

Multiplicity dependence of quarkonia production at ultrahigh energies

E. Levin^{*}

*Department of Particle Physics, School of Physics and Astronomy,
Raymond and Beverly Sackler Faculty of Exact Science, Tel Aviv University, Tel Aviv, 69978, Israel*



(Received 5 December 2025; accepted 22 January 2026; published 27 February 2026)

In this paper, we propose an approach in high-energy QCD, which allows us to calculate the inclusive quarkonia production at ultrahigh energies. This approach is based on t -channel unitarity and on the expressions for dipole densities from the procedure of summing of large Pomeron loops we have developed in our previous papers. In the framework of this approach, we discuss the dependence of quarkonia production on the multiplicity of the accompanying hadrons. In this paper, we used the three gluon- fusion mechanism for quarkonia production, without assuming the multiplicity dependence of the saturation scale. We found the multiplicity distribution of the produced gluon with $(\mathcal{P}_n^{J/\Psi})$ and without $(\mathcal{P}_n^{\text{gluon}})$ the quarkonia production. It turns out that $\frac{\mathcal{P}_n^{J/\Psi}}{\mathcal{P}_n^{\text{gluon}}} \propto n^2$, as has been expected and discussed in corresponding publications.

DOI: 10.1103/wskj-72dw

I. INTRODUCTION

In this paper, we continue our study of the three gluon fusion mechanism for J/Ψ production [1–5]. This mechanism is dominant for the interaction with nuclei [6–10], and it has been demonstrated in Ref. [3] that it makes a substantial contribution to hadron–hadron collisions.

The paper has three goals. The first is to develop an approach in high-energy QCD to calculate the inclusive cross section using the t -channel unitarity constraints and the expression for the dipole densities. The second is to take into account the general structure of the parton cascade. In particular, it accounts for the contribution of production of many Pomeron ladders in the parton cascade (see Fig. 1), which cancel each other in the inclusive production of J/Ψ . For example, production of gluons from extra Pomeron ladders in Fig. 1(a) does not contribute to the inclusive production which can be estimated using the diagram of Fig. 2(b). Such contributions have been neglected in all papers on the subject; see for example [4,5]. The third goal is to find the multiplicity distribution for J/Ψ production at ultra high energies in the framework of high-energy QCD; see Ref. [11] for a general review. Effective theory of high-energy QCD exists in two different formulations: the

color glass condensate (CGC) saturation approach [12–17], and the Balitsky, Fadin, Kuraev, and Lipatov (BFKL) Pomeron calculus [18–32]. In this paper, we restrict ourselves to the BFKL Pomeron calculus, which allows us to apply the Abramovsky, Gribov, and Kancheli (AGK) cutting rules [33] for finding the multiplicity of the produced gluons in the final state. Unfortunately, as far as we know, these observables are out of the reach for the CGC approach.

In our recent papers [35–37], we summed the large Pomeron loops at ultra high energy using the Mueller, Patel, Salam and Iancu (MPSI) [38] approach. It turns out that all three types of scattering, dilute-dilute dipole system (dipole–dipole scattering), dilute-dense dipole system (dipole–nucleus scattering) and dense-dense dipole system (ion–ion scattering), show the same energy dependence due to contributions of large BFKL Pomeron loops. In this paper, we concentrate our effort on the multiplicity distribution of all these processes at ultra high energies.

The recent experiments by ALICE [39–43] and STAR [44,45] have shown that the cross sections for J/Ψ production strongly depend on the multiplicity of accompanying hadrons. These data have stimulated theoretical discussions on the origin of such dependence; see Refs. [3–5,46–48].

In the next section of this paper, we will discuss the structure of the BFK parton cascade in the simple model for the BFKL pomeron calculus: the zero transverse dimension. Zero transverse dimension toy models can be viewed as a realization of the Pomeron calculus or, more generally, of Reggeon Field Theory (RFT). Over the

^{*}Contact author: leving@tauex.tau.ac.il

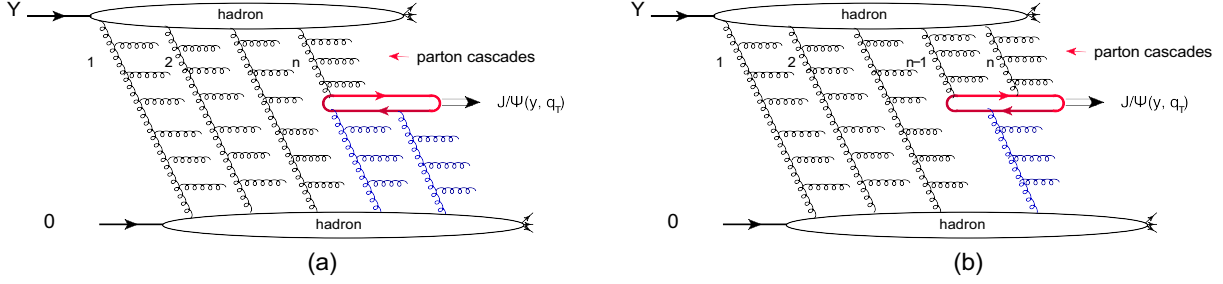


FIG. 1. The production of quarkonia in the three gluon fusion mechanism from n parton cascades. In (a) J/Ψ decays in two parton cascades, while in (b) it is produced from two parton cascades.

years, they have been intensively used to model high-energy collisions in QCD. These models [49–64] encode various fundamental features of QCD such as unitarity, but are much simpler than the latter and frequently solvable analytically. Hence they provide a valuable playground to gain intuition about the dynamics of real QCD. We believe that by using these models, we clarify all aspects of the multiplicity distribution for J/Ψ production.

In the rest of the paper, we discuss the calculation of the inclusive production in the framework of high-energy QCD. Our approach is based on the t -channel unitarity and on the expression for the dipole densities which we suggested in our previous papers [35–37]. We wish to emphasize that this is the first calculation in the framework of such an approach. Its advantage is in the clear understanding of the typical scales that govern this process. In this paper we restrict ourselves by calculating the cross sections integrated over transverse momenta.

In Sec. III, we review this approach for the scattering amplitude. In Secs. IV and V, we propose how to apply the t -channel unitarity to calculation of the inclusive production. We calculate the multiplicity distribution for the produced gluons using the AGK [33] cutting rules. In Sec. VI, we develop the approach for calculation of the multiplicity distribution for J/Ψ production in three gluon fusion mechanism. In the Conclusion, we review our results.

II. MULTIPLICITY DISTRIBUTION FOR J/Ψ PRODUCTION IN THE BFKL CASCADE IN ZERO TRANSVERSE DIMENSION

In the parton model [65–67], all partons have average transverse momentum which does not depend on energy. Therefore, we can obtain the parton model from the QCD cascade assuming that the unknown confinement of gluons leads to the QCD cascade for a dipole of fixed size. In this case, the QCD cascade equation [see equations in Ref. [28]] takes the following simple form:

$$\frac{dP_n(Y)}{dY} = -\Delta n P_n(Y) + (n-1)\Delta P_{n-1}(Y), \quad (2.1)$$

where $P_n(Y)$ denotes the probability to find n dipoles (of a fixed size in our model) at rapidity Y , and Δ is the intercept of the BFKL Pomeron.

We can introduce the generating function:

$$Z(Y, u) = \sum_n P_n(Y) u^n, \quad (2.2)$$

where u are arbitrary numbers. Using this generating function, we can rewrite Eq. (2.1) in the following form for the generating function:

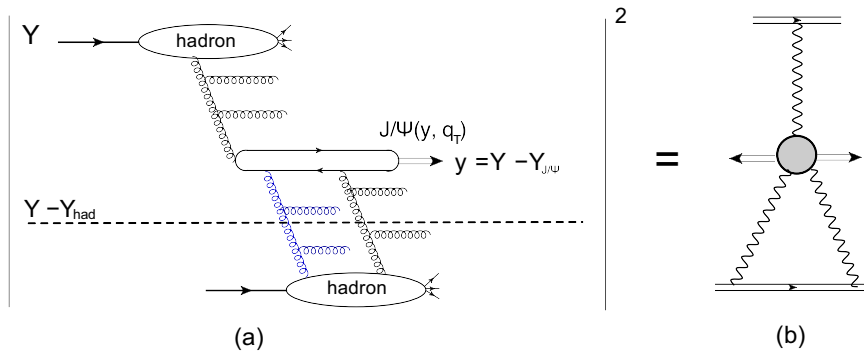


FIG. 2. (a) The three gluon fusion mechanism of J/Ψ production. (b) The Mueller diagram [34] for J/Ψ production, which illustrates the interrelation between three gluon fusion and the triple BFKL Pomeron interaction. The wavy lines describe the BFKL Pomerons, while the helical curves represent gluons.

$$\frac{\partial Z(Y, u)}{\partial Y} = -\Delta u(1 - u) \frac{\partial Z(Y, u)}{\partial u}. \quad (2.3)$$

At the initial rapidity $Y = 0$, we have only one dipole, so $P_1(Y = 0) = 1$ and $P_{n>1} = 0$ (so the state is the only one dipole); at $u = 1$, $Z(Y, u = 1) = \sum_n P_n = 1$. These two properties determine the initial and boundary conditions for the generating function

$$Z(Y = 0, u) = u; \quad Z(Y, u = 1) = 1. \quad (2.4)$$

The general solution to Eq. (2.3) is an arbitrary function ($Z(z)$) of the new variable: $z = \Delta Y + f(u)$, with $f(u)$ from the following equation:

$$1 = -u(1 - u)f'_u(u) \quad f(u) = \ln\left(\frac{u-1}{u}\right) + C_1. \quad (2.5)$$

The form of arbitrary function stems from the initial condition of Eq. (2.4):

$$Z(z(Y = 0)) = u; \quad (2.6)$$

Since $u = 1/(1 - e^z)$, we obtain that

$$Z_1(Y, u) = \frac{ue^{-\Delta Y}}{1 + u(e^{-\Delta Y} - 1)} = e^{-\Delta Y} \sum_{n=1}^{\infty} u^n (1 - e^{-\Delta Y})^{n-1}. \quad (2.7)$$

Note that $Z(Y, u = 1) = 1$, as it should be from Eq. (2.4). Recall that Z_1 describes the dipole cascade of the fast dipole.

From Eq. (2.7), we obtain

$$P_n(Y) = e^{-\Delta Y} (1 - e^{-\Delta Y})^{n-1} = \frac{1}{N(Y)} \left(1 - \frac{1}{N(Y)}\right)^{n-1} \quad (2.8)$$

at rapidity Y . In Eq. (2.9) $N(Y) = \exp(\Delta Y)$ is the average number of dipoles at rapidity Y .

