

Arf-Brown-Kervaire invariant on a lattice

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We propose a lattice formulation of the Arf-Brown-Kervaire (ABK) invariant which takes values in $\mathbb{Z}_8$. Compared to the standard $\mathbb{Z}$ -valued index, the ABK invariant is more involved in that it arises in Majorana fermion partition functions with reflection symmetry on two-dimensional nonorientable manifolds, and its definition contains an infinite sum over Dirac eigenvalues that requires proper regularization. We employ the massive Wilson-Dirac operator, with and without domain walls, on standard two-dimensional square lattices, and use its Pfaffian for the definition. Twisted boundary conditions and cross-caps, which reverse the orientation, are introduced to realize nontrivial topologies equipped with nontrivial Pin^- structures of Majorana fermions. We verify numerically (and partly analytically) that our formulation on a torus, Klein bottle, real projective plane (as well as its triple connected sum), and two types of Möbius strip reproduces the known values in continuum theory.

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I. INTRODUCTION

Topology is a key to understanding nonperturbative aspects of quantum field theories. Because of its stableness under perturbation of the fields, deformation of the base manifolds, renormalization group flow, and so on, topology has described many nontrivial low energy dynamics of gauge theories. In fermionic field theories, in particular, topology of gauge fields is closely connected via the Atiyah-Singer (AS) index theorem [1] to the Dirac operator index or quantum anomalies in more general setups, which appears in the complex phase of fermion partition functions. Recent developments highlight that topological properties are discussed not only on closed manifolds, but also in systems having boundaries, where the bulk-boundary correspondence of the anomaly plays an essential role in characterizing the symmetry-protected topological (SPT) phases of fermions [2].

In lattice gauge theory, which is another powerful nonperturbative tool for analyzing quantum field theories, topology is apparently a difficult property to formulate due to the absence of continuity. In fact, it was shown in [3,4] that the index can be formulated in a similar way to the continuum theory using lattice Dirac operators satisfying the Ginsparg-Wilson relation [5]. The index was defined by the Dirac operator zero modes, thanks to the exact

chiral symmetry on a lattice [6]. In recent studies [7], the Ginsparg-Wilson relation was generalized to wider fermionic systems such as those in odd dimensions, those with discrete symmetries/anomalies, and those having real structures, to describe more nontrivial indices of Dirac operators. However, these Dirac indices defined by the zero modes are formulated essentially on a flat periodic square lattice, whose continuum counterpart is limited to a flat torus. In particular, the Ginsparg-Wilson relation is known to be broken when we impose nontrivial boundary conditions [6].

In [8,9], in which three of the authors are involved, a new generalization was proposed using K theory [10,11]. By the so-called suspension isomorphism of K groups, the Dirac operator index is known to be expressed by a certain type of the spectral flow of massive Dirac operators in continuum theory. In [12–14], it was shown that the Atiyah-Patodi-Singer (APS) index [15] on a manifold with boundary is also mathematically equal to the spectral flow of Dirac operators with a domain wall in the mass term. The new formulation of the index was given by the lattice version of the corresponding spectral flow. Since the continuum target is a massive Dirac operator, we do not need the Ginsparg-Wilson relation or any other sophisticated symmetry on a lattice, and the Wilson-Dirac operator is good enough for the formulation. The new formulation of the index measured by the spectral flow of the Wilson-Dirac operator covers the standard AS index, APS index whose boundary may be curved, and their mod-two versions when the Dirac operator has a real structure. In Ref. [9], it was numerically shown that both $\mathbb{Z}$ and $\mathbb{Z}_2$ indices on a torus, a stripe with two flat boundaries, a disk, and a torus with a hole can be evaluated on a lattice.

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In this work, we try to formulate a further nontrivial topological quantity by fermion path integral on a lattice. In continuum theory, there exist more exotic $\mathbb{Z}_8$ - or $\mathbb{Z}_{16}$ -valued topological invariants¹ [21]. In physics, these invariants are related to the topological phases of Majorana fermions protected by the time-reversal symmetry. We will show that the $\mathbb{Z}_8$ -valued Arf-Brown-Kervaire (ABK) invariant [22] on a two-dimensional manifold can be described by the lattice Wilson-Dirac operator. Specifically, it appears in the phase factor of the fermion partition function Z [2,23],

$$Z \propto \exp\left(\frac{i\pi}{4}\beta\right), \quad (1)$$

where the integer-valued invariant β modulo-eight is the ABK invariant. Our goal is to regularize Z with the Wilson-Dirac operator on a lattice and show that the ABK invariant is obtained from the complex phase of Z . This requires a quite nontrivial setup on a lattice as listed below.

In continuum theory, the ABK invariant appears in the partition function of Majorana fermions having a reflection symmetry with $R^2 = (-1)^F$ where $(-1)^F$ is the fermion number parity. This indicates that we must have a real structure in the corresponding lattice fermion Dirac operator with the R symmetry and consider the Pfaffian of it, rather than its determinant.

The ABK invariant β can be viewed as an extension of the mod-two index. In contrast to the mod-two index, which is defined only for orientable manifolds, the ABK invariant is defined even on nonorientable manifolds. The reflection symmetry makes it possible to define the Majorana fermion on such manifolds. Mathematically, this corresponds to introducing a Pin^- structure [minus sign comes from $R^2 = (-1)^F$] on the manifolds. For this requirement, we must impose “twisted” boundary conditions [24,25] which reverse the orientation, rather than the standard periodic or antiperiodic lattice boundary conditions. We also note the recent related study [25] of staggered fermions on the torus and the Klein bottle, where the mod-eight anomaly (a similar mod-eight structure arises in our construction as an anomaly of the boundary fermions) were analyzed through the matching between lattice and continuum symmetries.

It is impossible to describe the ABK invariant with the zero modes of the Dirac operator alone. It is described in terms of η invariant [15] (spectral asymmetry between positive and negative eigenvalues) of the Dirac operator.

¹In previous studies, a state sum combinatorial definition of the ABK invariant on lattice using the Gu-Wen Grassmann integral [16] was formulated by [17,18], which corresponds to regularizing topological quantum field theories on the lattice. We also refer the readers to [19,20] in which the ABK invariant was discussed using the Hamiltonian formalism of fermions. In this work, we employ Euclidean fermion path-integral approach.

The $\mathbb{Z}_8$ quantization arises from the subtle spectral asymmetry [26–28], and a proper treatment of regularization is required. It is almost obvious from this fact that the overlap fermion having exact chiral zero modes would not be useful that much. On our two-dimensional lattices, it is manageable to numerically evaluate the Wilson fermion Pfaffians directly and examine if the lattice Wilson operator is adequate for the regularization of the ABK invariant.

Finally, the ABK invariant on a manifold with boundaries requires the APS boundary condition in the standard mathematical setup, which is nonlocal and difficult to realize on a lattice. In our formulation, we introduce the domain-wall mass term [29], which compensates the APS boundary condition, following the discussion in [12–14] and related argument in [30]. We note that, if the lattice version of this formulation is successful, it offers a nonperturbative path-integral regularization of the one-dimensional Majorana chain discussed in [31,32] and the anomaly inflow between its bulk and edge.

In this work, we evaluate the Majorana fermion Pfaffian employing the Wilson-Dirac operator on various types of two-dimensional lattices with and without domain walls. We find that the extracted ABK invariant is quantized into $\mathbb{Z}_8$ values on a conventional flat torus, Klein bottle, real projective plane, as well as its triple connected sum, and two different types of the Möbius strip. In the torus case, reproducing the continuum ABK invariant (i.e., the mod-two index, which is the special case for orientable manifolds) can be checked rather easily. In contrast, the corresponding statement on nonorientable manifolds is nontrivial. Performing the analysis on several lattices with different lattice spacings, we examine their convergence to the continuum values. In the Klein bottle case, we successfully decompose the Pfaffian into Fourier modes and identify a specific set of modes that contributes to the ABK invariant.

We expect that the analogous $\mathbb{Z}_{16}$ -valued invariant² in higher dimensions can be defined in a similar way, although there may be technical difficulties due to numerical costs.

The rest of this paper is organized as follows. In Sec. II, we review how the ABK invariants arise in the continuum theory of Majorana fermions. In Sec. III, we turn to the lattice path-integral formulation and introduce various unorientable manifolds with various Pin^- structures by twisted boundary conditions as well as cross-caps with and without domain walls. In Sec. IV, for the cases of a torus and Klein bottle, we analytically compute the ABK invariant on the corresponding lattices. Then we give our numerical results for more general manifolds in Sec. V. In Sec. VI, we give a summary and discussions.

²A state sum combinatorial definition of this invariant on lattice using the Gu-Wen Grassmann integral [16] is given by [17,33].

