha + \frac{1}{2}g^{\alpha\beta}\mathcal{L}_G/\sqrt{g}. \quad (1.13)$$

Section IV investigates matter Lagrangians of the form (1.6) with energy-momentum tensor

$$\frac{1}{2}\sqrt{g}T^{\alpha\beta} = \frac{\partial\mathcal{L}_M}{\partial g_{\alpha\beta}} - \partial_\mu \frac{\partial\mathcal{L}_M}{\partial(\partial_\mu g_{\alpha\beta})} + \partial_\mu \partial_\nu \frac{\partial\mathcal{L}_M}{\partial(\partial_\mu \partial_\nu g_{\alpha\beta})}. \quad (1.14)$$

As in the gravitational sector, there is a current $\mathcal{J}_{M2}^\alpha$ satisfying $\partial_\alpha \mathcal{J}_{M2}^\alpha = 0$ that can be expressed in terms of a superpotential $\mathcal{J}_{M2}^\alpha = \partial_\mu \tilde{\Phi}^{[\alpha\mu]}$. The superpotential has the same form as (1.12) in terms of a tensor $\tilde{M}$ analogous to (1.11). The matter analog of (1.13) is an expression for the energy-momentum tensor

$$T^{\alpha\beta} = \frac{1}{\sqrt{g}} \left[-\frac{\partial\mathcal{L}_M}{\partial(\partial_\alpha\phi)} \nabla^\beta \phi + g^{\alpha\beta} \mathcal{L}_M \right] + 4\nabla_\mu \nabla_\nu \tilde{M}^{\alpha\{\mu\nu\}\beta} - \tilde{M}^{\alpha\mu\nu\phi}R_{\mu\nu\phi}^\beta - \tilde{M}^{\beta\mu\nu\phi}R_{\mu\nu\phi}^\alpha \quad (1.15)$$

where the first line is the canonical energy-momentum tensor.

Section V summarizes the results when the metric contains N derivatives of the metric. The proof of the general results are contained in Appendixes A and B.

B. Notation

The signature of the metric is $(-+++)$, $g = -\det(g_{\mu\nu}) > 0$, and $R_{\sigma\tau\omega}^\alpha = \partial_\omega \Gamma_{\sigma\tau}^\alpha + \dots$.

II. DIFFEOMORPHISM INVARIANCE AND THE LIE DERIVATIVE

The infinitesimal coordinate transformation (1.1) produces an infinitesimal change in the metric and in the scalar field ϕ . Noether [1] denoted the variation in these quantities by $\bar{\delta}$ as have others [2,7,40,41]; however $\bar{\delta} = -\mathfrak{L}_\xi$, where $\mathfrak{L}_\xi$ is the Lie derivative, and $\mathfrak{L}_\xi$ will be used here to define infinitesimal variations. This means that the Noether current and superpotential has the opposite sign to that used by early authors.

The Lie derivative of any $\Psi(x)$ that changes to $\Psi'(x')$ under the infinitesimal coordinate transformation (1.1) is

$$\mathfrak{L}_\xi \Psi(x) = \Psi(x) - \Psi'(x') + \xi^\lambda \partial_\lambda \Psi. \quad (2.1)$$

Both Lagrangians $\mathcal{L}_G$ and $\mathcal{L}_M$ are scalar densities: either can be written as $\mathcal{L} = \sqrt{g}L$ where L is a scalar satisfying $L(x) = L'(x')$. The Lie derivative of L is

$$\mathfrak{L}_\xi L = \xi^\lambda \partial_\lambda L. \quad (2.2)$$

The transformation of the metric to first order is

$$g'_{\alpha\beta}(x') = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}(x) \approx g_{\alpha\beta}(x) - (\partial_\alpha \xi^\lambda) g_{\lambda\beta} - (\partial_\beta \xi^\lambda) g_{\alpha\lambda} \quad (2.3)$$

and fixes the Lie derivative as

$$\mathfrak{L}_\xi g_{\alpha\beta} = \xi^\lambda \partial_\lambda g_{\alpha\beta} + (\partial_\alpha \xi^\lambda) g_{\lambda\beta} + (\partial_\beta \xi^\lambda) g_{\alpha\lambda}. \quad (2.4)$$

This can also be written in terms of covariant derivatives

$$\mathfrak{L}_\xi g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha. \quad (2.5)$$

Because $\mathcal{L}$ is proportional to $\sqrt{g}$ one needs

$$\mathfrak{L}_\xi \sqrt{g} = \frac{1}{2} \sqrt{g} g^{\alpha\beta} \mathfrak{L}_\xi g_{\alpha\beta} = \partial_\lambda (\xi^\lambda \sqrt{g}). \quad (2.6)$$

Combining this with (2.2) gives

$$\mathfrak{L}_\xi \mathcal{L} = \partial_\lambda (\xi^\lambda \mathcal{L}). \quad (2.7)$$

This is the statement of diffeomorphism invariance that will be used repeatedly. It is Eq. (11) in Ref. [1] and will be called the Noether identity.

Since $\Psi'(x') - \xi^\lambda \partial_\lambda \Psi = \Psi'(x)$ the definition (2.1) can be stated as

$$\mathfrak{L}_\xi \Psi(x) = \Psi(x) - \Psi'(x). \quad (2.8)$$

This shows that when Ψ is the derivative of another function say $\partial_\mu \Omega$ then

$$\mathfrak{L}_\xi (\partial_\mu \Omega) = \partial_\mu \Omega(x) - \partial_\mu \Omega'(x) = \partial_\mu \mathfrak{L}_\xi \Omega, \quad (2.9)$$

which will be used without comment in the following.

One can rearrange (2.8) as

$$\Psi'(x) = \Psi(x) - \mathfrak{L}_\xi \Psi(x). \quad (2.10)$$

which shows that the Lie derivative can be viewed as an infinitesimal variation in Ψ . In the following Ψ can be either $g_{\alpha\beta}(x)$, $\phi(x)$, or their derivatives.

III. THE GRAVITATIONAL LAGRANGIAN $\mathcal{L}_G$

This section investigates gravitational Lagrangians (1.6) with arbitrary dependence on the Riemann tensor but with no derivatives of the Riemann tensor. Diffeomorphism invariance (2.7) implies

$$0 = \mathfrak{L}_\xi \mathcal{L}_G - \partial_\alpha (\xi^\alpha \mathcal{L}_G). \quad (3.1)$$

The Riemann tensor contains first and second derivatives of the metric and so this becomes

$$0 = \frac{\partial \mathcal{L}_G}{\partial g_{\alpha\beta}} \mathfrak{L}_\xi g_{\alpha\beta} + \frac{\partial \mathcal{L}_G}{\partial (\partial_\mu g_{\alpha\beta})} \partial_\mu \mathfrak{L}_\xi g_{\alpha\beta} + \frac{\partial \mathcal{L}_G}{\partial (\partial_\mu \partial_\nu g_{\alpha\beta})} \partial_\mu \partial_\nu \mathfrak{L}_\xi g_{\alpha\beta} - \partial_\alpha (\xi^\alpha \mathcal{L}_G). \quad (3.2)$$

It is useful to reorganize this as

$$0 = \sqrt{g} E^{\alpha\beta} \mathfrak{L}_\xi g_{\alpha\beta} + \partial_\alpha \mathcal{J}_G^\alpha, \quad (3.3)$$

where $E^{\alpha\beta}$ is given in (1.7) and the Noether current is

$$\mathcal{J}_G^\alpha = \left[\frac{\partial \mathcal{L}_G}{\partial (\partial_\alpha g_{\rho\sigma})} - \partial_\nu \frac{\partial \mathcal{L}_G}{\partial (\partial_\alpha \partial_\nu g_{\rho\sigma})} \right] \mathfrak{L}_\xi g_{\rho\sigma} + \frac{\partial \mathcal{L}_G}{\partial (\partial_\alpha \partial_\nu g_{\rho\sigma})} \partial_\nu \mathfrak{L}_\xi g_{\rho\sigma} - \xi^\alpha \mathcal{L}_G. \quad (3.4)$$

The current has terms with second-order derivatives of ξ , first-order derivatives of ξ , and no derivatives of ξ and can be expanded as

$$J_G^\alpha = A_\lambda^{\alpha\xi\lambda} + B_\lambda^{\alpha\mu} \partial_\mu \xi^\lambda + B_\lambda^{\alpha\{\mu_1\mu_2\}} \partial_{\mu_1} \partial_{\mu_2} \xi^\lambda \quad (3.5)$$

This type of derivative expansion is standard [2,5–7,9]. The coefficients can be expressed as functional derivatives of $\mathcal{L}_G$ but the explicit forms will not be needed until Sec. III B 1 and Appendix C. 1. The current will turn out to be a vector density; see (3.26).

A. The road to the superpotential

1. Proof that $\nabla_\alpha E_\lambda^\alpha = 0$

Because (3.3) holds for any ξ , substitution of the derivative expansion (3.5) into (3.3) results in four independent equations:

$$0 = B_\lambda^{\alpha\{\mu_1\mu_2\}} \partial_\alpha \partial_{\mu_1} \partial_{\mu_2} \xi^\lambda, \quad (3.6)$$

$$0 = \left\{ B_\lambda^{\mu_1\mu_2} + \partial_\alpha B_\lambda^{\alpha\{\mu_1\mu_2\}} \right\} \partial_{\mu_1} \partial_{\mu_2} \xi^\lambda, \quad (3.7)$$

$$0 = \left\{ \sqrt{g} 2E_\lambda^\alpha + A_\lambda^\alpha + \partial_\mu B_\lambda^{\mu\alpha} \right\} \partial_\alpha \xi^\lambda, \quad (3.8)$$

$$0 = \left\{ \sqrt{g} E^{\alpha\beta} \partial_\lambda g_{\alpha\beta} + \partial_\alpha A_\lambda^\alpha \right\} \xi^\lambda. \quad (3.9)$$

The first equation requires that the fully symmetrized, three-superscript coefficient vanish

$$B_\lambda^{\{\alpha\mu_1\mu_2\}} = 0. \quad (3.10)$$

The second equation (3.7) requires that the fully symmetrized, two-superscript coefficient satisfy

$$0 = B_{\lambda}^{\{\mu_1\mu_2\}} + \partial_{\alpha} B_{\lambda}^{\alpha\{\mu_1\mu_2\}}. \quad (3.11)$$

Application of $\partial_{\mu_1}\partial_{\mu_2}$ to this gives, because of (3.10),

$$0 = \partial_{\mu_1}\partial_{\mu_2} B_{\lambda}^{\{\mu_1\mu_2\}}. \quad (3.12)$$

The third equation (3.8) requires

$$A_{\lambda}^{\alpha} = -\sqrt{g}2E_{\lambda}^{\alpha} - \partial_{\mu} B_{\lambda}^{\mu\alpha}. \quad (3.13)$$

Application of ∂_{α} to this gives, because of (3.12),

$$\partial_{\alpha} A_{\lambda}^{\alpha} = -\partial_{\alpha} [\sqrt{g}2E_{\lambda}^{\alpha}]. \quad (3.14)$$

Substitution into the fourth equation (3.9) gives

$$\begin{aligned} 0 &= E^{\alpha\beta} \partial_{\lambda} g_{\alpha\beta} - \partial_{\alpha} [\sqrt{g}2E_{\lambda}^{\alpha}] \\ &= -2\sqrt{g}\nabla_{\alpha} E_{\lambda}^{\alpha}. \end{aligned} \quad (3.15)$$

This is true for any metric without imposing the field equations.

2. The conserved current $\mathcal{J}_{G2}^{\alpha}$

Substitution of (3.13) into the derivative expansion (3.5) suggests splitting $\mathcal{J}_G^{\alpha}$ into two parts

$$\mathcal{J}_G^{\alpha} = \mathcal{J}_{G1}^{\alpha} + \mathcal{J}_{G2}^{\alpha}, \quad (3.16)$$

where

$$\begin{aligned} \mathcal{J}_{G1}^{\alpha} &= -2\sqrt{g}E_{\lambda}^{\alpha}\xi^{\lambda}, \\ \mathcal{J}_{G2}^{\alpha} &= -(\partial_{\mu} B_{\lambda}^{\mu\alpha})\xi^{\lambda} + B_{\lambda}^{\alpha\mu}\partial_{\mu}\xi^{\lambda} + B_{\lambda}^{\alpha\{\mu_1\mu_2\}}\partial_{\mu_1}\partial_{\mu_2}\xi^{\lambda}. \end{aligned} \quad (3.17)$$

The Noether identity (3.3) is now

$$\begin{aligned} 0 &= 2\sqrt{g}E^{\alpha\beta}\nabla_{\alpha}\xi_{\beta} + \partial_{\alpha}[-2\sqrt{g}E_{\lambda}^{\alpha}\xi^{\lambda} + \mathcal{J}_{G2}^{\alpha}] \\ &= -2\sqrt{g}(\nabla_{\alpha}(E_{\lambda}^{\alpha})\xi^{\lambda} + \partial_{\alpha}\mathcal{J}_{G2}^{\alpha}) \\ &= \partial_{\alpha}\mathcal{J}_{G2}^{\alpha} \end{aligned} \quad (3.18)$$

because of (3.15). The result holds for any metric without using the field equations. This type of conservation law was termed “improper” by Noether [1].