Hence, from Eq. (2.9), the multiplicity distribution of dipoles at rapidity $Y_{J/\Psi}$ at which J/Ψ is produced is equal to

$$P_m(Y_{J/\Psi}) = \frac{1}{N(Y_{J/\Psi})} \left(1 - \frac{1}{N(Y_{J/\Psi})}\right)^{m-1}. \quad (2.9)$$

Each of this m dipoles produces its own cascade which can be found from Eq. (2.3) with the initial condition

$$Z(Y = Y_{J/\Psi}, u) = u^m \frac{1}{N(Y_{J/\Psi})} \left(1 - \frac{1}{N(Y_{J/\Psi})}\right)^{m-1}. \quad (2.10)$$

The solution for such initial condition has been found in Ref. [68] and it takes the form for dipoles at rapidity Y_{had}^1 :

$$\begin{aligned} Z_m(Y = Y_{\text{had}} - Y_{J/\Psi}, u) &= \sum_{n=m}^{\infty} u^n P_n^m(\delta Y = Y_{\text{had}} - Y_{J/\Psi}, Y_{J/\Psi}) \\ &= \frac{1}{N(Y_{J/\Psi})} \left(1 - \frac{1}{N(Y_{J/\Psi})}\right)^{m-1} \\ &\quad \times (Z_1(\delta Y))^m \\ \text{with } P_n^m(\delta Y, Y_{J/\Psi}) &= \frac{1}{N(Y_{J/\Psi})} \left(1 - \frac{1}{N(Y_{J/\Psi})}\right)^{m-1} \\ &\quad \times \frac{(n-1)!}{(m-1)!(n-m)!} \left(\frac{1}{N(\delta Y)}\right)^m \\ &\quad \times \left(1 - \frac{1}{N(\delta Y)}\right)^{n-m}. \end{aligned} \quad (2.11)$$

It is easy to see that

$$\sum_{m=1}^n P_n^m(\delta Y, Y_{J/\Psi}) = \frac{1}{N(Y_{\text{had}})} \left(1 - \frac{1}{N(Y_{\text{had}})}\right)^{n-1}. \quad (2.12)$$

In other words, summing over m , we obtain $Z_1(Y_{\text{had}})$ for dipole cascade.

For inclusive production of J/Ψ , we need to take into account: (i) that J/Ψ can be produced by any of m dipoles at $Y - Y_{J/\Psi}$, and (ii) the three gluon fusion mechanism of J/Ψ production [3–5] (see Fig. 2). The generating function, which includes the three gluon fusion mechanism, has the form

$$\begin{aligned} Z_{J/\Psi}(\delta Y, Y_{J/\Psi}, u) &= \sum_{m=1}^{\infty} P_m(Y_{J/\Psi}) \underbrace{(Z_1(\delta Y, u))^{m+1}}_{\text{three gluons fusion}} \\ &= \frac{\gamma(1 - \gamma)^2}{(1 + \gamma(N(\delta Y) - 1))(1 + \gamma(N(Y) - 1))} \\ &= Z_1(N(Y, u)) Z_1(N(\delta Y, u)), \end{aligned} \quad (2.13)$$

where $\gamma = 1 - u$.

For finding the multiplicity distribution of produced dipoles, we need to apply to Eq. (2.13) the AGK cutting rules [33]. For Z_1 , it has been done in Refs. [69,70]. It has been shown in these references that Z_1 generates the following inelastic cross sections for production of n cut Pomerons with average multiplicity $n_{IP} = n\Delta Y$:

¹ Y_{had} is the rapidity of hadrons that are measured.

$$\sigma_n(N(Y)) = \frac{(2\gamma N(Y))^n}{(1 + 2\gamma(N(Y) - 1))^{n-1}}. \quad (2.14)$$

Using Eq. (2.14), we can write the inelastic cross section for generating function of Eq. (2.13)²:

$$\sigma_n^{J/\Psi}(N(Y), N(\delta Y)) = \sum_{k=1}^{n-1} \sigma_k(N(Y)) \sigma_{n-k}(N(\delta Y)). \quad (2.15)$$

For calculating the inclusive cross section for J/Ψ production at fixed multiplicity n ($\frac{\sigma_n^{J/\Psi}}{dY_{J/\Psi}}$) by any of m dipoles at $Y - Y_{J/\Psi}$, we need

$$\frac{d\sigma_n^{J/\Psi}(N(Y), N(\delta Y))}{dY_{J/\Psi}} = \sum_{k=1}^{n-1} k \sigma_k(N(Y)) \sigma_{n-k}(N(\delta Y)). \quad (2.16)$$

Summing over all possible n , we obtain the inclusive cross section for J/Ψ production, which is equal to

$$\frac{d\sigma_{\text{incl}}^{J/\Psi}(N(Y), N(\delta Y))}{dY_{J/\Psi}} = \frac{4\gamma^2 N(Y) N(\delta Y)}{(1 - \gamma)^3 (1 + 2\gamma(N(\delta Y) - 1))}. \quad (2.17)$$

For small γ , Eq. (2.17) leads to $-\gamma^2 N(\delta Y) N(Y)$ which corresponds to the diagram of Fig. 2(b).

The analytic form of Eq. (2.16) is rather cumbersome. We write here the expression at small γ :

$$\begin{aligned} \frac{d\sigma_n^{J/\Psi}(N(Y), N(\delta Y))}{dY_{J/\Psi}} &= \frac{1}{(N(Y) - N(\delta Y))^2} \left\{ N(Y) \left(\left(\frac{2\gamma N(\delta Y)}{1 + 2\gamma N(\delta Y)} \right)^n - \left(\frac{2\gamma N(Y)}{1 + 2\gamma N(Y)} \right)^n \right) \right. \\ &\quad \left. + n \frac{(N(Y) - N(\delta Y))}{1 + 2\gamma N(\delta Y)} \left(\frac{2\gamma N(Y)}{1 + 2\gamma N(Y)} \right)^n \right\}. \end{aligned} \quad (2.18)$$

We can illustrate the main features of this equation, writing it for $N(Y) = N(\delta Y)^3$:

$$\frac{d\sigma_n^{J/\Psi}(N(Y), N(\delta Y))}{dY_{J/\Psi}} = \frac{n(n-1)}{2(1 + 2\gamma N(\delta Y))} \sigma_n(\delta Y). \quad (2.19)$$

For the ratio $\frac{d\sigma_n^{J/\Psi}(N(Y), N(\delta Y))}{dY_{J/\Psi}} / \frac{d\sigma_{\text{incl}}^{J/\Psi}(N(Y), N(\delta Y))}{dY_{J/\Psi}}$, we have

$$\frac{d\sigma_n^{J/\Psi}(N(Y), N(\delta Y))}{dY_{J/\Psi}} / \frac{d\sigma_{\text{incl}}^{J/\Psi}(N(Y), N(\delta Y))}{dY_{J/\Psi}} = \frac{n(n-1)}{2(2\gamma N(Y))^2} \sigma_n(\delta Y). \quad (2.20)$$

This result is the same as discussed in the previous publications [1–5]. Therefore, it indicates that the corrections from the production of multi-ladders in Fig. 1 leads to small corrections which can be treated perturbatively.

III. A RECAP: SUMMING LARGE POMERON LOOPS IN THE SATURATION REGION OF DIPOLE–DIPOLE SCATTERING

In our recent paper [35], we found the contribution to the dipole–dipole scattering amplitude of the sum of large Pomeron loops. Here, we give a brief review of this approach since we are going to use it below for finding the multiplicity distributions of produced gluons. Our approach is based on the t -channel unitarity, which has been written in the convenient

²In the one-dimensional model, σ_n are dimensionless. The natural generalization for QCD is $\sigma_n^{J/\Psi}(N(Y), N(\delta Y)) = \int d^2b \sum_{k=1}^{n-1} \sigma_k(N(Y); b) \sigma_{n-k}(N(\delta Y); b)$.

³Actually, this is a measured case when J/Ψ produced in the same rapidity window where the multiplicity distribution of produced dipoles is measured.

for the BFKL Pomeron calculus in Refs. [11,27,28,38,71–73] (see Fig. 3). The analytic expression for arbitrary Y_0 takes the form [27,28,72,73]:

$$A(Y, r, r'; \mathbf{b}) = \sum_{n=1}^{\infty} (-1)^{n+1} n! \int \prod_i^n d^2 r_i d^2 r'_i d^2 b'_i \times \int d^2 \delta b_i \gamma^{BA}(r_i, r'_i, \mathbf{b}_i - \mathbf{b}'_i \equiv \delta \mathbf{b}_i) \rho_n^P(Y - Y_0, r, b, \{\mathbf{r}_i, \mathbf{b}_i\}) \rho_n^T(Y_0, r', b', \{\mathbf{r}'_i, \mathbf{b}'_i\}), \quad (3.1)$$

where γ^{BA} is the scattering amplitude of two dipoles in the Born approximation of perturbative QCD. The dipole densities, $\rho_i^P(Y - Y_0, r, b, \{\mathbf{r}_i, \mathbf{b}_i\})$ for projectile and $\rho_i^T(Y_0, r', b', \{\mathbf{r}'_i, \mathbf{b}'_i\})$ for target, have been introduced in Ref. [27] as follows:

$$\rho_n(Y - Y_0, r, b, r_1, b_1, \dots, r_n, b_n) = \frac{1}{n!} \prod_{i=1}^n \frac{\delta}{\delta u_i} Z(Y - Y_0; [u])|_{u=1}, \quad (3.2)$$

where the generating functional Z is

$$Z(Y, \mathbf{r}, \mathbf{b}; [u_i]) = \sum_{n=1}^{\infty} \int P_n(Y, \mathbf{r}, \mathbf{b}; \{\mathbf{r}_i, \mathbf{b}_i\}) \prod_{i=1}^n u(\mathbf{r}_i, \mathbf{b}_i) d^2 r_i d^2 b_i, \quad (3.3)$$

where $u(\mathbf{r}_i, \mathbf{b}_i) \equiv u_i$ is an arbitrary function and P_n is the probability to have n dipoles with the given kinematics. The initial and boundary conditions for the BFKL cascade stemming from one dipole has the following form for the functional Z :

$$Z(Y=0, \mathbf{r}, \mathbf{b}; [u_i]) = u(\mathbf{r}, \mathbf{b}); \quad Z(Y, r, [u_i=1]) = 1; \quad (3.4a)$$

$$\rho_1(Y=0, r, b, r_1, b_1) = \delta^{(2)}(\mathbf{r} - \mathbf{r}_1) \delta^{(2)}(\mathbf{b} - \mathbf{b}_1);$$

$$\rho_n(Y=0, \mathbf{r}, \mathbf{b}; [r_i, b_i]) = 0 \quad \text{at } n \geq 2; \quad (3.4b)$$

In Eq. (3.1), $\mathbf{b}_i = \mathbf{b} - \mathbf{b}'_i$. Comparing Eqs. (3.2) and (3.3), one can see that $\rho_n = M_n/n!$ where M_n is the factorial moment for the scattering amplitude. Factorial

moments are the natural observables for the BFKL Pomeron calculus [see Ref. [71]].