II. TWO-DIMENSIONAL MAJORANA FERMION AND THE ABK INVARIANT

We first consider in continuum theory the Majorana fermion partition functions on a two-dimensional manifold X and introduce the ABK invariant.

Majorana fermions are obtained by imposing on the standard Dirac fermion ψ and the Majorana condition $\bar{\psi} = \psi^T C$ with the charge conjugation matrix C . For the Dirac matrices satisfying $\{\gamma^a, \gamma^b\} = \delta^{ab}$, C operates as $C\gamma^a C^{-1} = -(\gamma^a)^T$. Our action to be considered is

$$S = \int_X d^2x \sqrt{g} \psi^T C (D + m) \psi, \quad (2)$$

where $D = \gamma^a e_a^\mu D_\mu$ is the Dirac operator with the zweibein e_a^μ , which is related to the metric through $g^{\mu\nu} = e_a^\mu e_b^\nu \delta^{ab}$, $g = \det g_{\mu\nu}$, and $D_\mu = \partial_\mu + \omega_\mu$ is the covariant derivative with the spin connection ω_μ .

We take the standard representation of the Dirac matrices by the Pauli matrices,

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (3)$$

and

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4)$$

Note that C is antisymmetric, so that the mass term $m\psi^T C\psi$ is generally nonzero. Here we also define the chirality matrix $\bar{\gamma} = -i\gamma^1\gamma^2 = \text{diag}(1, -1)$.

In order to make the manifold X endowed with a Pin^- structure, we require the fermion action to have a reflection symmetry.

Here we take the x direction to be reversed under the reflection and the fermion fields are transformed as

$$\psi(x, y) \rightarrow R_x \psi(x, y) = \gamma^a e_a^1 \bar{\gamma} \psi(-x, y), \quad (5)$$

as well as the metric and zweibein accordingly. Note that the spin connection ω_μ transforms like an axial vector, since it is proportional to $[\gamma^1, \gamma^2]/2 = i\bar{\gamma}$.

Twice this transformation yields $R_x^2 = -1$, or $R_x^2 = (-1)^F$ with the fermion number F to be precise, in contrast to the Pin^+ case (where the reflections twice give identity), and the sign of the mass term $m\psi^T C\psi$ is unchanged. The chirality is no longer a good quantum number in the theory even when $m = 0$, since $\bar{\gamma}$ does not commute with R_x . Using this R_x symmetry together with appropriate boundary conditions, we can consider various

unorientable manifolds with various Pin^- structures and the Majorana fermions on them.³

The partition function of Majorana fermions is formally given by the Pfaffian of the operator $C(D + m)$ which needs to be properly regularized. Here we employ the Pauli-Villars regularization with a bosonic spinor with the mass value M . Note that for every eigenfunction χ with a pure imaginary eigenvalue $i\lambda$ of D , $C\chi^*$ is another eigenmode with the same eigenvalue $i\lambda$. Because of this property, the regularized Majorana fermion partition function can be expressed by

$$Z = \frac{\text{Pf}[C(D + m)]}{\text{Pf}[C(D + M)]} = \prod'_\lambda \frac{i\lambda + m}{i\lambda + M}, \quad (6)$$

where $\prod'$ means that the product is taken over only one eigenvalue from the doubly degenerate pair.

The relative sign of the fermion mass and the Pauli-Villars mass is important in that the ABK invariant appears only when $mM < 0$, where the system is in a nontrivial topological phase. Following [2], let us take the following limit $-m = M \rightarrow +\infty$ to obtain

$$Z = \lim_{M \rightarrow +\infty} \prod'_\lambda \frac{\lambda + iM}{\lambda} \frac{\lambda}{\lambda - iM}. \quad (7)$$

Intuitively, for large M , the phase of this infinite product can be rewritten as

$$\begin{aligned} \lim_{M \rightarrow +\infty} \prod'_\lambda \frac{\lambda + iM}{\lambda} \frac{\lambda}{\lambda - iM} &= \lim_{M \rightarrow +\infty} \prod'_\lambda \left(\frac{iM}{\lambda} \right) \left(\frac{-\lambda}{iM} \right) \\ &\propto \prod'_\lambda (-1). \end{aligned} \quad (8)$$

If the spectrum is paired as $\pm i\lambda$ (which happens when the spacetime manifold is orientable⁴), the infinite product reduces to the number of unpaired $\lambda = 0$ modes of the Dirac operators. Dividing this number by 2 (since the eigenvalues are always doubly degenerate) gives the mod-two index $I(D)$ of D , and the phase of the partition function can then be written as

$$Z \propto (-1)^{I(D)}. \quad (9)$$

³The partition function on nonorientable manifolds in the operator formalism was discussed in [19,20].

⁴On orientable manifolds, one can define a globally well-defined chirality operator that commutes with the free Dirac operator D , implying that the spectrum is paired as $\pm i\lambda$. For nonorientable manifolds, no globally well-defined chirality operator exists, and the $\pm$ pairing of the spectrum is lost.

However, in general situations, the eigenvalues are not paired as $\pm i\lambda$. Then the infinite product of the right-hand side of Eq. (8) is not well defined. Instead, we rewrite this phase in terms of an infinite sum over λ ,

$$\exp \left[i \frac{\pi}{2} \left(\sum_{\lambda \geq 0} 1 - \sum_{\lambda < 0} 1 \right) \right]. \quad (10)$$

From the left-hand side of Eq. (8), one can see that the complex phase of $\frac{\lambda+iM}{\lambda-iM}$ is mapped to slightly different regions depending on $\text{sign}(\lambda)$. Namely,

$$\begin{aligned} 0 < \arg \left(\frac{\lambda + iM}{\lambda - iM} \right) &\leq \pi (\lambda \geq 0), \\ -\pi < \arg \left(\frac{\lambda + iM}{\lambda - iM} \right) &< 0 (\lambda \leq 0). \end{aligned} \quad (11)$$

Therefore, this infinite sum can be regarded as accumulating the phase contributions according to the sign of λ . An appropriate regularization of this infinite sum yields the η invariant [2,15], which provides the total phase,

$$Z \propto \exp \left[i \frac{\pi}{2} \eta(iD) \right]. \quad (12)$$

The η invariant is usually defined by the ζ -function regularization as

$$\eta(iD) = \lim_{s \rightarrow +0} \sum_{\lambda} \text{sign}(\lambda) \frac{1}{|\lambda|^s}, \quad (13)$$

with the sign function defined by

$$\text{sign}(\lambda) = \begin{cases} +1 & \text{if } \lambda \geq 0, \\ -1 & \text{if } \lambda < 0. \end{cases} \quad (14)$$

Here the summation is taken over all eigenvalues λ of the Dirac operator. Note that the exponent in Eq. (6) has $O(1/M)$ corrections when M is finite.

According to Refs. [26–28], the ABK invariant⁵ multiplied by $1/2$ can be identified as a special version of the η invariant, which takes only integral or half-integral values on Pin^- manifolds. Namely, we can denote it by

⁵The original definition of the ABK invariant [22] was given in terms of $\mathbb{Z}_2$ cohomology of manifolds, and it is a topological invariant depending only on global topology of the manifold and its Pin^- structure, independent of the metric g . Furthermore, a correspondence with the two-dimensional Pin^- bordism group $\Omega_2^{\text{Pin}^-} \simeq \mathbb{Z}_8$ is known [34]. Indeed, the two-dimensional real projective plane ($\mathbb{RP}^2$) that generates the bordism group takes the minimal value $\beta = \pm 1$ (where the sign depends on the choice of Pin^- structure).

$$\beta = 2\eta(iD) \pmod{8}. \quad (15)$$

We also note that η takes even integers on orientable manifolds so that the Pfaffian is always real and the invariant reduces to $\mathbb{Z}_2$ values. Therefore, we need to consider unorientable manifolds to obtain nontrivial $\mathbb{Z}_8$ values of the ABK invariant.

For the Pfaffian of fermions in a two-dimensional condensed matter, the $\mathbb{Z}_8$ quantization classifies the SPT phases. When the number of fermion flavors is $k = 1, \dots, 7 \pmod{8}$, the complex phase $\exp(i \frac{\pi}{4} k \beta)$ for $m < 0$ cannot be modified to unity in the case of $m > 0$ without hitting singularity (or gap closing). In the $k = 0 \pmod{8}$ case, however, there is no relative phase difference and there is no topological obstruction. As Fidkowski and Kitaev [31,32] explicitly showed, the two partition functions ($m < 0$ and $m > 0$) are smoothly connected via additional four-Fermi interaction terms. Such an argument requires access to manifolds with odd β . If one looks only at orientable examples like the torus (with $\beta = 0, 4$), one can at best detect whether k is even or odd.⁶

So far, we have implicitly assumed that X is a closed manifold. Now let us discuss the case with boundaries. In mathematics, the η invariant of the massless Dirac operator iD can be defined on such a manifold with the so-called APS boundary condition [15]. As described in Ref. [2], this η invariant with the APS boundary condition describes the anomaly inflow or anomaly cancellation between the bulk and edge fermions in the SPT phases. However, the APS condition is nonlocal and unlikely to be realized in physics.

