

Properties of natural polynomials for Schwarzschild and Kerr black holes

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The quasinormal modes of black holes play various important roles in gravitational wave theory, signal modeling, and data analysis; however, there remain open questions about their mathematical properties. Aspects of classical polynomial theory have been proposed as a framework to investigate quasinormal mode orthogonality and completeness. We have recently presented a class of polynomials that are “natural” to quasinormal modes in that they are restricted by the quasinormal mode boundary conditions, and exactly tridiagonalize Teukolsky’s radial equation. In turn, these polynomials may be useful for (i) better understanding the vector space properties of quasinormal mode solutions (ii) serving as a basis for spectral methods to solve the Teukolsky and confluent Heun equations. Here, we provide an overview of these polynomials’ basic analytic properties: their three-term recurrence relation, ladder operators, and governing differential equation. We demonstrate that the natural polynomials for Schwarzschild and Kerr black holes are Pollaczek-Jacobi polynomials with complex valued parameters. We verify that, despite being constrained to the real line, key results for Pollaczek-Jacobi polynomials hold for black hole polynomials. Along the way, we observe a novel property that is particular to Schwarzschild: the polynomials’ three-term recurrence relation always peaks at the physical overtone index. This work supports the broader application of these polynomials to spectral methods for solving the Teukolsky equation, and other instances of the confluent Heun equation.

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I. INTRODUCTION

There is now ample evidence of stellar mass binary black holes (BBHs) that merge into a perturbed remnant, which *rings down*, radiating away its distortions predominantly via gravitational quasinormal modes (QNMs) [1–7]. The related *ringdown* radiation is increasingly well understood within single black hole (BH) perturbation theory, and this understanding has been instrumental in many tests of General Relativity (GR) [1–5,7,8]. However, the extent to which such tests can be carried out is influenced by practical limitations in our understanding of QNMs [9–14]. Although numerical simulations provide high accuracy predictions of postmerger gravitational waves, our incomplete understanding of the QNMs spatial completeness properties limits our ability to directly estimate their presence in simulations: rather than an unambiguous decomposition (i.e., projection), we must presently make heuristic and phenomenological fitting or filtering choices [6,13,15–26]. With future gravitational wave experiments (e.g., LISA and ET) expected to require unprecedentedly complete and accurate predictions of GR signals, there is

good reason to refine our understanding of the QNMs properties [21,27–30].

Some progress was made in Refs. [31,32], which we will henceforth refer to as Paper I and Paper II, respectively. Both focus on the QNMs of isolated Kerr and Schwarzschild BHs, which are of known astrophysical relevance [7,8,21]. This is also the scope of the present work.

In Paper I, it was found that QNM boundary conditions constrain a radial scalar product for the QNMs, and that this scalar product may be used to study a restricted version of their orthogonality and completeness properties. In Paper II, it was demonstrated that the scalar product may be used to define polynomials that are *natural* to the QNM problem in the following ways: (i) the polynomials may be defined by applying the scalar product from Paper I to the Gram-Schmidt algorithm; therefore, their properties are tied to the QNM boundary conditions, (ii) the polynomials enable the representation of QNM radial problem as a matrix eigenvalue problem in the simplest sense, and (iii) when represented in the basis of these polynomials, the QNM’s radial differential operator is exactly tridiagonal.

Since the polynomials in question are natural to BH solutions of Einstein’s equations, we will henceforth refer to the class of all such functions as *black hole polynomials*.¹

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¹To disambiguate between natural, BH, and confluent Heun polynomials, Appendix C provides a summary of the terminology used in this work.

While Chebyshev and Legendre polynomials are commonly used in spectral methods, the black hole polynomials offer a more tailored approach, allowing for spectral matrices that only require computation of tridiagonal elements [31–36]. Further, since the QNMs radial dependence is of the confluent Heun type, the polynomials in question are also natural to confluent Heun problems.² This implies that the black hole polynomials may be useful for other problems in physics and applied mathematics that are governed by the confluent Heun equation [31,32,37–39].

In Paper II, it was also noted that black hole polynomials for Kerr and Schwarzschild appear closely related to Pollaczek-Jacobi polynomials [40–43]. A key difference would appear to be that Pollaczek-Jacobi polynomials are defined on the real line, while black hole polynomials are defined on the complex plane [31]. There is therefore reason to carefully study the basic analytic properties of Pollaczek-Jacobi polynomials in the BH context. This motivates the present work.

In this paper, we present and review the basic properties of black hole polynomials for Schwarzschild and Kerr. The specific properties are the black hole polynomials’ governing differential equation, three-term recursion relation, and ladder operators (i.e., derivative rule). The main results of the present work are (i) the explicit demonstration that the analytic properties of Pollaczek-Jacobi polynomials apply to black hole polynomials for Schwarzschild and Kerr, and (ii) the black hole polynomials for Schwarzschild QNMs, unexpectedly, maximize the average value of the radial variable; in turn, this implies a deeper connection between the QNM quantization condition and the related BH polynomials.

These findings are built upon the results of Paper I, Paper II, and analogous results for Pollaczek-Jacobi polynomials [40–43]. In particular, our analysis relies heavily on Refs. [40,41], which provide a semianalytic description of the Pollaczek-Jacobi polynomials’ properties. The present work assumes some basic familiarity with classical polynomial theory, and common special functions, particularly concepts such as analytic continuation, ladder operators and orthogonality relations. Readers seeking background on these topics may find pedagogical introductions in Refs. [31,44–47].

This article is organized as follows. In Sec. II, we provide essential background on BH perturbation theory, including a review of the Teukolsky master equation and the formulation of the QNM radial eigenvalue problem. We then introduce a new choice of monomial moments for constructing black hole polynomials and demonstrate that this choice is equivalent to the construction presented in previous work, though with improved convergence

properties for specific physical regimes. In Sec. III, we establish mathematical properties satisfied by the black hole polynomials, analogous to those from classical polynomial theory, including their three-term recurrence relation, derivative rule, raising and lowering operators, and the differential equation they satisfy. In Sec. IV, we present the central result of this work: a special relationship between the overtone label and polynomial order in the black hole polynomials generated with respect to QNM boundary conditions, particularly in the Schwarzschild limit. Finally, in Sec. V, we discuss the implications of these results for understanding QNM behavior in BH mergers and outline future directions for extending this framework.

II. PRELIMINARIES

Einstein’s GR predicts gravitational waves that propagate with two polarizations: h_+ (“ h plus”) and $h_\times$ (“ h cross”) [48]. In gravitational wave theory, it is useful to define a complex strain, h , as the double time integral of the Newman-Penrose Weyl scalar ψ_4 [49,50].

$$h = h_+ - ih_\times \quad (1a)$$

$$= \int_{-\infty}^t \int_{-\infty}^{t'} \psi_4(t'', r, \theta, \phi) dt'' dt', \quad (1b)$$

where (t, r, θ, ϕ) are Boyer-Lindquist coordinates [51]. In Teukolsky’s formulation of linear BH perturbation theory, a rescaling of ψ_4 , namely,

$$\psi = -(r - ia \cos \theta)^4 \psi_4, \quad (2)$$

is known to solve Teukolsky’s master equation, which is itself one way of representing Einstein’s equations linearized about the Kerr solution [52–54]. In Eq. (2), $a = |J|/M$ is the black hole spin [52,55]. In Eq. (2), and henceforth, it is held that $G = c = 1$. While Eq. (2) is specific to outgoing gravitational radiation (with spin weight $s = -2$), ψ will generally refer to the Teukolsky wave quantity which may be defined for spin weights $-2 \leq s \leq 2$ according to Ref. [52].

