

Massive Dirac states bound to vortices by a boson-fermion interaction

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Results are presented from numerical simulations of the flat-space nonlinear Maxwell-Klein-Gordon-Dirac equations. The introduction of a boson-fermion interaction allows a scalar vortex to act as a harmonic trap that can confine massive Dirac bound states. A parametric analysis is performed to understand the range of boson-fermion coupling strengths, Ginzburg-Landau parameters, and fermion effective masses that support the existence of bound state solutions; results are shown to be comparable to quasiparticle bound states in gapped Dirac materials. Solutions are time evolved and are observed to be stable until the fermion field ψ becomes large enough to collapse the spontaneously broken vacuum of the condensate. Head-on scattering simulations are performed, and traditional vortex right-angle scattering is shown to break down with increased fermion field strength. For sufficiently large ψ and low velocity, the collision of two $m = 1$ vortices results in a pseudostable $m = 2$ bound state that eventually becomes unstable and decays back into two $m = 1$ vortices. For large ψ and collision velocity, vortex scattering is observed to produce nontopological (zero winding number) scalar bound states that are ejected from the collision. The scalar bubbles contain coherent fermion bound states in their interiors and interpolate between the spontaneously broken vacuum of the bulk and the modified vacuum induced by the boson-fermion interaction.

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I. INTRODUCTION

Vortices are localized configurations with topologically conserved charge that have been widely studied across many domains of physics including classical fluid dynamics, condensed matter physics, and particle theory. In the context of particle theory, Nielsen and Olesen explored vortices in the Abelian-Higgs model in an attempt to better understand the possible string nature of fundamental particles [1]. Many authors have since studied the existence, stability, scattering, and other aspects of Abelian-Higgs vortices in the context of early Universe cosmology, where they commonly appear as cosmic strings in theories that undergo spontaneous symmetry breaking [2–11]. While still unobserved experimentally, cosmic strings are believed by many to have been present in the early Universe, where they may have influenced the observed large-scale structure of the Universe [12–14]. Authors have also considered cosmic strings (vortices) that support the existence of bound fermion states. The pioneering work of Nohl explored massive fermion bound states on Abelian

vortex lines [15]. Jackiw and Rossi [16] explored similar solutions with an interaction that was dimensionally similar to a Yukawa potential but combined charge-conjugate states and inspired volumes of work on Majorana zero modes in both particle theory and condensed matter physics [5,17]. In the standard model, where string configurations are not inherently topologically stable and must be stabilized dynamically, Weigel *et al.* realized that the addition of an interaction term between heavy fermions and the boson field actually helps stabilize the string configuration [18].

While physically different models, when reduced to dimensionless variables, the model describing Abelian-Higgs vortices of particle theory is the same as the Ginzburg-Landau-Maxwell model of condensed matter physics, where vortices manifest themselves as magnetic flux tubes within superconductors. These vortices were predicted by Abrikosov in 1957 [19,20] and experimentally verified by Cribier *et al.* by means of neutron diffraction in 1964 [21]. This groundbreaking work spawned decades of research and continues to inspire work across many theoretical and experimental subfields of condensed matter physics. Of particular relevance to this work are the advances in Bose-Fermi mixtures and Dirac materials. There is a large body of research dedicated to studying Bose-Fermi mixtures where one uses a Bose-Fermi-Hubbard lattice model that can explain nanoscale interactions between the condensed boson field and interacting fermion fields [22–26]. Frequently, a mean field or

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semiclassical field theory approach is employed when one is interested in effects that are large compared to the underlying lattice [27–29]. Bose-Fermi mixtures describe a wide range of phenomena including charge and spin density waves and quasiparticles like polarons, excitons, and polaritons [30], and they are experimentally explored by many in cold-atomic physics using optical traps [24,25,31–34]. While there is much in the literature that describes Majorana zero modes and other massless Dirac states, there is also a rapidly growing interest in massive Dirac fermions as well, particularly in the context of tunable Dirac materials [35–39].

The model being investigated in this work builds upon the Ginzburg-Landau-Maxwell and Abelian-Higgs models by including a massive Dirac field that repulsively interacts with the scalar field. Stable massive Dirac bound state solutions are shown to exist, and a brief look at their scattering properties is provided. The remainder of this paper is organized as follows. In Sec. II, the formalism is defined and the fully general covariant equations of motion are presented. In Sec. III, time-independent stationary cylindrically symmetric solutions are obtained, approximations are used to obtain closed-form solutions, the full solutions are obtained numerically, and time-dependent equations are used to determine the stability of the stationary solutions. In Sec. IV, $(2 + 1)$ equations are used to perform scattering simulations by boosting the stationary charged vortex solutions at each other. New bubble-type solutions are observed and briefly discussed. In an attempt to balance clarity with detail, the equations of motion are included in the main body, while descriptions of numerical methods appear in Appendix A. Insight into the self-repulsion and dispersive properties of the one-dimensional (1D) free and interacting Dirac fields is provided in Appendix B to aid in understanding results from the model being explored in this work.

II. GENERAL FORMALISM, DEFINITIONS, AND CONVENTIONS

The model being studied is constructed by minimally coupling the Maxwell, spontaneously broken Ginzburg-Landau, massive Dirac, and boson-fermion interaction Lagrangians:

$$\mathcal{L} = \sqrt{-g}(L_M + L_{\text{GL}} + L_D + L_{\text{GLD}}), \quad (1)$$

where

$$L_M = -\frac{\epsilon_0 c^2}{4} F^{\mu\nu} F_{\mu\nu}, \quad (2)$$

$$L_{\text{GL}} = -\frac{\hbar^2}{2m_B} g^{\mu\nu} D_\mu \phi (D_\nu \phi)^* + \alpha_B \phi^2 - \frac{\beta_B}{2} \phi^4, \quad (3)$$

$$L_D = \frac{ic\hbar}{2} [\bar{\psi} \gamma^\mu D_\mu \psi - (D_\mu^* \bar{\psi}) \gamma^\mu \psi] - m_F c^2 \bar{\psi} \psi, \quad \text{and} \quad (4)$$

$$L_{\text{GLD}} = -\frac{1}{2} \mu_i (\phi^* \phi) (\bar{\psi} \psi); \quad (5)$$

the electromagnetic field strength tensor, scalar gauge covariant derivative, and fermion gauge covariant derivative are defined to be

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (6)$$

$$D_\mu \phi = \partial_\mu \phi - \frac{iq_2}{\hbar} A_\mu \phi, \quad \text{and} \quad (7)$$

$$D_\mu \psi = \partial_\mu \psi - \frac{iq_1}{\hbar} A_\mu \psi + \Gamma_\mu \psi; \quad (8)$$

and the spinor affine connection, spin connection, gamma matrices in a coordinate basis, and spinor adjoint are given by

$$\Gamma_\mu = -\frac{1}{8} \omega_{\mu AB} [\tilde{\gamma}^A, \tilde{\gamma}^B], \quad (9)$$

$$\omega_{\mu AB} = g_{\nu\alpha} e^\alpha_A (\partial_\mu e^\nu_B + \Gamma^\nu_{\mu\lambda} e^\lambda_B), \quad (10)$$

$$\gamma^\alpha = (e^\alpha_A) \tilde{\gamma}^A, \quad \text{and} \quad (11)$$

$$\bar{\psi} = \psi^\dagger \gamma^0. \quad (12)$$

For full generality, the following dimensionless variables are defined:

$$\hat{x}^\mu = \left(\frac{1}{\xi}\right) x^\mu, \quad (13)$$

$$\hat{A}_\mu = \left(\frac{q_2 \Lambda}{\hbar}\right) A_\mu, \quad (14)$$

$$\hat{\phi} = \left(\frac{\beta_B}{\alpha_B}\right)^{1/2} \phi, \quad \text{and} \quad (15)$$

$$\hat{\psi} = \left(\frac{c\xi}{\hbar}\right) \left(\frac{m_B m_F \beta_B}{\alpha_B}\right)^{1/2} \psi, \quad (16)$$

and physical parameters of condensate number density, London penetration depth, and coherence length, are defined to be