The massive fermion system follows a more natural boundary condition that is different from the APS condition. It is given by the projection with respect to the Dirac matrix $\gamma_{\perp}$, which is perpendicular to the surface. This boundary condition is the same as that of the domain-wall fermion [35,36] where the mass term flips its sign at the wall, on which the massless chiral modes are localized.

In Refs. [12,13], it was shown that the APS index [15] on a manifold with boundary is mathematically equal to the η invariant of the domain-wall fermion Dirac operator, where the original boundary is extended to form a closed manifold and the domain wall is instead put at the original location of the boundary. Interestingly, this expression of the index does not depend on the extended subregion at all. This physicist-friendly expression of the APS index is formulated in the lattice gauge theory [8,9], too. With a different discussion [30], a more general η invariant of the massless Dirac operator was shown to be equal to the phase of the massive fermion partition function with the physical boundary condition.

In this work, we examine if the phase of the lattice domain-wall fermion partition function [29,37] is consistent

⁶This is related to $\mathbb{Z}$ classification of SPT phases without the reflection symmetry.

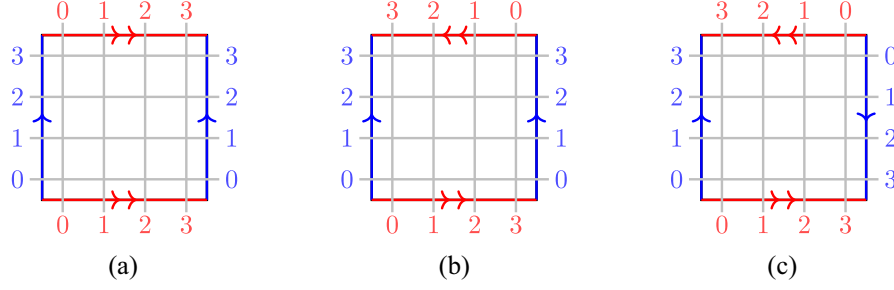


FIG. 1. Identification patterns of the link variables for a (a) torus, (b) Klein bottle, and (c) real projective plane are shown. The links across the boundaries with the same labels are connected.

with the $\mathbb{Z}_8$ -valued η invariant in the corresponding continuum theory, namely, that of a massless Dirac operator with the APS boundary condition. This is a nontrivial equality since it is impossible to define a massless Dirac operator on a lattice without breaking the Ginsparg-Wilson relation or any boundary condition that corresponds to the APS boundary condition. The spectrum of the domain-wall Dirac operator is totally different from that of the target massless Dirac operator in the continuum theory.

III. A LATTICE REGULARIZATION OF THE ABK INVARIANT

We consider standard two-dimensional square lattices whose size is denoted by $N_x a \times N_y a$ with the lattice spacing a . On each of these lattices we discretize the action Eq. (2) by that of the Wilson fermion

$$S_W = \sum_{x,y} a^2 \frac{i}{2} \psi(x,y)^T C D_W(m) \psi(x,y), \quad (16)$$

where x and y take integer multiples of a : $0, a, 2a \dots (N_{x/y} - 1)a$, and $D_W(m)$ denotes the massive Wilson-Dirac operator

$$D_W(m) = \sum_{\mu} \gamma^{\mu} \frac{\nabla_{\mu}^f + \nabla_{\mu}^b}{2} + m + \sum_{\mu} r \frac{\nabla_{\mu}^f - \nabla_{\mu}^b}{2}, \quad (17)$$

where ∇_{μ}^f and ∇_{μ}^b are the forward and backward difference operators, respectively. The definitions of γ^{μ} 's and C matrix are the same as those given in Eqs. (3) and (4). Here we take a constant mass term m , but later it will be generalized to the case with domain walls. Since the Majorana fermions considered in this work are neutral under gauge interactions, we set all link variables to the identity, except for those around the singularities explained below.

In the following, the Wilson parameter is set to $r = 1$. We have implicitly assumed that the zweibein in our target continuum theory is aligned with the coordinate ($e_a^{\mu} = \delta_a^{\mu}$).

The partition function of the lattice action is given by the Pfaffian

$$Z(m) = \text{Pf}(CD_W(m)). \quad (18)$$

For an antisymmetric matrix A of finite size $2n \times 2n$, its Pfaffian is defined as

$$\text{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sign}(\sigma) A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} \cdots A_{\sigma(2n-1)\sigma(2n)}, \quad (19)$$

where S_{2n} denotes the set of permutations of $\{1, 2, \dots, 2n\}$. The Pfaffian is also known as the square root of the determinant, $(\text{Pf}A)^2 = \det A$, yet the expression above defines it as a polynomial in the matrix elements with no sign ambiguity. Numerically, $\text{Pf}(CD_W(m))$ can be evaluated with a reasonable choice of the sizes $N_x, N_y < 30$. By imposing different boundary conditions, the operator $CD_W(m)$ becomes a finite-sized complex matrix so that $Z(m)$ can have a nontrivial phase, which defines our lattice formulation of the ABK invariant,

$$\beta^{\text{lat}}(m, a) := \frac{4}{\pi i} \arg \left(\frac{\text{Pf}(CD_W(m))}{\text{Pf}(CD_W(|m|))} \right). \quad (20)$$

Here, the Pauli-Villars partition function $\text{Pf}(CD_W(|m|)) = Z(|m|)$ is introduced to compare the results with the continuum theory.⁷

Our goal of this work is to examine if $\beta^{\text{lat}}(m, a)$ is consistent with the continuum value or not in the $a \rightarrow 0$ and $-m \rightarrow \infty$ limits. Our approach on the standard square lattice to set up nontrivial Pin^- manifolds by the boundary conditions is distinct from those employing triangular lattices [38] to express nonzero curvatures.

A. Flat torus T^2

As a warm-up, let us consider the simplest example: a flat torus [see Fig. 1(a)]. For each of two directions (x and y), there are two possible boundary conditions: periodic (P) or antiperiodic (A). Specifically, they are given by

⁷It is expected that the phase of $\text{Pf}(CD_W(|m|))$ is trivial, at least, in the continuum limit.

$$\psi(N_x a, y) = \pm \psi(0, y), \quad \psi(x, N_y a) = \pm \psi(x, 0), \quad (21)$$

for arbitrary x and y . There are four different Pin^- structures on the torus, PP , PA , AP , and AA , where each alphabet denotes the boundary condition in the order of x and y .

B. Klein bottle

The Klein bottle (KB) is obtained by gluing the upper and lower edges of a cylinder after reversing the orientation [see Fig. 1(b)]. First, we make a cylinder identifying $x = 0$ and $x = N_x a$, with P or A boundary conditions,

$$\psi(N_x a, y) = \pm \psi(0, y). \quad (22)$$

Then we glue the $y = 0$ with $y = N_y a$ edges with a group operation of $\text{Pin}^-(2)$ associated with the $O(2)$ operation that reverses the x direction. The operation is given by

$$\psi(x, N_y a) = s R_x \psi(x, 0) = s \gamma^1 \tilde{\gamma} \psi((N_x - 1)a - x, 0), \quad (23)$$

where we have two choices of the sign $s = \pm 1$. For the backward difference at the slice $y = 0$, which has to be the conjugate of the forward difference, it is understood that the inverse of the above boundary condition

$$\psi(x, -a) = s R_x^{-1} \psi(x, 0) \quad (24)$$

holds. Hence, the Klein bottle admits four distinct Pin^- structures we denote by $P+$, $P-$, $A+$, and $A-$, where the second $\pm$ corresponds to the sign s of Eq. (23). A similar approach for considering fermion partition functions on nonorientable manifolds is represented in [24].

C. Real projective plane $\mathbb{RP}^2$

In order to consider Majorana fermions on an $\mathbb{RP}^2$, we glue both the top and bottom and the left and right boundaries of the lattice with a twist in each direction [see Fig. 1(c)]. Explicit forms of the boundary conditions are

$$\begin{aligned} \psi(N_x a, y) &= s_x R_y \psi(0, y) = s_x \gamma^2 \tilde{\gamma} \psi(0, (N_y - 1)a - y), \\ \psi(x, N_y a) &= s_y R_x \psi(x, 0) = s_y \gamma^1 \tilde{\gamma} \psi((N_x - 1)a - x, 0), \end{aligned} \quad (25)$$

as well as their conjugates $\psi(-a, y) = s_x R_y^{-1} \psi(0, y)$ and $\psi(x, -a) = s_y R_x^{-1} \psi(x, 0)$. Here $s_y = \pm 1$ and $s_x = \pm 1$ can be taken independently.