Quasinormal modes are the discrete spectra of Teukolsky’s master equation. Of the many ways to define QNM solutions, the simplest begins by seeking solutions of the form

$$\psi(t, r, \theta, \phi) \propto e^{-i\omega t} e^{-im\phi} R(r) S(u), \quad (3)$$

that correspond to purely ingoing waves at the BH event horizon, and purely outgoing waves at spatial infinity. In Eq. (3), $R(r)$ and $S(u)$ are Teukolsky’s radial and angular functions, respectively.³ Equation (3) describes the functional

²It was found in Paper II that the radial dependence of Kerr QNMs is typically well approximated by a single confluent Heun polynomial, but that the QNM’s radial differential equation is generally well represented by black hole polynomials.

³Both $R(r)$ and $S(u)$ are known to be confluent Heun functions. Although $S(u)$ is often referred to as a spheroidal harmonic, the full spheroidal harmonic is $S(u)e^{im\phi}$ [56,57].

form of a single quasinormal mode. In Eq. (3), $\tilde{\omega}$ is a complex frequency,

$$\tilde{\omega} = \omega - i/\tau, \quad (4)$$

where ω is the central oscillatory frequency and $1/\tau$ the positive-value exponential decay. Each QNM's dependence on the polar angle, θ , is given by a spheroidal harmonic, $S(u)$.

The vector space properties of the spheroidal harmonics are well understood.⁴ The completeness of the QNM's spheroidal harmonics has been demonstrated for many physically relevant scenarios (see Ref. [57]), and while the spheroidal harmonics are not orthogonal, there have been shown to exist dual functions (adjoint-spheroidal harmonics) with which the spheroidal harmonics are *biorthogonal*.⁵ Completeness and biorthogonality make the spheroidal harmonics compatible with the universal projection (i.e., decomposition) of gravitational wave radiation into spheroidal multipole moments [21,57,58].

While the vector space properties of the spheroidal harmonics $S(u)$ are well established, properties of the radial functions, $R(r)$, remain an active area of research [20,57–59]. We now briefly review relevant technical results from Paper I and Paper II.

A radial scalar product. A radial scalar product for QNMs may be constructed via Sturm-Liouville theory to the radial part of Teukolsky's master equation [31]. In particular, the product may be defined such that Teukolsky's master equation is self-adjoint (though still non-Hermitian) [21,31,45,60]. We now outline how this was done in Paper I.

The radial Teukolsky function, $R(r)$, satisfies an eigenvalue problem,

$$\mathcal{L}_r R(r) = A R(r), \quad (5)$$

where $\mathcal{L}_r$ is a confluent Heun operator in natural general form.⁶ The eigenvalue of $\mathcal{L}_r$, A , may be complex valued. It is also the separation constant for Teukolsky's master equation.

Following Leaver (Ref. [61]), the use of a compactified radial coordinate, ξ , where

$$\xi = \frac{r - r_+}{r - r_-} \quad (6)$$

has the advantage of inducing polynomial coefficients in the radial operator that is derived from $\mathcal{L}_r$, once the QNM's

asymmetric boundary conditions are imposed. This process proceeds as follows.

The QNM boundary conditions are imposed on $R(r)$ by requiring that

$$R(r(\xi)) = \mu(\xi)f(\xi), \quad (7)$$

and that $f(\xi)$ is finite and nonzero near the BH event horizon ($\xi = 0$) and spacelike infinity ($\xi = 1$). Therefore, in those limits, the asymptotic behavior of $R(\xi)$ is determined by $\mu(\xi)$, which is chosen to ensure that the phase velocity of $\psi(t, r(\xi), \theta, \phi)$ is ingoing near the BH event horizon and outgoing near spacelike infinity. Applying Eqs. (6), (7) to (5) induces a similarity transformation on the eigenvalue problem. The result is

$$\mathcal{L}_\xi f(\xi) = A f(\xi). \quad (8)$$

Allowed values of A are constrained by the QNM boundary conditions [encapsulated by $\mu(\xi)$] and the requirement that $f(\xi)$ is analytic. In Eq. (8), the radial operator, $\mathcal{L}_\xi$, is

$$\mathcal{L}_\xi = (C_0 + C_1(1 - \xi)) \quad (9a)$$

$$+ (C_2 + C_3(1 - \xi) + C_4(1 - \xi)^2)\partial_\xi \quad (9b)$$

$$+ \xi(\xi - 1)^2\partial_\xi^2. \quad (9c)$$

The constants, $C_0 - C_4$, are

$$C_0 = -2am\tilde{\omega} - 2i\tilde{\omega}(-\delta + M(2s + 1)) + \tilde{\omega}^2(\delta + M)(\delta + 7M), \quad (10a)$$

$$C_1 = 8M^2\tilde{\omega}^2 + s(4iM\tilde{\omega} - 1) + 6iM\tilde{\omega} - 1 - (4M\tilde{\omega} + i)\frac{(am - 2M^2\tilde{\omega})}{\delta}, \quad (10b)$$

$$C_2 = 4i\delta\tilde{\omega}, \quad (10c)$$

$$C_3 = -2(s + 1) + 4i\tilde{\omega}(M - \delta), \quad (10d)$$

$$C_4 = s + 3 - 6iM\tilde{\omega} + \frac{i(am - 2M^2\tilde{\omega})}{\delta}, \quad (10e)$$

$$\delta = \sqrt{M^2 - a^2}. \quad (10f)$$

Given the transformed radial operator [Eq. (9)], Sturm-Liouville theory provides an algorithm for constructing a scalar product,

$$\langle a|b \rangle = \int_0^1 a(\xi)b(\xi) W(\xi)d\xi, \quad (11)$$

⁴Various proposals for the orthogonality properties of the spheroidal harmonics are described in detail in Sec. 2.5 of Ref. [21].

⁵A sequence with elements $a_j \in A$ is biorthogonal with another sequence, $b_k \in B$, if for every pair (j, k) there exists a bilinear form (e.g., scalar product), $\mathcal{G}(\cdot, \cdot)$, such that $\mathcal{G}(a_j, b_k) = \delta_{jk}$.

⁶See Appendix C of Paper I.

under which $\mathcal{L}_\xi$ is symmetric, meaning that $\langle a|\mathcal{L}_\xi b\rangle = \langle \mathcal{L}_\xi a|b\rangle$ [31]. In Eq. (11), $W(\xi)$ is called a weight function, and was found in Paper I to be

$$W(\xi) = \xi^{B_0}(1-\xi)^{B_1} e^{\frac{B_2}{1-\xi}}, \quad (12)$$

where,

$$B_0 = C_2 + C_3 + C_4 - 1, \quad (13a)$$

$$B_1 = -C_2 - C_3 - 2, \quad (13b)$$

$$B_2 = C_2. \quad (13c)$$

The functional form of $W(\xi)$, including the essential singularity at $\xi = 1$, reflects both the QNM boundary conditions and confluent nature of Teukolsky's radial equation. The potential for $W(\xi)$ to diverge exponentially as $\xi \rightarrow 1$ signals that Eq. (11) must be evaluated using analytic continuation (e.g., path integration in the complex plane, or residue calculus) [31].

Using this radial scalar product [Eq. (11)], the orthogonality and completeness properties of solutions to the radial problem were studied in Paper I. There, it was also pointed out that the scalar product implies the existence of polynomials whose properties are tailored to Teukolsky's radial functions.