$$\phi_0^2 = \frac{\alpha_B}{\beta_B}, \quad (17)$$

$$\Lambda = \left(\frac{m_B \epsilon_0 c^2}{q_2^2 \phi_0^2}\right)^{1/2}, \quad \text{and} \quad (18)$$

$$\xi = \left(\frac{\hbar^2}{2m_B \alpha_B} \right)^{1/2}, \quad (19)$$

which gives rise to the dimensionless model parameters

$$\kappa = \frac{\Lambda}{\xi}, \quad (20)$$

$$\kappa_d = \left(\frac{\hbar}{m_F c \xi} \right) = \frac{1}{2\pi} \left(\frac{\lambda_C}{\xi} \right) = \frac{\lambda_C}{\xi}, \quad \text{and} \quad (21)$$

$$\kappa_m = \frac{\mu_i \phi_0^2}{m_F c^2}, \quad (22)$$

where κ is the traditional Ginzburg-Landau (GL) parameter, κ_d is the scaled Compton length of the fermion quasiparticle, and κ_m measures the boson-fermion coupling strength. In practice, it is also helpful to use κ_d^{-1} , which functions as an effective dimensionless fermion mass. From these expressions, one can obtain the dimensionless actions

$$\hat{L}_M = -\frac{1}{4} \hat{F}^{\mu\nu} \hat{F}_{\mu\nu}, \quad (23)$$

$$\hat{L}_{\text{GL}} = -\frac{1}{2} g^{\mu\nu} \hat{D}_\mu \hat{\phi} (\hat{D}_\nu \hat{\phi})^* + \frac{1}{2} \hat{\phi}^2 - \frac{1}{4} \hat{\phi}^4, \quad (24)$$

$$\hat{L}_D = \frac{i}{2} \kappa_d [\hat{\psi} \gamma^\mu \hat{D}_\mu \hat{\psi} - (\hat{D}_\mu^* \hat{\psi}) \gamma^\mu \hat{\psi}] - \hat{\psi}^\dagger \hat{\psi}, \quad \text{and} \quad (25)$$

$$\hat{L}_{\text{GLD}} = -\frac{1}{2} \kappa_m (\hat{\phi}^* \hat{\phi}) (\hat{\psi}^\dagger \hat{\psi}), \quad (26)$$

where to preserve local $U(1)$ gauge invariance it is assumed that $q_2 = 2q_1$. While many normalizations of the Dirac field may be appropriate, the approach assumed in this work is

$$\int_V d^3x (\psi^\dagger \psi) = 1, \quad (27)$$

where ψ is the dimensionful field ($[\psi] = L^{-3/2}$) representing a single fermionic quasiparticle. Using (16) and (17), one gets

$$\int_V d^3x (\hat{\psi}^\dagger \hat{\psi}) = (m_F) \left(\frac{m_B}{\phi_0^2 \xi} \right) \left(\frac{c^2}{\hbar^2} \right), \quad (28)$$

which will be conserved over time by the equations of motion. Throughout this work, the amplitude of $\hat{\psi}$ will be parametrically varied, but it should be noted that changing the norm of the dimensionless field does not affect (27); changing the norm of $\hat{\psi}$ simply represents a change of the underlying model parameters. For example, one might set the properties of the bulk (m_B , ξ , and ϕ_0^2) to match a material of interest and use different values of $\hat{\psi}$ as a means

to explore what effective masses (m_F) support fermion bound state solutions.

Moving forward with dimensionless variables, the $\hat{\psi}$ will be omitted for clarity. The covariant equations of motion for the complex scalar field are given by

$$\square \phi = 2i\kappa^{-1} A^\rho \partial_\rho \phi + i\kappa^{-1} \phi \nabla_\rho A^\rho + \kappa^{-2} \phi A_\mu A^\mu - (1 - \kappa_m \hat{\psi}^2) \phi + (\phi^* \phi) \phi \quad \text{and} \quad (29)$$

$$\square \phi^* = -2i\kappa^{-1} A^\mu \partial_\mu \phi^* - i\kappa^{-1} \phi^* \nabla_\rho A^\rho + \kappa^{-2} \phi^* A_\mu A^\mu - (1 - \kappa_m \hat{\psi}^2) \phi^* + (\phi^* \phi) \phi^*, \quad (30)$$

where

$$\square \phi_i = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi). \quad (31)$$

The covariant Maxwell equations are given by

$$\nabla_\alpha F^{\beta\alpha} = -\frac{i}{2\kappa} g^{\beta\mu} (\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) - \kappa^{-2} A^\beta \phi^* \phi + \frac{1}{2} \kappa_d \kappa^{-1} (\bar{\psi} \gamma^\beta \psi) \quad \text{and} \quad (32)$$

$$\partial_{[\alpha} F_{\mu\nu]} = 0, \quad (33)$$

and the generally covariant Dirac equation is given by

$$i\gamma^\mu \left(\partial_\mu - \frac{i}{2\kappa} A_\mu + \Gamma_\mu \right) \psi - \kappa_d^{-1} \left(1 + \frac{\kappa_m \hat{\phi}^2}{2} \right) \psi = 0. \quad (34)$$

All spinors and gamma matrices are presented in the Dirac representation. The gamma matrices in an orthonormal basis are given by

$$\tilde{\gamma}^0 = \begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & -\mathbb{1}_2 \end{pmatrix}, \quad \tilde{\gamma}^k = \begin{pmatrix} 0_2 & \sigma^k \\ -\sigma^k & 0_2 \end{pmatrix}, \quad (35)$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (36)$$

and are converted to coordinate frames as needed using tetrads (11). Section III will use Cartesian-aligned tetrads with cylindrical coordinates, which allow the spin connection ($\omega_{\mu AB}$) and spinor affine connection (Γ_μ) to be zero, but it should be noted that $\gamma^\mu \neq \tilde{\gamma}^\mu$. Section IV will use Cartesian-aligned tetrads with Cartesian coordinates, which gives $\omega_{\mu AB} = \Gamma_\mu = 0$ and $\gamma^\mu = \tilde{\gamma}^\mu$.

The spacetime metric is assumed to be flat with negative signature $(-, +, +, +)$, which gives

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}\mathbb{1}_4 \quad (37)$$

for the Dirac algebra. Finally, only GL parameter values $\kappa \geq 1$ will be considered, which will focus the analysis on type-II superconductor types of behaviors, and all Maxwell equations are solved using the Lorentz gauge.

III. STATIONARY AXISYMMETRIC SOLUTIONS

This section explores the existence and stability of stationary massive Dirac bound states in vortices. Time-independent equations are derived, and closed-form approximate solutions and full numerical solutions are presented. A parametric analysis is performed to understand where bound states exist as a function of boson-fermion coupling strength, GL parameter, and effective fermion mass. Time dependent equations are derived and used to demonstrate stability of solutions and where they become unstable.

A. Stationary equations of motion

A general ansatz for a stationary fermion bound state confined to a scalar vortex can be given by

$$\phi(R, \theta) = \phi(R)e^{im\theta}, \quad \text{and} \quad (38)$$

$$\psi(t, R, \theta) = \begin{pmatrix} \Psi_1(R)e^{im_1\theta}e^{-i\omega_1 t} \\ \Psi_2(R)e^{im_2\theta}e^{-i\omega_2 t} \\ \pm i\Psi_3(R)e^{im_3\theta}e^{-i\omega_3 t} \\ \pm i\Psi_4(R)e^{im_4\theta}e^{-i\omega_4 t} \end{pmatrix}. \quad (39)$$