With the above boundary conditions, we have two pairs of singular points on the lattice: One pair is $(x, y) = (0, 0)$ and $((N_x - 1)a, (N_y - 1)a)$, which are connected by both of the x and y differences since

$$\begin{aligned} a \nabla_x^f \psi((N_x - 1)a, (N_y - 1)a) &= s_x \gamma^2 \tilde{\gamma} \psi(0, 0) - \psi((N_x - 1)a, (N_y - 1)a), \\ a \nabla_y^f \psi((N_x - 1)a, (N_y - 1)a) &= s_y \gamma^1 \tilde{\gamma} \psi(0, 0) - \psi((N_x - 1)a, (N_y - 1)a). \end{aligned} \quad (26)$$

Another pair is $(x, y) = ((N_x - 1)a, 0)$ and $(0, (N_y - 1)a)$ to which

$$\begin{aligned} a \nabla_x^f \psi((N_x - 1)a, 0) &= s_x \gamma^2 \tilde{\gamma} \psi(0, (N_y - 1)a) - \psi((N_x - 1)a, 0), \\ a \nabla_y^f \psi(0, (N_y - 1)a) &= s_y \gamma^1 \tilde{\gamma} (\psi((N_x - 1)a, 0)) - \psi(0, (N_y - 1)a) \end{aligned} \quad (27)$$

hold. We understand this as a consequence of the fact that the $\mathbb{RP}^2$ cannot be realized as a globally flat manifold. In fact, we can interpret that there are two different links from $((N_x - 1)a, (N_y - 1)a)$ in the x direction and y direction, which are curved to end up with the same point $(0, 0)$ as shown in Fig. 2. For the two forward difference operators at $((N_x - 1)a, (N_y - 1)a)$, the two factors $U_x = s_x \gamma^2 \tilde{\gamma}$ and $U_y = s_y \gamma^1 \tilde{\gamma}$ in front of $\psi(0, 0)$ in Eq. (26) can be understood as the corresponding link variables of the local Lorentz $SO(2)$ gauge field. This interpretation similarly works for Eq. (27).

It is then interesting to consider the Wilson loop W_1 formed by the above nontrivial link variables starting from $(0, 0)$ in the counterclockwise direction, as well as W_2 from $(0, (N_y a - 1)a)$. We have

$$W_1 = W_2 = U_x U_y^{-1} = -s_x s_y \gamma_1 \gamma_2 = \exp[-\pi s_x s_y T], \quad (28)$$

with the local Lorentz generator $T = [\gamma_1, \gamma_2]/4$. If our lattice setup is a good regularization of a smooth continuum $\mathbb{RP}^2$, the Riemann curvature may be identified from $W_{1,2}$ as $R_1 = R_2 = -\pi s_x s_y / A$, where A is the area of each Wilson loop. Then we can compute the Euler characteristic of the lattice by

$$\chi = \frac{1}{2\pi} \int R = \frac{(R_1 + R_2)A}{2\pi} = -s_x s_y. \quad (29)$$

Since χ of $\mathbb{RP}^2$ is unity, our assumption of the smoothness requires $s_x s_y = -1$. Therefore, we expect that two choices $(s_x, s_y) = (+, -)$ and $(-, +)$ will represent the two Pin^-

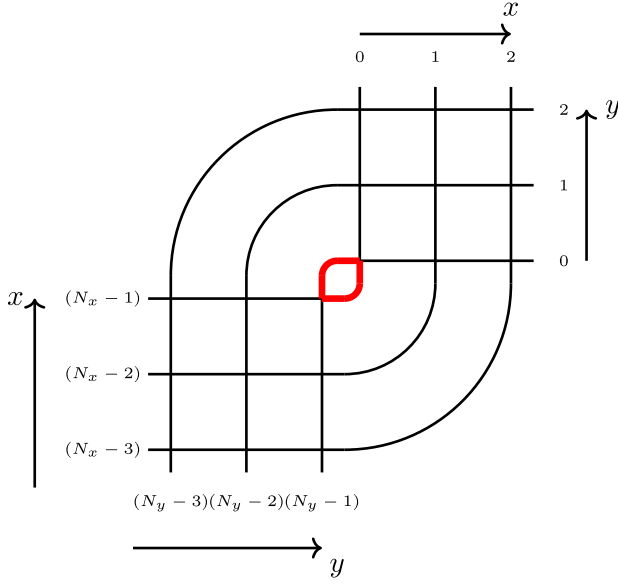


FIG. 2. The lattice network around the singularity of $\mathbb{RP}^2$. Note that $(0, 0)$ and $((N_x - 1)a, (N_y - 1)a)$ points are connected by two different link variables (denoted by thick lines), which form a nontrivial Wilson loop discussed in the main text.

structures of $\mathbb{RP}^2$, while the other $(+, +)$ and $(-, -)$ seem to fail in regulating the singularity.

D. Introducing cross-caps

Introducing cross-caps, we can consider more complicated manifolds. For instance, adding a cross-cap to a torus yields a manifold whose topology is a triple connected sum of $\mathbb{RP}^2$, which we denote by $\mathcal{N}^3\mathbb{RP}^2$. A cross cap is obtained by cutting the manifold along a line segment and identifying the two edges with reversed orientation as in Fig. 3.

Let us consider a line segment (x_L, y_c) and (x_R, y_c) . In the continuum case, a cross-cap is obtained by gluing two patches, the lower ($y \leq y_c$) and the upper ($y \geq y_c$), by the following boundary conditions:

$$\psi_{\text{lower}}(x, y_c) = \begin{cases} \pm \gamma^1 \bar{\gamma} \psi_{\text{upper}}(x_L + x_R - x, y_c) & \text{if } x_L < x < x_R, \\ \psi_{\text{upper}}(x, y_c) & \text{otherwise.} \end{cases} \quad (30)$$

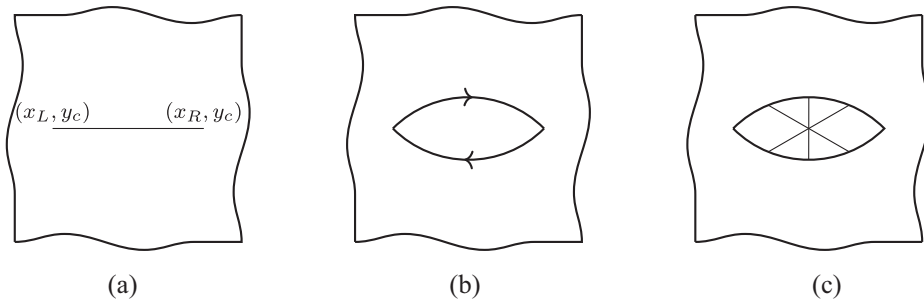


FIG. 3. Introducing a cross-cap (c), obtained by (a) cutting the manifold and (b) identifying edges reversing the orientation.

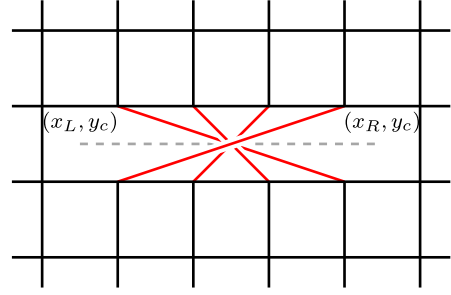


FIG. 4. Lattice network around the cross-cap.

Both choices of the sign $\pm$ are allowed, and hence the introduction of one cross-cap doubles the number of possible Pin^- structures. To realize this construction on the lattice, we impose the rule that, when the forward difference operator ∇_y^f refers to a point across the cross-cap segment from the point (x, y) [i.e., the cross-cap lies between (x, y) and $(x, y + 1)$], it refers to $\pm \gamma^1 \bar{\gamma} \psi(x_L + x_R - x, y + 1)$ instead of $\psi(x, y + 1)$ and vice versa for the opposite direction. This lattice version of the cross-cap is presented in Fig. 4.

E. Including domain walls

In order to consider the ABK (or η invariant in more general setups) on open manifolds, let us introduce a domain-wall fermion mass term [29,37]. We consider a closed continuum subspace X on the periodic or antiperiodic lattice, and the position-dependent mass is defined by

$$m_X(x, y) = \begin{cases} -m_0 & \text{if } (x, y) \text{ in } X, \\ +m_0 & \text{otherwise,} \end{cases} \quad (31)$$

where we assume that $m_0 > 0$. In the conventional lattice simulations, one often takes X with flat boundaries, but here we allow the cases with curved domain walls [39–44].