The main ingredient of Eq. (3.1) is the dipole densities, which have been found in Ref. [35]⁴:

$$\rho_n^P(Y - Y_0; r, b, \{\mathbf{r}_i, \mathbf{b}_i\}) = C \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\omega}{2\pi i} e^{\frac{\bar{\gamma} \kappa (\omega - 1)^2}{2}} \frac{1}{n!} \frac{\Gamma(\omega + n)}{\Gamma(\omega)} \prod_{i=1}^n \rho_1(z_i), \quad (3.5)$$

where

$$z_i = \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} (Y - Y_0) + \xi_{r, r_i}; \quad \xi_{r, r_i} = \ln \left(\frac{r^2 r_i^2}{(\mathbf{b} + \frac{1}{2}(\mathbf{r} - \mathbf{r}_i))^2 (\mathbf{b} - \frac{1}{2}(\mathbf{r} - \mathbf{r}_i))^2} \right). \quad (3.6)$$

Here, $\rho_1(z_i)$ satisfies the linear BFKL equation which has a simple solution in the saturation region [20,74]:

$$N_{IP}(z_i) = N_0 (r_i^2 Q_s^2(Y - Y_0, b))^{\bar{\gamma}} = N_0 e^{\bar{\gamma} z_i} \equiv N_0 G_{IP}(z_i), \quad (3.7)$$

where $\bar{\gamma} = 1 - \gamma_{cr}$, N_0 is a constant, and z_i is defined as

$$z_i = \ln(r_i^2 Q_s^2(Y - Y_0, b)). \quad (3.8)$$

In Eq. (3.7), $G_{IP}(z_i)$ is the BFKL Pomeron Green's function.

⁴In Ref. [35], Eq. (3.5) is written with ω instead of $\omega - 1$. We will discuss this issue below in Sec. IV.

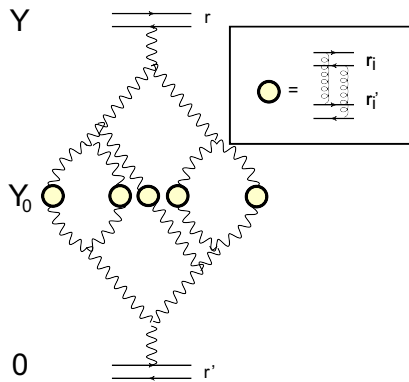


FIG. 3. Summing large Pomeron loops for dipole-dipole scattering. The wavy lines denote the BFKL Pomeron exchanges. The circles denote the amplitude γ in the Born approximation of perturbative QCD.

It can be seen from Eq. (3.8) that BK equation leads to a new dimensional scale: saturation momentum [20], which has the following Y dependence [20,74,75]:

$$Q_s^2(Y, b) = Q_s^2(Y=0, b) e^{\bar{\alpha}_S \kappa Y - \frac{3}{2\gamma_{cr}} \ln Y}, \quad (3.9)$$

where $Y=0$ is the initial value of rapidity and κ and γ_{cr} are determined by the following equations:

$$\kappa \equiv \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} = -\frac{d\chi(\gamma_{cr})}{d\gamma_{cr}}, \quad (3.10)$$

where $\chi(\gamma)$ is given by

$$\omega(\bar{\alpha}_S, \gamma) = \bar{\alpha}_S \chi(\gamma) = \bar{\alpha}_S (2\psi(1) - \psi(\gamma) - \psi(1-\gamma)). \quad (3.11)$$

Here, $\psi(z)$ is the Euler ψ function [see Ref. [76] formulae 8.36]. As one can see, Eq. (3.8) gives the physics meaning of Eq. (3.6). In Eq. (3.5),

$$\rho_1(z_i) = G_{IP}(z_i). \quad (3.12)$$

Coming back to Eq. (3.5), one can see that this equation reconciles the exact solution to the Balitsky-Kovchegov (BK) equation [77], which describes the rare fluctuation in the dipole–target scattering, with the fact that this equation sums the “fan” Pomeron diagrams.

The t -channel unitarity for the Green’s function of the BFKL Pomeron has the form [see Refs. [55,71]]

$$G_{IP}(Y, r, r'; \mathbf{b}) = \int d^2 r_1 d^2 b_1 d^2 r'_1 d^2 b'_1 G_{IP}(Y - Y_0; r, b - b_1, r_1) \gamma^{BA}(r_1, r'_1, \delta b = \mathbf{b}_1 - \mathbf{b}'_1) G_{IP}(Y_0; r', 0, b'_1, r'_1). \quad (3.13)$$

Plugging Eq. (3.5) into Eq. (3.1), we can calculate the scattering amplitude:

$$S(z) = C^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \int_{\epsilon_i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\bar{\gamma}^2 \kappa}{2}((\omega-1)^2 + (\omega_1-1)^2)} \times \frac{\Gamma(\omega+n)}{\Gamma(\omega)} \frac{\Gamma(\omega_1+n)}{\Gamma(\omega_1)} \int d^2 r_i d^2 r'_i d^2 b_i d^2 b'_i \prod_{i=1}^n G_{IP}(z_i) \gamma^{BA}(r_i, r'_i, \delta b) G_{IP}(z'_i). \quad (3.14)$$

In Eq. (3.14), z_i is given in Eq. (3.6) while $z'_i = \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}}(Y_0) + \xi_{r',r'_i}$. ξ_{r',r'_i} is equal to ξ_{r,r_i} of Eq. (3.6) where r and r_i are replaced by r' and r'_i , respectively.

Using Eq. (3.13), we reduce Eq. (3.14) to the following expression:

$$S(z') = C^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \int_{\epsilon_i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\bar{\gamma}^2 \kappa}{2}((\omega-1)^2 + (\omega_1-1)^2)} \frac{\Gamma(\omega+n)}{\Gamma(\omega)} \frac{\Gamma(\omega_1+n)}{\Gamma(\omega_1)} G_{IP}^n(z') \quad (3.15)$$

with

$$z' = \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} Y + \xi_{r,r'}. \quad (3.16)$$

$\xi_{r,r'}$ is equal to ξ_{r,r_1} of Eq. (3.6), where r and r_1 are replaced by r and r' , respectively.

Replacing $\Gamma(\omega_1+n)$ by the integrals over t , viz.:

$$\Gamma(\omega_1+n) = \int_0^\infty dt t^{\omega_1+n-1} e^{-t}, \quad (3.17)$$

we obtain

$$S(z') = C^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\bar{\gamma}^2 \kappa}{2}((\omega-1)^2 + (\omega_1-1)^2)} \int_0^\infty dt e^{-t} t^{\omega_1-1} \frac{\Gamma(\omega+n)}{\Gamma(\omega)\Gamma(\omega_1)} (t G_{IP}(z'))^n. \quad (3.18)$$

After summation over n , we have

$$S(z') = C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\bar{\gamma}^2 \kappa}{2}((\omega-1)^2 + (\omega_1-1)^2)} \frac{1}{\Gamma(\omega_1)} \int_0^\infty dt e^{-t} (1 + t G_{IP}(z'))^{-\omega} t^{\omega_1-1}, \quad (3.19a)$$

$$G_{IP}(z') \gg 1 \quad C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\tilde{\gamma}^2 \kappa}{2}((\omega-1)^2 + (\omega_1-1)^2)} \int_0^\infty dt e^{-t} \exp(-\omega \ln(G_{IP}(z'))) \frac{t^{\omega_1-\omega-1}}{\Gamma(\omega_1)}, \quad (3.19b)$$

$$= C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\tilde{\gamma}^2 \kappa}{2}((\omega-1)^2 + (\omega_1-1)^2)} \frac{\Gamma(\omega_1 - \omega)}{\Gamma(\omega_1)} \exp(-\omega \ln(G_{IP}(z'))). \quad (3.19c)$$

Closing contour of integration over ω_1 on the poles of $\Gamma(\omega - \omega_1)$, one can see that the main contribution gives the pole at $\omega = \omega_1$ while all other poles lead to the amplitude that decreases as $\exp(-\tilde{\gamma} n z')$ for the pole $\omega_1 = \omega - n$. Taking integral over ω , we obtain

$$S(z') = C'^2 \exp\left(-\frac{z'^2}{4\kappa}\right). \quad (3.20)$$

Hence, we can conclude that the sum of the large Pomeron loops leads to the S-matrix in perfect agreement with the estimates of “rare” fluctuation given in the papers of Iancu and Mueller in Ref. [38]. C' is a smooth function of z which we cannot guarantee. Note that in our case it contains the factor $1/(1 + e^{\tilde{\gamma} z})$, which could provide the matching at $z \rightarrow 0$ with the solution of Ref. [74].

IV. MULTIPLICITY DISTRIBUTION OF PRODUCED GLUONS

Equation (3.18) is suited for using the AGK [33] cutting rules for finding the multiplicity distributions of produced gluons. We recall that an application of AGK cutting rules is based on two main equations:

$$2 \operatorname{Im} G_{IP}(z) = \sigma_{\text{in}}^{\text{BFKL}}(z); \quad (4.1a)$$

$$\sigma_n(z) = \sum_j \underbrace{\sigma_j^{\text{AGK}}(z)}_{\propto (\operatorname{Im} G_{IP})^j} \underbrace{\frac{(j\tilde{\gamma}z)^n}{n!} e^{-j\tilde{\gamma}z}}_{\text{Poisson distribution}}. \quad (4.1b)$$

Equation (4.1) is the s -channel unitarity constraint for the BFKL Pomeron, which clarifies the structure of the BFKL Pomeron. $\sigma_{\text{in}}^{\text{BFKL}}(z)$ is the inelastic cross section of produced gluons with mean multiplicity $\bar{n} = \tilde{\gamma}z$, where $\tilde{\gamma} = 1 - \gamma_{\text{cr}}$ [see Eq. (3.10)]. We also know that produced gluons have the Poisson distribution with this mean multiplicity [35]. Eq. (4.1b) shows how to calculate the inelastic cross section with the given multiplicity using the AGK cutting rules. These rules, which we discuss below, allow us to find the imaginary part of the scattering amplitude that determines the cross sections, through the powers of $\operatorname{Im} G^{\text{BFKL}}(z)$.⁵

The AGK cutting rules [33] allow us to calculate the contributions of n -cut Pomerons if we know F_k : the contribution of the exchange of k -Pomerons to the cross section. They take the form

$$n \geq 1: \sigma_n^k(Y, \xi_{r,r'}) = (-1)^{n-k} \frac{k!}{(n-k)!n!} 2^k F_k(Y, \xi_{r,r'}), \quad (4.2a)$$

$$n = 0: \sigma_0^k(Y) = (-1)^k (2^k - 2) F_k(Y, \xi_{r,r'}), \quad (4.2b)$$

$$\sigma_{\text{tot}} = 2 \sum_{k=1}^{\infty} (-1)^{k+1} F_k(Y, \xi_{r,r'}). \quad (4.2c)$$

$\sigma_{\text{tot}} = 2 \operatorname{Im} A(z)$, where A is the scattering amplitude. σ_0 is the cross section of diffractive production of small numbers of gluons, which is much smaller than $\bar{n} = \tilde{\gamma}z$.