The desire to have stable scalar vortices leads to selecting a vortex winding number of $m = 1$. The ω_i are all set to ω so that the time-dependent equations become separable in time. The conditions

$$m_4 = m_1 + 1 \quad \text{and} \quad (40)$$

$$m_2 = m_3 + 1 \quad (41)$$

allow for angular separation of the partial differential equations and also lead to the fermion solutions being eigenstates of total angular momentum in the z direction, $\hat{J}_z$. The addition of $\pm i$ to ψ_3 and ψ_4 is simply an overall phase that simplifies the field equations that follow. One can then assume $A_z = 0$, which decouples (ψ_1, ψ_4) and (ψ_2, ψ_3) , which allows one to set $\psi_2 = \psi_3 = 0$. Finally, setting $m_1 = 0$ leads to the fermion solutions being $j_z = +\frac{1}{2}$ eigenstates. These assumptions lead to the stationary equations of motion

$$\begin{aligned} \partial_R \chi &= -2\kappa^{-1} \frac{\tilde{A}_\theta}{R} \phi + \kappa^{-2} \phi (-A_t^2 + \tilde{A}_\theta^2) \\ &\quad - (1 - \kappa_m \bar{\psi} \psi) \phi + \phi^3, \end{aligned} \quad (42)$$

$$\frac{\partial}{\partial R^2} (R\phi) = \frac{1}{2} \chi, \quad (43)$$

$$\partial_R B^z = -\frac{\phi^2}{\kappa R} + \kappa^{-2} \tilde{A}_\theta \phi^2 - \frac{\kappa_d}{\kappa} (\Psi_1 \Psi_4), \quad (44)$$

$$\frac{1}{R} \partial_R (R \tilde{A}_\theta) = B_z, \quad (45)$$

$$\frac{1}{R} \partial_R (R E^R) = \kappa^{-2} A_t \phi^2 + \frac{\kappa_d}{2\kappa} (\Psi_1^2 + \Psi_4^2), \quad (46)$$

$$\partial_R A_t = E_R, \quad (47)$$

$$\begin{aligned} \partial_R \Psi_1 &= -\left(\omega + \frac{1}{2\kappa} A_t + \kappa_d^{-1} \left(1 + \frac{\kappa_m \hat{\phi}^2}{2}\right)\right) \Psi_4 \\ &\quad - \frac{1}{2\kappa} \tilde{A}_\theta \Psi_1, \quad \text{and} \end{aligned} \quad (48)$$

$$\begin{aligned} \frac{1}{R} \partial_R (R \Psi_4) &= \left(\omega + \frac{1}{2\kappa} A_t - \kappa_d^{-1} \left(1 + \frac{\kappa_m \hat{\phi}^2}{2}\right)\right) \Psi_1 \\ &\quad + \frac{1}{2\kappa} \tilde{A}_\theta \Psi_4, \end{aligned} \quad (49)$$

where

$$\bar{\psi} \psi = \Psi_1^2 - \Psi_4^2 \quad \text{and} \quad (50)$$

$$\tilde{A}_\theta = A_\theta / R = A^\theta R. \quad (51)$$

The field χ was introduced to help with regularity at the origin when solving these equations numerically. While it is helpful to be able to refer to the fermion bound states (ψ_1 and ψ_4) and the scalar vortex solutions (ϕ) as distinct entities, when being discussed as a single entity their combination will hereafter be referred to as a “charged vortex.”

B. Approximate closed-form solutions

To get an approximate understanding of when bound states exist, an effective potential is obtained by taking an additional spatial derivative of (48) and substituting repeatedly to put the resulting equation in the form

$$\frac{\partial^2 \Psi_1}{\partial R^2} \approx f_1 \Psi_1, \quad (52)$$

where

$$f_1 = \kappa_d^{-2} \left(1 + \frac{1}{2} \kappa_m \hat{\phi}^2 \right)^2 - \left(\omega + \frac{1}{2\kappa} A_t \right)^2 + \frac{\tilde{A}_\theta^2}{4\kappa^2} - \frac{1}{2\kappa} \left(B_z - \frac{\tilde{A}_\theta}{R} \right) \quad (53)$$

and terms that multiply ψ_4 are neglected since for most solutions, especially for large effective fermion mass, the Ψ_4 terms are small relative to Ψ_1 . Based on the linear properties of the scalar field at the origin of the vortex, the following form for the scalar field is a reasonable approximation:

$$\phi \approx \frac{R}{a}, \quad (54)$$

which to leading order in R creates a spatially harmonic potential for a positive (repulsive) boson-fermion interaction. In the large κ limit, the gauge field contributions are small, which leads to

$$f_1 \approx \kappa_d^{-2} \left(1 + \kappa_m \left(\frac{R}{a} \right)^2 \right) - \omega^2. \quad (55)$$

Equations (52) and (55) are known to have Gaussian solutions

$$\Psi_1(R) = \Psi_{1,0} \exp \left(-\frac{R^2}{2\sigma^2} \right) \quad (56)$$

for real constant $\Psi_{1,0}$ and when subjected to the constraints

$$\omega^2 = \sigma^{-2} + \kappa_d^{-2} \quad \text{and} \quad (57)$$

$$\sigma^{-4} = \frac{\kappa_m \kappa_d^{-2}}{a^2}. \quad (58)$$

Equation (57) is the dispersion relation and corresponds to the energy-momentum relation one might use in condensed matter systems, $E(p)$, where $\omega \propto E$, $\sigma^{-2} \propto p^2$, $(\kappa_d^{-1})^2 \propto m_F^2$, and $\kappa_d^{-1} > 0$ describes a gapped material.

Knowing that bound states can exist when f_1 is negative within the trap and positive outside the trap, using $f_1(0) < 0$ and $f_1(R_{\max}) > 0$ allows one to obtain a condition on ω that should support fermion bound states:

$$\omega^2 > \kappa_d^{-2} \quad \text{and} \quad (59)$$

$$\omega^2 < \kappa_d^{-2} \left(1 + \frac{1}{2} \kappa_m \right)^2, \quad (60)$$

or, more simply,

$$\kappa_d^{-1} < \omega < \kappa_d^{-1} \left(1 + \frac{1}{2} \kappa_m \right). \quad (61)$$

Solving (60) for the roots of κ_m , one gets

$$\kappa_m^+ \geq 2\delta_\omega \quad \text{and} \quad (62)$$

$$\kappa_m^- \leq -4 - 2\delta_\omega, \quad (63)$$

where one can define

$$\delta_\omega = \frac{\omega}{\kappa_d^{-1}} - 1. \quad (64)$$

The focus is on the positive (repulsive) root for fermion states so they can be bound within the vortex, and it can be seen that there are no bound states when $\kappa_m = 0$. Equation (61) describes the energy band of allowable bound states and is bounded from below by the effective mass ($\kappa_d^{-1} \propto m_F$) and bounded from above by the effective mass increased by a term proportional to the boson-fermion interaction strength (κ_m). The size of the energy band goes to zero in the limit that the interaction strength goes to zero ($\kappa_m \rightarrow 0$), implying a lack of bound states when there is no boson-fermion interaction.

Another aspect of these solutions that can be understood in closed form is the impact of the fermion field on the spontaneously broken scalar vacuum. The shape of the scalar field vortex and the location of its vacuum expectation value are determined by the potential

$$V(\phi) = -\frac{1}{2} (1 - \kappa_m \bar{\psi} \psi) \phi^2 + \frac{1}{4} \phi^4, \quad (65)$$

where for small $\kappa_m \bar{\psi} \psi$ the expectation value approaches the uncharged vortex solution with expectation value $\phi_0 = 1$. For larger fermion field strength, the shape of the potential will change, eventually reaching a point where the spontaneously broken vacuum collapses back to $\phi_0 = 0$ as $\kappa_m \bar{\psi} \psi \rightarrow 1$. This effectively destroys superconductivity in a larger region than just the center of the vortex, flattening out the harmonic trap for the fermion field, and suggests a loss of containment of the fermion field by the trap. Using (58), the condition on the fermion field for when this occurs becomes

$$\psi_{1,0}^2 = \frac{\sigma^4 \chi_0^2}{4} (\kappa_d^{-1})^2, \quad (66)$$

where χ_0 is the value of χ at the origin and $\sigma^4 \chi_0^2 / 4$ is close to unity when the width of the fermion bound state is close to the width of the vortex. Returning briefly to dimensionful variables, (21), (28), and (66) imply that the range of effective masses that support the existence of bound states has an upper limit determined by the parameters of the bulk:

$$0 \leq m_F \lesssim \left(\frac{m_B}{\phi_0^2 \xi^3} \right). \quad (67)$$

For typical bulk parameters $\phi_0^2 \approx 10^{30} \text{ m}^{-3}$, $\xi \approx 10^{-9} \text{ m}$, and $m_B = 2m_e$, this condition becomes

$$0 \leq m_F \lesssim 2 \times 10^{-3} m_e, \quad (68)$$

which is comparable to the effective masses of quasiparticles in many gapped Dirac materials [40,41]. In summary, for the approximate bound states discussed here, the effective mass $\kappa_d^{-1} \propto m_F$ sets the mass gap, while the interaction strength κ_m determines the energy band of allowed bound states.