We define the lattice version of the ABK invariant on open manifolds by

$$\beta^{\text{lat}}(m_X, a; X) := \frac{4}{\pi i} \arg \left(\frac{\text{Pf}(CD_W(m_X))}{\text{Pf}(CD_W(m_0))} \right). \quad (32)$$

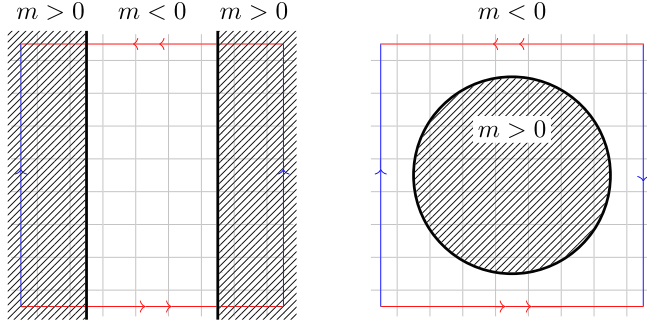


FIG. 5. In order to realize a Möbius strip, we consider the two types of the domain-wall mass term shown above. The unshaded regions have a negative mass, while the shaded regions are assigned to have a positive mass. The unshaded regions possess the Möbius strip topology. The left panel represents two domain walls to form a straight Möbius strip on a Klein bottle, and the right one corresponds to a circular domain wall removing a disk region from an $\mathbb{RP}^2$, which forms a curved Möbius strip.

If our lattice calculation of the ABK invariant is consistent with the continuum result, the values depend only on the topology of the subregion X and not on the local properties in the $a \rightarrow 0$ and $m_0 \rightarrow \infty$ limits as in the closed manifold case.

A typical example of two-dimensional nonorientable open manifolds is the Möbius strip, which can be identified as half of the Klein bottle discussed above. The corresponding domain-wall mass term is given by

$$m_{\text{MS}}(x, y) = \begin{cases} -m_0 & \text{if } \frac{(N_y+1)a-w}{2} < y < \frac{(N_y+1)a+w}{2}, \\ +m_0 & \text{otherwise,} \end{cases} \quad (33)$$

where w is the width and we set the same boundary condition as Eq. (23). Another way to realize a Möbius strip is to remove a disk from an $\mathbb{RP}^2$. When we introduce a domain-wall mass term

$$m_{\text{disk}}(x, y) = \begin{cases} +m_0 & \text{if } \sqrt{(x-x_0)^2 + (y-y_0)^2} \leq r, \\ -m_0 & \text{otherwise,} \end{cases} \quad (34)$$

the subregion outside of the disk ($\overline{\text{disk}}$) has the same topology as the Möbius strip introduced in Eq. (33). The two domain-wall configurations used to realize the Möbius strip are illustrated in Fig. 5. If our formulation of the ABK invariant is topological, these two different lattice geometries should give the same value of $\beta^{\text{lat}}(m_X, a; X)$ in the $a \rightarrow 0$ and $m_0 \rightarrow \infty$ limits.

Now we are ready to compute $\text{Pf}(CD_W(m_X))$ with or without domain walls to extract the complex phase on various types of nonorientable manifolds discussed above. Some of them have translational symmetry so that we can analytically evaluate the Pfaffian via Fourier transformation, which will be discussed in the next section. For more

general manifolds, we will perform numerical evaluation on a finite lattice with $N_x, N_y \lesssim 30$ in Sec. V.

IV. FOURIER ANALYSIS ON FLAT CLOSED MANIFOLDS

Since the Wilson-Dirac operator $D_W(m)$ is a finite complex matrix, it is not difficult to compute, at least numerically, the partition function $Z = \text{Pf}(CD_W(m))$. For the torus and Klein bottle cases defined in the previous section, we can analytically perform the program thanks to the translational symmetry. By the Fourier transformation, we can microscopically identify specific eigenmodes that contribute to the phase of Z and thus to the ABK invariant.

In the momentum space, the Wilson-Dirac operator $D_W(m)$ is expressed as

$$i\gamma^1 \frac{1}{a} \sin(ap_x) + i\gamma^2 \frac{1}{a} \sin(ap_y) + m + \frac{1}{a} (2 - \cos(ap_x) - \cos(ap_y)), \quad (35)$$

and it has eigenvalues

$$\mu^\pm(p_x, p_y) = \pm i \frac{1}{a} \sqrt{\sin(ap_x)^2 + \sin(ap_y)^2} + m + \frac{1}{a} (2 - \cos(ap_x) - \cos(ap_y)), \quad (36)$$

with eigenvectors

$$u^\pm(x, y; p_x, p_y) = \frac{1}{\sqrt{2}} \left(\pm \frac{1}{\sqrt{\sin(ap_x)^2 + \sin(ap_y)^2}} \right) e^{i(p_x x + p_y y)}. \quad (37)$$

The boundary conditions on the target manifold constrain the allowed values of the momentum and form of the eigenfunctions.

A. Flat torus T^2

In continuum theory, it is known for the two-dimensional torus T^2 that among the four Pin^- structures, only the PP boundary condition gives a nontrivial phase (-1) to the partition function; in other words, the ABK invariant $\beta(PP) = 4$, while the others (PA, AP, AA) have $\beta = 0$. This $\mathbb{Z}_2$ structure reflects the fact that T^2 is an orientable spin manifold and the ABK invariant reduces to the standard mod-two index. These results can be checked straightforwardly as follows, in contrast to the nontriviality of the Klein bottle case discussed in the next section.

Let us examine if our lattice setup reproduces this simplest result. Under the boundary conditions (21), the allowed momenta are

$$p_{x/y} = \begin{cases} \frac{2\pi}{aN_{x/y}} k & (k = 0, \dots, N_{x/y}) \text{ if periodic,} \\ \frac{2\pi}{aN_{x/y}} (k + \frac{1}{2}) & (k = 0, \dots, N_{x/y}) \text{ if antiperiodic,} \end{cases} \quad (38)$$

and the $u^\pm(x, y; p_x, p_y)$ set forms a linearly independent basis. The partition function can be written as

$$Z = \text{Pf}(CD_W(m)) = \prod_{p_x, p_y, \pm} \mu^\pm(p_x, p_y), \quad (39)$$

where $\prod'$ takes one eigenvalue from the doubly degenerate eigenvalues.

When $\sin(ap_x), \sin(ap_y) \neq 0$, i.e., $p_x, p_y \neq 0$ and $\neq \pi/a$, the eigenvalues appear in complex conjugate pairs. Therefore, these eigenvalues do not contribute to the phase of the partition function. In addition, when p_x or p_y is π/a (corresponding to the doubler modes) and $-2 < ma$, the eigenvalues become positive and do not contribute to the phase. The only nontrivial contribution to the phase comes from the (one of doubly degenerate) $p_x = p_y = 0$ eigenvalue $\mu(0, 0) = m$ and this eigenvalue exists only when the Pin^- structure is PP .

In the PP case, the ratio of the partition function in the topological phase ($-2 < ma < 0$) to that in the trivial vacuum ($0 < M$) exhibits the phase

$$\frac{Z(m; PP)}{Z(M; PP)} \propto \frac{m}{M} \propto -1, \quad (40)$$

$$\beta^{\text{latt}}(PP) = 4. \quad (41)$$

This result is consistent with the continuum theory. The feature that the phase arises solely from the zero mode of the massless Dirac operator (mod-two index) is also reproduced.

B. Klein bottle

On a Klein bottle, the nontrivial boundary condition given in Eqs. (22) and (23) mixes the modes with $\pm p_x$ to form the eigenvectors

$$u_{\text{KB}}^\pm(x, y; p_x, p_y) = \frac{1}{2} u^\pm(x, y; p_x, p_y) + s \frac{1}{2} i (-1)^{k_y} \times \frac{\sin(ap_x) + i \sin(ap_y)}{\sqrt{\sin(ap_x)^2 + \sin(ap_y)^2}} \times u^\pm(x, y; -p_x, p_y), \quad (42)$$

with the eigenvalues $\mu^\pm(p_x, p_y)$ in Eq. (36), where $s = \pm$ is the sign that corresponds to the choice of the boundary conditions Ps, As , and the allowed momenta are given by

$$p_x = \begin{cases} \frac{2\pi}{N_x a} k_x & (k_x = 0, \dots, \lfloor \frac{N_x}{2} \rfloor) \text{ if periodic,} \\ \frac{2\pi}{N_x a} (k_x + \frac{1}{2}) & (k_x = 0, \dots, \lfloor \frac{N_x-1}{2} \rfloor) \text{ if antiperiodic,} \end{cases} \quad (43)$$

$$p_y = \frac{\pi}{N_y a} \left(k_y + \frac{1}{2} \right) \quad (k_y = 0, \dots, 2N_y - 1). \quad (44)$$

To derive this result, it is useful to start from the eigenfunctions on a torus with twice the length $2N_y a$ in the y direction, which is the double cover of the Klein bottle. We impose the antiperiodic boundary condition in the y direction. The Klein bottle is identified as a $\mathbb{Z}_2$ quotient of this torus. The above eigenfunctions are obtained by projecting those on the torus onto the $\mathbb{Z}_2$ -invariant subspace. See the Appendix for the details.