Using Eqs. (4.2a) and (4.2c), we obtain from Eq. (3.18):

$$\sigma_n^{\text{AGK}}(z') = C^2 \sum_{k=n}^{\infty} \frac{(-1)^{k-n}}{k!} \left(\frac{k!}{(k-n)!n!} \right) \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\tilde{\gamma}^2 \kappa}{2}((\omega-1)^2 + (\omega_1-1)^2)} \int_0^\infty dt e^{-t} \frac{t^{\omega_1-1}}{\Gamma(\omega)} \frac{\Gamma(\omega+k)}{\Gamma(\omega_1)} (2t G_{IP}(z'))^k \quad (4.3)$$

$$= \frac{C^2}{n!} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\tilde{\gamma}^2 \kappa}{2}((\omega-1)^2 + (\omega_1-1)^2)} \int_0^\infty dt e^{-t} \frac{t^{\omega_1-1}}{\Gamma(\omega)\Gamma(\omega_1)} \Gamma(n+\omega) \frac{(2t G_{IP}(z'))^n}{(1+2t G_{IP}(z'))^{\omega+n}}.$$

⁵In the widespread slang, this contribution is called by cut Pomeron.

Integrating over t , we have

$$\sigma_n^{\text{AGK}}(z') = \frac{C^2}{n!} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\gamma^2 \kappa}{2}((\omega-1)^2 + (\omega_1-1)^2)} \frac{\Gamma(\omega+n)\Gamma(\omega_1+n)}{\Gamma(\omega)\Gamma(\omega_1)} \frac{1}{N^{\omega_1}(z')} U\left(n+\omega, \omega_1-\omega+1, \frac{1}{N(z')}\right), \quad (4.4)$$

where $N(z') = 2G_{IP}(z')$ and $U(a, b, z)$ is Tricomi confluent hypergeometric function [see Ref. [78], formula 13.1.3]. For large $N(z')$,

$$\Gamma(\omega+n)U\left(n+\omega, \omega_1-\omega+1, \frac{1}{N(z')}\right) \xrightarrow{N(z') \gg 1} 2(\omega+n)^{-\Delta} \left(\frac{1}{N(z')}\right)^{\Delta} K_{2\Delta}\left(2\sqrt{\frac{n+\omega}{N(z')}}\right), \quad (4.5)$$

where $\Delta = \frac{1}{2}(\omega - \omega_1)$. For large n Eq. (4.5) gives

$$\Gamma(\omega+n)U\left(n+\omega, \omega_1-\omega+1, \frac{1}{N(z')}\right) \xrightarrow{n > N(z')} 2(\omega+n)^{-\Delta-1/4} \left(\frac{1}{N(z')}\right)^{-\Delta-1/4} \exp\left(-\left(2\sqrt{\frac{n+\omega}{N(z')}}\right)\right). \quad (4.6)$$

where $\Delta = \frac{1}{2}(\omega - \omega_1)$. For large n Eq. (4.6) gives

$$\Gamma(\omega+n)U\left(n+\omega, \omega_1-\omega+1, \frac{1}{N(z')}\right) \xrightarrow{n > N(z')} 2(\omega+n)^{-\Delta-1/4} \left(\frac{1}{N(z')}\right)^{-\Delta-1/4} \exp\left(-\left(2\sqrt{\frac{n+\omega}{N(z')}}\right)\right). \quad (4.7)$$

The expression for σ_n^{AGK} takes the form

$$\sigma_n^{\text{AGK}}(z') = \frac{C^2}{n!} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\gamma^2 \kappa}{2}((\omega-1)^2 + (\omega_1-1)^2)} \frac{\Gamma(\omega_1+n)}{\Gamma(\omega)\Gamma(\omega_1)} \frac{(\omega+n)^{-\Delta-1/4}}{N^{\Sigma-1/4}(z')} 2 \exp\left(-\left(2\sqrt{\frac{n+\omega}{N(z')}}\right)\right). \quad (4.8)$$

Taking the integral using the method of steepest descent, we can see that $\omega_1 \sim \ln N(z')$ is large. Hence, Eq. (4.8) for $n < \omega_1$ can be rewritten as

$$\sigma_n^{\text{AGK}}(z') = \frac{C^2}{n!} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\Sigma}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\Delta}{2\pi i} e^{\frac{\gamma^2 \kappa}{2}((\Sigma-1)^2 + \Delta^2)} \frac{(\Sigma+\Delta)^n}{\Gamma(\Sigma+\Delta)} \frac{2(n+\Sigma+\Delta)^{-\Delta-1/4}}{N^{\Sigma-1/4}(z')} \exp\left(-\left(2\sqrt{\frac{n+\Sigma+\Delta}{N(z')}}\right)\right), \quad (4.9)$$

where $\Sigma = \frac{1}{2}(\omega + \omega_1)$.

The equations for saddle points have the following forms, assuming that $\Delta^{\text{SP}} \ll \Sigma^{\text{SP}}$:

$$2\bar{\gamma}^2 \kappa (\Sigma^{\text{SP}} - 1) - \ln N(z') - \ln \Sigma^{\text{SP}} = 0, \quad (4.10a)$$

$$2\bar{\gamma}^2 \kappa \Delta^{\text{SP}} - \ln(n + \Sigma^{\text{SP}}) = 0. \quad (4.10b)$$

The solutions are

$$\Sigma^{\text{SP}} - 1 = \frac{1}{2\bar{\gamma}^2 \kappa} \left\{ \ln N(z') + \ln \left(\frac{1}{2\bar{\gamma}^2 \kappa} \ln N(z') \right) \right\}, \quad (4.11a)$$

$$\Delta^{\text{SP}} = \frac{1}{2\bar{\gamma}^2 \kappa} \ln(n + \Sigma^{\text{SP}}). \quad (4.11b)$$

From these equations, one can see that for $n > N(z')$ Δ^{SP} starts to exceed Σ^{SP} . Therefore, we will consider this case separately below. Using Eqs. (4.11a) and (4.11b), we integrate over Σ and Δ and find σ_n in the first region:

$$(I) \quad n < \frac{1}{2\bar{\gamma}^2 \kappa} \ln N(z') \equiv \frac{1}{2\bar{\gamma}^2 \kappa} \zeta.$$

In this region σ_n are equal:

$$\sigma_n^{\text{AGK}}(z') = \frac{C^2 e^{-\frac{3}{4}\zeta}}{n!} e^{-\frac{1}{4\bar{\gamma}^2\kappa}\zeta^2} \frac{\left(\frac{1}{2\bar{\gamma}^2\kappa}\zeta\right)^n}{\Gamma\left(\frac{1}{2\bar{\gamma}^2\kappa}\zeta\right)} \exp\left(-2\sqrt{\frac{1}{2\bar{\gamma}^2\kappa}\zeta}\right). \quad (4.12)$$

Therefore, in this region, the produced gluons have the Poisson distribution with the mean multiplicity $\bar{n}^I = \frac{1}{2\bar{\gamma}^2\kappa}\zeta$.

We have to return to Eq. (4.8) to find σ_n in the second region:

$$(II) \quad N(z') > n > \frac{1}{2\bar{\gamma}^2\kappa} \ln N(z') \equiv \frac{1}{2\bar{\gamma}^2\kappa}\zeta.$$

For $n > \omega_1$ Eq. (4.8) takes the form:

$$\begin{aligned} \sigma_n^{\text{AGK}}(z') &= C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\Sigma}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\Delta}{2\pi i} e^{\bar{\gamma}^2\kappa((\Sigma-1)^2+\Delta^2)} \\ &\times \frac{n^{\Sigma+\Delta-1}}{\Gamma(\Sigma+\Delta)\Gamma(\Sigma-\Delta)} \frac{2}{N^{\Sigma-1/4}(z')} (n+\Sigma+\Delta)^{-\Delta-1/4} \exp\left(-\left(2\sqrt{\frac{n+\Sigma+\Delta}{N(z')}}\right)\right). \end{aligned} \quad (4.13)$$

For Eq. (4.13), the saddle points for $\Delta^{\text{SP}} \ll \Sigma^{\text{SP}}$ satisfy the following equations:

$$2\bar{\gamma}^2\kappa(\Sigma^{\text{SP}} - 1) - \ln N(z') - 2 \ln \Sigma^{\text{SP}} + \ln n = 0, \quad (4.14a)$$

$$2\bar{\gamma}^2\kappa\Delta^{\text{SP}} + \ln(n + \Sigma^{\text{SP}}) = 0, \quad (4.14b)$$

with solutions

$$\Sigma^{\text{SP}} - 1 = \frac{1}{2\bar{\gamma}^2\kappa} \left\{ \ln N(z') + 2 \ln \left(\frac{1}{2\bar{\gamma}^2\kappa} \ln N(z') \right) - \ln n \right\}, \quad (4.15a)$$

$$\Delta^{\text{SP}} = -\frac{1}{2\bar{\gamma}^2\kappa} \ln(n + \Sigma^{\text{SP}}). \quad (4.15b)$$

Taking integrals, we obtain:

$$\sigma_n^{\text{AGK}}(z') = C^2 e^{-\zeta} \frac{1}{n} \left(\frac{N(z')}{n} \right)^{1/4} \exp \left\{ -\frac{1}{4\bar{\gamma}^2\kappa} \left(\zeta + 2 \ln \left(\frac{\zeta}{2\bar{\gamma}^2\kappa} \right) - \ln n \right)^2 - \frac{1}{4\bar{\gamma}^2\kappa} \ln^2 n \right\} \exp \left(-2\sqrt{\frac{n}{N(z')}} \right). \quad (4.16)$$