Black hole polynomials. Black hole polynomials for Kerr and Schwarzschild⁷ may be constructed by applying Eqs. (11), (12) to the Gram-Schmidt algorithm.⁸ Henceforth, the resulting polynomials of order k will be referred to as $u_k(\xi)$. Since $W(\xi)$ is defined in Eq. (12) as a deformation of the (classical) Jacobi weight, $u_k(\xi)$ are *semiclassical polynomials* [44].

In the context of the radial scalar product [Eq. (11)], it will sometimes be appropriate to refer to the polynomials as bras or kets, i.e., $\langle u_j|$ or $|u_k\rangle$, respectively. For example, as detailed in Paper II, black hole polynomials for Kerr and Schwarzschild are typically complete [Eq. (14a)] and orthonormal [Eq. (14b)]:

$$\mathbb{I} = \sum_k |u_k\rangle\langle u_k|, \quad (14a)$$

$$\langle u_j|u_k\rangle = \delta_{jk}. \quad (14b)$$

Concurrently, weight functions of the form of $W(\xi)$ have been studied in applied math, and are known to be linked to Pollaczek-Jacobi polynomials (see, e.g., [40,41]). However, the use of ξ [Eq. (6)], as well as the normalization

of the polynomials [Eq. (14b)], are a departure from conventions used in orthogonal polynomial theory, namely, Refs. [40–43]. In turn, these differences impact the representation of the Pollaczek-Jacobi polynomial and black hole polynomial's properties.

Summary of conventions. The black hole polynomials and Pollaczek-Jacobi polynomials are closely related, but differ in the conventions chosen for their domain variables and weight functions. To begin comparing the two sets of polynomials, it is first useful to describe how to convert between these different choices of convention.

The primary difference lies in the choice of domain variable: the black hole polynomials, $u_k(\xi)$, are defined with respect to $\xi \in [0, 1]$, while the Pollaczek-Jacobi polynomials, $q_k(x)$, are defined with respect to $x \in [1, 0]$. The two domains are related by

$$\xi := 1 - x. \quad (15)$$

Black hole polynomials exist within a scalar product space which uses $W(\xi)$ from Eq. (12),

$$W(\xi) = \xi^{B_0}(1-\xi)^{B_1} e^{\frac{B_2}{1-\xi}}. \quad (16)$$

The analogous scalar product for the Pollaczek-Jacobi polynomials is

$$w(x) := x^\alpha(1-x)^\beta e^{-t/x}. \quad (17)$$

In Eq. (17), α , β , and t are typically defined as real valued parameters for which $t \geq 0$, $\alpha > 0$, and $\beta > 0$. This is unlike the B_0 , B_1 , and B_2 [see Eqs. (10)–(13)], which are generally complex valued.

Equation (15) requires the following relationship between the two weight functions,

$$W(\xi) = w(1-x). \quad (18)$$

In Eq. (18), it has been used that a weight function may be defined up to an overall constant scale factor. Equation (18) requires that $\{B_0, B_1, B_2\}$ and $\{\alpha, \beta, t\}$ are related as follows:

$$\alpha = B_1, \quad \beta = B_0, \quad t = -B_2. \quad (19)$$

In polynomial theory, a key use of the weight function is in computing monomial moments, which have varied uses, from deriving polynomial properties, to assessing the application of polynomials in approximation theory [31,44]. For black hole polynomials, the p th monomial moment is defined as

$$\langle \xi^p \rangle = \int_0^1 \xi^p W(\xi) d\xi \quad (20a)$$

$$= \int_0^1 \xi^{B_0+p} (1-\xi)^{B_1} e^{\frac{B_2}{1-\xi}} d\xi \quad (20b)$$

⁷The QNMs radial equation for both Kerr and Schwarzschild BHs is of the confluent Heun type. Consequently, the following discussion applies to both equally.

⁸See also Paper II, Sec. V.

$$= e^{B_2} \Gamma(B_0 + p + 1) U(B_0 + p + 1, -B_1, -B_2). \quad (20c)$$

In Eq. (20c), $\Gamma(z)$ is the Euler gamma function, and $U(a, b, c)$ the Tricomi confluent hypergeometric function, both of which may be evaluated with complex valued inputs by common analytic continuation techniques (See, e.g., Sec. V of Paper I) [47,62]. Equation (20c) follows from Eq. (20b) by application of the integral form of $U(a, b, c)$.

For the Pollaczek-Jacobi polynomials, the p th monomial moment is defined as

$$\langle x^p \rangle = \int_0^1 x^p w(x) dx \quad (21a)$$

$$= \int_0^1 x^{B_1+p} (1-x)^{B_0} e^{\frac{B_2}{x}} dx \quad (21b)$$

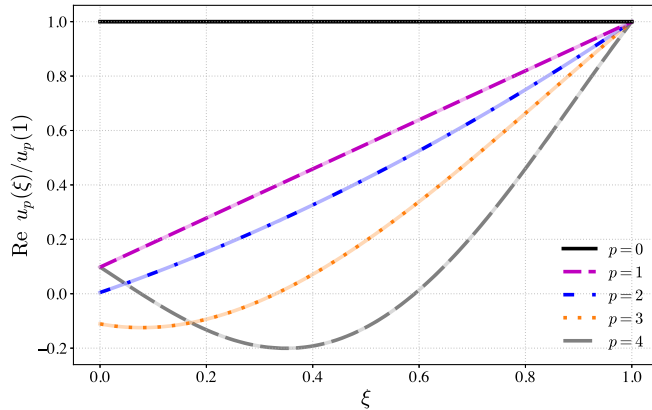
$$= e^{B_2} \Gamma(B_0 + 1) U(B_0 + 1, -B_1 - p, -B_2). \quad (21c)$$

In this way, the difference in domain impacts the related scalar product. This, in turn, impacts monomial moments.

Since the BH and Pollaczek-Jacobi weights differ by a change of variable, application of the Gram-Schmidt process results in the same polynomials. This is as shown in Fig. 1, where BH and Pollaczek-Jacobi polynomials are shown for a Kerr BH with spin of $a = 0.7$, and for $(\ell, m, n, s) = (2, 2, 2, -2)$. There, since the values of B_0 , B_1 , and B_2 are complex, so too are the related polynomials.

Converting between monomial moments. Since ξ and x are related by a constant shift [Eq. (15)], the $\langle \xi^p \rangle$ and $\langle x^p \rangle$ are related by the binomial theorem,

$$\langle \xi^p \rangle = \langle (1-x)^p \rangle = \sum_{j=0}^p \binom{p}{j} (-1)^j \langle x^j \rangle. \quad (22)$$



In Eq. (22), each $\langle \xi^p \rangle$ is expressed as a linear combination of $\langle x^j \rangle$, with binomial coefficients, $\binom{p}{j} = \frac{p!}{j!(p-j)!}$. For example, $\langle \xi^1 \rangle$ may be expanded according to

$$\langle \xi \rangle = \langle 1-x \rangle \quad (23a)$$

$$= \langle x^0 \rangle - \langle x^1 \rangle. \quad (23b)$$

The binomial theorem may also be used to write $\langle x^p \rangle$ in terms of $\langle \xi^p \rangle$, via $x = 1 - \xi$.

Since the polynomials are equivalent when evaluated with the same underlying parameters [Eq. (19)], it is reasonable to assume that they share generic properties: governing differential equation, three-term recursion relation, and ladder operators. To verify this assumption, it is important to note a final difference in convention.

Monic polynomials. While black hole polynomials are typically constructed to have unit modulus [Eq. (14)], meaning a norm of 1, this is typically not the case for the Pollaczek-Jacobi polynomials, whose normalization is chosen to facilitate the derivation of their analytic properties.