Another proof: A further check is to differentiate the derivative expansion of $\mathcal{J}_{G2}^{\alpha}$ in (3.17):

$$\begin{aligned} \partial_{\alpha}\mathcal{J}_{G2}^{\alpha} &= -(\partial_{\alpha}\partial_{\mu} B_{\lambda}^{\mu\alpha})\xi^{\lambda} \\ &\quad - (\partial_{\mu} B_{\lambda}^{\mu\alpha})(\partial_{\alpha}\xi^{\lambda}) + (\partial_{\alpha} B_{\lambda}^{\alpha\mu})(\partial_{\mu}\xi^{\lambda}) \\ &\quad + B_{\lambda}^{\alpha\mu}\partial_{\alpha}\partial_{\mu}\xi^{\lambda} + (\partial_{\alpha} B_{\lambda}^{\alpha\{\mu_1\mu_2\}})\partial_{\mu_1}\partial_{\mu_2}\xi^{\lambda} \\ &\quad + B_{\lambda}^{\alpha\{\mu_1\mu_2\}}\partial_{\alpha}\partial_{\mu_1}\partial_{\mu_2}\xi^{\lambda}. \end{aligned} \quad (3.19)$$

The first term vanishes because of (3.12); the two terms on the second line cancel automatically; the two terms on the third line cancel because of (3.11); the fourth line vanishes because of (3.10).

Comment: Though different physically, a simple example of a superpotential is the field strength tensor $\sqrt{g}F^{\alpha\mu}$ of electrodynamics. The electromagnetic current $\mathcal{J}_{em}^{\alpha} = \partial_{\mu}(\sqrt{g}F^{\alpha\mu})$ is conserved because of the antisymmetry of $F^{\alpha\mu}$ and $\mathcal{J}_{em}^{\alpha}$ can be expressed in terms of the vector potential: A^{λ} , $\partial_{\mu}A^{\lambda}$, and $\partial_{\mu_1}\partial_{\mu_2}A^{\lambda}$ as in (3.17).

3. Formula for the superpotential

The superpotential satisfying $\partial_{\mu}\Phi^{[\alpha\mu]} = \mathcal{J}_{G2}^{\alpha}$ is

$$\begin{aligned} \Phi^{[\alpha\mu_1]} &= \left[B_{\lambda}^{[\alpha\mu_1]} - \frac{1}{3}(\partial_{\mu_2} C_{\lambda}^{[\alpha\mu_1]\mu_2}) \right] \xi^{\lambda} + \frac{2}{3} C_{\lambda}^{[\alpha\mu_1]\mu_2} \partial_{\mu_2} \xi^{\lambda}, \\ C_{\lambda}^{[\alpha\mu_1]\mu_2} &= B_{\lambda}^{\alpha\{\mu_1\mu_2\}} - B_{\lambda}^{\mu_1\{\alpha\mu_2\}}. \end{aligned} \quad (3.20)$$

The proof is given in Appendix B. 1.

B. Manifestly covariant results

From the previous formulas for $\mathcal{J}_{G2}^{\alpha}$ and $\Phi^{[\alpha\mu]}$ it is unclear how they behave under general coordinate transformations and whether they can be easily computed from the Lagrangian. This section addresses those matters.

The coefficients in the derivative expansion of the current (3.5) are determined by the dependence of the Lagrangian on the first and second derivatives of the metric. The Lagrangian under consideration depends on the Riemann tensor but not on its derivatives and so derivatives of the metric occur only from the dependence of the Lagrangian on the Riemann tensor. It is natural to define

$$\frac{\partial \mathcal{L}_G}{\partial R_{\alpha\mu\nu\beta}} = \sqrt{g} M^{\alpha\mu\nu\beta}. \quad (3.21)$$

The tensor M has the properties

$$M^{\alpha\mu\nu\beta} = -M^{\mu\alpha\nu\beta} = -M^{\alpha\mu\beta\nu} = M^{\nu\beta\alpha\mu}, \quad (3.22)$$

$$0 = M^{\alpha\mu\nu\beta} + M^{\alpha\nu\beta\mu} + M^{\alpha\beta\mu\nu}. \quad (3.23)$$

The Noether current (3.4) requires the derivative

$$\begin{aligned} \frac{\partial \mathcal{L}_G}{\partial(\partial_{\alpha}\partial_{\beta}g_{\mu\nu})} &= \frac{\partial \mathcal{L}_G}{\partial R_{\rho\sigma\tau\omega}} \frac{\partial R_{\rho\sigma\tau\omega}}{\partial(\partial_{\alpha}\partial_{\beta}g_{\mu\nu})} \\ &= \sqrt{g}[M^{\alpha\mu\nu\beta} + M^{\alpha\nu\beta\mu}] \\ &= 2\sqrt{g}M^{\alpha\{\mu\nu\}\beta}. \end{aligned} \quad (3.24)$$

Note that $M^{\mu\{\alpha\beta\}\nu} = M^{\alpha\{\mu\nu\}\beta}$ and thus is symmetric in both $\alpha\beta$ and $\mu\nu$. The dependence of $\mathcal{L}_G$ on the first derivative of the metric is also due to the Riemann tensor:

$$\begin{aligned} \frac{\partial \mathcal{L}_G}{\partial(\partial_\alpha g_{\mu\nu})} &= \frac{\partial \mathcal{L}_G}{\partial R_{\rho\sigma\tau\omega}} \frac{\partial R_{\rho\sigma\tau\omega}}{\partial(\partial_\alpha g_{\mu\nu})} \\ &= 2\sqrt{g} \left[\Gamma_{\rho\omega}^\mu M^{\rho\{\alpha\omega\}\nu} + \Gamma_{\rho\omega}^\nu M^{\rho\{\alpha\mu\}\omega} - \Gamma_{\rho\omega}^\alpha M^{\rho\{\mu\nu\}\omega} \right]. \end{aligned} \quad (3.25)$$

1. Covariant Noether current

The Noether current is determined by the tensor M . The remaining obstacle is to convert the ordinary derivatives ∂_ν in (3.4) into covariant derivatives ∇_ν . This is performed in Appendix C.1 with the result

$$\begin{aligned} \mathcal{J}_G^\alpha &= \sqrt{g} \left[-2(\nabla_\beta M^{\alpha\{\mu\nu\}\beta}) \xi_\xi g_{\mu\nu} \right. \\ &\quad \left. + 4M^{\alpha\{\beta\nu\}\mu} \nabla_\mu \nabla_\nu \xi_\beta - \xi^\alpha L_G \right]. \end{aligned} \quad (3.26)$$

2. Covariant superpotential

The next step is to express $\mathcal{J}_G^\alpha$ in terms of ordinary derivatives of ξ as in (3.5), identify the coefficients $B_\lambda^{\alpha\mu}$ and $B_\lambda^{\alpha\{\mu\nu\}}$, and substitute into formula (3.20) for the superpotential. These steps are performed in Appendix C.2 and yield [39]

$$\Phi^{[\alpha\mu]} = 4\sqrt{g}(\nabla_\nu M^{\alpha\mu\nu\beta})\xi_\beta - 2\sqrt{g}M^{\alpha\mu\nu\beta}\nabla_\nu \xi_\beta. \quad (3.27)$$

This can be expressed in the more general form

$$\begin{aligned} \Phi^{[\alpha\mu]} &= \sqrt{g} \left[-r\nabla_\nu (M^{\alpha\mu\nu\beta}\xi_\beta) \right. \\ &\quad \left. + (r+4)(\nabla_\nu M^{\alpha\mu\nu\beta})\xi_\beta + (r-2)M^{\alpha\mu\nu\beta}\nabla_\nu \xi_\beta \right] \end{aligned} \quad (3.28)$$

for any real parameter r .

3. Covariant form for $\mathcal{J}_{G2}^\alpha$

The divergence of the superpotential is

$$\begin{aligned} \mathcal{J}_{G2}^\alpha &= \partial_\mu \Phi^{[\alpha\mu]} \\ &= \sqrt{g} \left[4(\nabla_\mu \nabla_\nu M^{\alpha\mu\nu\beta})\xi_\beta - 2(\nabla_\beta M^{\alpha\{\mu\nu\}\beta})\xi_\xi g_{\mu\nu} \right. \\ &\quad \left. - 2M^{\alpha\mu\nu\beta}\nabla_\mu \nabla_\nu \xi_\beta \right] \end{aligned} \quad (3.29)$$

independent of r . As expected it has first and second derivatives of ξ .

4. Covariant form of the generalized Einstein tensor

The covariant form for $\mathcal{J}_{G1}^\alpha$ is obtained by subtraction:

$$\mathcal{J}_{G1}^\alpha = \mathcal{J}_G^\alpha - \mathcal{J}_{G2}^\alpha. \quad (3.30)$$

The terms proportional to $\xi_\xi g_{\alpha\beta}$ in (3.26) and (3.29) cancel. Subtraction of the terms containing $\nabla_\mu \nabla_\nu \xi_\beta$ gives

$$\begin{aligned} &\sqrt{g}[4M^{\alpha\{\beta\nu\}\mu} + 2M^{\alpha\mu\nu\beta}]\nabla_\mu \nabla_\nu \xi_\beta \\ &= -2\sqrt{g}M^{\alpha\beta\mu\nu}(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\xi_\beta \\ &= 2\sqrt{g}M^{\alpha\mu\nu\phi}R_{\mu\nu\phi}^\beta \xi_\beta. \end{aligned} \quad (3.31)$$

The difference between (3.26) and (3.29) is therefore

$$\mathcal{J}_{G1}^\alpha = -2\sqrt{g}S^{\alpha\beta}\xi_\beta, \quad (3.32)$$

where the coefficient is

$$S^{\alpha\beta} = 2\nabla_\mu \nabla_\nu M^{\alpha\mu\nu\beta} - M^{\alpha\mu\nu\phi}R_{\mu\nu\phi}^\beta + \frac{1}{2}g^{\alpha\beta}L_G. \quad (3.33)$$

Despite its appearance, $S^{\alpha\beta}$ is symmetric and has zero covariant divergence.

Divergence of $S^{\alpha\beta}$: To compute the covariant divergence of the first term in (3.33) use the $\alpha\mu$ antisymmetry of $M^{\alpha\mu\nu\beta}$ to obtain

$$\begin{aligned} \nabla_\alpha S^{\alpha\beta} &= (\nabla_\alpha \nabla_\mu - \nabla_\mu \nabla_\alpha)\nabla_\nu M^{\alpha\mu\nu\beta} \\ &\quad - (\nabla_\alpha M^{\alpha\mu\nu\phi})R_{\mu\nu\phi}^\beta \\ &\quad - M^{\alpha\mu\nu\phi}\nabla_\alpha R_{\mu\nu\phi}^\beta + \frac{1}{2}\nabla^\beta L_G. \end{aligned} \quad (3.34)$$

The commutator of the covariant derivatives gives a Riemann tensor that cancels the second line. Only the last line remains and it is convenient to lower the β index and to again use the $\alpha\mu$ antisymmetry of $M^{\alpha\mu\nu\beta}$:

$$\begin{aligned} \nabla_\alpha S_\beta^\alpha &= -\frac{1}{2}M^{\alpha\mu\nu\phi}(\nabla_\alpha R_{\beta\mu\nu\phi} + \nabla_\mu R_{\alpha\beta\nu\phi}) + \frac{1}{2}\nabla_\beta L_G \\ &= -\frac{1}{2}M^{\alpha\mu\nu\phi}\nabla_\beta R_{\alpha\mu\nu\phi} + \frac{1}{2}\nabla_\beta L_G \end{aligned} \quad (3.35)$$

after using the Bianchi identity. In view of the definition of M this is

$$\nabla_\alpha S_\beta^\alpha = -\frac{1}{2}\frac{\partial L_G}{\partial R_{\alpha\mu\nu\phi}}\nabla_\beta R_{\alpha\mu\nu\phi} + \frac{1}{2}\nabla_\beta L_G. \quad (3.36)$$

Application of ∇_β to L_G cancels the first term and shows that $\nabla_\alpha S_\beta^\alpha = 0$.

Symmetry of $S^{\alpha\beta}$: The antisymmetric part of $S^{\alpha\beta}$ is

$$S^{\alpha\beta} - S^{\beta\alpha} = 2(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) M^{\alpha\mu\nu\beta} - M^{\alpha\mu\nu\phi} R_{\mu\nu\phi}^\beta + M^{\beta\mu\nu\phi} R_{\mu\nu\phi}^\alpha. \quad (3.37)$$

This vanishes because the commutator of the covariant derivatives cancels the second line.