In region III:

$$(III) \quad n > N(z').$$

$\Delta \gg \Sigma$ and using Eq. (4.13), we obtain the equations for the saddle points: For Eq. (4.13), the saddle points for $\Delta^{\text{SP}} \ll \Sigma^{\text{SP}}$ satisfy the following equations:

$$2\bar{\gamma}^2\kappa(\Sigma^{\text{SP}} - 1) - \ln N(z') = 0, \quad (4.17a)$$

$$2\bar{\gamma}^2\kappa\Delta^{\text{SP}} + 2 \ln \left(\frac{n}{\Delta^{\text{SP}}} \right) = 0. \quad (4.17b)$$

They have solutions

$$\Sigma^{\text{SP}} - 1 = \frac{1}{2\bar{\gamma}^2\kappa} \ln N(z') = \frac{1}{2\bar{\gamma}^2\kappa}\zeta, \quad (4.18a)$$

$$\Delta^{\text{SP}} = -\frac{1}{\bar{\gamma}^2\kappa} \ln \left(\frac{n}{\frac{1}{\bar{\gamma}^2\kappa} \ln n} \right). \quad (4.18b)$$

Taking the integrals, we have:

$$\begin{aligned} \sigma_n^{\text{AGK}}(z') &= C^2 e^{-\zeta} \frac{1}{n} \left(\frac{N(z')}{n} \right)^{1/4} \\ &\times \exp \left\{ -\frac{1}{4\bar{\gamma}^2\kappa} \left(\zeta^2 + \ln^2 \left(\frac{n}{\frac{1}{\bar{\gamma}^2\kappa} \ln n} \right) \right) \right\} \\ &\times \exp \left(-2\sqrt{\frac{n}{N(z')}} \right). \end{aligned} \quad (4.19)$$

Now we are going to calculate the total inelastic cross section:

$$\sigma_{\text{in}} = \sum_{n=1}^{\infty} \sigma_n = \sum_{n=1}^{\infty} \sigma_n^{\text{AGK}}. \quad (4.20)$$

We return to Eq. (4.3) and sum over n , which reduces this equation to the form:

$$\sigma_{\text{in}} = C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\tilde{\gamma}^2 \kappa}{2}((\omega-1)^2 + (\omega_1-1)^2)} \int_0^\infty dt e^{-t} \frac{t^{\omega_1-1}}{\Gamma(\omega_1)} \left\{ 1 - \frac{1}{(1+tN(z'))^\omega} \right\}. \quad (4.21)$$

Integrating over t leads to the following expression:

$$\sigma_{\text{in}} = C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\tilde{\gamma}^2 \kappa}{2}((\omega-1)^2 + (\omega_1-1)^2)} \left\{ 1 - \left(\frac{1}{N(z')} \right)^{\omega_1} U\left(\omega_1, \omega_1 - \omega, \frac{1}{N(z')}\right) \right\}. \quad (4.22)$$

Using Eq. (4.8) we obtain:

$$\sigma_{\text{in}}(z') = C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\Sigma}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\Delta}{2\pi i} e^{\tilde{\gamma}^2 \kappa((\Sigma-1)^2 + \Delta^2)} \left\{ 1 - \frac{1}{\Gamma(\Sigma + \Delta)} \frac{2}{N^\Sigma(z')} (\Sigma + \Delta)^\Delta \exp\left(-2\sqrt{\frac{\Sigma + \Delta}{N(z')}}\right) \right\}. \quad (4.23)$$

Taking integrals over Σ and Δ using the method of steepest descent, we have:

$$\sigma_{\text{in}}(z') = C^2 \left\{ 1 - e^{-\zeta} e^{-\frac{1}{4\tilde{\gamma}^2 \kappa} \left(\zeta + \ln\left(\frac{\zeta}{2\tilde{\gamma}^2 \kappa}\right) \right)^2 - \frac{1}{4\tilde{\gamma}^2 \kappa} \ln^2\left(\frac{\zeta}{2\tilde{\gamma}^2 \kappa}\right)} \exp\left(-2\sqrt{\frac{\frac{1}{4\tilde{\gamma}^2 \kappa} \zeta}{N(z')}}\right) \right\}. \quad (4.24)$$

For the cross section of the inclusive production of the produced gluons, we have

$$\begin{aligned} \sigma_{\text{incl}}(z') &= \sum_{n=1}^{\infty} n \sigma_n^{\text{AGK}} = C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\Sigma}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\Delta}{2\pi i} e^{\tilde{\gamma}^2 \kappa((\Sigma-1)^2 + \Delta^2)} \omega \int_0^\infty dt e^{-t} N(z') \frac{t^{\omega_1}}{\Gamma(\omega_1)} \\ &= C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{\omega d\omega}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{\omega_1 d\omega_1}{2\pi i} e^{\tilde{\gamma}^2 \kappa((\omega-1)^2 + (\omega_1-1)^2)} = C^2 N(z'). \end{aligned} \quad (4.25)$$

One can see that the integrals over ω and ω_1 would lead to zero if we do not put $\omega - 1$ and $\omega_1 - 1$ in the exponent. The defect of the expression for the dipole densities in Eq. 12a of Ref. [35] is that in the series of this equation, the contribution of one Pomeron exchange vanishes due to ω integration. Actually, Eq. (4.25) follows directly from the AGK cutting rules at any value of z with $C^2 = 1$. Note that the multiplicity distribution $\mathcal{P}_n = \sigma_n/\sigma_{\text{in}}$ does not depend on the value of the smooth function C .

We introduce the ration $\mathcal{P}_n^{\text{gluon}} = \frac{\sigma_n^{\text{AGK}}(z')}{\sigma_{\text{incl}}^{\text{AGK}}(z')}$ to get rid of the unknown smooth function C^2 , which is equal to

$$\mathcal{P}_n^{\text{gluon}} = \frac{\sigma_n^{\text{AGK}}(z')}{\sigma_{\text{incl}}^{\text{AGK}}(z')} = \frac{1}{n} \frac{1}{N(z)} \left(\frac{n}{N(z)} \right)^{-1/4} \exp\left\{ -\frac{1}{4\tilde{\gamma}^2 \kappa} \left(\zeta^2 + \ln^2\left(\frac{n}{\frac{1}{\tilde{\gamma}^2 \kappa} \ln n}\right) \right) \right\} \exp\left(-2\sqrt{\frac{n}{N(z')}}\right) \quad (4.26)$$

V. INCLUSIVE PRODUCTION OF J/Ψ IN THE BFKL CASCADE

First, we rewrite in our notations the cross section of J/Ψ production due to three gluon fusion that has been written in Ref. [3]. This cross section is shown in Fig. 2(a) and in Fig. 4, where our main notations are shown. For simplicity, we consider the inclusive cross section integrated over the transverse momentum q_T of produced J/Ψ . This cross section is equal to [see Eq. 9 of Ref. [3]]:

$$\begin{aligned} \frac{d\sigma_{J/\Psi}(y)}{dy} &= 2 \int_0^1 dz \int_0^1 dz' \int \frac{d^2 r}{4\pi} \frac{d^2 r'}{4\pi} (2N_{IP}(Y-y, r)) \\ &\quad \times \Psi_{J/\Psi}(r, z) \Psi_{J/\Psi}(r', z') \left(2N\left(y; \frac{\mathbf{r} + \mathbf{r}'}{2}\right) - 2N\left(y; \frac{\mathbf{r} - \mathbf{r}'}{2}\right) \right)^2. \end{aligned} \quad (5.1)$$

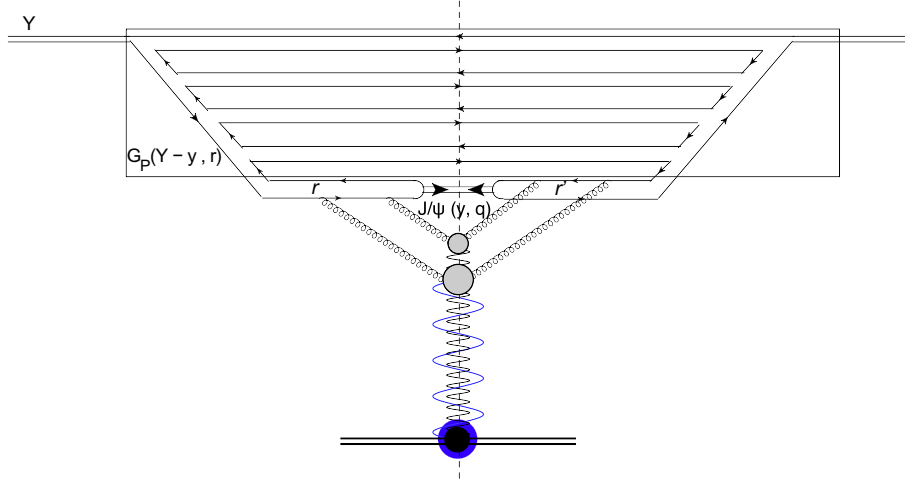


FIG. 4. The cross-section corresponding to the first diagram of Fig. 2(a) in the BFKL Pomeron calculus. The vertical wavy lines of different colors and shapes passing through unitarity cut are BFKL Pomerons [described by the Green functions $G_{IP}(Y_{J/\psi}, r)$ or $G_{IP}(Y_{J/\psi}, r')$ in Eq. (5.1)], and helical lines correspond to the gluons.