According to Eq. (14b), the black hole polynomials are orthonormal,

$$\langle u_j | u_k \rangle = \int_0^1 u_j(\xi) u_k(\xi) W(\xi) d\xi = \delta_{j,k}. \quad (24)$$

In contrast, the Pollaczek-Jacobi polynomials are typically defined to be *monic*, meaning that their leading coefficient is 1,

$$q_n(x) = x^n + p_1(n)x^{n-1} + \dots \quad (25)$$

In Eq. (25), $q_n(x)$ is a monic Pollaczek-Jacobi polynomial, and $p_1(n)$ is the next to leading coefficient. The monic

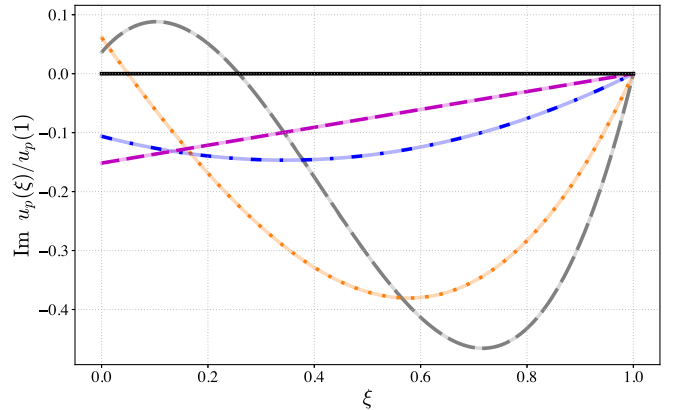


FIG. 1. The first five black hole polynomials, i.e., u_p for $p \in \{0, 1, 2, 3, 4\}$. Polynomials are constructed using spin weight $s = -2$, BH spin $a/M = 0.7$, and QNM frequency $M\bar{\omega}_{222} = 0.4999 - 0.4123i$. To make comparison more straightforward, polynomials have been normalized by their value at $\xi = 1$ (i.e., spatial infinity). The solid lines plot the monic orthogonal polynomials and the dashed lines the normalized polynomials, showing they are one and the same once different choices of conventions have been accounted for. Left panel: Real part of each polynomial. Right panel: imaginary part of each polynomial. See discussion in Sec. II.

polynomials are orthogonal, but not orthonormal, meaning that

$$\langle q_j | q_k \rangle = \int_0^1 q_j(x) q_k(x) w(x) dx = h_k \delta_{j,k}. \quad (26)$$

In Eq. (26), h_k is the monic polynomial's normalization constant. Consequently, the monic polynomials are related to the normalized polynomials via

$$u_k(x) = q_k(x) / \sqrt{h_k}, \quad (27)$$

where each h_k is simply the k^{th} entry of the diagonal Gramian matrix of the monic polynomials,

$$h_k = \langle q_k | q_k \rangle. \quad (28)$$

In Eq. (27), and henceforth, we refer to monic Pollaczek-Jacobi polynomial of order n as $q_n(x)$, and its normalized counterpart as $u_n(x)$. The latter is not to be confused with $u_n(\xi)$, given $u_n(x) \neq u_n(\xi)$ unless $\xi = x = \frac{1}{2}$.

In Eq. (26), the next to leading coefficient, $p_1(n)$, plays a central role in the derivation of the Pollaczek-Jacobi polynomials analytic properties (see, e.g., Refs. [40,44]).

III. CLASSICAL PROPERTIES OF NATURAL POLYNOMIALS FOR KERR BLACK HOLES

The semiclassical black hole polynomials share a subset of the properties of their classical counterparts (namely, orthogonality, a three-term recurrence relation, a derivative rule, and ladder operators) but do not obey a classical Rodrigues rule, and their derivatives satisfy a five-term rather than a three-term recurrence relation. Having established in Sec. II that the black hole polynomials are equivalent to the Pollaczek-Jacobi polynomials, the analytic properties derived in Refs. [40,41] for the latter apply directly to the former. This section presents the following properties in detail: a three-term recurrence relation, a

derivative rule and its associated ladder operators, and a second-order linear differential equation satisfied by the black hole polynomials.

A. Three-term recurrence relation

The black hole polynomials satisfy a three-term recurrence relation whose coefficients α_n and β_n encode a connection between polynomial order and overtone label. A three-term recurrence relation exists for any set of orthogonal, complete, polynomials ordered by degree [44,46], and generally takes the form,

$$x q_n(x) = q_{n+1}(x) + \alpha_n q_n(x) + \beta_n q_{n-1}(x), \quad (29)$$

for $n > 0$. This relation follows from the fact that multiplying q_n by x raises its order by one, so $x q_n$ must be expressible as a linear combination of q_{n+1} , q_n , and q_{n-1} , the only three basis elements of the same or adjacent order.

The recurrence coefficient α_n is determined by the sub-leading coefficients of consecutive polynomials,

$$\alpha_n = p_1(n) - p_1(n+1), \quad (30)$$

where $p_1(n)$ denotes the coefficient of x^{n-1} in the monic polynomial, as shown in Eq. (25).

Recall the monic polynomials q_p and the orthonormal polynomials u_p are related by a normalization factor, given by Eq. (27), namely, $q_n = \sqrt{h_n} u_n$, where $h_n = \langle q_n | q_n \rangle$ is the squared norm. Since α_n arises from the ratio of leading coefficients between q_n and u_n , it follows that

$$h_n = \alpha_n^{-2}. \quad (31)$$

The coefficients α_n and h_n thus carry information about how the polynomial normalization evolves with order, and, as shown in panel (b) of Fig. 2, their diagonal entries in the Gramian of the three-term recurrence peak at polynomial

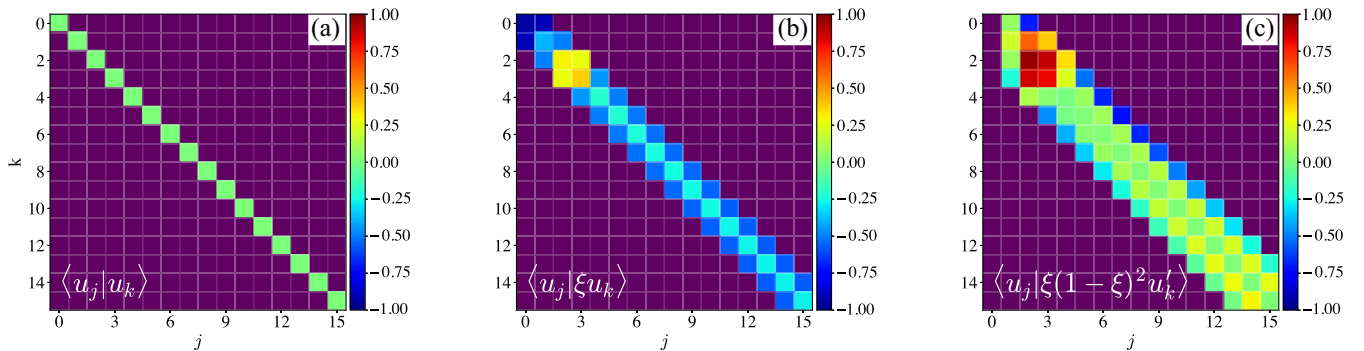


FIG. 2. Component matrices that show the scalar product calculations resulting in (a) orthogonality, (b) a three-term recurrence, and (c) a five-term recurrence are plotted on a logarithmic scale. All panels are computed for the $\ell, m = (2, 2)$ mode of a Schwarzschild black hole with spin weight $s = -2$ and overtone $n = 3$. Note the maximum values for panels (b) and (c) occur when overtone n matches polynomial order k .

order $k = n$ when the polynomials are evaluated at physical QNM frequencies.