C. Stationary numerical solutions

This section presents the numerical solutions to the full stationary equations of motion, demonstrating the existence of bound state solutions across a wide range of GL parameters (κ), effective fermion masses (κ_d^{-1}), and boson-fermion interaction strengths (κ_m). Starting with a single point in this large parameter space, Fig. 1 displays the basic attributes of typical fermion bound states confined to the core of a scalar vortex. The scalar field, fermion field components, and electric and magnetic fields are shown for a range of GL parameters, $\kappa = 1, 10, 100$, and effective fermion mass $\kappa_d^{-1} = 1$. The confinement strength κ_m is tuned to obtain a unit radius at half max for the ψ_1 component of the fermion field for $\kappa = 1$. The scaling of the spatial coordinates by the coherence length (13) ensures that the vortex core will remain roughly on the order of unity. However, as one increases the GL parameter, the London penetration depth increases, which increases the penetration of the field into the bulk while keeping the topologically conserved magnetic flux constant. This reduces the field strength within the vortex core so that, in the large- κ limit, the shape of the at-rest bound state fermion becomes independent of the electromagnetic fields. Hereafter, the radius of $|\psi_1(R)|$ at half max is denoted R_ψ .

The volume of parameter space where bound states exist is first explored by varying the boson-fermion interaction strength (κ_m). Figure 2 displays plots of the fermion bound state radius R_ψ as a function of κ_m across a range of effective fermion masses κ_d^{-1} . The results show that increasing the interaction strength decreases the radius of a given bound state; each curve (distinct κ_d^{-1}) is observed to approach a small R_ψ , limiting behavior of $R_\psi \propto b(\kappa_m)^{-1/4}$ with best-fit parameters captured in Table I. This is precisely the behavior predicted by the condition (58) on the closed-form approximate solution (56) that assumes the vortex acts as a spatially harmonic trap. However, as the interaction strength is decreased, the radius of the bound state increases, eventually extending beyond the size of the

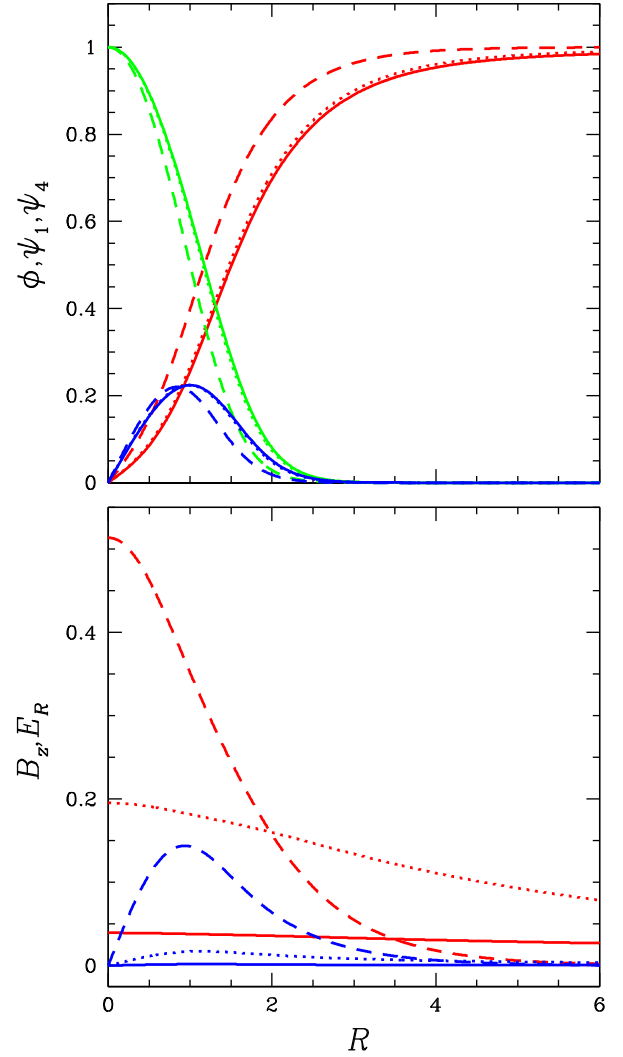


FIG. 1. Plots of scalar field ϕ (top, red lines), fermion field components ψ_1 (top, green lines), ψ_4 (top, blue lines), electric field E_R (bottom, red lines), and magnetic field B_z (bottom, blue lines) for a bound state solution with effective mass $\kappa_d^{-1} = 1$ and GL parameter values $\kappa = 1, 10, 100$ in dashed, dotted, and solid lines, respectively. The confinement strength κ_m is chosen to give a fermion radius of unity for $\kappa = 1$.

trap. As κ_m is decreased even further, the trap can no longer contain the fermion field, and bound states cease to exist. Although κ_m can become arbitrarily small for arbitrarily large κ_d^{-1} , for any specific value of κ_d^{-1} there is a nonzero finite κ_m below which bound states do not exist. The roughly log-periodic spacing of the plots with different κ_d^{-1} and the best-fit values of b captured in Table I suggest that, for a given R_ψ , the amount of interaction strength needed to confine the fermion bound state scales as $\kappa_m \propto \kappa_d^2$. To more definitively determine this relationship, Fig. 3 presents κ_m as a function of κ_d^{-1} for a range of GL parameters (κ) and selected fermion bound state radii (R_ψ). For each κ and R_ψ , a curve is generated by fixing κ_d^{-1} and varying the

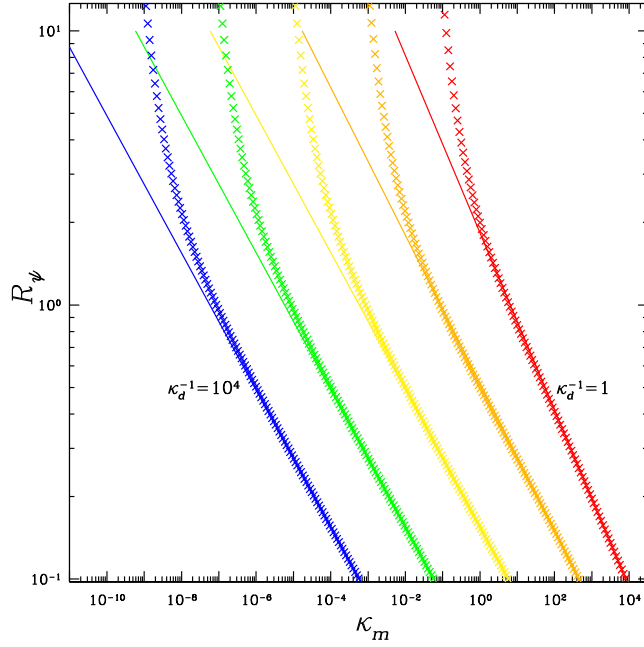


FIG. 2. Plots of fermion bound state radius R_ψ as a function of boson-fermion interaction strength κ_m for $\kappa_d^{-1} = 1, 10, 10^2, 10^3, 10^4$ in red, orange, yellow, green, and blue x's, respectively. The solid lines represent best-fit approximations to the small- R solutions that are in good agreement with the approximate closed-form solutions (see Table I).

interaction strength κ_m until the radius of the resulting bound state is the desired R_ψ . The curves are shown to very closely demonstrate the expected $\kappa_m \propto \kappa_d^2$ relationship.