The wave function of J/Ψ can be found in the appendix to Ref. [3] as well as other details of the derivation. One can recognize that $2N$ in this equation corresponds to the unitarity constraints of Eq. (4.20). We can simplify Eq. (5.1) by integrating over angle between r and r' using that $r = r'$ with good accuracy since both are equal to r_d : the typical size of $c\bar{c}$ -dipole of the J/Ψ meson. Using Eq. (3.7) for N we obtain:

$$\frac{d\sigma_{J/\Psi}(y)}{dy} = 2 \int_0^1 dz \int_0^1 dz' \int \frac{d^2r}{4\pi} \frac{d^2r'}{4\pi} (2N_{IP}(Y-y, r)), \quad (5.2a)$$

$$\begin{aligned} & \Psi_{J/\psi}(r, z) \Psi_{J/\psi}(r', z') 4Q^{2\bar{\gamma}}(Y-y) \\ & \times \left(4\sqrt{\pi} \left(2^{4\bar{\gamma}} \frac{\Gamma(\frac{1}{2} + 4\bar{\gamma})}{\Gamma(1 + 4\bar{\gamma})} - \frac{\Gamma(\frac{1}{2} + 2\bar{\gamma})}{\Gamma(1 + 2\bar{\gamma})} \right) \right) (r^2 r'^2)^{\bar{\gamma}} \\ & \times \text{Const } G_{IP}^{\text{cut}}(z_d) (G_{IP}^{\text{cut}}(z'_d))^2. \end{aligned} \quad (5.2b)$$

$G_{IP}^{\text{cut}}(z_d)$ and $G_{IP}^{\text{cut}}(z'_d)$ denote the Green's function of the cut BFKL Pomeron. In Eq. (5.2b), we do not specify the value of the Const since we cannot guarantee this value in our approach. z_d and z'_d are equal to

$$\begin{aligned} z_d &= \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} (Y-y) + \xi_{r, r_d}; & z'_d &= \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} y + \xi_{r_d, r'}; \\ z &= \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} Y + \xi_{r, r'}; \end{aligned} \quad (5.3)$$

Using our approach of Ref. [35], we can rewrite Eq. (5.2b) as the first diagram of the BFKL Parton cascade for σ_n with produced J/Ψ :

$$\text{Const } 2\rho_1(z_d) 4\rho_1(z'_{d,i}) \rho_1(z'_{d,j}) \equiv \text{Const } 2\rho_1(z_i) 4\rho_1(z'_{d,j});$$

$$\text{with } z_{d,i} = \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} y + \xi_{r_d, r_i};$$

$$z_{d,j} = \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} y + \xi_{r_d, r_j};$$

$$z_i = \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} Y + \xi_{r, r_i}. \quad (5.4)$$

Note that ρ_1 in this equation is given by Eq. (3.12).

The general structure of the J/Ψ production in our approach is shown in Fig. 5. Choosing $Y_0 = y$ in Eq. (3.1), we see from Fig. 5 that we can calculate the inclusive production of J/Ψ using Eq. (3.1) for the scattering amplitude but introducing a new vertex (blue circle in Fig. 5) for the J/Ψ production and taking into account that in Eq. (3.1), at least one Pomeron is a cut Pomeron. Bearing this in mind, we can rewrite Eq. (3.7) for the projectile densities in the following form for Fig. 5(a):

$$\begin{aligned} & \rho_n^P(Y-y; r, b, \{r_i, b_i\}) \\ &= \frac{1}{n!} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} e^{\frac{\bar{\gamma}^2 \kappa(\omega_1-1)^2}{2}} \frac{\Gamma(\omega_1+n)}{\Gamma(\omega_1)} \prod_{i=1}^n G_{IP}(z_{d,i}). \end{aligned} \quad (5.5)$$

For the target densities, we have:

$$\begin{aligned} & \rho_n^T(y; r, b, \{r_i, b_i\}) = \frac{1}{(n+1)!} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_2}{2\pi i} e^{\frac{\bar{\gamma}^2 \kappa(\omega_2-1)^2}{2}} \\ & \times \frac{\Gamma(\omega_2+n+1)}{\Gamma(\omega_2)} \prod_{i=1}^{n+1} G_{IP}(z_{d,i}). \end{aligned} \quad (5.6)$$

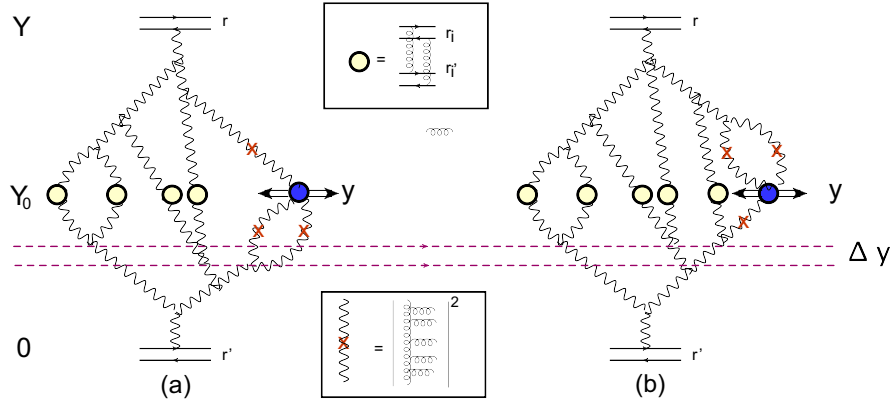


FIG. 5. Summing large Pomeron loops for J/Ψ production in dipole-dipole scattering. The wavy lines denote the BFKL Pomeron exchanges. The circles denote the amplitude γ in the Born approximation of perturbative QCD. The blue blob denotes the J/Ψ production due to three gluon fusion. The wavy lines with the cross indicate the cut Pomerons. The dashed lines denote the rapidity window, where the multiplicity of produced gluons is measured. The picture reflects the Mueller diagram technique for inclusive production [see Ref. [34]]. (a) and (b) describe the parton cascade structure of Fig. 1(a) and 1(b) respectively.

VI. MULTIPLICITY DISTRIBUTION FOR J/Ψ PRODUCTION IN DIPOLE-DIPOLE SCATTERING

Using Eqs. (3.5) and (5.6) for the dipole densities, we can calculate the scattering amplitude for the J/Ψ production from Eq. (3.1), viz.:

$$S_{J/\Psi}(z', z'_d) = C^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\epsilon_i \infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} \int_{\epsilon_i \infty}^{\epsilon+i\infty} \frac{d\omega_2}{2\pi i} e^{\frac{i^2 \kappa}{2} ((\omega_1-1)^2 + (\omega_2-1)^2)} \frac{\Gamma(\omega+n)}{\Gamma(\omega)} \times \Gamma(\omega_2+1+n) \Gamma(\omega_1) \frac{t_2^{\omega_2-1+n-k}}{\Gamma(\omega_2)} \prod_{i=1}^k \int d^2 r_i d^2 r'_i d^2 b_i d^2 b'_i \prod_{i=1}^{n-1} G_{IP}(z_{d,i}) \gamma^{BA}(r_i, r'_i, \delta b) G_{IP}(z_{d,j}) G_{IP}^2(z'_{d,j}). \quad (6.1)$$

We wish to emphasize that this is our main equation for inclusive production. In this equation, we assume that $Y_0 = y$ is the central region of rapidity and we can use the dipole densities of Eqs. (5.5) and (5.6).

Making integrations using Eq. (3.13) we obtain:

$$S_{J/\Psi}(z', z'_d) = C^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\epsilon_i \infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} \int_{\epsilon_i \infty}^{\epsilon+i\infty} \frac{d\omega_2}{2\pi i} e^{\frac{i^2 \kappa}{2} ((\omega_1-1)^2 + (\omega_2-1)^2)} \frac{\Gamma(\omega_1+n)}{\Gamma(\omega_1)} \frac{\Gamma(\omega_2+n+1)}{\Gamma(\omega_2)} G_{IP}^n(z) G_{IP}(z'_d). \quad (6.2)$$

We use the AGK cutting rules [see Eqs. (4.3) and (4.2a)—Eq. (4.2c)] to find the cross section for k cut Pomerons with the Green's functions $G_{IP}^k(z')$ ($\sigma_k^{\text{AGK}}(z')$) and for $n-k$ cut Pomerons the Green's functions $G_{IP}^{n-k}(z'_s)$ ($\sigma_{n-k}^{\text{AGK}}(z')$). For this goal, we have to consider the scattering amplitude with l Pomerons in t -channel [see Eq. (3.1)] and apply the AGK cutting rules to this amplitude. The expression for $\sigma_n^{\text{AGK}}(z', z'_d)$ ⁶ takes the form

$$\sigma_n^{\text{AGK}}(z', z'_d) = C^2 \frac{1}{n!} \int_{\epsilon_i \infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} \int_{\epsilon_i \infty}^{\epsilon+i\infty} \frac{d\omega_2}{2\pi i} e^{\frac{i^2 \kappa}{2} ((\omega_1-1)^2 + (\omega_2-1)^2)} n \frac{\Gamma(\omega_1+l)}{\Gamma(\omega_1)} \times \sum_{l=k}^{\infty} \frac{(-1)^{l-n}}{l!} \frac{l!}{(l-n)! n!} \underbrace{(2G_{IP}(z))^l 2G_{IP}(z_d)}_{\text{AGK cutting rules}} \frac{\Gamma(\omega_2+l+1)}{\Gamma(\omega_2)}. \quad (6.3)$$

$\sigma_k^{\text{AGK}}(z') \text{ (see Eq. (III.1))}$

The extra factor of n in Eq. (6.3) reflects the possibility to produce J/Ψ from each of n BFKL Pomerons.

⁶We hope that using the same notation as in Sec. III $\sigma_n^{\text{AGK}}(z', z'_d)$ will be not confusing since we have different arguments.

Making summations over l and replacing $\Gamma(\omega_1 + l)$ by Eq. (3.17) we have:

$$\begin{aligned}\sigma_n^{\text{AGK}}(z', z'_d) &= \frac{C^2}{(n-1)!} \int_{\epsilon_i \infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} \int_{\epsilon_i \infty}^{\epsilon+i\infty} \frac{d\omega_2}{2\pi i} e^{\frac{\tilde{\gamma}^2 \kappa}{2}((\omega_1-1)+(\omega_2-1)^2)} \frac{\Gamma(\omega_2 + n + 1)}{\Gamma(\omega_1)\Gamma(\omega_2)} \int_0^\infty e^{-t} dt t^{\omega_1-1} \frac{(tN(z))^n N(z'_d)}{(1+tN(z))^{\omega_2+n+1}} \\ &= \frac{C^2 N(z'_d)}{(n-1)!} \int_{\epsilon_i \infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} \int_{\epsilon_i \infty}^{\epsilon+i\infty} \frac{d\omega_2}{2\pi i} e^{\frac{\tilde{\gamma}^2 \kappa}{2}((\omega_1-1)+(\omega_2-1)^2)} \\ &\quad \times \frac{\Gamma(\omega_2 + n + 1)\Gamma(\omega_1 + n)}{\Gamma(\omega_1)\Gamma(\omega_2)} \left(\frac{1}{N(z)}\right)^{\omega_1} U\left(\omega_1 + n, \omega_1 - \omega_2, \frac{1}{N(z)}\right),\end{aligned}\quad (6.4)$$

where $N(z) = 2G_{IP}(z)$ and $N(z'_d) = 2G_{IP}(z'_d)$.