B. Derivative rule and ladder operators

The black hole polynomials satisfy an analytic derivative rule that connects polynomials of consecutive order, and from which ladder operators and a five-term recurrence relation follow directly. The derivative rule takes the form

$$\frac{d}{dx} q_n(x) = -V_n(x)q_n(x) + \beta_n U_n(x)q_{n-1}(x), \quad (32)$$

where the quantities U_n , V_n , and β_n are defined in Appendix B, Eqs. (B12), (B13), and (B6), respectively. Note β_n should not be confused with β , the weight function parameter, but rather identified as the three-term recurrence relation coefficient from Eq. (29). A five-term recurrence relation follows from substituting the three-term recurrence relation of Eq. (29) into the right-hand side of Eq. (32), since each order of q appearing there itself satisfies a three-term relation,

$$x(x^2 - 1) \frac{d}{dx} q_k(x) = c_{0,k} q_{k-2} + c_{1,k} q_{k-1} + c_{2,k} q_k + c_{3,k} q_{k+1} + c_{4,k} q_{k+2}, \quad (33)$$

where the recurrence coefficients $c_{i,k}$ are calculated numerically. As shown in panel (c) of Fig. 2, the Gramian of the five-term recurrence relation exhibits a similar peak structure to that of Eq. (29), with maximum values again occurring when overtone n matches polynomial order k . The derivative rule of Eq. (32) may be recast as a lowering operator by rearranging,

$$L_n = \beta_n^{-1} U_n(x)^{-1} \left(\frac{d}{dx} + V_n(x) \right), \quad (34)$$

such that applying L_n to a polynomial of order n results in a polynomial of order $n - 1$,

$$L_n q_n(x) = q_{n-1}(x). \quad (35)$$

The corresponding raising operator, R_n , follows from self-adjointness (see Ref. [46]) and has the form,

$$R_n = U_n(x)^{-1} \left(-\frac{d}{dx} + V_n(x) + \mathcal{L}(w) \right), \quad (36)$$

where $\mathcal{L}(w) = \frac{d}{dx} \ln w(x)$ and applying R_n to a polynomial of order n results in a polynomial of order $n + 1$,

$$R_n q_n(x) = q_{n+1}(x). \quad (37)$$

Since the first monic polynomial is $q_1 = 1$, repeated

application of R_n generates the entire sequence and applying the raising operator n times yields q_n .

C. Differential equation

It was shown in Ref. [42] that the Pollaczek–Jacobi polynomials satisfy a linear, second-order differential equation, which, by the equivalence established in Sec. II, implies black hole polynomials are also solutions. In particular, both sets of polynomials are solutions to the homogeneous differential equation,

$$\frac{d^2 q_n}{dx^2} - M(x) \frac{dq_n}{dx} + N(x) q_n = 0, \quad (38)$$

where the coefficient functions are given by

$$M(x) = \frac{d}{dx} \ln(w(x) U_n(x)) \quad (39)$$

and

$$N(x) = U_n(x) \cdot \frac{d}{dx} \left(\frac{V_n(x)}{U_n(x)} \right) + \sum_{j=0}^{n-1} U_j(x). \quad (40)$$

The quantities U_n and V_n are defined in Appendix B, Eqs. (B12), (B13), and the sum U_j runs over consecutive terms. The coefficient functions $M(x)$ and $N(x)$ are rational functions, and may be written schematically,

$$M(x) = \frac{m_0}{x} + \frac{m_1}{x^2} + \frac{m_2}{1-x} + \frac{m_3}{m_4 + m_5 x}, \quad (41)$$

and

$$N(x) = \frac{n_0}{x} + \frac{n_1}{x^2} + \frac{n_2}{1-x} + \frac{n_3}{(1-x)^2}, \quad (42)$$

where the coefficients m_i and n_i are rational combinations of α , β , t , and $p_1(n)$, and may be computed explicitly using the definitions of U_n and V_n in Appendix B, Eqs. (B12), (B13).

A rigorous proof that the polynomials satisfy this differential equation may be found in Ref. [42]. As a numerical check, Eq. (38) is verified here for physically relevant cases with complex-valued coefficients, namely, the outgoing $\ell, m = (2, 2)$ mode of a Schwarzschild black hole, with weight function coefficients B_0 , B_1 , and B_2 computed using the positive repository [63]. The residual is found to be of order

$$\left| \frac{d^2 q_n}{dx^2} - M(x) \frac{dq_n}{dx} + N(x) q_n \right| \approx 10^{-17}. \quad (43)$$

Although the black hole polynomials tridiagonalize the Teukolsky radial equation rather than solving it directly,

they are themselves solutions to this different second-order linear differential equation. Whether this equation describes a physical quantity related to linearized Kerr remains an open question.

IV. A SPECIAL PROPERTY FOR SCHWARZSCHILD

For Schwarzschild black holes, the recurrence coefficient α_n of the three-term recurrence relation peaks at polynomial order $k = n$ when evaluated for physical QNM frequencies, providing a direct correspondence between polynomial order and overtone label. This section describes this property, its dependence on black hole spin, and its implications for ordering QNM solutions.

To understand this result, recall that the overtone label n is traditionally determined by the magnitude of the imaginary part of the QNM frequency, which lacks a strict algebraic hierarchy. The black hole polynomials offer an alternative approach to ordering overtones. Since the black hole polynomials are generated directly from the QNM boundary conditions and tri-diagonalize the radial Teukolsky operator, their ordering emerges naturally from the physics of the problem rather than relying on an *ad hoc* sorting of eigenvalues. The observation that α_n peaks at $k = n$ makes this connection precise, at least in the Schwarzschild limit.

As shown in Fig. 3, the peak is clearly visible for multiple overtones in the zero spin limit, and this behavior holds across various physical combinations of ℓ, m indices. Representative cases are shown in the left ($n = 0$), center ($n = 2$), and right ($n = 6$) panels. The inverse relationship holds (the recurrence relation achieves a *minimum* rather than a maximum) for spin weight $s = +2$. As spin increases, the peak migrates: for $a = 0.40$ and $a = 0.70$, it shifts to $k = n - 1$ when $m = 1, 2$ and, in the case of $a = 0.70$, the same shift occurs when $m = 0$ as well. A

similar pattern holds for $\ell = 3$ modes, though these are not shown.

The precise mechanism behind this peak, and whether an exact relationship between k and n can be established for general Kerr black holes, remains an open question that warrants further study. Nevertheless, the correspondence is robust in the Schwarzschild limit and suggests the black hole polynomials may provide a more robust basis for ordering QNM solutions.

V. DISCUSSION AND CONCLUSIONS

Motivated by the open questions surrounding QNM orthogonality and completeness, this work presents a study of the semiclassical properties of black hole polynomials. Building on Paper I and Paper II, which established a radial scalar product and demonstrated the existence of polynomials natural to the QNM boundary value problem, the present work has shown that these black hole polynomials are closely related to the semiclassical Pollaczek-Jacobi polynomials, and that their properties follow directly from those derived in Refs. [40–43]. While Refs. [40–43] treat Pollaczek-Jacobi polynomials with real valued parameters, we have found that, despite their complex nature, the black hole polynomials satisfy the same three-term recurrence relation, derivative rule, and second-order linear differential equation as the Pollaczek-Jacobi polynomials.