For $p_x \neq 0$ and $\neq \pi/a$, the eigenfunctions u_{KB}^+ and u_{KB}^- are independent, and therefore their eigenvalues $\mu^+(p_x, p_y)$ and $\mu^-(p_x, p_y) = \mu^+(p_x, p_y)^*$ do not contribute to the complex phase of the partition function. At $p_x = 0$ or $= \pi/a$, the situation is different. Substituting these values into Eq. (42), one finds that either u_{KB}^+ or u_{KB}^- is absent. The $\pm$ label of surviving eigenfunctions $u_{\text{KB}}^\pm$ alternate with k_y . More precisely, in the $P+$ and $A+$ case, surviving eigenfunctions are $u_{\text{KB}}^-, u_{\text{KB}}^+, u_{\text{KB}}^-, \dots$, for $k_y = 0, 1, 2, \dots$, respectively, whereas for the $P-$ and $A-$ case, the opposite sequence appears.

Therefore, if $p_x = 0$ or $p_x = \pi/a$ appear in Eq. (43), the corresponding factors

$$\prod_{p_y} \mu^{s(-1)^{k_y+1}}(0, p_y), \quad \prod_{p_y} \mu^{s(-1)^{k_y+1}}(\pi/a, p_y) \quad (45)$$

contribute to the phase of the partition function. To evaluate these factors, we set

$$\begin{aligned} \theta(p_y; ma) &\equiv \arg(\mu^+(0, p_y)) \\ &= \arg(+i \sin(ap_y) + ma + 1 - \cos(ap_y)). \end{aligned} \quad (46)$$

Using this function θ , the complex phase of Eq. (45) can be rewritten as

$$\begin{aligned} &\exp \left(-s \frac{i}{2} \sum_{p_y} (-1)^{k_y} \theta(p_y; ma) \right), \\ &\exp \left(-s \frac{i}{2} \sum_{p_y} (-1)^{k_y} \theta(p_y; ma + 2) \right). \end{aligned} \quad (47)$$

Here, the factor $1/2$ arises from taking only half of the degenerate eigenvalues. Note that there are no eigenvalues at $p_y = 0, 2\pi/a$ and no real eigenvalue, so that θ can be

taken in the range $-\pi < \theta(p_y; m) < \pi$ and there is no ambiguity in multiplying $\pm 1/2$ to it.

It is not difficult to show that the contribution from $p_x = \pi/a$ can be neglected in the continuum limit. Therefore, we can express the ABK invariant for the $P\pm$ case by a finite sum,

$$\begin{aligned} \beta^{\text{latt}}(Ps; m, a) &= -\frac{2s}{\pi} \sum_{p_y} (-1)^{k_y} \theta(p_y; ma) \\ &= -\frac{2s}{\pi} \sum_{n=0}^{N_y-1} \left\{ \theta\left(\frac{\pi}{aN_y} \left(2n + \frac{1}{2}\right); ma\right) \right. \\ &\quad \left. - \theta\left(\frac{\pi}{aN_y} \left(2n + 1 + \frac{1}{2}\right); ma\right) \right\}. \end{aligned} \quad (48)$$

In the continuum limit $N_y \rightarrow \infty$, together with the $m \rightarrow \infty$ limit fixing ma finite, the summation can be evaluated as

$$\begin{aligned} &\sum_{p_y} (-1)^{k_y} \theta(p_y; ma) \\ &= \sum_{n=0}^{N_y-1} \left\{ -\theta'\left(\frac{\pi}{aN_y} (2n+1); ma\right) \frac{\pi}{aN_y} + O\left(\frac{1}{N_y^2}\right) \right\} \\ &\xrightarrow{N_y \rightarrow \infty} -\frac{1}{2} \int_0^{2\pi/a} \theta'(p_y; ma) dp_y \\ &= \frac{\theta(0; ma) - \theta(2\pi/a; ma)}{2}. \end{aligned} \quad (49)$$

When $m > 0$, $\theta(0; ma) - \theta(2\pi/a; ma)$ makes no phase contribution because $\theta(0; ma) = \theta(2\pi/a; ma) = 0$. On the other hand, for $m < 0$, the $e^{i\theta(p_y; ma)}$ moves clockwise on the complex plane around the origin from $e^{i\theta(0; ma)} = -1$ to $e^{i\theta(2\pi/a; ma)} = -1$, and hence the phase difference is $\theta(0; ma) - \theta(2\pi/a; ma) = 2\pi$. Thus, we can neglect the $p_x = \pi/a$ case, as well as the positively massive fermion phase.

Summarizing the above result, in the limit $a \rightarrow 0$, $m \rightarrow \infty$, and $N_{x/y} \rightarrow \infty$, with $N_{x/y}a$ and ma fixed, we have

$$Z(m; P\pm) \propto \begin{cases} 1 & \text{if } (ma > 0), \\ \exp(\mp i \frac{\pi}{2}) = \mp i & \text{if } (-2 < ma < 0), \end{cases} \quad (50)$$

$$Z(m; AP\pm) \propto 1 \quad \text{if } (-2 < ma), \quad (51)$$

and the corresponding lattice ABK invariant $\beta^{\text{latt}} = \frac{4}{\pi i} \arg \frac{Z(m<0)}{Z(|m|)}$ becomes

$$\beta^{\text{latt}}(m \rightarrow \infty, a \rightarrow 0; P\pm) = \mp 2, \quad (52)$$

$$\beta^{\text{latt}}(m \rightarrow \infty, a \rightarrow 0; A\pm) = 0. \quad (53)$$

This result is consistent with the continuum theory.

TABLE I. Numerical evaluation of the lattice ABK invariant on the Klein bottle $\beta^{\text{latt}}(P+)$ at $|ma| = 1$ and its error from the continuum value $\beta = -2$.

$N_x \times N_y$	$\beta^{\text{latt}}(m, a; P+)$	Error $ \beta^{\text{latt}}(P+) - \beta(P+) $
10×10	$-1.99751\dots$	2.49×10^{-3}
20×20	$-1.99999757\dots$	2.43×10^{-6}
30×30	$-1.9999999762\dots$	2.37×10^{-9}

In Table I, we give numerical values of $\beta^{\text{latt}}(m, a; P+)$ with various lattice spacings. We can see that the value quickly converges to the known continuum results, with errors that are exponentially suppressed as $N_{x/y}$ increases.

V. NUMERICAL ANALYSIS ON GENERAL MANIFOLDS

In two-dimensional lattice gauge theory, it is numerically feasible to directly evaluate the Pfaffian⁸ of the Wilson-Dirac operator with a reasonable lattice size $N < 30$. Setting $m = -1/a$ on isotropic lattices with $N_x = N_y = N$ in the range $4 \leq N \leq 24$, we compute the ratio of the Wilson-Dirac Pfaffian and that of the Pauli-Villars contribution, from which the ABK invariant β^{latt} is extracted. Fixing the physical volume size $L = Na$, increasing N corresponds to approaching the continuum limit $a \rightarrow 0$ together with the large mass limit $m = 1/a \rightarrow \infty$, at the same time.

Using the setup for the boundary condition described in Sec. III, we compute the ABK invariant on various manifolds with various Pin^- structures.

First, we consider the following closed manifolds: a torus (T^2 ; PP, AP, PA, AA), Klein bottle (KB; $P\pm, A\pm$), real projective plane ($\mathbb{RP}^2$; $\pm \mp$), and triple connected sum of $\mathbb{RP}^2$ or $\mathcal{N}^3\mathbb{RP}^2$. For $\mathcal{N}^3\mathbb{RP}^2$, it is constructed by introducing a cross-cap given by a twisted gluing explained in Sec. III D along a horizontal line segment of the length $L/2$ at the center of the torus. Together with four choices of the boundary conditions in the x and y directions and the sign choice in Eq. (30), there exist eight different Pin^- structures. Figure 6 represents our numerical results for the various manifolds above. It is obvious from the plot that the values of β^{latt} are quantized into eight discrete values, which becomes stable already at $N \sim 10$. All the obtained results at large N are consistent with the known values in the continuum theory. It is remarkable that the data for $\mathbb{RP}^2$ and $\mathcal{N}^3\mathbb{RP}^2$ show correct values even though the curvature on them is almost singular or localized in some small-scale regions of the size a .