Collectively, these results clearly demonstrate the existence of fermion bound states within the core of a vortex across a wide range of the $(\kappa, \kappa_d^{-1}, \kappa_m)$ parameter space. It has been shown that, when the bound state is sufficiently confined to the core of the vortex, the solutions closely resemble the closed-form Gaussian solutions confined by a spatially harmonic trap. It should be noted, however, that up to this point all solutions were generated with sufficiently

TABLE I. Table of best-fit parameters for relationship $R_\psi \approx b(\kappa_m)^p$ for a range fermion effective masses κ_d^{-1} . When comparing the numerically calculated fermion radius R_ψ to the width of the closed-form solutions, σ , the numerically obtained solutions are in close agreement with the closed-form approximate solutions (58) that predict a $R_\psi \propto (\kappa_m)^{-1/4}$ relationship, especially for larger effective fermion mass.

κ_d^{-1}	b	p
1	1.8	-0.32
10	5.2×10^{-1}	-0.27
10^2	1.5×10^{-1}	-0.25
10^3	4.9×10^{-2}	-0.25
10^4	1.5×10^{-2}	-0.25

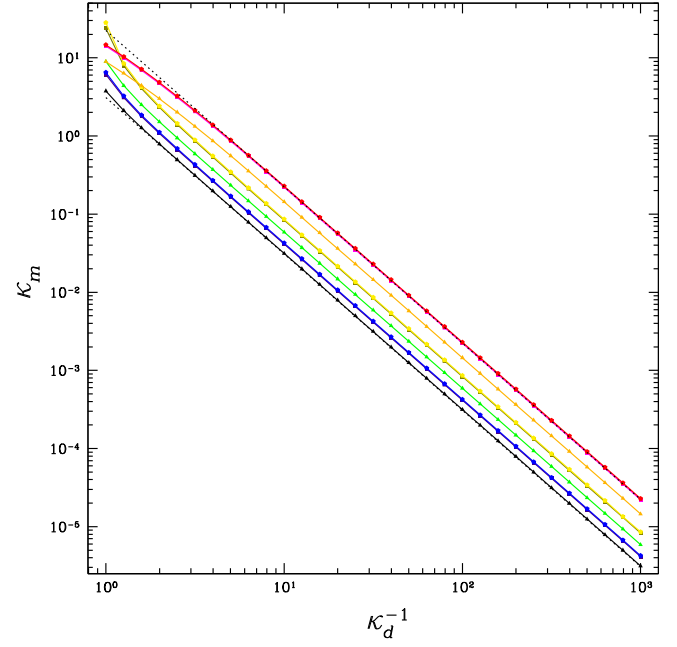


FIG. 3. Plots of boson-fermion interaction strength κ_m required to keep a fermion bound state of mass κ_d^{-1} confined to a particular radius R_ψ . From top to bottom, plots for $R_\psi = 0.75$ and $\kappa = 100, 10, 1$ are in red, magenta, and orange, respectively; plots for $R_\psi = 1$ and $\kappa = 100, 10, 1$ are in yellow, olive, and green, respectively; and plots for $R_\psi = 1.25$ and $\kappa = 100, 10, 1$ are in blue, purple, and black, respectively. Plots are bounded above and below by curves demonstrating $\kappa_m \propto (\kappa_d^{-1})^{-2}$ (dotted gray).

low fermion field strength as not to significantly adjust the spontaneously broken vacuum of the condensate. Guided again by the closed-form results in Sec. III B and the scalar potential (65), one expects a shift in the scalar vacuum and a loss of containment as $\kappa_m \bar{\psi}\psi \rightarrow 1$, which occurs when the fermion field approaches $\psi_{1,0}^2 \approx \kappa_d^{-2}$.

Figure 4 plots the fermion bound state radius (R_ψ) and the measured vortex radius (R_ϕ) as a function of fermion field strength for a range of effective fermion masses (κ_d^{-1}) and interaction strengths (κ_m). Baseline values of κ_m are determined for each that result in $R_\psi = 1$ in the limit of weak fermion field; in this limit, the measured vortex radius is close to unity and approaches the traditional (uncharged) vortex solution. As $\psi_{1,0}$ approaches κ_d^{-1} , the force between the fermion bound state and the confining scalar vortex becomes significant and the wall of the scalar vortex begins to expand so that the outward self-repulsion of the fermion field can balance with the inward force of the vortex wall. The vortex is able to accommodate some pressure, but, as the fermion bound state expands beyond the vortex, sufficient confinement force is no longer present, and solutions to the stationary equations can no longer be obtained.

While not explored parametrically in detail here, it is worth briefly discussing how the addition of a fermion

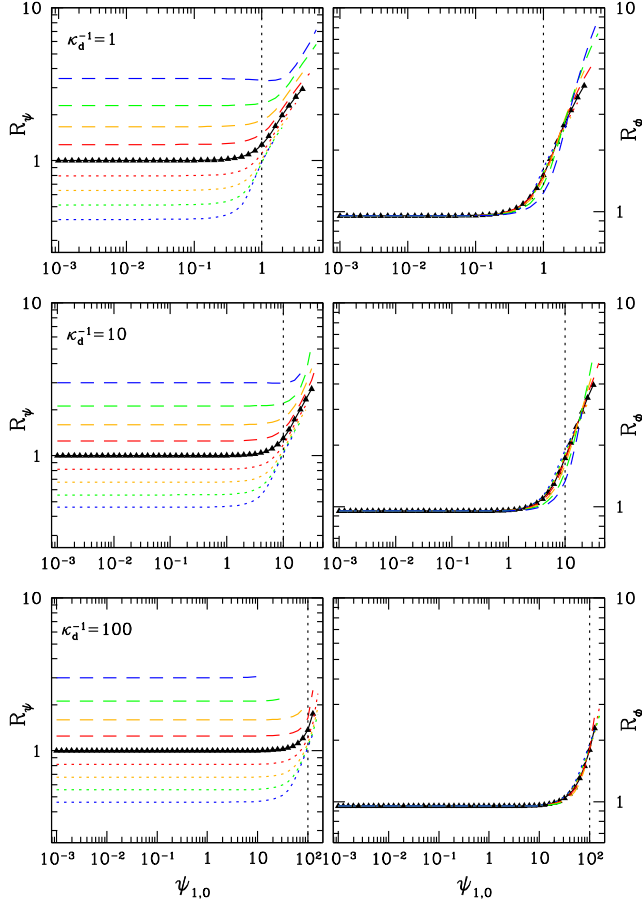


FIG. 4. Plots of fermion bound state radius R_ψ and vortex core radius R_ϕ as a function of fermion field strength for $\kappa_d^{-1} = 1, 10, 100$ (top, middle, and bottom, respectively). Each plot contains a curve (black) with the boson-fermion interaction strength κ_m that confines the fermion bound state to $R_\psi = 1$ for field strength $\psi_{1,0} = 1 \times 10^{-3}$. Additional solutions are provided by increasing κ_m by a factor of 2, 4, 8, and 16 (dotted red, orange, green, and blue curve, respectively) and decreasing κ_m by a factor of 2, 4, 8, and 16 (dashed red, orange, green, and blue curve, respectively). A $\psi_{1,0} = \kappa_d^{-1}$ vertical dotted line is drawn on each graph, and all plots have GL parameter $\kappa = 100$.

bound state impacts the overall mass and energy of the charged vortex solution. In Fig. 5, one can see that for $\psi_{1,0}$ significantly less than unity the energy from the fermion field does not significantly contribute to the overall energy. However, when the fermion field strength at the origin grows larger than unity, the contribution by the fermion bound state to the overall energy can become quite large. For solutions with $\kappa_d^{-1} = 100$, the ratio of the fermion energy to an uncharged ($\psi = 0$) vortex energy was observed to exceed 10^4 .