Relation to $E^{\alpha\beta}$: Using the manifestly symmetric form of $S^{\alpha\beta}$ the current (3.32) is

$$\mathcal{J}_{G1}^\alpha = -\sqrt{g}(S^{\alpha\beta} + S^{\beta\alpha})\xi_\beta. \quad (3.38)$$

From (3.17) $\mathcal{J}_{G1}^\alpha = -2\sqrt{g}E^{\alpha\beta}\xi_\beta$ and so the generalized Einstein tensor is given by $E^{\alpha\beta} = (S^{\alpha\beta} + S^{\beta\alpha})/2$. The explicitly symmetric form is

$$E^{\alpha\beta} = 2\nabla_\mu \nabla_\nu M^{\alpha\{\mu\nu\}\beta} - \frac{1}{2}M^{\alpha\mu\nu\phi} R_{\mu\nu\phi}^\beta - \frac{1}{2}M^{\beta\mu\nu\phi} R_{\mu\nu\phi}^\alpha + \frac{1}{2}g^{\alpha\beta}L_G. \quad (3.39)$$

C. Examples

1. $f(R)$ gravity

The Lagrangian $\mathcal{L}_G = \sqrt{g}f(R)/16\pi G$ has attractive features for cosmology [21–27]. The tensor M is

$$M^{\alpha\mu\nu\beta} = \frac{f'(R)}{32\pi G}(g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\mu\nu}) \quad (3.40)$$

which gives a superpotential [39]

$$\Phi^{[\alpha\mu]} = \frac{\sqrt{g}}{16\pi G}[f'(R)(\nabla^\mu \xi^\alpha - \nabla^\alpha \xi^\mu) + 2f''(R)((\partial^\alpha R)\xi^\mu - (\partial^\mu R)\xi^\alpha)]. \quad (3.41)$$

2. Quadratic gravity

The Lagrangian for quadratic gravity [28–35] is

$$\mathcal{L}_G = \sqrt{g}[aR^2 + bR^{\mu\nu}R_{\mu\nu} + cR^{\rho\sigma\tau\omega}R_{\rho\sigma\tau\omega}] \quad (3.42)$$

where a, b, c are parameters that distinguish various subcases. The superpotential (3.27) requires the tensor M

$$M^{\alpha\mu\nu\beta} = a(g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\mu\nu})R + \frac{b}{2}(g^{\alpha\nu}R^{\beta\mu} + g^{\beta\mu}R^{\alpha\nu} - g^{\alpha\beta}R^{\mu\nu} - g^{\mu\nu}R^{\alpha\beta}) + 2cR^{\alpha\mu\nu\beta} \quad (3.43)$$

and its covariant derivative

$$\nabla_\nu M^{\alpha\mu\nu\beta} = \left(a + \frac{b}{4}\right)(g^{\beta\mu}\nabla^\alpha R - g^{\alpha\beta}\nabla^\mu R) + \frac{b}{2}(\nabla^\alpha R^{\beta\mu} - \nabla^\mu R^{\alpha\beta}) + 2c\nabla_\nu R^{\alpha\mu\nu\beta}. \quad (3.44)$$

The Einstein tensor is

$$E^{\alpha\beta} = \left(a + \frac{b}{2} + c\right)(\nabla^\alpha \nabla^\beta + \nabla^\beta \nabla^\alpha)R - \left(2a + \frac{b}{2}\right)g^{\alpha\beta}\nabla_\mu \nabla^\mu R - (b + 4c)\nabla_\mu \nabla^\mu R^{\alpha\beta} + -2aRR^{\alpha\beta} + 4cR_\lambda^\alpha R^{\lambda\beta} + 2(b + 2c)R_{\mu\nu}R^{\alpha\{\mu\nu\}\beta} - 2cR^{\alpha\mu\nu\lambda}R_{\mu\nu\lambda}^\beta + \frac{g^{\alpha\beta}}{2}(aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR^{\rho\sigma\tau\omega}R_{\rho\sigma\tau\omega}). \quad (3.45)$$

- (a) *Conformal gravity*: A special case of (3.42) is conformal gravity. The Lagrangian is

$$\mathcal{L}_G = c\sqrt{g}C^{\rho\sigma\tau\omega}C_{\rho\sigma\tau\omega}, \quad (3.46)$$

where C is the Weyl tensor

$$C_{\rho\sigma\tau\omega} = \frac{1}{6}(g_{\rho\tau}g_{\sigma\omega} - g_{\rho\omega}g_{\sigma\tau})R + R_{\rho\sigma\tau\omega} - \frac{1}{2}(g_{\rho\tau}R_{\sigma\omega} + g_{\sigma\omega}R_{\rho\tau} - g_{\rho\omega}R_{\sigma\tau} - g_{\sigma\tau}R_{\rho\omega}). \quad (3.47)$$

The parameters in (3.42) are

$$a = \frac{1}{3}c; \quad b = -2c \quad (3.48)$$

Neither the superpotential nor the Einstein tensor are particularly simple.

- (b) *Gauss-Bonnet gravity*: For the values

$$a = c; \quad b = -4c \quad (3.49)$$

the second-order derivatives of R and $R^{\alpha\beta}$ in (3.45) are absent and

$$E^{\alpha\beta} = c \left[-2RR^{\alpha\beta} + 4R_\lambda^\alpha R^{\beta\lambda} - 4R_{\mu\nu}R^{\alpha\{\mu\nu\}\beta} - 2R^{\alpha\mu\nu\lambda}R_{\mu\nu\lambda}^\beta + \frac{1}{2}g^{\alpha\beta}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R^{\rho\sigma\tau\omega}R_{\rho\sigma\tau\omega}) \right]. \quad (3.50)$$

This is Gauss-Bonnet gravity [21,22,36–38]. The Lagrangian is a total derivative in four dimensions and so the

Einstein tensor should vanish. As expected (3.50) does vanish due to a known identity in four dimensions [37,38]. This is a stringent check of (3.45) and of the basic formula (3.39).

Even though the Gauss-Bonnet term is a total derivative, the tensor M is not zero:

$$M^{\alpha\mu\nu\beta} = c \left[(g^{\alpha\nu} g^{\beta\mu} - g^{\alpha\beta} g^{\mu\nu}) R + 2R^{\mu\alpha\beta\nu} - 2(g^{\alpha\nu} R^{\beta\mu} + g^{\beta\mu} R^{\alpha\nu} - g^{\alpha\beta} R^{\mu\nu} - g^{\mu\nu} R^{\alpha\beta}) \right], \quad (3.51)$$

and the superpotential (3.27) is nonzero. This, because the Noether identity (3.1) is a property of the Lagrangian not of the action.

IV. THE MATTER LAGRANGIAN $\mathcal{L}_M$

Analysis similar to Sec. III will be applied to a matter Lagrangian with arbitrary dependence on the Riemann tensor and on a scalar fields ϕ :

$$\mathcal{L}_M(g_{\alpha\beta}, R_{\rho\sigma\tau\omega}, \phi, \partial_\mu \phi). \quad (4.1)$$

Matter Lagrangians with no Riemann tensor but with second-order derivatives of the scalar field will contain first-order derivatives of the metric from the Christoffel connection and are treated in Sec. IV.D.

A. Diffeomorphism invariance of $\mathcal{L}_M$

Diffeomorphism invariance of (4.1) requires

$$0 = \xi_\epsilon \mathcal{L}_M - \partial_\alpha (\xi^\alpha \mathcal{L}_M) \quad (4.2)$$

or more explicitly

$$\begin{aligned} 0 = & \frac{\partial \mathcal{L}_M}{\partial \phi} \xi_\epsilon \phi + \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu \phi)} \partial_\mu \xi_\epsilon \phi - \partial_\alpha (\xi^\alpha \mathcal{L}_M) \\ & + \frac{\partial \mathcal{L}_M}{\partial g_{\alpha\beta}} \xi_\epsilon g_{\alpha\beta} + \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu g_{\alpha\beta})} \partial_\mu \xi_\epsilon g_{\alpha\beta} \\ & + \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu \partial_\nu g_{\alpha\beta})} \partial_\mu \partial_\nu \xi_\epsilon g_{\alpha\beta}. \end{aligned} \quad (4.3)$$

This can be organized as

$$\begin{aligned} 0 = & \frac{1}{2} \sqrt{g} T^{\alpha\beta} \xi_\epsilon g_{\alpha\beta} + \left[\frac{\partial \mathcal{L}_M}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu \phi)} \right] \xi_\epsilon \phi \\ & + \partial_\alpha \mathcal{J}_M^\alpha, \end{aligned} \quad (4.4)$$

where the energy-momentum tensor of matter is given in (1.14) and the Noether current is

$$\begin{aligned} \mathcal{J}_M^\alpha = & \frac{\partial \mathcal{L}_M}{\partial (\partial_\alpha \phi)} \xi_\epsilon \phi - \xi^\alpha \mathcal{L}_M \\ & + \left[\frac{\partial \mathcal{L}_M}{\partial (\partial_\alpha g_{\rho\sigma})} - \partial_\nu \frac{\partial \mathcal{L}_M}{\partial (\partial_\nu \partial_\alpha g_{\rho\sigma})} \right] \xi_\epsilon g_{\rho\sigma} \\ & + \frac{\partial \mathcal{L}_M}{\partial (\partial_\alpha \partial_\nu g_{\rho\sigma})} \partial_\nu \xi_\epsilon g_{\rho\sigma}. \end{aligned} \quad (4.5)$$

The first line is $-\sqrt{g} \Theta_\lambda^\alpha \xi^\lambda$ where Θ_λ^α is the canonical energy-momentum tensor.

B. Analysis of $\mathcal{J}_M^\alpha$

1. Derivative expansion of $\mathcal{J}_M^\alpha$

As in Sec. III A the next step is to parametrize the current in terms of derivatives of ξ :

$$J_M^\alpha = \tilde{A}_\lambda^\alpha \xi^\lambda + \tilde{B}_\lambda^{\alpha\mu} \partial_\mu \xi^\lambda + \tilde{B}_\lambda^{\alpha\{\mu_1\mu_2\}} \partial_{\mu_1} \partial_{\mu_2} \xi^\lambda. \quad (4.6)$$

The expansion coefficients are analogous to those in (3.5) but depend on both the metric and the field ϕ .

Substitution of (4.6) into the Noether identity (4.4) gives three equations involving derivatives of ξ ,

$$0 = \tilde{B}_\lambda^{\alpha\{\mu_1\mu_2\}} \partial_\alpha \partial_{\mu_1} \partial_{\mu_2} \xi^\lambda, \quad (4.7)$$

$$0 = \left\{ \tilde{B}_\lambda^{\mu_1\mu_2} + \partial_\alpha \tilde{B}_\lambda^{\alpha\{\mu_1\mu_2\}} \right\} \partial_{\mu_1} \partial_{\mu_2} \xi^\lambda, \quad (4.8)$$

$$0 = \left\{ \sqrt{g} T_\lambda^\alpha + \tilde{A}_\lambda^\alpha + \partial_\mu \tilde{B}_\lambda^{\mu\alpha} \right\} \partial_\alpha \xi^\lambda, \quad (4.9)$$

and a fourth equation with no derivatives of ξ ,

$$\begin{aligned} 0 = & \left\{ \frac{1}{2} \sqrt{g} T^{\alpha\beta} \partial_\lambda g_{\alpha\beta} \right. \\ & \left. + \left[\frac{\partial \mathcal{L}_M}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu \phi)} \right] \partial_\lambda \phi + \partial_\alpha \tilde{A}_\lambda^\alpha \right\} \xi^\lambda. \end{aligned} \quad (4.10)$$

The first equation (4.7) requires

$$\tilde{B}_\lambda^{\{\alpha\mu_1\mu_2\}} = 0. \quad (4.11)$$

The second (4.8) requires

$$0 = \tilde{B}_\lambda^{\{\mu_1\mu_2\}} + \partial_\alpha \tilde{B}_\lambda^{\alpha\{\mu_1\mu_2\}}, \quad (4.12)$$

which implies

$$0 = \partial_{\mu_1} \partial_{\mu_2} \tilde{B}_\lambda^{\{\mu_1\mu_2\}}. \quad (4.13)$$

The third equation (4.9) requires

$$\tilde{A}_\lambda^\alpha = -\sqrt{g} T_\lambda^\alpha - \partial_\mu \tilde{B}_\lambda^{\mu\alpha}, \quad (4.14)$$

which leads to

$$\partial_\alpha \tilde{A}_\lambda^\alpha = -\partial_\alpha [\sqrt{g} T_\lambda^\alpha]. \quad (4.15)$$

Substitution of (4.14) into (4.10) gives

$$0 = -\sqrt{g} \nabla_\alpha T_\lambda^\alpha + \left[\frac{\partial \mathcal{L}_M}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu \phi)} \right] \partial_\lambda \phi. \quad (4.16)$$

This relation is the analogue of $\nabla_\alpha E_\lambda^\alpha = 0$; it holds for any metric and any scalar field ϕ . Only when the ϕ satisfies the Euler-Lagrange equation does the energy-momentum tensor satisfy covariant conservation.