Using Eqs. (4.6) and (4.8), we reduce Eq. (6.4) to the form:

$$\begin{aligned}\sigma_n^{\text{AGK}}(z', z'_d) &= \frac{C^2 N(z'_d)}{(n-1)!} \int_{\epsilon_i \infty}^{\epsilon+i\infty} \frac{d\Sigma}{2\pi i} \int_{\epsilon_i \infty}^{\epsilon+i\infty} \frac{d\Delta}{2\pi i} e^{\tilde{\gamma}^2 \kappa((\Sigma-1)^2 + \Delta^2)} \frac{\Gamma(\Sigma - \Delta + n + 1)}{\Gamma(\Sigma + \Delta)\Gamma(\Sigma - \Delta)} \\ &\quad \times \left(\frac{1}{N(z)}\right)^{\Sigma-1/4} (\Sigma - \Delta + n - 1)^{\Delta-1/4} \exp\left(-2\sqrt{\frac{\Sigma + \Delta + n - 1}{N(z)}}\right).\end{aligned}\quad (6.5)$$

In the region

- (I) $n < \frac{1}{2\tilde{\gamma}^2 \kappa} \ln N(z') \equiv \frac{1}{2\tilde{\gamma}^2 \kappa} \zeta$
we have, after taking the integral over Σ using the method of steepest descent,

$$\sigma_n^{\text{AGK}}(z', z'_d) = \frac{C^2 N(z'_d)}{(n-1)!} \left(\frac{1}{N(z)}\right)^{3/4} e^{-\frac{1}{4\tilde{\gamma}^2 \kappa} \zeta^2} \frac{\left(\frac{1}{2\tilde{\gamma}^2 \kappa} \zeta\right)^{n+1}}{\Gamma\left(\frac{1}{2\tilde{\gamma}^2 \kappa} \zeta\right)} \exp\left(-2\sqrt{\frac{\frac{1}{2\tilde{\gamma}^2 \kappa} \zeta}{N(z')}}\right). \quad (6.6)$$

- (II) For $N(z') > n > \frac{1}{2\tilde{\gamma}^2 \kappa} \ln N(z') \equiv \frac{1}{2\tilde{\gamma}^2 \kappa} \zeta$
one can see that we have the same equations for the saddle points as have been discussed above [see Eqs. (4.17a), (4.17b), (4.15a), and (4.15b)]. Using these inputs, we obtain for $\sigma_n^{\text{AGK}}(z')$:

$$\sigma_n^{\text{AGK}}(z, z'_d) = C^2 N(z'_d) \left(\frac{n}{N(z)}\right)^{3/4} \exp\left\{-\frac{1}{4\tilde{\gamma}^2 \kappa} \left(\zeta + 2 \ln\left(\frac{\zeta}{2\tilde{\gamma}^2 \kappa}\right) - \ln n\right)^2 - \frac{1}{4\tilde{\gamma}^2 \kappa} \ln^2 n\right\} \exp\left(-2\sqrt{\frac{n}{N(z')}}\right). \quad (6.7)$$

In region III:

- (III) $n > N(z')$

we have the same situation as in region II: the solutions for the saddle points of Σ and Δ coincide with Eq. (4.18a) and, leading to the following $\sigma_n^{\text{AGK}}(z, z'_d)$:

$$\sigma_n^{\text{AGK}}(z, z'_d) = C^2 N(z'_d) \left(\frac{n}{N(z)}\right)^{3/4} \exp\left\{-\frac{1}{4\tilde{\gamma}^2 \kappa} \left(\zeta^2 + \ln^2\left(\frac{n}{\frac{1}{\tilde{\gamma}^2 \kappa} \ln n}\right)\right)\right\} \exp\left(-2\sqrt{\frac{n}{N(z')}}\right). \quad (6.8)$$

To find the multiplicity distributions, we need to calculate the total inclusive cross section, viz.:

$$\sigma_{\text{incl}}^{\text{AGK}}(z', z'_d) = \sum_{n=1}^{\infty} \sigma_n^{\text{AGK}}(z', z'_d). \quad (6.9)$$

Summing over n Eq. (6.4), we obtain:

$$\sigma_{\text{incl}}^{\text{AGK}}(z', z'_d) = C^2 N(z) N(z'_d). \quad (6.10)$$

Therefore, we have the same situation as with Eq. (4.25): the inclusive cross section has the same form as it follows from the general features of the AGK cutting rules. From this, we can also conclude that $C^2 = 1$ or better to say that we can use the ratio $\sigma_n/\sigma_{\text{incl}}$ in which there is no unknown constants. Hence,

$$\begin{aligned} \mathcal{P}_n^{J/\Psi} &= \frac{\sigma_n^{\text{AGK}}(z, z'_d)}{\sigma_{\text{incl}}^{\text{AGK}}(z', z'_d)} = \frac{1}{N(z)} \left(\frac{n}{N(z)} \right)^{3/4} \\ &\times \exp \left\{ -\frac{1}{4\bar{\gamma}^2 \kappa} \left(\zeta^2 + \ln^2 \left(\frac{n}{\frac{1}{\bar{\gamma}^2 \kappa} \ln n} \right) \right) \right\} \\ &\times \exp \left(-2\sqrt{\frac{n}{N(z')}} \right). \end{aligned} \quad (6.11)$$

One can see that $\mathcal{P}_n$ has the KNO scaling behavior [79].

From Eqs. (6.11) and (4.26), we obtain that

$$\frac{\mathcal{P}_n^{J/\Psi}}{\mathcal{P}_n^{\text{gluon}}} = n \frac{n}{N(z)} \quad (6.12)$$

has a simple form. We need to draw the reader's attention to two features of Eq. (6.12). First, it does not reproduce the result of Sec. II, where this ratio has been calculated in the one-dimensional model. Second, Eq. (6.12) has the same form discussed in Ref. [5]. Summing the two diagrams of Fig. 5, we reproduce Eq. 33 of this reference. Bearing this in mind, we refer our reader to this paper for comparison with the available experimental data.

VII. CONCLUSIONS

The main result of the paper is Eq. (6.3) (see Fig. 5), which gives the procedure of calculation of the inclusive production using t -channel unitarity in high-energy QCD [see Eq. (3.1)]. We demonstrated this procedure using the dipole densities suggested in our previous publications [35–37]. The expressions for these densities stem from our attempts to reconcile the exact solution to the Balitsky-Kovchegov (BK) equation [11,14,15], which describes the rare fluctuation [38] in the dipole–target scattering, with the fact that this equation sums the “fan” Pomeron diagrams. We believe that this equation has a general application for the dipole densities which can be found with more refined procedure. For example, we can use the dipole densities given by JIMWLK equation [16], since they

can be expressed through the BFKL Pomeron contributions [30,31].

We calculate the multiplicity distributions of produced gluons both with and without J/Ψ inclusive production. These distributions are defined as the following ratios:

$$\text{For } J/\Psi \text{ production: } \mathcal{P}_n^{J/\Psi} = \frac{\sigma_n^{\text{AGK}}(z, z'_d)}{\sigma_{\text{incl}}^{\text{AGK}}(z', z'_d)}, \quad (7.1a)$$

$$\text{Without } J/\Psi: \mathcal{P}_n^{\text{gluon}} = \frac{\sigma_n^{\text{AGK}}(z)}{\sigma_{\text{incl}}^{\text{AGK}}(z)}. \quad (7.1b)$$

The general feature of these distributions is that they show the KNO scaling behavior [79], being the function of $n/N(z)$ where $N(z') = 2 \exp(\bar{\gamma}z)$ is the average multiplicity of dipoles in the BFKL cascade.

The ratio of Eq. (6.12) has been predicted in almost all previous publications on the subject [1–5]. Hence, we refer our reader to these papers for comparison with the experimental data. We hope that $\mathcal{P}_n^{J/\Psi}$ itself can be measured and, in comparison with these measurements, it will allow us to find the range of energy for our predictions.

In this paper, we also found the multiplicity distribution in the simple one-dimensional model which corresponds to the BFKL cascade in QCD. The main question that we answered is how the production of many Pomeron ladders in Fig. 1 contribute to the production of J/Ψ at fixed n . They all cancel after summing over n . Our result is that the influence of final n is not very important and they can be treated as corrections.

We hope that this paper will contribute to the further study of the Pomeron calculus in QCD, especially to the understanding of typical scales for inclusive production. In this paper we find the inclusive cross section integrated over transverse momenta. We are planning to study the production of the gluons and quarkonia with fixed transverse momenta (p_T) to learn more about suppression at $p_T < Q_s$ and about violation of k_T -factorization [80–82].

ACKNOWLEDGMENTS

We thank our colleagues at Tel Aviv University for discussions. Special thanks go A. Kovner and M. Lublinsky for stimulating and encouraging discussions on the subject of this paper. This research was supported by BSF grant 2022132.

DATA AVAILABILITY

No data were created or analyzed in this study.