D. Time evolution and stability

Previous observations about the existence of stationary solutions were based on the ability to find a solution to the

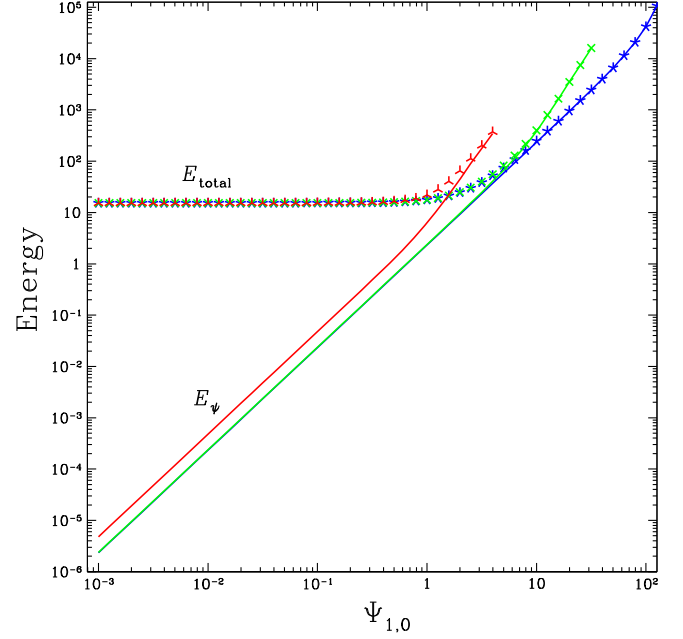


FIG. 5. Plots of fermion energy (solid lines) and total energy (points) as a function of fermion field strength $\psi_{1,0}$ for $\kappa = 100$ and $\kappa_d^{-1} = 1, 10, 100$ in red, green, and blue, respectively. The fermion energy is observed to exceed the uncharged vortex energy when $\psi_{1,0} \gtrsim 1$, and for $\kappa_d^{-1} = 100$, total energies exceeding 10^4 times the uncharged energy were observed.

time-independent stationary equations of motion; however, these solutions may not be long-term stable. This section presents the time evolution of stationary charged vortices in Sec. III C with the intent of better understanding their stability properties. The time-evolution equations for the scalar field are given by

$$\begin{aligned} \partial_t \Pi_1 = & \partial_R \chi_1 + \frac{2}{\kappa} \left(-A_t \Pi_2 + A_R \left(\chi_2 - \frac{\phi_2}{R} \right) + \frac{1}{R} \tilde{A}_\theta \phi_1 \right) \\ & - \kappa^{-2} \phi_1 (-A_t^2 + A_R^2 + \tilde{A}_\theta^2) \\ & + \phi_1 (1 - \kappa_m (a_1^2 + a_2^2 - b_1^2 - b_2^2)) \\ & - \phi_1 (\phi_1^2 + \phi_2^2), \end{aligned} \quad (69)$$

$$\begin{aligned} \partial_t \Pi_2 = & \partial_R \chi_2 - \frac{2}{\kappa} \left(-A_t \Pi_1 + A_R \left(\chi_1 - \frac{\phi_1}{R} \right) - \frac{1}{R} \tilde{A}_\theta \phi_2 \right) \\ & - \kappa^{-2} \phi_2 (-A_t^2 + A_R^2 + \tilde{A}_\theta^2) \\ & + \phi_2 (1 - \kappa_m (a_1^2 + a_2^2 - b_1^2 - b_2^2)) \\ & - \phi_2 (\phi_1^2 + \phi_2^2), \end{aligned} \quad (70)$$

$$\partial_t \chi_1 = 2 \frac{\partial}{\partial R^2} (R \Pi_1), \quad (71)$$

$$\partial_t \chi_2 = 2 \frac{\partial}{\partial R^2} (R \Pi_2), \quad (72)$$

$$\partial_t \phi_1 = \Pi_1, \quad \text{and} \quad (73)$$

$$\partial_t \phi_2 = \Pi_2. \quad (74)$$

The evolution equations for the electromagnetic fields are

$$\begin{aligned} \partial_t E^R &= \frac{1}{\kappa} (\phi_2 \chi_1 - \phi_1 \chi_2) + \kappa^{-2} A_R (\phi_1^2 + \phi_2^2) \\ &\quad - \frac{\kappa_d}{\kappa} (a_1 b_1 + a_2 b_2), \end{aligned} \quad (75)$$

$$\begin{aligned} \partial_t \tilde{E}^\theta &= -\partial_R B^z + \left(\frac{\tilde{A}_\theta}{\kappa^2} - \frac{1}{\kappa R} \right) (\phi_1^2 + \phi_2^2) \\ &\quad - \frac{\kappa_d}{\kappa} (a_1 b_2 - a_2 b_1), \end{aligned} \quad (76)$$

$$\partial_t B_z = -\frac{1}{R} \partial_R (R \tilde{E}_\theta), \quad (77)$$

$$\partial_t A_t = \frac{1}{R} \partial_R (R A_R), \quad (78)$$

$$\partial_t A_R = \partial_R A_t - E_R, \quad \text{and} \quad (79)$$

$$\partial_t \tilde{A}_\phi = -\tilde{E}_\theta. \quad (80)$$

And, finally, for the fermion field,

$$\begin{aligned} \partial_t a_1 &= -\frac{1}{R} \partial_R (R b_1) + \frac{1}{2\kappa} [-A_t a_2 - A_R b_2 + \tilde{A}_\theta b_1] \\ &\quad + \frac{1}{\kappa_d} \left(1 + \frac{1}{2} \kappa_m \hat{\phi}^2 \right) a_2, \end{aligned} \quad (81)$$

$$\begin{aligned} \partial_t a_2 &= -\frac{1}{R} \partial_R (R b_2) + \frac{1}{2\kappa} [A_t a_1 + A_R b_1 + \tilde{A}_\theta b_2] \\ &\quad - \frac{1}{\kappa_d} \left(1 + \frac{1}{2} \kappa_m \hat{\phi}^2 \right) a_1, \end{aligned} \quad (82)$$

$$\begin{aligned} \partial_t b_1 &= -\partial_R a_1 + \frac{1}{2\kappa} [-A_t b_2 - A_R a_2 - \tilde{A}_\theta a_1] \\ &\quad - \frac{1}{\kappa_d} \left(1 + \frac{1}{2} \kappa_m \hat{\phi}^2 \right) b_2, \quad \text{and} \end{aligned} \quad (83)$$

$$\begin{aligned} \partial_t b_2 &= -\partial_R a_2 + \frac{1}{2\kappa} [A_t b_1 + A_R a_1 - \tilde{A}_\theta a_2] \\ &\quad + \frac{1}{\kappa_d} \left(1 + \frac{1}{2} \kappa_m \hat{\phi}^2 \right) b_1. \end{aligned} \quad (84)$$

Figure 6 displays four stability parameters obtained by time evolving solutions to the stationary equations of motion for a wide range of GL parameters (κ) and effective fermion mass (κ_d^{-1}); each parameter measures the variation in a quantity that should remain constant over time for a stable bound state. The first stability parameter considered is the

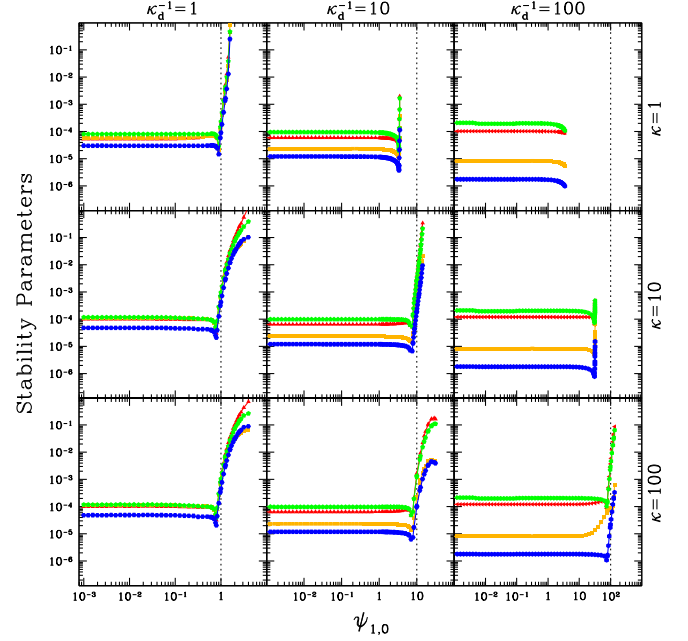


FIG. 6. Plots of stability parameters as a function of fermion field strength $\psi_{1,0}$ for a range of κ_d^{-1} and κ . The L2 norms of the difference in max/min envelopes for $|\psi_1|$ and $|\psi_4|$ normalized by $\psi_{1,0}$ are plotted in green and blue, respectively. The standard deviation in the fermion radius R_ψ is plotted in red, and the standard deviation in the frequency of the fermion field at $R = 0$ is plotted in orange. Evolutions with $\kappa_d^{-1} = 100$ were evolved to $t = 200$, and all other evolutions to $t = 40$, and a $\psi_{1,0} = \kappa_d^{-1}$ vertical dotted line is drawn on each graph.