2. The conserved current $\mathcal{J}_{M2}^\mu$

Substitution of (4.14) into the derivative expansion (4.6) suggests splitting $\mathcal{J}_M^\alpha$ into two parts

$$\mathcal{J}_M^\alpha = \mathcal{J}_{M1}^\alpha + \mathcal{J}_{M2}^\alpha, \quad (4.17)$$

where

$$\begin{aligned} \mathcal{J}_{M1}^\alpha &= -\sqrt{g} T_\lambda^\alpha \xi^\lambda, \\ \mathcal{J}_{M2}^\alpha &= -(\partial_\mu \tilde{B}^{\alpha\mu}) \xi^\lambda + \tilde{B}_\lambda^{\alpha\mu} \partial_\mu \xi^\lambda + \tilde{B}_\lambda^{\alpha\{\mu_1\mu_2\}} \partial_{\mu_1} \partial_{\mu_2} \xi^\lambda. \end{aligned} \quad (4.18)$$

The Noether identity (4.4) is now

$$0 = \left\{ -\sqrt{g} \nabla_\alpha T_\lambda^\alpha + \left[\frac{\partial \mathcal{L}_M}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu \phi)} \right] \partial_\lambda \phi \right\} \xi^\lambda + \partial_\alpha \mathcal{J}_{M2}^\alpha.$$

The quantity in curly braces always vanishes as shown in (4.16) and therefore

$$0 = \partial_\alpha \mathcal{J}_{M2}^\alpha. \quad (4.19)$$

One can also compute the divergence from (4.18).

C. Covariant form

Since $\mathcal{J}_{M2}^\alpha$ is determined by $\tilde{B}_\lambda^{\alpha\mu}$ and $\tilde{B}_\lambda^{\alpha\{\mu_1\mu_2\}}$ it can be expressed in terms of the tensor

$$\sqrt{g} \tilde{M}^{\alpha\mu\nu\beta} = \frac{\partial \mathcal{L}_M}{\partial R_{\alpha\mu\nu\beta}} \quad (4.20)$$

analogous to (3.21) except for its dependence on ϕ . The conserved current is

$$\begin{aligned} \mathcal{J}_{M2}^\alpha &= \sqrt{g} \left[4(\nabla_\mu \nabla_\nu \tilde{M}^{\alpha\mu\nu\beta}) \xi_\beta - 2(\nabla_\beta \tilde{M}^{\alpha\{\mu\nu\}\beta}) \xi_\xi g_{\mu\nu} \right. \\ &\quad \left. - 2\tilde{M}^{\alpha\mu\nu\beta} \nabla_\mu \nabla_\nu \xi_\beta \right]. \end{aligned} \quad (4.21)$$

The superpotential satisfying $\partial_\mu \tilde{\Phi}^{[\alpha\mu]} = \mathcal{J}_{M2}^\alpha$ is

$$\tilde{\Phi}^{[\alpha\mu]} = 4\sqrt{g}(\nabla_\nu \tilde{M}^{\alpha\mu\nu\beta}) \xi_\beta - 2\sqrt{g} \tilde{M}^{\alpha\mu\nu\beta} \nabla_\nu \xi_\beta. \quad (4.22)$$

The dependence of the energy-momentum tensor on the Riemann tensor is

$$\begin{aligned} T^{\alpha\beta} &= \frac{1}{\sqrt{g}} \left[-\frac{\partial \mathcal{L}_M}{\partial (\partial_\alpha \phi)} \nabla^\beta \phi + g^{\alpha\beta} \mathcal{L}_M \right] \\ &\quad + 4\nabla_\mu \nabla_\nu \tilde{M}^{\alpha\{\mu\nu\}\beta} - \tilde{M}^{\alpha\mu\nu\phi} R_{\mu\nu}^\beta - \tilde{M}^{\beta\mu\nu\phi} R_{\mu\nu}^\alpha. \end{aligned} \quad (4.23)$$

The first line is the canonical energy-momentum tensor.

1. Example

An natural possibility is to allow the scalar field to couple to the Ricci scalar:

$$\mathcal{L}_M = \mathcal{L}_0(g_{\alpha\beta}, \phi, \partial_\mu \phi) + \sqrt{g} \gamma \phi^2 R \quad (4.24)$$

where γ is a real coupling constant. The tensor $\tilde{M}$ is

$$\tilde{M}^{\alpha\mu\nu\beta} = \frac{\gamma}{2} \phi^2 (g^{\alpha\nu} g^{\beta\mu} - g^{\alpha\beta} g^{\mu\nu}), \quad (4.25)$$

and the superpotential is

$$\begin{aligned} \tilde{\Phi}^{[\alpha\mu]} &= \sqrt{g} \gamma \phi^2 (\nabla^\mu \xi^\alpha - \nabla^\alpha \xi^\mu) \\ &\quad + 4\sqrt{g} \gamma \phi (\xi^\mu \nabla^\alpha \phi - \xi^\alpha \nabla^\mu \phi). \end{aligned} \quad (4.26)$$

The first line is essentially the original Komar form (1.4). It is simpler to compute the energy-momentum tensor from (4.23) than from the definition (1.14):

$$\begin{aligned} T^{\alpha\beta} &= \frac{1}{\sqrt{g}} \left[-\frac{\partial \mathcal{L}_0}{\partial (\partial_\alpha \phi)} \nabla^\beta \phi + g^{\alpha\beta} \mathcal{L}_0 \right] \\ &\quad + \gamma \phi^2 (g^{\alpha\beta} R - 2R^{\alpha\beta}) \\ &\quad + 4\gamma [(\nabla^\alpha \phi)(\nabla^\beta \phi) - g^{\alpha\beta} (\nabla_\mu \phi)(\nabla^\mu \phi)] \\ &\quad + 4\gamma \phi \left[\frac{1}{2} (\nabla^\alpha \nabla^\beta + \nabla^\beta \nabla^\alpha) \phi - g^{\alpha\beta} \nabla_\mu \nabla^\mu \phi \right]. \end{aligned} \quad (4.27)$$

D. Matter Lagrangians with no Riemann tensor

The previous discussion in Sec. IV treated matter Lagrangians in which first-order and second-order derivatives of the metric occur through the Riemann tensor. In the first category below the Lagrangian contains only first-order derivatives of the metric but it results in a conserved current with a superpotential. The second category below has no derivatives of the metric and contains no conserved current.

1. Matter Lagrangians second-order derivatives of ϕ but no Riemann tensor

If one allows second-order derivatives of the scalar field $\nabla_{\{\mu} \nabla_{\nu\}} \phi$ then the necessary Christoffel connection

contains first-order derivatives of the metric and these will lead to a superpotential even in the absence of the Riemann tensor. Thus consider

$$\mathcal{L}_M(g_{\alpha\beta}, \phi, \nabla_\mu \phi, \nabla_{\{\mu} \nabla_{\nu\}} \phi). \quad (4.28)$$

Diffeomorphism invariance leads to

$$\begin{aligned} 0 = & \frac{1}{2} \sqrt{g} T^{\alpha\beta} \xi_\xi g_{\alpha\beta} \\ & + \left[\frac{\partial \mathcal{L}_M}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu \phi)} + \partial_\mu \partial_\nu \frac{\mathcal{L}_M}{\partial (\partial_\mu \partial_\nu \phi)} \right] \xi_\xi \phi \\ & + \partial_\alpha \mathcal{J}_M^\alpha. \end{aligned} \quad (4.29)$$

The Noether current $\mathcal{J}_M^\alpha$ is simpler than (4.5) since now $\mathcal{L}_M$ has no second-order derivatives of the metric. The derivative expansion of $\mathcal{J}_M^\alpha$ is simpler than (4.6): $\tilde{B}_\lambda^{\alpha\{\mu_1\mu_2\}} = 0$ and $\tilde{B}_\lambda^{\{\alpha\mu\}} = 0$; thus $\tilde{B}_\lambda^{\alpha\mu} = \tilde{B}_\lambda^{[\alpha\mu]}$. The energy-momentum tensor satisfies

$$\begin{aligned} 0 = & -\sqrt{g} \nabla_\alpha T^\alpha_\lambda \\ & + \left[\frac{\partial \mathcal{L}_M}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu \phi)} + \partial_\mu \partial_\nu \frac{\mathcal{L}_M}{\partial (\partial_\mu \partial_\nu \phi)} \right] \partial_\lambda \phi. \end{aligned} \quad (4.30)$$

The Noether identity (4.29) reduces as usual to current conservation $0 = \partial_\alpha \mathcal{J}_{M2}^\alpha$ where now the superpotential form is automatic:

$$\mathcal{J}_{M2}^\alpha = -(\partial_\mu \tilde{B}_\lambda^{\mu\alpha}) \xi^\lambda + \tilde{B}_\lambda^{\alpha\mu} \partial_\mu \xi^\lambda = \partial_\mu (\tilde{B}_\lambda^{[\alpha\mu]}). \quad (4.31)$$

In terms of functional derivatives of the Lagrangian

$$\tilde{B}_\lambda^{[\alpha\mu]} = g_{\lambda\sigma} \left[\frac{\partial \mathcal{L}_M}{\partial (\partial_\alpha g_{\mu\sigma})} - \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu g_{\alpha\sigma})} \right]. \quad (4.32)$$

Since the derivatives of the metric come entirely from the Christoffel connection in $\nabla_{\{\mu} \nabla_{\nu\}} \phi$ it is natural to define

$$\sqrt{g} M^{\alpha\beta} = \frac{\partial \mathcal{L}_M}{\partial (\nabla_{\{\alpha} \nabla_{\beta\}} \phi)} \quad (4.33)$$

which allows the superpotential to be written covariantly:

$$\Phi^{[\alpha\mu]} = \sqrt{g} [(\nabla^\alpha \phi) M_\lambda^\mu - (\nabla^\mu \phi) M_\lambda^\alpha] \xi^\lambda. \quad (4.34)$$

2. Mater Lagrangians with no derivatives of the metric and no Riemann tensor

The simplest matter Lagrangian has no Riemann tensor and only first-order derivatives of the scalar field:

$$\mathcal{L}_M(g_{\alpha\beta}, \phi, \nabla_\mu \phi). \quad (4.35)$$

Diffeomorphism invariance gives

$$\begin{aligned} 0 = & \frac{1}{2} \sqrt{g} T^{\alpha\beta} \xi_\xi g_{\alpha\beta} \\ & + \left[\frac{\partial \mathcal{L}_M}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu \phi)} \right] \xi_\xi \phi + \partial_\alpha \mathcal{J}_M^\alpha. \end{aligned} \quad (4.36)$$

The Noether current is

$$\mathcal{J}_M^\alpha = \left[\frac{\partial \mathcal{L}_M}{\partial (\partial_\alpha \phi)} \xi_\xi \phi - \delta_\lambda^\alpha \mathcal{L}_M \right] \xi^\lambda. \quad (4.37)$$

One recovers the previous definition of the energy-momentum tensor (4.23) but without the tensor $\tilde{M}$ and the previous formula for the divergence of the energy-momentum tensor (4.16). There is no conserved current and consequently no superpotential.

V. DISCUSSION

Appendices A and B generalize the analysis to Lagrangians with any number of derivatives of the Riemann tensor such as $g^{\mu\nu}(\partial_\mu R)(\partial_\nu R)$ or $(\nabla^\mu R^{\alpha\beta})(\nabla_\mu R_{\alpha\beta})$ and more generally with $N-2$ derivatives of the Riemann tensor [42,43]:

$$\mathcal{L}_G(g_{**}, R_{****}, \nabla_{\mu_1} R_{****}, \dots, \nabla_{\{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_{N-2}\}} R_{****}) \quad (5.1)$$

and thus N derivatives of the metric. As before, applying Noether's identity to $\mathcal{L}_G$ and $\mathcal{L}_M$ separately produces more results than if applied to the sum.

For the gravitational Lagrangian Appendix A shows that the generalized Einstein tensor

$$\sqrt{g} E^{\alpha\beta} = \sum_{s=0}^N (-1)^s \partial_{\mu_1} \dots \partial_{\mu_s} \frac{\partial \mathcal{L}_G}{\partial (\partial_{\mu_1} \dots \partial_{\mu_s} g_{\alpha\beta})} \quad (5.2)$$

satisfies $\nabla_\alpha E^{\alpha\beta} = 0$ for any value of the metric.