In Sec. II, the black hole polynomials were shown to be structurally equivalent to the Pollaczek-Jacobi polynomials, allowing their properties to be determined from those of the latter. In Sec. III, the basic semiclassical properties of the black hole polynomials were presented and verified numerically: (i) a three-term recurrence relation [Eq. (29)], (ii) an analytic derivative rule [Eq. (32)] from which raising and lowering operators follow, and (iii) a second-order linear differential equation [Eq. (38)]

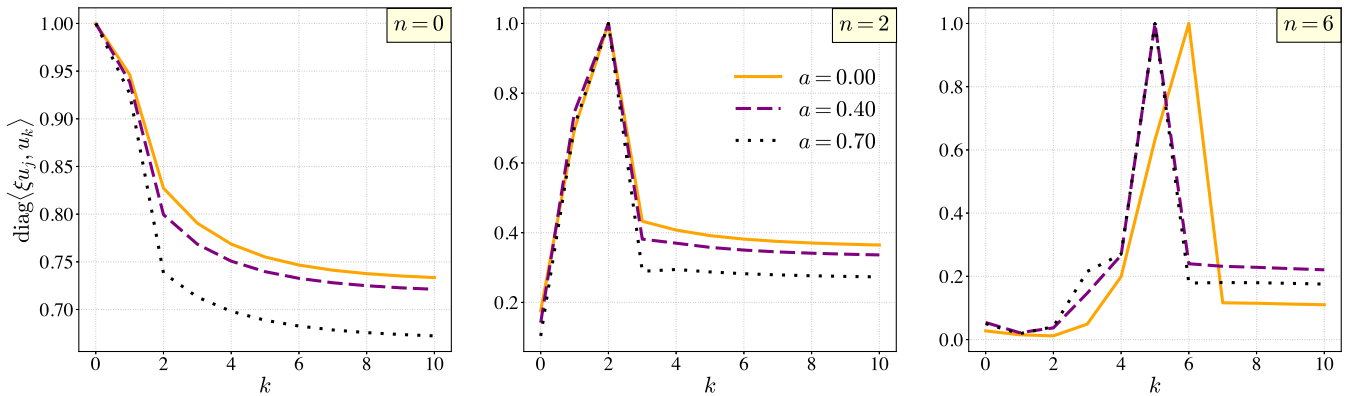


FIG. 3. The normalized three-term recurrence peaks [corresponding to the diagonal entries of the Gramian of Eq. (29)] are plotted here for the $(\ell, m) = (2, 2)$ mode and spin weight $s = -2$. The left panel shows the peak for $n = 0$, the fundamental overtone, the center panel for $n = 2$, and the right panel for $n = 6$. Notably, the peak tracks polynomial order k of the corresponding recurrence relation. This behavior is robust to overtone number for Schwarzschild black holes, for which the peak is always observed to occur around $k = n$. For astrophysically relevant Kerr black holes with spin of around $a = 0.40$ and $a = 0.70$, the peak shifts to $k = n - 1$.

verified numerically to floating-point precision. In all cases, we have numerically verified that the basic properties of Pollaczek-Jacobi polynomials hold for physically relevant parameters of Kerr BHs. This is the main result of the present work, and it motivates further exploration of the Pollaczek-Jacobi polynomials use in BH physics, particularly in spectral methods in single BH perturbation theory [33–36,39,64–66].

An unexpected result of this work was presented in Sec. IV. For Schwarzschild BHs, the recurrence coefficient α_n peaks at polynomial order $k = n$ (i.e., the QNM overtone index) when evaluated at physical QNM frequencies, as shown in Fig. 3 and Fig. 4. This correspondence is robust across all combinations of ℓ and m , and persists for lower overtones at nonzero spin, though the peak migrates as $a \rightarrow 1$. This results prompts the following question: *For Schwarzschild black holes BHs, why should α_n know about the overtone label n ?* At present, we may only conjecture that this results is related to that of Paper II, where it was found that QNMs with overtone label n are typically well approximated by a single confluent Heun polynomial of order n .

Developing a precise answer to that question, and the use of our BH polynomials in spectral algorithms for linear perturbation theory (within and beyond GR), motivates future work.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: COMPARISON OF MONOMIAL MOMENTS

The two sets of monomial moments, $\langle \xi^p \rangle$ and $\langle x^p \rangle$, differ in their asymptotic behavior and convergence properties, which has practical consequences for the numerical implementation of the black hole polynomials. The moments exhibit different behaviors relative to spin weight s , for example, with fast, monotonically decreasing convergence in some instances, and slow, nonmonotonic convergence in the case of $\langle x^p \rangle$ and spin weight $s = 2$. Furthermore, at higher overtones, the moments develop a nonmonotonic kink near small p before settling into their asymptotic trend, as seen in the $n = 6$ panel. These comparisons between the two sets of monomial moments are needed because their asymptotic behavior directly impacts the convergence properties of the sums and integrals they describe. In both analytic and numerical contexts, whether a choice of monomial moments grows or decays (and how quickly) can slow or even prevent convergence, whereas well-controlled asymptotics allow for efficient and stable evaluation. Understanding how different monomial sequences behave, particularly at large order, provides insight into the stability and tractability of the polynomial systems studied.

The analysis that follows is motivated by the fact that real-valued orthogonal polynomials tend to have strongly monotonic monomial moments, which in turn support rapidly convergent sums and reliable numerical behavior. However, as was found in Ref. [31], the polynomials examined in this work do not follow this typical pattern:

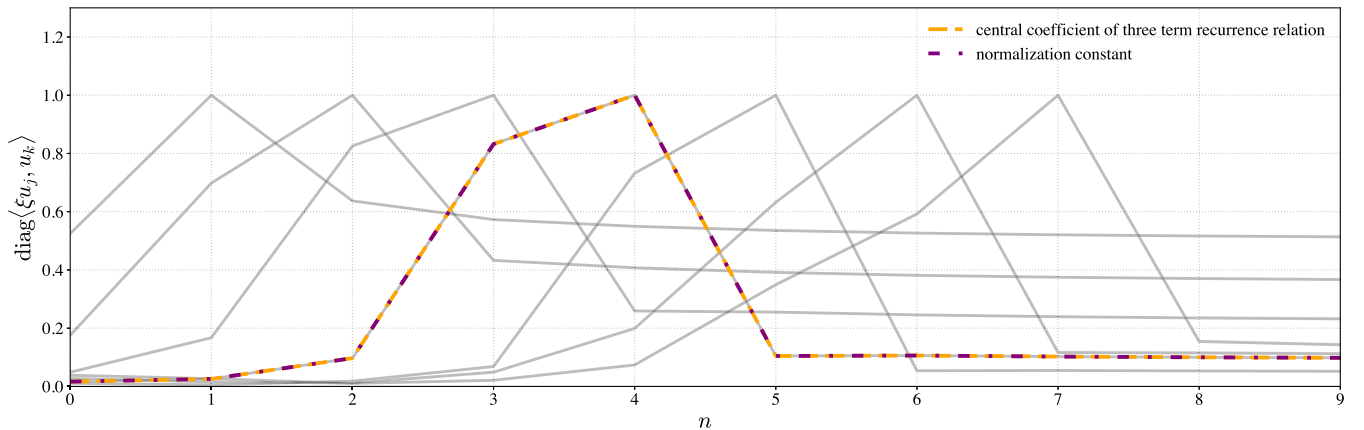


FIG. 4. Normalized peaks of the three-term recurrence relation are plotted for a sequence of overtones $n \in [1, 7]$. All polynomials used to generate the Gramians are calculated around the Schwarzschild solution, where $a = 0$, and for spin $s = -2$, and the $(2, 2)$ mode. For each gray curve, the maximum value reached occurs at the index corresponding to overtone n . Two example quantities, α_4^{-1} and $\sqrt{h_4}$, are calculated and plotted for $n = 4$ to show these track the peak in the same way the three-term recurrence relation does.

their monomial moments display markedly different structure and asymptotics. This deviation from the expected monotonic behavior necessitates a more careful study, both to gain a better understanding of the polynomials' analytic features and to assess numerical considerations.