standard deviation of radius of the bound state (R_ψ) over the duration of the simulation. Also considered is the standard deviation of the instantaneous angular frequency of the ψ_1 component of the fermion field

$$\omega = (a_2 \partial_t a_1 - a_1 \partial_t a_2) / (a_1^2 + a_2^2) \quad (85)$$

at the core of the vortex. The last two stability parameters are obtained by calculating the difference between the maximum and minimum of $|\psi_1(t, R)|$ and $|\psi_4(t, R)|$ over time at each point and then taking the L2 norm over a spatial domain containing the vortex and bound state. The results show that bound state solutions are stable when $\tilde{\psi}\psi$ does not significantly change the vacuum of the scalar field. For small $\tilde{\psi}\psi$, the trap remains harmonic, but when $\tilde{\psi}\psi \rightarrow \kappa_m^{-1}$, the harmonic trap deforms and flattens out. For $\kappa \gtrsim \kappa_d^{-1}$, the fermion bound states are observed to be stable up to $\psi_{1,0} \approx \kappa_d^{-1}$ as suggested by the closed-form analysis. For $\kappa \ll \kappa_d^{-1}$, time-independent stationary bound states were not even obtainable up to the $\psi_{1,0} \approx \kappa_d^{-1}$ threshold, but the stationary states that do exist are observed to be stable up to that threshold, with a small region of instability on the boundary. When $\tilde{\psi}\psi$ is large, two distinct effects lead to instability. First, the boson-

fermion interaction becomes strong enough to change the shape of the trap (65), flattening the trap and lowering the confinement force on the bound state. Second, for $\kappa \ll \kappa_d^{-1}$, the gauge potential influences the effective potential (53) by reducing the potential barrier that would confine the bound state, thereby lowering the fermion energy that can be supported by the trap.

IV. HEAD-ON SCATTERING

Previous sections have established the conditions for the existence and stability of fermion bound states within a vortex core when a repulsive boson-fermion interaction is present. This section offers a brief look at the scattering of two such charged vortices. For each simulation, initial data are created using the time-independent stationary equations of motion to generate at-rest vortex solutions. The at-rest solutions are individually boosted head on at one another (zero impact parameter) and superimposed to create the initial data. The time-dependent $(2+1)$ equations of motion are then evolved to simulate the scattering event.

The boost equations are described in Appendix A, while the time-evolution equations for the complex scalar field are given by

$$\begin{aligned} \partial_t \Pi_1 = & \partial_x \Phi_{1x} + \partial_y \Phi_{1y} - \kappa^{-2} \phi_1 (-A_t^2 + A_x^2 + A_y^2) \\ & + 2\kappa^{-1} (-A_t \Pi_2 + A_x \Phi_{2x} + A_y \Phi_{2y}) \\ & + \phi_1 (1 - \kappa_m (a_1^2 + a_2^2 - b_1^2 - b_2^2)) \\ & - \phi_1 (\phi_1^2 + \phi_2^2), \end{aligned} \quad (86)$$

$$\begin{aligned} \partial_t \Pi_2 = & \partial_x \Phi_{2x} + \partial_y \Phi_{2y} - \kappa^{-2} \phi_2 (-A_t^2 + A_x^2 + A_y^2) \\ & - 2\kappa^{-1} (-A_t \Pi_1 + A_x \Phi_{1x} + A_y \Phi_{1y}) \\ & + \phi_2 (1 - \kappa_m (a_1^2 + a_2^2 - b_1^2 - b_2^2)) \\ & - \phi_2 (\phi_1^2 + \phi_2^2), \end{aligned} \quad (87)$$

$$\partial_t \Phi_{1x} = \partial_x \Pi_1, \quad (88)$$

$$\partial_t \Phi_{1y} = \partial_y \Pi_1, \quad (89)$$

$$\partial_t \Phi_{2x} = \partial_x \Pi_2, \quad (90)$$

$$\partial_t \Phi_{2y} = \partial_y \Pi_2, \quad (91)$$

$$\partial_t \phi_1 = \Pi_1, \quad \text{and} \quad (92)$$

$$\partial_t \phi_2 = \Pi_2. \quad (93)$$

The equations for the electromagnetic fields and vector potential are given by

$$\begin{aligned} \partial_t E_x = & \partial_y B_z + \frac{1}{\kappa} (\phi_2 \Phi_{1x} - \phi_1 \Phi_{2x}) + \kappa^{-2} A_x (\phi_1^2 + \phi_2^2) \\ & - \kappa_d \kappa^{-1} (a_1 b_1 + a_2 b_2), \end{aligned} \quad (94)$$

$$\begin{aligned} \partial_t E_y = & -\partial_x B_z + \frac{1}{\kappa} (\phi_2 \Phi_{1y} - \phi_1 \Phi_{2y}) + \kappa^{-2} A_y (\phi_1^2 + \phi_2^2) \\ & - \kappa_d \kappa^{-1} (a_1 b_2 - a_2 b_1), \end{aligned} \quad (95)$$

$$\partial_t E_z = \partial_x B_y - \partial_y B_x, \quad (96)$$

$$\partial_t B_x = -\partial_y E_z, \quad (97)$$

$$\partial_t B_y = \partial_x E_z, \quad (98)$$

$$\partial_t B_z = -(\partial_x E_y - \partial_y E_x), \quad (99)$$

$$\partial_t A_x = \partial_x A_t - E_x, \quad (100)$$

$$\partial_t A_y = \partial_y A_t - E_y, \quad \text{and} \quad (101)$$

$$\partial_t A_t = \partial_x A_x + \partial_y A_y. \quad (102)$$

And, finally, the evolution equations for the fermion field components are given by

$$\begin{aligned} \partial_t a_1 = & -\partial_x b_1 - \partial_y b_2 + \frac{1}{2} \kappa^{-1} (-A_t a_2 - A_x b_2 + A_y b_1) \\ & + \kappa_d^{-1} \left(1 + \frac{1}{2} \kappa_m (\phi_1^2 + \phi_2^2) \right) a_2, \end{aligned} \quad (103)$$

$$\begin{aligned} \partial_t a_2 = & -\partial_x b_2 + \partial_y b_1 + \frac{1}{2} \kappa^{-1} (A_t a_1 + A_x b_1 + A_y b_2) \\ & - \kappa_d^{-1} \left(1 + \frac{1}{2} \kappa_m (\phi_1^2 + \phi_2^2) \right) a_1, \end{aligned} \quad (104)$$

$$\begin{aligned} \partial_t b_1 = & -\partial_x a_1 + \partial_y a_2 + \frac{1}{2} \kappa^{-1} (-A_t b_2 - A_x a_2 - A_y a_1) \\ & - \kappa_d^{-1} \left(1 + \frac{1}{2} \kappa_m (\phi_1^2 + \phi_2^2) \right) b_2, \quad \text{and} \end{aligned} \quad (105)$$

$$\begin{aligned} \partial_t b_2 = & -\partial_x a_2 - \partial_y a_1 + \frac{1}{2} \kappa^{-1} (A_t b_1 + A_x a_1 - A_y a_2) \\ & + \kappa_d^{-1} \left(1 + \frac{1}{2} \kappa_m (\phi_1^2 + \phi_2^2) \right) b_1. \end{aligned} \quad (106)$$

The fact that vortices scatter at right angles when colliding head on and above a critical velocity is well known and discussed throughout the literature [7,8,10,42,43]. This work focuses on new phenomenology that arises with the addition of a fermion bound state. As an in-depth scattering analysis will be saved for future work, this initial look at scattering is limited to effective fermion mass $\kappa_d^{-1} = 1$ to keep the frequency of oscillation of the fermion