Appendix B examines the derivative expansion of the current $\mathcal{J}_{G2}^\alpha$:

$$\mathcal{J}_{G2}^\alpha = B_\lambda^{\alpha\xi^\lambda} + \sum_{s=1}^N B_\lambda^{\alpha\{\mu_1 \dots \mu_s\}} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^\lambda \quad (5.3)$$

where the coefficients are related to the Lagrangian by expanding (A5) in terms of derivatives of ξ . The superpotential antisymmetric in $\alpha\mu_1$ must have the expansion

$$\Phi^{[\alpha\mu_1]} = W_\lambda^{[\alpha\mu_1]} \xi^\lambda + \sum_{s=2}^N W_\lambda^{[\alpha\mu_1]\mu_2 \dots \mu_s} \partial_{\mu_2} \dots \partial_{\mu_s} \xi^\lambda. \quad (5.4)$$

Appendix B.4 shows that $\partial_{\mu_1} \Phi^{[\alpha\mu_1]} = \mathcal{J}_{G2}^\alpha$ is solved by the W 's given in Eq. (B24).

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: THE CONSERVED CURRENT WHEN $\mathcal{L}_G$ CONTAINS N DERIVATIVES OF THE METRIC

This Appendix and the next one treat Lagrangians with $N - 2$ covariant derivatives of the Riemann tensor and thus N derivatives of the metric. Either $\mathcal{L}_G$ or $\mathcal{L}_M$ could contain N derivatives of the metric. Here $\mathcal{L}_G$ will be analyzed:

$$\mathcal{L}_G(g_{**}, R_{****}, \nabla_{\mu_1} R_{****}, \dots, \nabla_{\{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_{N-2}}\} R_{****}).$$

Applying the results to $\mathcal{L}_M$ is straightforward. Appendix A shows the following: (i) the generalized Einstein tensor $E^{\alpha\beta}$ defined in (5.2) satisfies $\nabla_\alpha E^{\alpha\beta} = 0$ and (ii) there is a conserved current satisfying $\partial_\alpha \mathcal{J}_G^\alpha = 0$. Neither result requires the metric to satisfy the field equations.

Diffeomorphism invariance requires

$$0 = \xi_\xi \mathcal{L}_G - \partial_\mu (\xi^\mu \mathcal{L}_G). \quad (\text{A1})$$

The more explicit statement is

$$0 = \frac{\partial \mathcal{L}_G}{\partial g_{\alpha\beta}} \xi_\xi g_{\alpha\beta} + \sum_{s=1}^N \frac{\partial \mathcal{L}_G}{\partial (\partial_{\mu_1} \dots \partial_{\mu_s} g_{\alpha\beta})} \partial_{\mu_1} \dots \partial_{\mu_s} \xi_\xi g_{\alpha\beta} - \partial_\mu (\xi^\mu \mathcal{L}_G), \quad (\text{A2})$$

which can be reorganized as

$$0 = \sqrt{g} E^{\alpha\beta} \xi_\xi g_{\alpha\beta} + \partial_\alpha \mathcal{J}_G^\alpha. \quad (\text{A3})$$

The Noether current is

$$\mathcal{J}_G^\alpha = \sum_{\ell=0}^N \mathcal{J}_\ell^\alpha - \xi^\alpha \mathcal{L}_G, \quad (\text{A4})$$

$$\mathcal{J}_\ell^\alpha = \sum_{p=0}^{\ell} (-1)^p \partial_{\mu_1} \dots \partial_{\mu_p} \frac{\partial \mathcal{L}_G}{\partial (\partial_\alpha \partial_{\mu_1} \dots \partial_{\mu_\ell} g_{\rho\sigma})} \times \partial_{\mu_{p+1}} \dots \partial_{\mu_\ell} \xi_\xi g_{\rho\sigma}. \quad (\text{A5})$$

1. Proof that $\nabla_\alpha E^{\alpha\beta} = 0$

The highest derivative of ξ in (A5) comes from the order N derivative of $\xi_\xi g_{\alpha\beta}$ and so the current has a derivative expansion

$$\mathcal{J}_G^\alpha = A_\lambda^\alpha \xi^\lambda + \sum_{s=1}^N B_\lambda^{\alpha\{\mu_1 \dots \mu_s\}} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^\lambda. \quad (\text{A6})$$

The coefficients A_λ^α and $B_\lambda^{\alpha\{\mu_1 \dots \mu_s\}}$ can be expressed as functional derivatives of $\mathcal{L}_G$ using (A5).

Since ξ is arbitrary, substitution of this into (A3) gives $N + 2$ independent equations:

$$0 = B_\lambda^{\alpha\{\mu_N \dots \mu_1\}} \partial_\alpha \partial_{\mu_N} \dots \partial_{\mu_1} \xi^\lambda, \quad (\text{A7})$$

$$0 = \left[B_\lambda^{\mu_N \{\mu_{N-1} \dots \mu_1\}} + \partial_\alpha B_\lambda^{\alpha \{\mu_N \dots \mu_1\}} \right] \partial_{\mu_N} \dots \partial_{\mu_1} \xi^\lambda \quad (\text{A8})$$

$\vdots$

$$0 = \left[B_\lambda^{\mu_3 \{\mu_2 \mu_1\}} + \partial_\alpha B_\lambda^{\alpha \{\mu_3 \mu_2 \mu_1\}} \right] \partial_{\mu_3} \partial_{\mu_2} \partial_{\mu_1} \xi^\lambda, \quad (\text{A9})$$

$$0 = \left[B_\lambda^{\mu_2 \mu_1} + \partial_\alpha B_\lambda^{\alpha \{\mu_2 \mu_1\}} \right] \partial_{\mu_2} \partial_{\mu_1} \xi^\lambda, \quad (\text{A10})$$

$$0 = \left[2\sqrt{g} E_\lambda^\alpha + A_\lambda^\alpha + \partial_\mu B_\lambda^{\mu\alpha} \right] \partial_\alpha \xi^\lambda, \quad (\text{A11})$$

$$0 = \left[\sqrt{g} E^{\alpha\beta} \partial_\lambda g_{\alpha\beta} + \partial_\alpha A_\lambda^\alpha \right] \xi^\lambda. \quad (\text{A12})$$

The first equation (A7) requires the fully symmetrized B with $N + 1$ superscripts to vanish:

$$0 = B_\lambda^{\{\alpha \mu_N \dots \mu_1\}}. \quad (\text{A13})$$

The second equation (A8) requires

$$0 = B_\lambda^{\{\mu_N \dots \mu_1\}} + \partial_\alpha B^{\alpha \{\mu_N \dots \mu_1\}}. \quad (\text{A14})$$

Applying N partial derivatives and using (A13) gives

$$0 = \partial_{\mu_N} \dots \partial_{\mu_1} B_\lambda^{\{\mu_N \dots \mu_1\}}. \quad (\text{A15})$$

The next equation (not shown above) is

$$0 = B_\lambda^{\{\mu_{N-1} \dots \mu_1\}} + \partial_\alpha B^{\alpha \{\mu_{N-1} \dots \mu_1\}}. \quad (\text{A16})$$

Applying $N - 1$ partial derivatives and using (A15) gives

$$0 = \partial_{\mu_{N-1}} \dots \partial_{\mu_1} B_\lambda^{\{\mu_{N-1} \dots \mu_1\}}. \quad (\text{A17})$$

The general result for $2 \leq s \leq N$ is

$$0 = B_\lambda^{\{\mu_s \dots \mu_1\}} + \partial_\alpha B^{\alpha \{\mu_s \dots \mu_1\}}, \quad (\text{A18})$$

$$0 = \partial_{\mu_s} \dots \partial_{\mu_1} B_\lambda^{\{\mu_s \dots \mu_1\}}. \quad (\text{A19})$$

The last two equations (A11) and (A12) are different. The first of these requires

$$A_\lambda^\alpha = -2\sqrt{g}E_\lambda^\alpha - \partial_\mu B^{\mu\alpha}. \quad (\text{A20})$$

Application of ∂_α and using (A19) with $s = 2$ gives

$$\partial_\alpha A_\lambda^\alpha = -2\partial_\alpha(\sqrt{g}E_\lambda^\alpha). \quad (\text{A21})$$

Substitution into (A12) gives

$$0 = \sqrt{g}E^{\alpha\beta}\partial_\lambda g_{\alpha\beta} - 2\partial_\alpha(\sqrt{g}E^{\alpha\lambda}) \quad (\text{A22})$$

which simplifies to

$$0 = -2\sqrt{g}\nabla_\alpha E_\lambda^\alpha. \quad (\text{A23})$$

This holds for any metric.

2. The conserved current J_{G2}^α

The formula for A_λ^α in (A20) makes it convenient to separate the derivative derivative expansion (A6) of the current into two parts:

$$\mathcal{J}_{G1}^\alpha = -2\sqrt{g}E_\lambda^\alpha \xi^\lambda, \quad (\text{A24})$$

$$\mathcal{J}_{G2}^\alpha = -(\partial_\mu B_\lambda^{\mu\alpha})\xi^\lambda + \sum_{s=1}^N B_\lambda^{\alpha\{\mu_1\cdots\mu_s\}}\partial_{\mu_1}\cdots\partial_{\mu_s}\xi^\lambda. \quad (\text{A25})$$

Their sum is $\mathcal{J}_G^\alpha = \mathcal{J}_{G1}^\alpha + \mathcal{J}_{G2}^\alpha$. The Noether identity (A3) becomes

$$0 = 2\sqrt{g}E^{\alpha\beta}\nabla_\alpha \xi_\beta + \partial_\alpha \left[-2\sqrt{g}E^{\alpha\beta}\xi_\beta + J_{G2}^\alpha \right], \quad (\text{A26})$$

which simplifies to

$$0 = -2\sqrt{g}(\nabla_\alpha E^{\alpha\beta})\xi_\beta + \partial_\alpha J_{G2}^\alpha. \quad (\text{A27})$$

Because $\nabla_\alpha E^{\alpha\beta} = 0$ this reduces to

$$0 = \partial_\alpha J_{G2}^\alpha. \quad (\text{A28})$$

Another proof: If one applies ∂_α to the series expansion (A25) the term with no derivative of ξ is $-(\partial_\alpha \partial_\mu B_\lambda^{\mu\alpha})$ which vanishes because of (A19). The terms with one derivative of ξ are

$$-(\partial_\mu B_\lambda^{\mu\alpha})\partial_\alpha \xi^\lambda + (\partial_\alpha B_\lambda^{\alpha\mu_1})\partial_{\mu_1} \xi^\lambda \quad (\text{A29})$$

which vanish by inspection. The term in the divergence with $N + 1$ derivatives of ξ vanishes

$$B_\lambda^{\alpha\{\mu_1\cdots\mu_N\}}\partial_\alpha \partial_{\mu_1}\cdots\partial_{\mu_N}\xi^\lambda = 0 \quad (\text{A30})$$

because of (A13). The terms remaining are

$$\begin{aligned} \partial_\alpha \mathcal{J}_{G2}^\alpha &= \sum_{s=1}^{N-1} \left[B_\lambda^{\alpha\{\mu_1\cdots\mu_s\}}\partial_\alpha \partial_{\mu_1}\cdots\partial_{\mu_s}\xi^\lambda \right. \\ &\quad \left. + (\partial_\alpha B_\lambda^{\alpha\{\mu_1\cdots\mu_{s+1}\}})\partial_{\mu_1}\cdots\partial_{\mu_{s+1}}\xi^\lambda \right]. \end{aligned} \quad (\text{A31})$$

In the first term relabel α as μ_{s+1} . The $s + 1$ derivatives symmetrize the $s + 1$ superscripts and so

$$\begin{aligned} \partial_\alpha \mathcal{J}_{G2}^\alpha &= \sum_{s=1}^{N-1} \left[B_\lambda^{\{\mu_1\cdots\mu_{s+1}\}} + \partial_\alpha B_\lambda^{\alpha\{\mu_1\cdots\mu_{s+1}\}} \right] \\ &\quad \times \partial_{\mu_1}\cdots\partial_{\mu_{s+1}}\xi^\lambda. \end{aligned} \quad (\text{A32})$$

This vanishes because of (A18).

Comment: In (A25) the coefficient of ξ^λ is $-\partial_\mu B_\lambda^{\mu\alpha}$. One could begin with the derivative expansion

$$\mathcal{J}_{G2}^\alpha = B_\lambda^{\alpha\lambda}\xi^\lambda + \sum_{s=1}^N B_\lambda^{\alpha\{\mu_1\cdots\mu_s\}}\partial_{\mu_1}\cdots\partial_{\mu_s}\xi^\lambda. \quad (\text{A33})$$

The condition $\partial_\alpha \mathcal{J}_{G2}^\alpha = 0$ requires $B_\lambda^\alpha = -\partial_{\mu_1} B_\lambda^{\mu_1\alpha}$.