The scalar product used to generate the natural polynomials, via the Gram-Schmidt algorithm, makes use of the fact that it is generally possible to describe any scalar product as a linear combination of monomial moments:

$$\langle a|b \rangle = \sum_{j,k} a_j b_k \langle x^{j+k} \rangle. \quad (A1)$$

The convergence of the sequences involving these moments (for example, the natural polynomials) is directly tied to the convergence of the coefficients $a_j b_k$. Using the definition of Eq. (21c), it is possible to perform a qualitative analysis of the large order behavior in p . The hypergeometric function U is defined via a series expansion in its third term, and to zeroth order has an approximate form

$$U(x, y, z) \approx \frac{\Gamma(1-y)}{\Gamma(x-y+1)} + \mathcal{O}(z). \quad (A2)$$

Combining this approximate form with Eq. (21c) allows $\langle x^p \rangle$ to be expressed in terms of the Euler gamma function,

$$\langle x^p \rangle \approx \frac{\Gamma(1+B_1+p)}{\Gamma(2+B_0+B_1+p)} \sim p^{-1-B_0} \Gamma(1+B_0). \quad (A3)$$

This representation makes use of the fact that for $p \rightarrow \infty$, $\Gamma(p+\alpha) \sim p^\alpha \Gamma(p)$ [67]. The physical quantities represented by B_0 may then be substituted for by using Eqs. (10) and (13), to conclude the exponent of p goes as

$$\langle x^p \rangle \sim p^{-i(am-2M^2\omega)/\delta+2iM\omega+s-1}. \quad (A4)$$

In the zero dimensionless black hole spin limit, i.e., $a = 0$, $M = \delta$, this simplifies to

$$\langle x^p \rangle \sim p^{-1+s+4iM\omega}. \quad (A5)$$

This behavior may be more easily compared to the asymptotics of $\langle \xi^p \rangle$ monomial moments, which obey

$$\langle \xi^p \rangle \sim p^{-1-2s+4iM\omega}. \quad (A6)$$

Indeed, in this particular limit, the behavior has very similar dependencies with a $+s$ rather than $-2s$ factor. In general, however, the relationship is more complicated, and varies with overtone n . A few example cases are plotted in Fig. 5, for nonzero spin and varying overtone n , which demonstrate the two monomial moment definitions have opposite spin s dependencies and converge at different rates. In particular, the $\langle x^p \rangle$ moments grow monotonically for $s = +2$ and decay for $s = -2$, while the $\langle \xi^p \rangle$ moments exhibit the opposite spin dependence, as shown in panels (a) and (b). Both sets develop a nonmonotonic feature [see panel (c) of Fig. 5] at higher overtones near small p before settling into their asymptotic trend.

Because of the fact that the natural polynomials may be generated using either choice of monomial moments, it is important to consider the convergence rate for both. Although the $\langle x^p \rangle$ can be generated very quickly (only U and not Γ must be calculated for each p), the resulting polynomial coefficients need higher numerical precision to avoid telephasing problems. Telephasing occurs when consecutive terms in a sum nearly cancel, causing loss of significant digits and numerical instability in finite-precision calculations. As a result, different physical

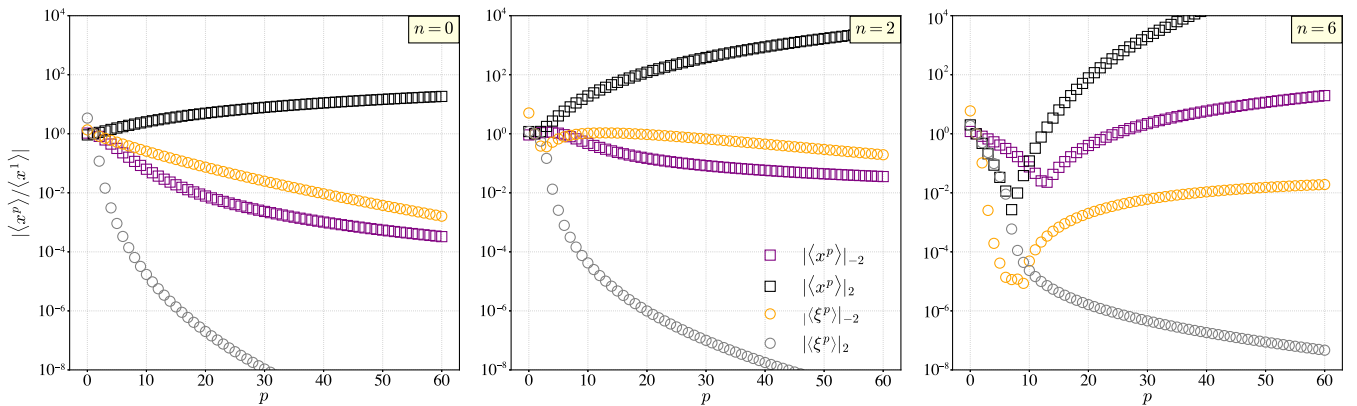


FIG. 5. Example distributions of absolute, normalized monomial moments are plotted against monomial order p for the $(\ell, m) = (2, 2)$ mode using the two definitions [see Eq. (20a) and Eq. (21a)] corresponding to two related domain variables, ξ and x , for multiple overtones and spin weights. The square markers correspond to the real-valued moments and the circles to complex-valued moments, whereas the colorful markers track spin weight $s = -2$ and the gray scale $s = 2$. The left panel shows the fundamental overtone $n = 0$, the center panel $n = 2$, and the right panel $n = 6$. This is done to compare the convergence properties of the two sets of monomial moments and to study the effect of negating spin weight for either case.

scenarios call for different choices: $\langle x^p \rangle$ converges faster for $s = -2$ and lower overtone, $\langle \xi^p \rangle$ for $s = +2$ and lower overtone, and $\langle \xi^p \rangle$ for higher overtone and either spin s .

The behavior of monomial moments has a direct impact on the numerical implementation of all classical polynomial properties. When monomial moments fail to decay asymptotically, numerical evaluations of related quantities can become unstable or ill conditioned. To address this, extended-precision arithmetic is used to ensure reliable computation in regimes where standard precision is insufficient. This approach is consistent with recent numerical studies, in which precision levels as high as $\text{dps} = 400$ have been used [39].

APPENDIX B: ON THE ANALYTIC EVALUATION OF POLLACZEK-JACOBI POLYNOMIALS

Throughout this work, the results of Ref. [40] have been used in detail. Here, a brief practical guide to their results is provided.

The two main methods for generating the monic orthogonal polynomials $q_k(x)$ analytically are the three-term recurrence relation of Eq. (29) and the raising operator of Eq. (36). Additionally, the classical properties discussed in Sec. III—orthogonality, three-term recurrence, raising and lowering operators, and a second-order differential equation—involve combinations of $U_n(x)$, $V_n(x)$, and β_n , which are stated here for completeness. Together, these expressions provide a complete analytic construction of the black hole polynomials.