For $\mathbb{RP}^2$, we also try the ‘‘incorrect’’ boundary conditions $(++)$ and $(--)$ in Eq. (25). We find that they give very

⁸We use the Julia package `SkewLinearAlgebra.jl` for our numerical computation in this section.

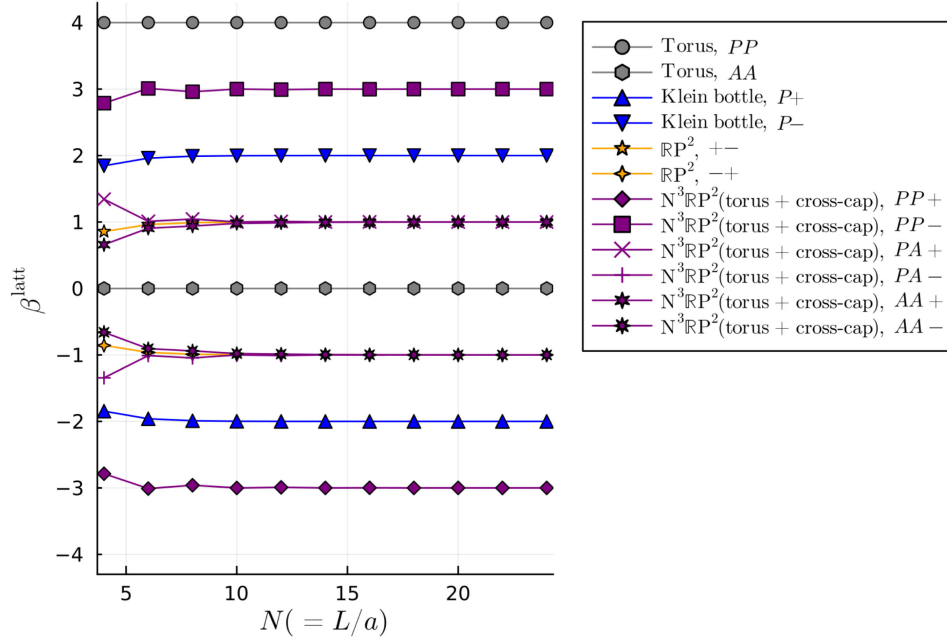


FIG. 6. The lattice ABK invariant β^{latt} is plotted as a function of $N = 1/a$ for various closed manifolds.

small values $|\beta^{\text{latt}}(\mathbb{RP}^2, ++)|, |\beta^{\text{latt}}(\mathbb{RP}^2, --)| < 10^{-9}$, which do not match any values of the ABK invariant on $\mathbb{RP}^2$ in the continuum theory. As discussed in Sec. III C these cases would contain singularities that cannot be smoothed in the continuum limit, while the correct $(+-)$ and $(-+)$ cases seem to be successful in reproducing strong but finite curvatures on the two plaquettes at the corners.

Next, we introduce the domain-wall mass term. If our conjecture in Sec. III E is correct, the Pfaffian of the

domain-wall fermion Dirac operator gives the ABK invariant of the negative mass region with boundaries located at the domain wall. Here we examine this on Möbius strips, for which the continuum value is $\beta = \pm 1$, with two different setups of the domain walls. One is to put two straight domain walls on a Klein bottle [given by Eq. (33)] where the negative mass region forms a strip with width $L/2$ (strip on KB; $P\pm, A\pm$). Another setup is to put a circular domain wall with radius $3L/8$ on an $\mathbb{RP}^2$ [given by

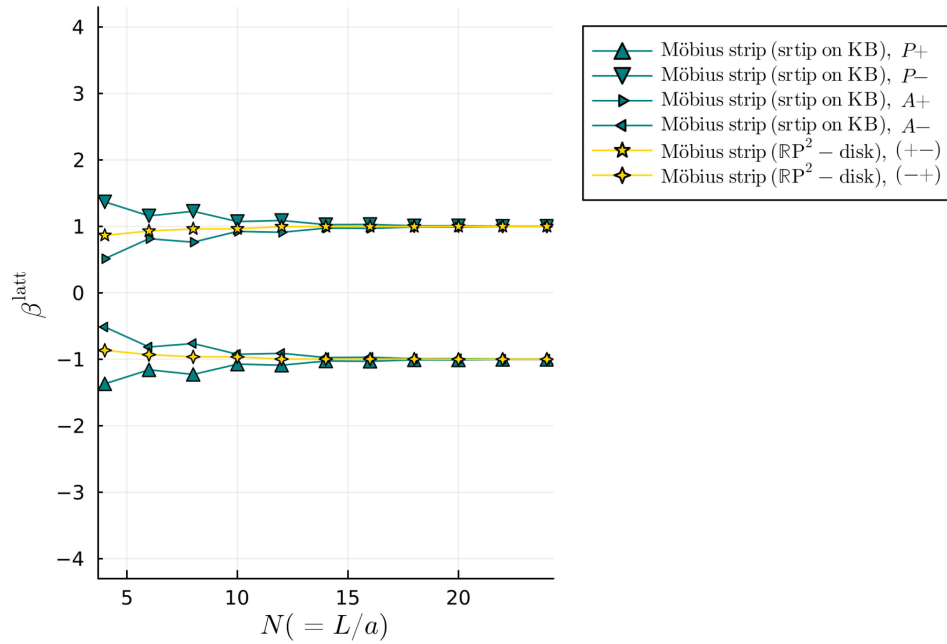


FIG. 7. The lattice ABK invariant β^{latt} is plotted as a function of $N = 1/a$ for various setups of a Möbius strip.

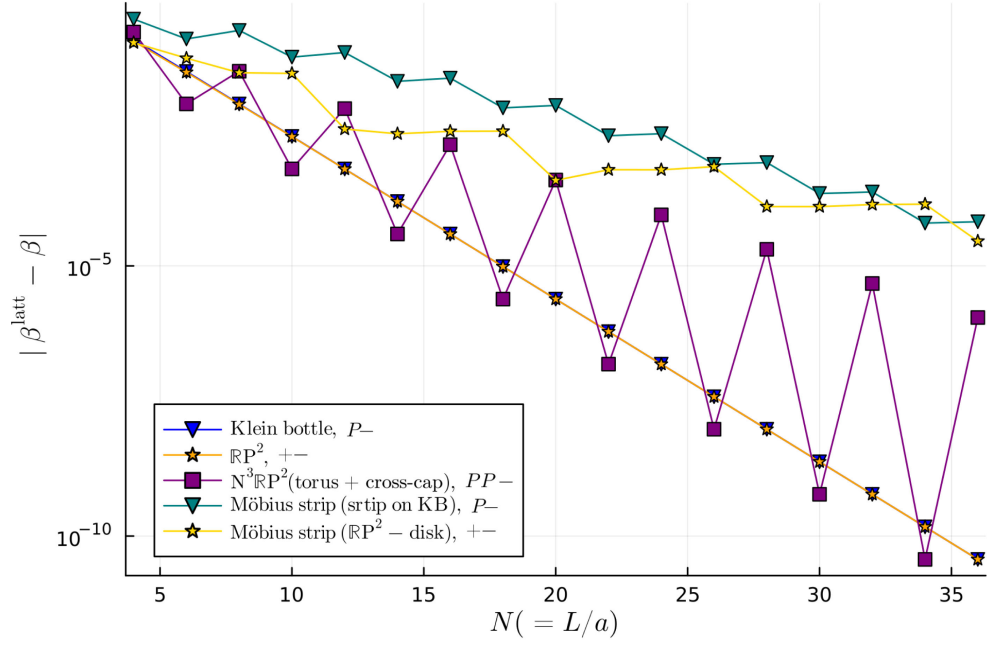


FIG. 8. Deviation of the lattice ABK invariant from the continuum value $|\beta^{\text{latt}} - \beta|$ is plotted as a function of $N = 1/a$ for various lattice setups.