APPENDIX B: THE SUPERPOTENTIAL WHEN $\mathcal{L}_G$ CONTAINS N DERIVATIVES OF THE METRIC

This Appendix contains the major proofs: Sec. B.1 proves the correctness of the $N = 2$ superpotential in (3.17); Sec. B.2 displays the $N = 3$ superpotential without proof; Sec. B.3 displays the superpotential for any N ; and Sec. B.4 proves the correctness of the superpotential shown in Sec. B.3.

Since $\mathcal{J}_{G2}^\alpha$ (A25) has derivatives of ξ up to order N , the superpotential must have derivatives up to order $N - 1$ and have the structure

$$\Phi^{[\alpha\mu_1]} = W_\lambda^{[\alpha\mu_1]}\xi^\lambda + \sum_{s=2}^N W_\lambda^{[\alpha\mu_1]\mu_2\cdots\mu_s}\partial_{\mu_2}\cdots\partial_{\mu_s}\xi^\lambda, \quad (\text{B1})$$

where each W is $\alpha\mu_1$ antisymmetric. The derivative is

$$\begin{aligned} \partial_{\mu_1} \Phi^{[\alpha\mu_1]} &= (\partial_{\mu_1} W_\lambda^{[\alpha\mu_1]})\xi^\lambda \\ &\quad + \sum_{s=2}^N \left[W_\lambda^{[\alpha\mu_2]\mu_3\cdots\mu_s} + \partial_{\mu_1} W_\lambda^{[\alpha\mu_1]\mu_2\cdots\mu_s} \right] \partial_{\mu_2}\cdots\partial_{\mu_s}\xi^\lambda \\ &\quad + W_\lambda^{[\alpha\mu_1]\mu_2\cdots\mu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}\xi^\lambda. \end{aligned} \quad (\text{B2})$$

The W 's must be such that $\partial_{\mu_1} \Phi^{[\alpha\mu_1]}$ is equal to (A25). The solution for the W 's will be expressed in terms of the quantity

$$C_\lambda^{[\alpha\mu_1]\mu_2\cdots\mu_s} = B_\lambda^{\alpha\{\mu_1\cdots\mu_s\}} - B_\lambda^{\mu_1\{\alpha\mu_2\cdots\mu_s\}}, \quad (\text{B3})$$

which is antisymmetric in $\alpha\mu_1$ and symmetric under all permutations of $\mu_2.. \mu_s$. Note that for $s = 1$

$$\frac{1}{2}C_\lambda^{[\alpha\mu_1]} = \frac{1}{2}[B_\lambda^{\alpha\mu_1} - B_\lambda^{\mu_1\alpha}] = B_\lambda^{[\alpha\mu_1]}. \quad (\text{B4})$$

1. Proof of the $N=2$ superpotential in (3.20)

The main text treated $\mathcal{L}_G$ having no derivatives of the Riemann tensor, i.e., $N = 2$. The superpotential for this case is displayed in (3.20) without proof. The proof will now be given.

The Noether current $\mathcal{J}_{G2}^\alpha$ in (3.17) contains second-order derivatives of ξ ; the superpotential must contain first-order derivatives:

$$\Phi^{[\alpha\mu_1]} = W_\lambda^{[\alpha\mu_1]} \xi^\lambda + W_\lambda^{[\alpha\mu_1]\mu_2} \partial_{\mu_2} \xi^\lambda. \quad (\text{B5})$$

The divergence is

$$\partial_{\mu_1} \Phi^{[\alpha\mu_1]} = X_0^\alpha + X_1^\alpha + X_2^\alpha, \quad (\text{B6})$$

$$X_0^\alpha = (\partial_{\mu_1} W_\lambda^{[\alpha\mu_1]}) \xi^\lambda,$$

$$X_1^\alpha = [W^{[\alpha\mu_1]} + \partial_{\mu_2} W_\lambda^{[\alpha\mu_2]\mu_1}] \partial_{\mu_1} \xi^\lambda, \quad (\text{B7})$$

$$X_2^\alpha = W_\lambda^{[\alpha\mu_1]\mu_2} \partial_{\mu_1} \partial_{\mu_2} \xi^\lambda. \quad (\text{B8})$$

This section will show that the solution is

$$W_\lambda^{[\alpha\mu_1]\mu_2} = \frac{2}{3} C_\lambda^{[\alpha\mu_1]\mu_2}, \quad (\text{B9})$$

$$W_\lambda^{[\alpha\mu_1]} = \frac{1}{2} C_\lambda^{[\alpha\mu_1]} - \frac{1}{3} \partial_{\mu_2} C_\lambda^{[\alpha\mu_1]\mu_2}. \quad (\text{B10})$$

The C 's are defined by (B3) and the three-superscript B is constrained by

$$0 = B_\lambda^{\{\alpha\mu_1\mu_2\}} = \frac{1}{3} [B_\lambda^{\alpha\{\mu_1\mu_2\}} + B_\lambda^{\mu_1\{\alpha\mu_2\}} + B_\lambda^{\mu_2\{\alpha\mu_1\}}], \quad (\text{B11})$$

which will be used repeatedly. The term with no derivative of ξ is

$$\begin{aligned} X_0^\alpha &= \left[\frac{1}{2} \partial_{\mu_1} C_\lambda^{[\alpha\mu_1]} - \frac{1}{3} \partial_{\mu_1} \partial_{\mu_2} C_\lambda^{[\alpha\mu_1]\mu_2} \right] \xi^\lambda \\ &= \left[\partial_{\mu_1} B_\lambda^{[\alpha\mu_1]} + \frac{2}{3} \partial_{\mu_1} \partial_{\mu_2} B_\lambda^{\mu_1\{\alpha\mu_2\}} + \frac{1}{3} \partial_{\mu_1} \partial_{\mu_2} B_\lambda^{\mu_2\{\alpha\mu_1\}} \right] \xi^\lambda \\ &= [\partial_{\mu_1} B_\lambda^{[\alpha\mu_1]} - \partial_{\mu_1} B_\lambda^{\{\alpha\mu_1\}}] \xi^\lambda \\ &= (-\partial_{\mu_1} B^{\mu_1\alpha}) \xi^\lambda. \end{aligned} \quad (\text{B12})$$

Passage from the second equality to the third is due to (A18) with $s = 2$. The term with first-order derivatives of ξ is

$$\begin{aligned} X_1^\alpha &= \left[\frac{1}{2} C_\lambda^{[\alpha\mu_1]} - \frac{1}{3} \partial_{\mu_2} C_\lambda^{[\alpha\mu_1]\mu_2} + \frac{2}{3} \partial_{\mu_2} C_\lambda^{[\alpha\mu_2]\mu_1} \right] \partial_{\mu_1} \xi^\lambda \\ &= [B_\lambda^{[\alpha\mu_1]} - \partial_{\mu_2} B_\lambda^{\mu_2\{\alpha\mu_1\}}] \partial_{\mu_1} \xi^\lambda \\ &= [B_\lambda^{[\alpha\mu_1]} + B_\lambda^{\{\alpha\mu_1\}}] \partial_{\mu_1} \xi^\lambda \\ &= B_\lambda^{\alpha\mu_1} \partial_{\mu_1} \xi^\lambda. \end{aligned} \quad (\text{B13})$$

Passage from the second to the third line uses (A18) with $s = 2$. The term with second-order derivatives of ξ is

$$\begin{aligned} X_2^\alpha &= \frac{2}{3} [B_\lambda^{\alpha\{\mu_1\mu_2\}} - B_\lambda^{\mu_1\{\alpha\mu_2\}}] \partial_{\mu_1} \partial_{\mu_2} \xi^\lambda \\ &= \frac{1}{3} [2B_\lambda^{\alpha\{\mu_1\mu_2\}} - B_\lambda^{\mu_1\{\alpha\mu_2\}} - B_\lambda^{\mu_2\{\alpha\mu_1\}}] \partial_{\mu_1} \partial_{\mu_2} \xi^\lambda \\ &= B_\lambda^{\alpha\{\mu_1\mu_2\}} \partial_{\mu_1} \partial_{\mu_2} \xi^\lambda, \end{aligned} \quad (\text{B14})$$

where the second line results from symmetrizing in $\mu_1\mu_2$ and the third line follows from (B11). The sum of (B12), (B13), and (B14) is

$$\begin{aligned} \partial_{\mu_1} \Phi^{[\alpha\mu_1]} &= -(\partial_{\mu_1} B^{\mu_1\alpha}) \xi^\lambda + B_\lambda^{\alpha\mu_1} \partial_{\mu_1} \xi^\lambda \\ &\quad + B_\lambda^{\alpha\{\mu_1\mu_2\}} \partial_{\mu_1} \partial_{\mu_2} \xi^\lambda \end{aligned} \quad (\text{B15})$$

and this agrees with (3.17) or (A25) with $N = 2$.

2. The $N=3$ superpotential

A derivative of the Ricci scalar $\nabla_\mu R$ or of the Ricci tensor $\nabla_\mu R_{\alpha\beta}$ is ultimately a covariant derivative of the Riemann tensor. If $\mathcal{L}$ contains such terms the Noether current contains third-order derivatives of ξ and the superpotential must contain second-order derivatives:

$$\begin{aligned} \Phi^{[\alpha\mu_1]} &= W_\lambda^{[\alpha\mu_1]} \xi^\lambda + W_\lambda^{[\alpha\mu_1]\mu_2} \partial_{\mu_2} \xi^\lambda \\ &\quad + W_\lambda^{[\alpha\mu_1]\mu_2\mu_3} \partial_{\mu_2} \partial_{\mu_3} \xi^\lambda. \end{aligned} \quad (\text{B16})$$

The solution for the W 's is

$$W_\lambda^{[\alpha\mu_1]\mu_2\mu_3} = \frac{3}{4} C_\lambda^{[\alpha\mu_1]\mu_2\mu_3} \quad (\text{B17})$$

$$W_\lambda^{[\alpha\mu_1]\mu_2} = \frac{2}{3} C_\lambda^{[\alpha\mu_1]\mu_2} - \frac{1}{2} \partial_{\mu_3} C_\lambda^{[\alpha\mu_1]\mu_2\mu_3} \quad (\text{B18})$$

$$\begin{aligned} W_\lambda^{[\alpha\mu_1]} &= \frac{1}{2} C_\lambda^{[\alpha\mu_1]} - \frac{1}{3} \partial_{\mu_2} C_\lambda^{[\alpha\mu_1]\mu_2} \\ &\quad + \frac{1}{4} \partial_{\mu_2} \partial_{\mu_3} C_\lambda^{[\alpha\mu_1]\mu_2\mu_3}. \end{aligned} \quad (\text{B19})$$

The proof that $\partial_{\mu_1} \Phi^{[\alpha\mu_1]}$ agrees with the current (A25) for $N = 3$ is a special case of the general proof in B.4.