Both analytic constructions depend on a calculation of the logarithmic derivative of the determinant of the Hankel matrix, a matrix involving the monomial moments of the domain variable (see Sec. VI of Ref. [40]). In the case of the polynomials studied, the Hankel matrix has symmetric entries consisting of consecutive monomial moments along each row and column, that is,

$$\mathcal{H} = \begin{pmatrix} \mu(0, t) & \mu(1, t) & \mu(2, t) & \cdots \\ \mu(1, t) & \mu(2, t) & \mu(3, t) & \cdots \\ \mu(2, t) & \mu(3, t) & \mu(4, t) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (\text{B1})$$

up to a set order n . The determinant of $\mathcal{H}$ is calculated in the typical way, and denoted by $D_n(t)$. It is then possible to define an explicit form for H_n , the logarithmic derivative of the Hankel matrix,

$$H_n = t \frac{d}{dt} \ln D_n(t). \quad (\text{B2})$$

This logarithmic derivative, H_n , can be expressed in terms of the weight function coefficients and the subleading order polynomial coefficient $p_1(n)$ of the monic polynomial,

$$H_n = (2n + \alpha + \beta)p_1(n) + n(n + \alpha - t). \quad (\text{B3})$$

It is possible to solve for $p_1(n)$ by rearranging Eq. (B3),

$$p_1(n) = \frac{1}{\gamma_n} (H_n - n(n + \alpha - t)), \quad (\text{B4})$$

where $\gamma_n = 2n + \alpha + \beta$.

Recall the recurrence coefficient α_n , which appears in the three-term recurrence relation in front of the q_n term, was defined in Eq. (30) as the difference of consecutive order $p_1(n)$,

$$\alpha_n = p_1(n) - p_1(n + 1). \quad (\text{B5})$$

Therefore the Hankel determinant provides H_n , which in turn provides $p_1(n)$ and α_n .

From this information, it is also possible to calculate the second recurrence coefficient β_n , which appears in front of the q_{n-1} term,

$$\beta_n = \frac{-2p_1(n)^3 + (3t - \alpha + 2\beta - 2n)p_1(n)^2 - (t^2 - 2(n - \beta)t - (2n + \alpha)\beta)p_1(n) - (t + \beta)nt}{\gamma_n(\gamma_n^2 - 1) + 2p_1(n) + \gamma_n(\gamma_n - 1)(\gamma_n + 2)p_1(n + 1) - \gamma_n(\gamma_n + 1)(\gamma_n - 2)p_1(n - 1) - (t + \beta)(\gamma_n^2 + 1)}. \quad (\text{B6})$$

The $q_k(x)$ follow from substituting Eqs. (B5), (B6) into Eq. (29) and noting $q_1 = 1$. In particular, Eq. (29) may be rearranged to explicitly solve for the next polynomial in the sequence,

$$q_{n+1}(x) = x q_n(x) - \alpha_n q_n(x) - \beta_n q_{n-1}(x). \quad (\text{B7})$$

Next, consider the following equations comprised of the weight function coefficients, α , β , and t , and the subleading order coefficients $p_1(n)$. The goal in defining these functions is to express $U_n(x)$ and $V_n(x)$ as rational

functions and offer an alternative method for calculating the $q_n(x)$ analytically. These functions are

$$S_n = \gamma_n p_1(n) - (\gamma_n + 2)p_1(n + 1) + t + \beta \quad (\text{B8})$$

and its helper function,

$$\tilde{S}_n = S_n - \gamma_n - 1, \quad (\text{B9})$$

as well as

$$T_n = -(p_1(n))^2 - (\gamma_n - \beta - t)p_1(n) + nt - \frac{(1 - \gamma_n^2)\beta_n}{\gamma_n}, \quad (\text{B10})$$

and its helper function,

$$\tilde{T}_n = T_n + p_1(n). \quad (\text{B11})$$

Note that S_n , $\tilde{S}_n$, T_n , and $\tilde{T}_n$ correspond to the functions R_n , R_n^* , r_n , and r_n^* , respectively, in Ref. [40].

As a result, $U_n(x)$ and $V_n(x)$ may now be written as simple rational functions,

$$U_n(x) = \frac{\tilde{S}_n}{x^2} + \frac{S_n}{x} - \frac{S_n}{x-1} \quad (\text{B12})$$

and

$$V_n(x) = \frac{\tilde{T}_n}{x^2} - \frac{n - T_n}{x} - \frac{T_n}{x-1}. \quad (\text{B13})$$

The $q_n(x)$ follow from substituting Eqs. (B12), (B13), and (B6) into the raising operator defined in Eq. (36) and noting $q_1 = 1$. The rational forms of U_n and V_n also provide insight into why the five-term recurrence relation Eq. (33) has $x(x^2 - 1)$ as its leading factor. Rearranging Eqs. (B12) and (B13), the rational functions assume the form,

$$x^2(x-1)U_n(x) = (\tilde{S}_n - S_n)x - \tilde{S}_n, \quad (\text{B14})$$

and

$$x^2(x-1)V_n(x) = (\tilde{T}_n - T_n + n)x - nx^2 - \tilde{T}_n. \quad (\text{B15})$$

Since U_n and V_n appear in the derivative rule Eq. (32), substituting these forms into the derivative rule introduces

the factor $x^2(x-1)$, which is precisely the operator appearing on the left-hand side of the five-term recurrence Eq. (33).

Together, the quantities S_n , $\tilde{S}_n$, T_n , $\tilde{T}_n$, and β_n provide a complete and efficient characterization of the black hole polynomials' derivative rule and recurrence structure, expressing all relevant quantities as rational functions of x with coefficients determined entirely by the weight function parameters and the Hankel determinant. If the parameter mapping Eq. (19) is applied, these terms depend instead on the coefficients of Eq. (13), which are determined by combinations of the physical parameters of the system: δ , M , s , and $\tilde{\omega}$.

APPENDIX C: HEUN, CONFLUENT HEUN, AND BLACK HOLE POLYNOMIALS

Here and in Table I we clarify the terminology used in this work and in Paper II regarding Heun polynomials, confluent Heun polynomials, and black hole polynomials.

Heun polynomials are solutions of the Heun equation, and confluent Heun polynomials are solutions of its confluent form [37]. The “black hole polynomials” are solutions to the differential equation of Sec. III C, i.e., Eq. (38). Black hole polynomials arise from application of the Gram-Schmidt algorithm to a weight function from the Teukolsky equation, and as such are precisely the “natural polynomials” studied in Paper II.

Paper II also describes the natural polynomials as *canonical confluent Heun polynomials*, since they may be derived from confluent Heun systems while retaining the canonical properties of classical polynomials. In practice, the use of black hole polynomials (equivalently, natural polynomials) to represent the radial Teukolsky operator results in an exactly tridiagonal form, whereas confluent Heun polynomials do not.

TABLE I. Terminology used for polynomial classes discussed in this work.

Term	Definition
Natural polynomials	Polynomials generated by applying Gram-Schmidt with the QNM-motivated radial scalar product (or corresponding weight function) from the Teukolsky framework; in this manuscript they are the same object as black hole polynomials.
Black hole polynomials	Solutions to the differential equation of Sec. III C [i.e., Eq. (38)]; equivalent to the natural polynomials of Paper II, and constructed to exactly tridiagonalize the radial Teukolsky operator.
Confluent Heun polynomials	Polynomial solutions of confluent forms of the Heun equation (special-function solutions tied to confluent Heun systems).
Canonical confluent Heun polynomials	Terminology used in Paper II for natural/black hole polynomials: they arise from confluent Heun systems while retaining canonical classical-polynomial properties (e.g., classical recurrence and operator structure).

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