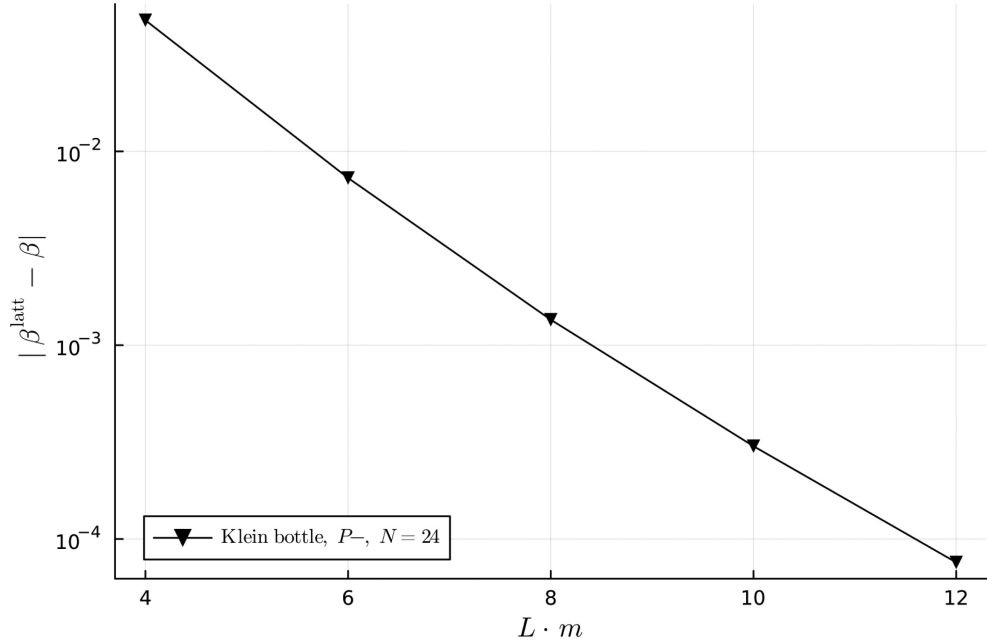


FIG. 9. Deviation of the lattice ABK invariant from the continuum value $|\beta^{\text{latt}} - \beta|$ for the Klein bottle case is plotted as a function of mL at a fixed value of $N = 1/a = 24$.

Eq. (34)], where the negative mass region is assigned to out of the domain wall ($\mathbb{RP}^2 - \text{disk}; \pm \mp$). As Fig. 7 shows, our numerical results all converge to the expected values ± 1 in the continuum theory where two distinct Pin[−] structures are known. Note that the above two setups have completely different shapes although they share the same

topology. Our observation supports that our formulation of the ABK invariant β^{latt} is a topological invariant, being insensitive to details of the lattice geometries.

Figure 8 shows deviation of the lattice ABK invariants from the corresponding continuum theory values in a logarithmic plot. Although some of them show oscillating

behaviors, they all show an exponential decay to zero, which confirms that the lattice discretization systematics in our formulation of β^{latt} is well under control.

In order to see the finite fermion mass corrections, we also investigate the mass dependence of β^{latt} with a fixed $a = 1/24$. As Fig. 9 shows, the deviation of β^{latt} from its continuum value in the case of the Klein bottle exponentially decreases as a function of mL .

VI. SUMMARY AND DISCUSSION

We have proposed a lattice formulation of the ABK invariant β^{latt} using the Pfaffian of the Wilson-Dirac operator that is a finite product of complex numbers. We have successfully equipped various nontrivial topology with nontrivial Pin^- structures of Majorana fermions by introducing twisted boundary conditions that reverse the orientation, as well as introducing cross-caps with twisted link variables. Moreover, our formulation with the domain-wall mass term enables us to compute the ABK invariant on open manifolds without introducing the nonlocal APS boundary conditions.

On two-dimensional flat lattices, we have numerically verified that β^{latt} on a torus, Klein bottle, $\mathbb{RP}^2$, $\mathcal{N}^3\mathbb{RP}^2$, and two types of Möbius strips is quantized to the $\mathbb{Z}_8$ values all consistent with those in continuum theory. It is remarkable that the convergence to the continuum limit is exponential with respect to the lattice spacing a , rather than linear. The underlying reason deserves further investigation. It is also notable that $\mathbb{RP}^2$ and $\mathcal{N}^3\mathbb{RP}^2$ have nontrivial curvatures in continuum theory. It seems that the singular plaquettes at the corners of our lattice have managed to represent the nonflat nature of these manifolds. Moreover, the two different shapes of the Möbius strips have shared the same β^{latt} , which supports that our formulation correctly captures the topology of the target manifold in the continuum limit, which is insensitive to details of the lattice geometry.

From the viewpoint of the lattice Dirac operator spectrum, it is rather nontrivial that the total complex phase of the lattice eigenvalues (including those of doublers) converges to a quantized value in $\mathbb{Z}_8$. For two cases that have the translational symmetries, we have analytically identified the Fourier modes that contribute to β^{latt} . On a torus, the value is essentially determined by the zero modes, while on a Klein bottle, $p_x = 0$ but all $p_y \neq 0$ modes contribute. For the latter case, we have explicitly shown that the summation of the complex phases forms a good approximation of an integral that ends up with 2π .

From the above observations, our lattice formulation gives a well-defined regularization of the ABK invariant with the discretization systematics well under control. We would like to emphasize that our formulation employs the Wilson-Dirac operator, without relying on any lattice chiral symmetry and the associated index theorem.

Finally, we would like to comment on possible applications of this work. The ABK invariant is a special version of the η invariant, and our lattice formulation in this work would naturally apply to general types of the η invariants, such as the $\mathbb{Z}_{16}$ -valued invariant that appears in the four-dimensional Majorana fermion theory with Pin^+ structures, although there may be technical difficulties due to numerical costs. In this work, we have focused on free Majorana fermion systems. It would be interesting to investigate what role β^{latt} plays in the interacting theories, in particular, when the number of fermions is eight, with which the anomaly cancels. When there is a domain wall, like the Möbius strips we have studied in this work, there would be a non-trivial bulk-edge anomaly cancellation or anomaly inflow. Our lattice formulation would offer a nonperturbative tool to explore this issue.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the author upon reasonable request.

APPENDIX: FERMIONS ON A KLEIN BOTTLE

A Klein bottle of size $L_x \times L_y$ can be constructed as a quotient of its double cover, a torus of size $L_x \times 2L_y$ by an equivalence relation $(x, y + L_y) \sim (L_x - x, y)$. The Majorana fermion field $\psi_{\text{KB}}(x, y)$ defined on the Klein bottle can be obtained by projecting the field $\psi_T(x, y)$ defined on the torus to the quotient. For simplicity, we consider this program in continuum theory but it is straightforward to extend the discussion to the lattice gauge theory.

Let us start with imposing the boundary conditions on $\psi_T(x, y)$. We take the antiperiodic condition in the y direction,⁹ while both periodic (P) and antiperiodic (A) conditions are allowed in the x direction,

$$\begin{aligned}\psi_T(x, y + 2L_y) &= -\psi_T(x, y), \\ \psi_T(x + L_x, y) &= s_x \psi_T(x, y),\end{aligned}\tag{A1}$$

where the sign s_x is $+$ for P and $-$ for A .

⁹With periodic boundary conditions in the y direction, it is difficult to maintain Pin^- structures.

Next we introduce a glide reflection transformation by

$$G_x \psi_T(x, y) = \gamma^1 \tilde{\gamma} \psi_T(L_x - x, y - L_y), \quad (\text{A2})$$

which satisfies $G_x^2 = 1$ and the free Majorana fermion action is invariant under G_x , since G_x commutes with $(D + m)$. In fact, we can define the fermion fields on the Klein bottle by the projection of $\psi_T(x, y)$ onto the $G_x = s (= \pm 1)$ subspace,

$$\psi_{\text{KB}}^s(x, y) := \frac{G_x + s}{2} \psi_T(x, y). \quad (\text{A3})$$

It is not difficult to confirm that $\psi_{\text{KB}}^s(x, y)$ satisfy the required boundary conditions on the Klein bottle,

$$\psi_{\text{KB}}^s(x, L_y) = s R_x \psi_{\text{KB}}^s(x, 0) = s \gamma^1 \tilde{\gamma} \psi_{\text{KB}}^s(L_x - x, 0), \quad (\text{A4})$$

$$\psi_{\text{KB}}^s(L_x, y) = s_x \psi_{\text{KB}}^s(0, y), \quad (\text{A5})$$

and the fundamental region for the quotient by G_x can be taken as half of the torus in the range $0 \leq y < L_y$. In particular, if $\psi_T(x, y)$ is an eigenfunction of $(D + m)$ (or more general operators that commute with G_x) then the projected $\psi_{\text{KB}}^s(x, y)$ is also an eigenfunction of $(D + m)$ with the same eigenvalue.

It is then not difficult to perform the Fourier transformation of $\psi_{\text{KB}}^s(x, y)$ from that of $\psi_T(x, y)$ to obtain the eigenvalue decomposition of $\text{Pf}[C(D + m)]$ as explicitly presented in Sec. IV B.

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