3. The superpotential for any N

If $\mathcal{L}$ contains $N - 2$ covariant derivatives of the Riemann tensor the conserved current given in (A18) has N derivatives of ξ and the superpotential must have $N - 1$ derivatives of ξ :

$$\Phi^{[\alpha\mu_1]} = W_{\lambda}^{[\alpha\mu_1]} \xi^{\lambda} + \sum_{s=2}^N W_{\lambda}^{[\alpha\mu_1]\mu_2 \dots \mu_s} \partial_{\mu_2} \dots \partial_{\mu_s} \xi^{\lambda}. \quad (\text{B20})$$

The W with $N + 1$ superscripts is

$$W_{\lambda}^{[\alpha\mu_1]\mu_2 \dots \mu_N} = \frac{N}{N+1} C_{\lambda}^{[\alpha\mu_1]\mu_2 \dots \mu_N}. \quad (\text{B21})$$

W 's with fewer superscripts are given by the recursion relation

$$W_{\lambda}^{[\alpha\mu_1]\mu_2 \dots \mu_s} = \frac{s}{s+1} \left[C_{\lambda}^{[\alpha\mu_1]\mu_2 \dots \mu_s} - \partial_{\mu_{s+1}} W_{\lambda}^{[\alpha\mu_1]\mu_2 \dots \mu_{s+1}} \right] \quad (\text{B22})$$

that relates the W with $s + 1$ superscripts to the derivative of the W with $s + 2$ superscripts. Using this once gives

Substitution of (B24) gives the explicit value

$$Y_{\lambda}^{\alpha; \mu_1 \dots \mu_s} = \frac{s}{s+1} C_{\lambda}^{[\alpha\mu_1] \dots \mu_s} + \sum_{p=1}^{N-s} \frac{(-1)^p}{s+p+1} \partial_{\mu_{s+1}} \dots \partial_{\mu_{s+p}} \times \left[s C_{\lambda}^{[\alpha\mu_1] \dots \mu_{s+p}} - (s+1) C_{\lambda}^{[\alpha\mu_{s+p}] \mu_1 \dots \mu_{s+p-1}} \right]. \quad (\text{B27})$$

For (B25) one needs the products of Y 's with s derivatives of ξ and it is convenient to organize them according to the number of derivatives p acting on C :

$${}^{\alpha}Z_s^0 = \frac{s}{s+1} C_{\lambda}^{[\alpha\mu_1]\mu_2 \dots \mu_s} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda}, \quad (\text{B28})$$

$${}^{\alpha}Z_s^p = \frac{(-1)^p}{s+p+1} \partial_{\mu_{s+1}} \dots \partial_{\mu_{s+p}} \left[s C_{\lambda}^{[\alpha\mu_1] \dots \mu_{s+p}} - (s+1) C_{\lambda}^{[\alpha\mu_{s+p}] \mu_1 \dots \mu_{s+p-1}} \right] \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda}, \quad (\text{B29})$$

$$Y_{\lambda}^{\alpha; \mu_1 \dots \mu_s} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda} = \sum_{p=0}^{N-s} {}^{\alpha}Z_s^p. \quad (\text{B30})$$

The divergence of the superpotential is the double sum

$$\partial_{\mu_1} \Phi^{[\alpha\mu_1]} = (\partial_{\mu_1} W_{\lambda}^{[\alpha\mu_1]}) \xi^{\lambda} + \sum_{s=1}^N \sum_{p=0}^{N-s} {}^{\alpha}Z_s^p. \quad (\text{B31})$$

In ${}^{\alpha}Z_s^p$ substitution of the definition of the C 's gives

$$W_{\lambda}^{[\alpha\mu_1]\mu_2 \dots \mu_s} = \frac{s}{s+1} C_{\lambda}^{[\alpha\mu_1] \dots \mu_s} - \frac{s}{s+2} \partial_{\mu_{s+1}} C_{\lambda}^{[\alpha\mu_1]\mu_2 \dots \mu_{s+1}} + \frac{s}{s+2} \partial_{\mu_{s+1}} \partial_{\mu_{s+2}} W_{\lambda}^{[\alpha\mu_1]\mu_2 \dots \mu_{s+2}}. \quad (\text{B23})$$

Iteration leads to an explicit formula for $1 \leq s \leq N$:

$$W_{\lambda}^{[\alpha\mu_1] \dots \mu_s} = \frac{s}{s+1} C_{\lambda}^{[\alpha\mu_1] \dots \mu_s} + \sum_{p=1}^{N-s} \frac{s(-1)^p}{s+p+1} \partial_{\mu_{s+1}} \dots \partial_{\mu_{s+p}} C_{\lambda}^{[\alpha\mu_1] \dots \mu_{s+p}}. \quad (\text{B24})$$

4. Proof of the superpotential formula

The derivative of the superpotential (B20) is

$$\partial_{\mu_1} \Phi^{[\alpha\mu_1]} = (\partial_{\mu_1} W_{\lambda}^{[\alpha\mu_1]}) \xi^{\lambda} + \sum_{s=1}^N Y_{\lambda}^{\alpha; \mu_1 \dots \mu_s} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda}, \quad (\text{B25})$$

where the coefficients are

$$Y_{\lambda}^{\alpha; \mu_1 \dots \mu_s} = W_{\lambda}^{[\alpha\mu_1]\mu_2 \dots \mu_s} + \partial_{\mu_{s+1}} W_{\lambda}^{[\alpha\mu_{s+1}]\mu_1 \dots \mu_s}. \quad (\text{B26})$$

Y is symmetric in $\mu_2 \dots \mu_s$ but has no special property under $\alpha\mu_1$ exchange.

$$\left[sC_\lambda^{[\alpha\mu_1].. \mu_{s+p}} - (s+1)C_\lambda^{[\alpha\mu_{s+p}]\mu_1.. \mu_{s+p-1}} \right] \partial_{\mu_1} \dots \partial_{\mu_s} \xi^\lambda = \left[-B_\lambda^{\alpha\{\mu_1.. \mu_{s+p}\}} - sB_\lambda^{\mu_1\{\alpha\mu_2.. \mu_{s+p}\}} + (s+1)B_\lambda^{\mu_{s+p}\{\alpha\mu_1.. \mu_{s+p-1}\}} \right] \partial_{\mu_1} \dots \partial_{\mu_s} \xi^\lambda. \quad (\text{B32})$$

The second term in the square brackets on the right can be symmetrized over $\mu_1 \dots \mu_s$:

$$sB_\lambda^{\mu_1\{\alpha\mu_2.. \mu_{s+p}\}} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^\lambda = \sum_{j=1}^s B_\lambda^{\mu_j\{\alpha.. \mu_{s+p}\}'} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^\lambda \quad (\text{B33})$$

where the prime indicates that μ_j is omitted from the enclosed list. This allows ${}^\alpha Z_s^p$ to be expressed as

$${}^\alpha Z_s^p = \frac{(-1)^p}{s+p+1} \partial_{\mu_{s+1}} \dots \partial_{\mu_{s+p}} \left[-B_\lambda^{\alpha\{\mu_1.. \mu_{s+p}\}} - \sum_{j=1}^s B_\lambda^{\mu_j\{\alpha\mu_2.. \mu_{s+p}\}'} + (s+1)B_\lambda^{\mu_{s+p}\{\alpha\mu_1.. \mu_{s+p-1}\}} \right] \partial_{\mu_1} \dots \partial_{\mu_s} \xi^\lambda. \quad (\text{B34})$$

a. The largest p

When $p = N - s$ all the B 's in (B34) have $N + 1$ superscripts and satisfy the constraint

$$0 = B_\lambda^{\{\alpha\mu_1.. \mu_N\}} = \frac{1}{N+1} \left[B_\lambda^{\alpha\{\mu_1.. \mu_N\}} + \sum_{j=1}^N B_\lambda^{\mu_j\{\alpha.. \mu_N\}'} \right]. \quad (\text{B35})$$

Consequently

$${}^\alpha Z_p^{N-s} = \frac{(-1)^{N-s}}{N+1} \partial_{\mu_{s+1}} \dots \partial_{\mu_N} \times \left[\sum_{j=s+1}^N B_\lambda^{\mu_j\{\alpha.. \mu_N\}'} + (s+1)B_\lambda^{\mu_N\{\alpha\mu_1.. \mu_{N-1}\}} \right] \partial_{\mu_1} \dots \partial_{\mu_s} \xi^\lambda. \quad (\text{B36})$$

In the sum on j , the derivatives impose symmetry in $\mu_{s+1} \dots \mu_N$ which implies

$$\partial_{\mu_{s+1}} \dots \partial_{\mu_N} \sum_{j=s+1}^N B_\lambda^{\mu_j\{\alpha.. \mu_N\}'} = \partial_{\mu_{s+1}} \dots \partial_{\mu_N} (N-s) B_\lambda^{\mu_N\{\alpha\mu_1.. \mu_{N-1}\}} \quad (\text{B37})$$

and leads to

$$\begin{aligned} {}^\alpha Z_s^{N-s} &= \frac{(-1)^{N-s}}{N+1} \partial_{\mu_{s+1}} \dots \partial_{\mu_N} (N+1) B_\lambda^{\mu_N\{\alpha\mu_1.. \mu_{N-1}\}} \\ &= (-1)^{N-s-1} \partial_{\mu_{s+1}} \dots \partial_{\mu_{N-1}} B_\lambda^{\{\alpha\mu_1.. \mu_{N-1}\}}. \end{aligned} \quad (\text{B38})$$

Before proceeding it is useful to rewrite this. The B 's with fewer than $N + 1$ superscripts do not satisfy a constraint like (B35). Fortunately all one needs is to express the B with N symmetrized superscripts as a sum over N terms each of which has $N - 1$ symmetrized superscripts

$${}^\alpha Z_s^{N-s} = \frac{(-1)^{N-s-1}}{N} \partial_{\mu_{s+1}} \dots \partial_{\mu_{N-1}} \left[B_\lambda^{\alpha\{\mu_1.. \mu_{N-1}\}} + \sum_{j=1}^{N-1} B_\lambda^{\mu_j\{\alpha.. \mu_{N-1}\}'} \right]. \quad (\text{B39})$$

This will be used in the following.

b. The next largest p

For the case $p = N - s - 1$, Eq. (B34) gives

$${}^\alpha Z_s^{N-s-1} = \frac{(-1)^{N-s-1}}{N} \partial_{\mu_{s+1}} \dots \partial_{\mu_{N-1}} \left[-B_\lambda^{\alpha\{\mu_1.. \mu_{N-1}\}} - \sum_{j=1}^s B_\lambda^{\mu_j\{\alpha.. \mu_{N-1}\}'} + (s+1)B_\lambda^{\mu_{N-1}\{\alpha\mu_1.. \mu_{N-2}\}} \right] \partial_{\mu_1} \dots \partial_{\mu_s} \xi^\lambda. \quad (\text{B40})$$

The sum of this with (B39) is

$$\alpha Z_s^{N-s-1} + \alpha Z_s^{N-s} = \frac{(-1)^{N-s-1}}{N} \partial_{\mu_{s+1}} \dots \partial_{\mu_{N-1}} \left[- \sum_{j=s+1}^{N-1} B_{\lambda}^{\mu_j \{\alpha \dots\}'} + (s+1) B_{\lambda}^{\mu_{N-1} \{\alpha \mu_1 \dots \mu_{N-2}\}} \right] \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda}. \quad (\text{B41})$$

In the sum on j the derivatives impose symmetry in $\mu_{s+1} \dots \mu_{N-1}$ and allows the sum to be replaced by a single term $(N-s-1) B_{\lambda}^{\mu_{N-1} \{\alpha \mu_1 \dots \mu_{N-2}\}}$. When combined with the second term the result is $N B_{\lambda}^{\mu_{N-1} \{\alpha \mu_1 \dots \mu_{N-2}\}}$ and gives

$$\begin{aligned} \alpha Z_s^{N-s-1} + \alpha Z_s^{N-s} &= (-1)^{N-s-1} \partial_{\mu_{s+1}} \dots \partial_{\mu_{N-1}} B_{\lambda}^{\mu_{N-1} \{\alpha \mu_1 \dots \mu_{N-2}\}} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda} \\ &= (-1)^{N-s-2} \partial_{\mu_{s+1}} \dots \partial_{\mu_{N-2}} B_{\lambda}^{\{\alpha \mu_1 \dots \mu_{N-2}\}} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda} \end{aligned} \quad (\text{B42})$$

after using (A18).

c. The sum on p

The pattern set in (B42) continues and leads to the sum

$$\begin{aligned} \sum_{p=1}^{N-s} \alpha Z_s^p &= -\partial_{\mu_{s+1}} B_{\lambda}^{\mu_{s+1} \{\alpha \mu_1 \dots \mu_s\}} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda} \\ &= B_{\lambda}^{\{\alpha \mu_1 \dots \mu_s\}} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda} \\ &= \frac{1}{s+1} \left[B_{\lambda}^{\alpha \{\mu_1 \dots \mu_s\}} + s B_{\lambda}^{\mu_1 \{\alpha \mu_2 \dots \mu_s\}} \right] \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda}. \end{aligned} \quad (\text{B43})$$

From Eq. (B28) the $p = 0$ term is

$$\alpha Z_s^0 = \frac{s}{s+1} \left[B_{\lambda}^{\alpha \{\mu_1 \dots \mu_s\}} - B_{\lambda}^{\mu_1 \{\alpha \mu_2 \dots \mu_s\}} \right] \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda} \quad (\text{B44})$$

and so the full sum on p is

$$\sum_{p=0}^{N-s} \alpha Z_s^p = B_{\lambda}^{\alpha \{\mu_1 \dots \mu_s\}} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda}. \quad (\text{B45})$$

with the next highest value of p ; continue adding terms with successively smaller values of p . The sum from $p = 1$ to $p = N-1$ gives $-\partial_{\mu_1} B_{\lambda}^{\{\alpha \mu_1\}}$ and so

$$\begin{aligned} \partial_{\mu_1} W_{\lambda}^{[\alpha \mu_1]} &= \partial_{\mu_1} B_{\lambda}^{[\alpha \mu_1]} - \partial_{\mu_1} B_{\lambda}^{\{\alpha \mu_1\}} \\ &= -\partial_{\mu_1} B_{\lambda}^{\mu_1 \alpha}. \end{aligned} \quad (\text{B47})$$

The final result is

d. The complete sum

The divergence of the superpotential as given by (B31) is now

$$\partial_{\mu_1} \Phi^{[\alpha \mu_1]} = (\partial_{\mu_1} W_{\lambda}^{[\alpha \mu_1]}) \xi^{\lambda} + \sum_{s=1}^N B_{\lambda}^{\alpha \{\mu_1 \dots \mu_s\}} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda}.$$