- [1] V. A. Khoze, A. D. Martin, M. G. Ryskin, and W. J. Stirling, *Eur. Phys. J. C* **39**, 163 (2005).
- [2] L. Motyka and M. Sadzikowski, *Eur. Phys. J. C* **75**, 213 (2015).
- [3] E. Levin and M. Siddikov, *Eur. Phys. J. C* **79**, 376 (2019).
- [4] E. Levin, I. Schmidt, and M. Siddikov, *Eur. Phys. J. C* **80**, 560 (2020).
- [5] E. Gotsman and E. Levin, *Eur. Phys. J. C* **81**, 99 (2021).
- [6] D. Kharzeev and K. Tuchin, *Nucl. Phys. A* **770**, 40 (2006).
- [7] D. Kharzeev, E. Levin, M. Nardi, and K. Tuchin, *Phys. Rev. Lett.* **102**, 152301 (2009); *Nucl. Phys. A* **826**, 230 (2009).
- [8] F. Dominguez, D. E. Kharzeev, E. Levin, A. H. Mueller, and K. Tuchin, *Phys. Lett. B* **710**, 182 (2012).
- [9] D. E. Kharzeev, E. M. Levin, and K. Tuchin, *Nucl. Phys. A* **924**, 47 (2014).
- [10] Z. B. Kang, Y. Q. Ma, and R. Venugopalan, *J. High Energy Phys.* **01** (2014) 056.
- [11] Y. V. Kovchegov and E. Levin, *Cambridge Monogr. Part. Phys., Nucl. Phys., Cosmol.* **33**, 1 (2012).
- [12] L. McLerran and R. Venugopalan, *Phys. Rev. D* **49**, 2233 (1994); **49**, 3352 (1994); **50**, 2225 (1994); **59**, 09400 (1999).
- [13] A. H. Mueller, *Nucl. Phys. B* **415**, 373 (1994); **437**, 107 (1995).
- [14] I. Balitsky, *Nucl. Phys. B* **463**, 99 (1996); *Phys. Rev. D* **60**, 014020 (1999).
- [15] Y. V. Kovchegov, *Phys. Rev. D* **60**, 034008 (1999).
- [16] J. Jalilian-Marian, A. Kovner, A. Leonidov, and H. Weigert, *Nucl. Phys. B* **504**, 415 (1997); *Phys. Rev. D* **59**, 014014 (1998); A. Kovner, J. G. Milhano, and H. Weigert, *Phys. Rev. D* **62**, 114005 (2000); E. Iancu, A. Leonidov, and L. D. McLerran, *Nucl. Phys. A* **692**, 583 (2001); *Phys. Lett. B* **510**, 133 (2001); E. Ferreira, E. Iancu, A. Leonidov, and L. McLerran, *Nucl. Phys. A* **703**, 489 (2002); H. Weigert, *Nucl. Phys. A* **703**, 823 (2002).
- [17] F. Gelis, E. Iancu, J. Jalilian-Marian, and R. Venugopalan, *Annu. Rev. Nucl. Part. Sci.* **60**, 463 (2010).
- [18] V. S. Fadin, E. A. Kuraev, and L. N. Lipatov, *Phys. Lett. B* **60B**, 50 (1975); E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, *Zh. Eksp. Teor. Fiz.* **72**, 377 (1977) [*Sov. Phys. JETP* **45**, 199 (1977)]; I. I. Balitsky and L. N. Lipatov, *Yad. Fiz.* **28**, 1597 (1978) [*Sov. J. Nucl. Phys.* **28**, 822 (1978)].
- [19] L. N. Lipatov, *Zh. Eksp. Teor. Fiz.* **90**, 1536 (1986) [*Sov. Phys. JETP* **63**, 904 (1986)].
- [20] L. V. Gribov, E. M. Levin, and M. G. Ryskin, *Phys. Rep.* **100**, 1 (1983).
- [21] E. M. Levin and M. G. Ryskin, *Phys. Rep.* **189**, 267 (1990).
- [22] A. H. Mueller and J. Qiu, *Nucl. Phys. B* **268**, 427 (1986).
- [23] A. H. Mueller and B. Patel, *Nucl. Phys. B* **425**, 471 (1994).
- [24] J. Bartels, M. Braun, and G. Vacca, *Eur. Phys. J. C* **40**, 419 (2005); J. Bartels and C. Ewerz, *J. High Energy Phys.* **09** (1999) 026; J. Bartels and M. Wusthoff, *Z. Phys. C* **66**, 157 (1995); J. Bartels, *Z. Phys. C* **60**, 471 (1993).
- [25] M. Braun, *Phys. Lett. B* **632**, 297 (2006); *Eur. Phys. J. C* **33**, 113 (2004); *Phys. Lett. B* **483**, 115 (2000); *Eur. Phys. J. C* **16**, 337 (2000); **6**, 321 (1999); M. Braun and G. Vacca, *Eur. Phys. J. C* **6**, 147 (1999).
- [26] Y. V. Kovchegov and E. Levin, *Nucl. Phys. B* **577**, 221 (2000).
- [27] E. Levin and M. Lublinsky, *Nucl. Phys. A* **763**, 172 (2005).
- [28] E. Levin and M. Lublinsky, *Phys. Lett. B* **607**, 131 (2005); *Nucl. Phys. A* **730**, 191 (2004).
- [29] E. Levin, J. Miller, and A. Prygarin, *Nucl. Phys. A* **806**, 245 (2008).
- [30] T. Altinoluk, C. Contreras, A. Kovner, E. Levin, M. Lublinsky, and A. Shulkin, *Int. J. Mod. Phys. Conf. Ser.* **25**, 1460025 (2014); T. Altinoluk, N. Armesto, A. Kovner, E. Levin, and M. Lublinsky, *J. High Energy Phys.* **08** (2014) 007.
- [31] T. Altinoluk, A. Kovner, E. Levin, and M. Lublinsky, *J. High Energy Phys.* **04** (2014) 075; T. Altinoluk, C. Contreras, A. Kovner, E. Levin, M. Lublinsky, and A. Shulkin, *J. High Energy Phys.* **09** (2013) 115.
- [32] E. Levin, *J. High Energy Phys.* **11** (2013) 039.
- [33] V. A. Abramovsky, V. N. Gribov, and O. V. Kancheli, *Yad. Fiz.* **18**, 595–616 (1973) [*Sov. J. Nucl. Phys.* **18**, 308 (1974)].
- [34] A. H. Mueller, *Phys. Rev. D* **2**, 2963 (1970).
- [35] E. Levin, *Phys. Rev. D* **110**, 116021 (2024).
- [36] E. Levin, *Phys. Rev. D* **111**, 096004 (2025).
- [37] E. Levin, *Phys. Rev. D* **112**, 054033 (2025).
- [38] A. H. Mueller and B. Patel, *Nucl. Phys. B* **425**, 471 (1994); A. H. Mueller and G. Salam, *Nucl. Phys. B* **475**, 293 (1996); G. Salam, *Nucl. Phys. B* **461**, 512 (1996); E. Iancu and A. Mueller, *Nucl. Phys. A* **730**, 494 (2004); **730**, 460 (2004).
- [39] S. Acharya *et al.* (ALICE Collaboration), *Phys. Lett. B* **810**, 135758 (2020).
- [40] D. Thakur (ALICE Collaboration), *Springer Proc. Phys.* **234**, 217 (2019).
- [41] C. Jahnke (ALICE Collaboration), [arXiv:1805.00841](https://arxiv.org/abs/1805.00841).
- [42] J. Adam *et al.* (ALICE Collaboration), *J. High Energy Phys.* **09** (2015) 148.
- [43] B. Abelev *et al.* (ALICE Collaboration), *Phys. Lett. B* **712**, 165 (2012).
- [44] B. Trzeciak (STAR Collaboration), *J. Phys. Conf. Ser.* **668**, 012093 (2016).
- [45] R. Ma (STAR Collaboration), *Nucl. Part. Phys. Proc.* **276–278**, 261 (2016).
- [46] B. Kopeliovich, H. Pirner, I. Potashnikova, K. Reygers, and I. Schmidt, *Phys. Rev. D* **88**, 116002 (2013).
- [47] E. Ferreira and C. Pajares, *Phys. Rev. C* **86**, 034903 (2012).
- [48] Y. Q. Ma, P. Tribedy, R. Venugopalan, and K. Watanabe, *Phys. Rev. D* **98**, 074025 (2018).
- [49] D. Amati, L. Caneschi, and R. Jengo, *Nucl. Phys. B* **101**, 397 (1975).
- [50] V. Alessandrini, D. Amati, and R. Jengo, *Nucl. Phys. B* **108**, 425 (1976).
- [51] R. Jengo, *Nucl. Phys. B* **108**, 447 (1976).
- [52] D. Amati, M. Le Bellac, G. Marchesini, and M. Ciafaloni, *Nucl. Phys. B* **112**, 107 (1976).
- [53] M. Ciafaloni, M. Le Bellac, and G. C. Rossi, *Nucl. Phys. B* **130**, 388 (1977).
- [54] M. Ciafaloni, *Nucl. Phys. B* **146**, 427 (1978).
- [55] A. H. Mueller and G. P. Salam, *Nucl. Phys. B* **475**, 293 (1996); G. P. Salam, *Nucl. Phys. B* **461**, 512 (1996).
- [56] N. Armesto, S. Bondarenko, J. G. Milhano, and P. Quiroga, *J. High Energy Phys.* **05** (2008) 103.
- [57] P. Rembiesa and A. M. Stasto, *Nucl. Phys. B* **725**, 251 (2005).

- [58] A. I. Shoshi and B. W. Xiao, *Phys. Rev. D* **73**, 094014 (2006).
- [59] M. Kozlov and E. Levin, *Nucl. Phys.* **A779**, 142 (2006).
- [60] J.-P. Blaizot, E. Iancu, and D. N. Triantafyllopoulos, *Nucl. Phys.* **A784**, 227 (2007).
- [61] E. Levin and A. Prygarin, *Eur. Phys. J. C* **53**, 38 (2008).
- [62] A. Kovner, E. Levin, and M. Lublinsky, *J. High Energy Phys.* **08** (2016) 031.
- [63] A. Kovner, E. Levin, and M. Lublinsky, *J. High Energy Phys.* **05** (2022) 019.
- [64] A. Kovner, E. Levin, and M. Lublinsky, *J. High Energy Phys.* **10** (2024) 127.
- [65] R. P. Feynman, *Phys. Rev. Lett.* **23**, 1415 (1969); *Photon-Hadron Interactions* (W. A. Benjamin, Reading, 1972).
- [66] J. D. Bjorken and E. A. Paschos, *Phys. Rev.* **185**, 1975 (1969).
- [67] V. N. Gribov, *Yad. Fiz.* **9**, 640 (1969) [*Sov. J. Nucl. Phys.* **9**, 369 (1969)]; Space-time description of hadron interactions at high-energies, *Proceedings of the ITEP School on Elementary Particle Physics* (1973), Vol. 1, p. 65, [arXiv: hep-ph/0006158](#).
- [68] A. Kovner, E. Levin, and M. Lublinsky, *J. High Energy Phys.* **05** (2022) 019.
- [69] E. Levin, *Phys. Rev. D* **111**, 016019 (2025).
- [70] E. Levin and A. Prygarin, *Eur. Phys. J. C* **53**, 385 (2008).
- [71] A. H. Mueller, *Nucl. Phys.* **B415**, 373 (1994); **437**, 107 (1995).
- [72] Y. V. Kovchegov, *Phys. Rev. D* **72**, 094009 (2005).
- [73] E. Levin, *Nucl. Phys.* **A763**, 140 (2005).
- [74] A. H. Mueller and D. N. Triantafyllopoulos, *Nucl. Phys.* **B640**, 331 (2002).
- [75] S. Munier and R. B. Peschanski, *Phys. Rev. D* **69**, 034008 (2004); *Phys. Rev. Lett.* **91**, 232001 (2003).
- [76] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. (Academic Press, London, 1994).
- [77] E. Levin and K. Tuchin, *Nucl. Phys.* **B573**, 833 (2000); **A691**, 779 (2001); **693**, 787 (2001).
- [78] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (United States Department of Commerce, National Bureau of Standards, 1964).
- [79] Z. Koba, H. B. Nielsen, and P. Olesen, *Nucl. Phys.* **B40**, 317 (1972).
- [80] S. Catani, M. Ciafaloni, and F. Hautmann, *Nucl. Phys.* **B366**, 135 (1991).
- [81] J. C. Collins and R. K. Ellis, *Nucl. Phys.* **B360**, 3 (1991).
- [82] E. M. Levin, M. G. Ryskin, Y. M. Shabelski, and A. G. Shuvaev, *Sov. J. Nucl. Phys.* **53**, 657 (1991).