One needs $W_{\lambda}^{[\alpha \mu_1]}$ from (B24)

$$\begin{aligned} \partial_{\mu_1} W_{\lambda}^{[\alpha \mu_1]} &= \frac{1}{2} \partial_{\mu_1} C_{\lambda}^{[\alpha \mu_1]} \\ &+ \sum_{p=1}^{N-1} \frac{(-1)^p}{p+2} \partial_{\mu_1} \dots \partial_{\mu_{p+1}} C_{\lambda}^{[\alpha \mu_1] \mu_2 \dots \mu_{p+1}}. \end{aligned} \quad (\text{B46})$$

The first term is $\partial_{\mu_1} B_{\lambda}^{[\alpha \mu_1]}$ as noted in (B4). Evaluating the sum over p requires the same approach as above: compute the term with the highest value of p ; add to that the term

$$\partial_{\mu_1} \Phi^{[\alpha \mu_1]} = -(\partial_{\mu_1} B_{\lambda}^{\mu_1 \alpha}) \xi^{\lambda} + \sum_{s=1}^N B_{\lambda}^{\alpha \{\mu_1 \dots \mu_s\}} \partial_{\mu_1} \dots \partial_{\mu_s} \xi^{\lambda}. \quad (\text{B48})$$

This is the current $\mathcal{J}_{G2}^{\alpha}$ given in (A25) which proves that the superpotential given in (B20) with coefficients (B24) is correct.

e. Covariant form

To express the generalized Einstein tensor or the superpotential or various other quantities covariantly requires introducing for each covariant derivative of order s of the Riemann tensor in $\mathcal{L}_G$ a tensor

$$\sqrt{g} M^{\{\mu_1 \dots \mu_s\} \alpha_1 \dots \alpha_4} = \frac{\partial \mathcal{L}_G}{\partial (\nabla_{\{\mu_1} \dots \nabla_{\mu_s\}} R_{\alpha_1 \dots \alpha_4})}. \quad (\text{B49})$$

APPENDIX C: COVARIANT RESULTS FOR $N=2$

1. Tensor form for the $N=2$ Noether current

This section will prove the result (3.26) starting from Eq. (3.4) for the Noether current:

$$\mathcal{J}_G^\alpha = \left[\frac{\partial \mathcal{L}_G}{\partial(\partial_\alpha g_{\rho\sigma})} - \partial_\nu \frac{\partial \mathcal{L}_G}{\partial(\partial_\alpha \partial_\nu g_{\rho\sigma})} \right] \mathfrak{L}_\xi g_{\rho\sigma} + \frac{\partial \mathcal{L}_G}{\partial(\partial_\alpha \partial_\nu g_{\rho\sigma})} \partial_\nu \mathfrak{L}_\xi g_{\rho\sigma} - \xi^\alpha \mathcal{L}_G. \quad (\text{C1})$$

The current is a sum of three functional derivatives

$$\mathcal{J}_G^\alpha = \mathcal{V}_1^\alpha + \mathcal{V}_2^\alpha + \mathcal{V}_3^\alpha - \xi^\alpha \mathcal{L} \quad (\text{C2})$$

which will now be calculated in terms of the tensor $M^{\alpha\mu\nu\beta}$ defined in (3.21).

The first functional derivative gives

$$\begin{aligned} \mathcal{V}_1^\alpha &= \frac{\partial \mathcal{L}}{\partial(\partial_\alpha g_{\mu\nu})} \mathfrak{L}_\xi g_{\mu\nu} \\ &= \sqrt{g} \left[2\Gamma_{\rho\omega}^\mu M^{\rho\{\nu\alpha\}\omega} + 2\Gamma_{\rho\omega}^\nu M^{\rho\{\mu\alpha\}\omega} \right. \\ &\quad \left. - 2\Gamma_{\rho\omega}^\alpha M^{\rho\{\mu\nu\}\omega} \right] \mathfrak{L}_\xi g_{\mu\nu}. \end{aligned} \quad (\text{C3})$$

The second functional derivative gives

$$\begin{aligned} \mathcal{V}_2^\alpha &= - \left[\partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \partial_\beta g_{\mu\nu})} \right] \mathfrak{L}_\xi g_{\mu\nu} \\ &= - \left[\partial_\beta (2\sqrt{g} M^{\alpha\{\mu\nu\}\beta}) \right] \mathfrak{L}_\xi g_{\mu\nu} \\ &= 2\sqrt{g} \left[-\nabla_\beta M^{\alpha\{\mu\nu\}\beta} + \Gamma_{\rho\omega}^\alpha M^{\rho\{\mu\nu\}\omega} \right. \\ &\quad \left. + \Gamma_{\rho\omega}^\mu M^{\alpha\{\rho\nu\}\omega} + \Gamma_{\rho\omega}^\nu M^{\alpha\{\rho\mu\}\omega} \right] \mathfrak{L}_\xi g_{\mu\nu}. \end{aligned} \quad (\text{C4})$$

The third functional derivative gives

$$\begin{aligned} \mathcal{V}_3^\alpha &= \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \partial_\beta g_{\mu\nu})} \partial_\beta \mathfrak{L}_\xi g_{\mu\nu} \\ &= 2\sqrt{g} M^{\alpha\{\mu\nu\}\beta} \partial_\beta \mathfrak{L}_\xi g_{\mu\nu} \\ &= 2\sqrt{g} M^{\alpha\{\mu\nu\}\beta} \nabla_\beta \mathfrak{L}_\xi g_{\mu\nu} \\ &\quad + 2\sqrt{g} \left[\Gamma_{\rho\omega}^\mu M^{\alpha\{\rho\nu\}\omega} + \Gamma_{\rho\omega}^\nu M^{\alpha\{\rho\mu\}\omega} \right] \mathfrak{L}_\xi g_{\mu\nu}. \end{aligned} \quad (\text{C5})$$

The sum of (C3), (C4), and (C5) is

$$\begin{aligned} &\mathcal{V}_1^\alpha + \mathcal{V}_2^\alpha + \mathcal{V}_3^\alpha \\ &= 2\sqrt{g} \left[-\nabla_\beta M^{\alpha\{\mu\nu\}\beta} \mathfrak{L}_\xi g_{\mu\nu} + M^{\alpha\{\mu\nu\}\beta} \nabla_\beta \mathfrak{L}_\xi g_{\mu\nu} \right] \\ &\quad + \sqrt{g} \Gamma_{\rho\omega}^\mu \{ 2M^{\rho\{\alpha\nu\}\omega} + 4M^{\alpha\{\rho\nu\}\omega} \} \mathfrak{L}_\xi g_{\mu\nu} \\ &\quad + \sqrt{g} \Gamma_{\rho\omega}^\nu \{ 2M^{\rho\{\alpha\mu\}\omega} + 4M^{\alpha\{\rho\mu\}\omega} \} \mathfrak{L}_\xi g_{\mu\nu}. \end{aligned} \quad (\text{C6})$$

By using the relations (3.22) the first quantity in braces can be simplified to

$$\Gamma_{\rho\omega}^\mu \{ M^{\alpha\omega\rho\nu} - M^{\alpha\rho\omega\nu} + 2M^{\alpha\nu\rho\omega} \}. \quad (\text{C7})$$

This vanishes because the quantity in braces is antisymmetric under $\rho\omega$ interchange. The second quantity in braces in (C6) vanishes similarly.

The total Noether current is therefore

$$\begin{aligned} \mathcal{J}_G^\alpha &= -2\sqrt{g} (\nabla_\beta M^{\alpha\{\mu\nu\}\beta}) \mathfrak{L}_\xi g_{\mu\nu} \\ &\quad + 2\sqrt{g} M^{\alpha\{\mu\nu\}\beta} \nabla_\beta \mathfrak{L}_\xi g_{\mu\nu} \\ &\quad - \xi^\alpha \sqrt{g} \mathcal{L}_G \end{aligned} \quad (\text{C8})$$

and is explicitly a vector density. Some relabeling in the second line converts this to

$$\begin{aligned} \mathcal{J}_G^\alpha &= -2\sqrt{g} (\nabla_\beta M^{\alpha\{\mu\nu\}\beta}) \mathfrak{L}_\xi g_{\mu\nu} \\ &\quad + 4\sqrt{g} M^{\alpha\{\beta\nu\}\mu} \nabla_\mu \nabla_\nu \mathfrak{L}_\xi g_{\mu\nu} \\ &\quad - \xi^\alpha \sqrt{g} \mathcal{L}_G, \end{aligned} \quad (\text{C9})$$

which is the result used in (3.26).

2. Tensor form for the $N=2$ superpotential

The superpotential given in Eq. (3.32)

$$\Phi^{[\alpha\mu]} = \left[B_\lambda^{[\alpha\mu]} - \frac{1}{3} \partial_\nu C_\lambda^{[\alpha\mu]\nu} \right] \xi^\lambda + \frac{2}{3} C_\lambda^{[\alpha\mu]\nu} \partial_\nu \xi^\lambda \quad (\text{C10})$$

was proven correct in Appendix B.1. This section will express the superpotential in a manifestly covariant form. The first step is to expand the covariant expression for the current (C9) in a derivative expansion:

$$\mathcal{J}_G^\alpha = A_\lambda^{\alpha\xi\lambda} + B_\lambda^{\alpha\mu} \partial_\mu \xi^\lambda + B_\lambda^{\alpha\{\mu\nu\}} \partial_\mu \partial_\nu \xi^\lambda. \quad (\text{C11})$$

Only the antisymmetric part $B_\lambda^{[\alpha\mu]} = (B_\lambda^{\alpha\mu} - B_\lambda^{\mu\alpha})/2$ is needed:

$$\begin{aligned} B_\lambda^{[\alpha\mu]} &= 3\sqrt{g} (\nabla_\nu M^{\alpha\mu\nu\beta}) g_{\beta\lambda} - 3\Gamma_{\beta,\rho\lambda} \mathcal{M}^{\alpha\mu\rho\beta} \\ &\quad + \sqrt{g} (M^{\alpha\rho\omega\nu} \Gamma_{\rho\omega}^\mu - M^{\mu\rho\omega\nu} \Gamma_{\rho\omega}^\alpha) g_{\nu\lambda}. \end{aligned} \quad (\text{C12})$$

Also needed is

$$\begin{aligned} C_{\lambda}^{[\alpha\mu]\nu} &= B_{\lambda}^{\alpha\{\mu\nu\}} - B_{\lambda}^{\mu\{\alpha\nu\}} \\ &= -3\sqrt{g}M^{\alpha\mu\nu\beta}g_{\beta\lambda}. \end{aligned} \quad (\text{C13})$$

The derivative of this is

$$\begin{aligned} -\frac{1}{3}\partial_{\nu}C_{\lambda}^{[\alpha\mu]\nu} &= \sqrt{g}(\nabla_{\nu}M^{\alpha\mu\nu\beta})g_{\beta\lambda} + \sqrt{g}\Gamma_{\beta,\rho\lambda}M^{\alpha\mu\rho\beta} \\ &+ \sqrt{g}(\Gamma_{\rho\omega}^{\alpha}M^{\mu\rho\omega\beta} - \Gamma_{\rho\omega}^{\mu}M^{\alpha\rho\omega\beta})g_{\beta\lambda}. \end{aligned} \quad (\text{C14})$$

The sum of (C12) and (C14) when multiplied by ξ^{λ} is

$$\begin{aligned} \left[B_{\lambda}^{[\alpha\mu]} - \frac{1}{3}\partial_{\nu}C_{\lambda}^{[\alpha\mu]\nu} \right] \xi^{\lambda} &= 4\sqrt{g}(\nabla_{\nu}M^{\alpha\mu\nu\beta})\xi_{\beta} \\ &- 2\sqrt{g}M^{\alpha\mu\rho\beta}\Gamma_{\beta,\rho\lambda}\xi^{\lambda}. \end{aligned} \quad (\text{C15})$$

Equation (C10) requires the combination

$$\frac{2}{3}C_{\lambda}^{[\alpha\mu]\nu}\partial_{\nu}\xi^{\lambda} = 2\sqrt{g}\left[-M^{\alpha\mu\nu\beta}\nabla_{\nu}\xi_{\beta} + M^{\alpha\mu\rho\beta}\Gamma_{\beta,\rho\lambda}\xi^{\lambda}\right]. \quad (\text{C16})$$

The sum of (C15) and (C16) gives the tensor result

$$\Phi^{[\alpha\mu]} = 4\sqrt{g}(\nabla_{\nu}M^{\alpha\mu\nu\beta})\xi_{\beta} - 2\sqrt{g}M^{\alpha\mu\nu\beta}\nabla_{\nu}\xi_{\beta} \quad (\text{C17})$$

which is used in Eq. (3.27) and subsequently.

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