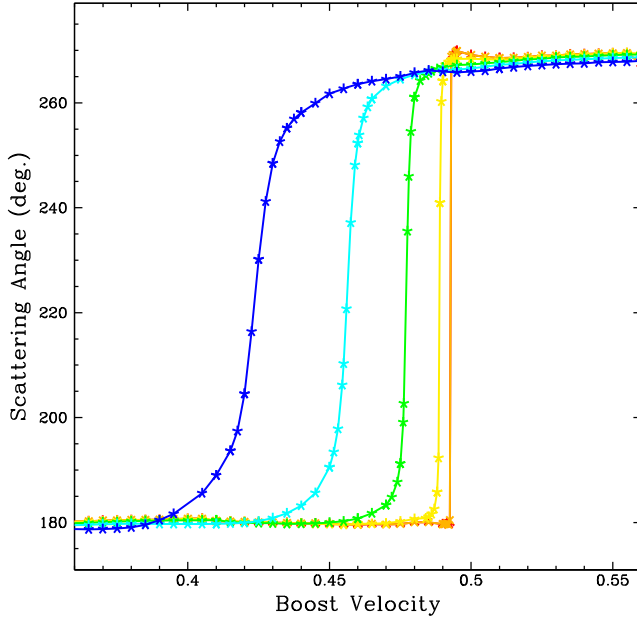


FIG. 7. Plots of scattering angle as a function of boost velocity for head-on collisions of identical vortices. Curves are presented for fermion field strengths of $\psi_{1,0} = 10^{-3}, 10^{-2}, 10^{-1}, 2 \times 10^{-1}, 3 \times 10^{-1}, 4 \times 10^{-1}$ in red, orange, yellow, green, cyan, and blue, respectively. Vortices are boosted at each other with zero initial impact parameter. All solutions are for $\kappa_d^{-1} = 1$ and $\kappa = 2$ and use a κ_m that gives $R_\psi = 1$ for $\psi_{1,0} = 10^{-3}$.

field on the order of unity and $\kappa = 2$ to allow a transition (from back- to right-angle scattering) boost velocity on the order of $v_b \approx 0.5$; such a velocity allows the dynamics to unfold quickly while also not significantly Lorentz contracting the solutions and requiring very fine numerical grids [43].

Figure 7 displays results of scattering angle as a function of boost velocity for a range of fermion field strengths. In the limit of weak fermion field strength, one sees nearly right-angle scattering, consistent with previous results. However, even in the weak-field limit, electromagnetic forces result in a slight deviation in the vortex trajectory, resulting in a small effective impact parameter that precludes perfect right-angle scattering. As the field strength increases, both sharpness of the transition (with respect to boost velocity) and the value at which the transition occurs decreases. For low-velocity collisions, the vortices are observed to elastically backscatter similar to traditional vortices. For high-velocity collisions, the vortices undergo deep inelastic scattering and new phenomenology is observed.

Figure 8 demonstrates high-velocity ($v_b = 0.75$) scattering in the strong (fermion) field regime ($\psi_{1,0} = 0.4$). The vortices can be seen to scatter at near right angles like uncharged vortices, but one also sees the production of new localized bound states. These bound states are not vortices, as they do not have a topological charge; the winding number of the complex scalar field is zero. The solutions

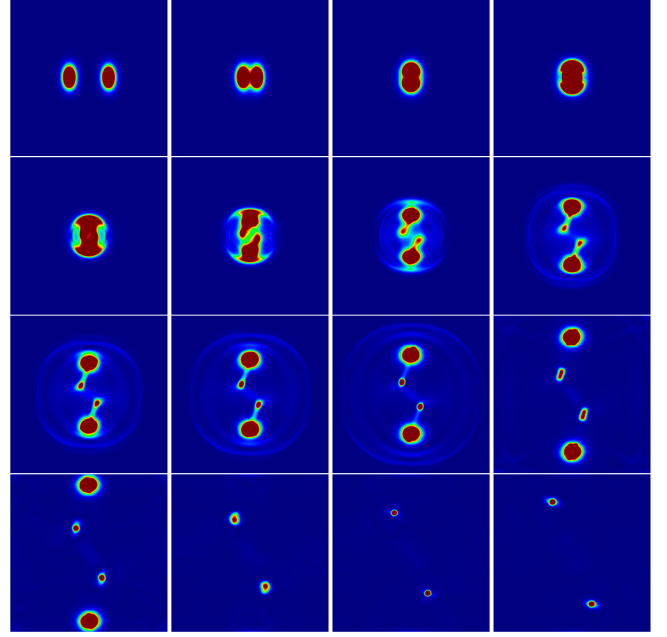


FIG. 8. Time slices of the energy density for the head-on collision of two charged vortices with $v_x(0) = \pm 0.75$ and $\psi_{1,0} = 0.4$. The plots represent times for $n = 0, 175, 270, 315, 365, 420, 500, 600, 750, 900, 1000, 1400, 1650, 2100, 2400$, and 2944 , where $t = n\Delta t$ and $\Delta t = 0.025$, displayed from top left to bottom right, respectively. The displayed simulations were conducted on an 801×801 grid with domain spanning $\{x: -20 < x < 20\}$ and $\{y: -20 < y < 20\}$. The sequence shows approximate right-angle scattering of the charged vortices and the generation of two stable bubbles.

are “bubblelike” states where the scalar field interpolates between the spontaneously broken vacuum of the bulk, ϕ_0 , and the dynamically induced vacuum created by the boson-fermion interaction (65) that becomes significant when $\bar{\psi}\psi \rightarrow \kappa_m^{-1}$. Figure 9 displays similar vortex dynamics for a slightly higher fermion field strength ($\psi_{1,0} = 0.5$). The vortices still undergo nearly right-angle scattering, but significantly more interaction can be observed; four bubbles are produced and ejected from the collision, while one large bubble remains in the system center of mass. As one continues to increase the fermion field strength of the scattering vortices, one reaches a point at which right-angle scattering breaks down. Figure 10 displays vortex dynamics for $\psi_{1,0} = 1.5$. The vortices are observed to coalesce for a period of time, creating a pseudostable $m = 2$ vortex that eventually decays into two stable $m = 1$ vortices while producing bubbles in the process. Two stable bubbles can be seen ejected orthogonal to the original collision axis, while one bubble is again seen remaining in the system center of mass. Unstable bubbles can also be seen to appear (most clearly around $n = 1720$) but eventually disperse, seemingly when the surface tension of the scalar bubble outweighs the outward repulsive force from the fermion bound state.

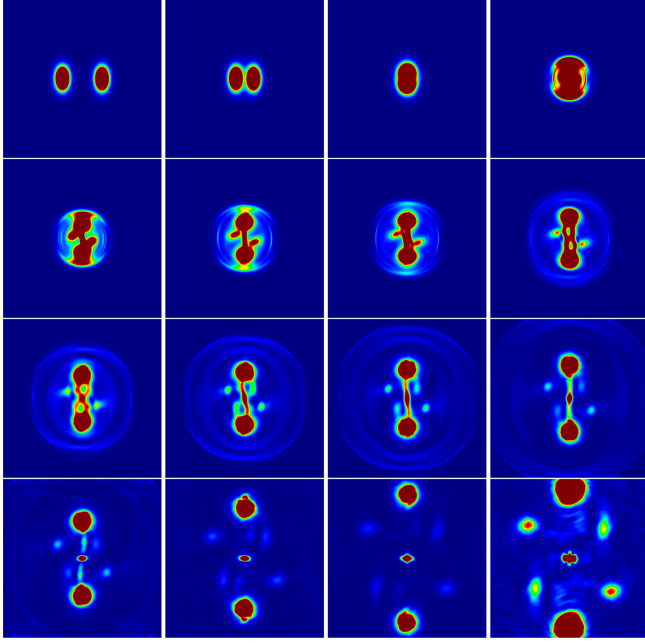


FIG. 9. Time slices of the energy density for the head-on collision of two charged vortices with $v_x(0) = \pm 0.75$ and $\psi_{1,0} = 0.5$. The plots represent times for $n = 0, 150, 300, 425, 500, 557, 600, 700, 800, 900, 1000, 1150, 1300, 1800$, and 2300 , where $t = n\Delta t$ and $\Delta t = 0.025$, displayed from top left to bottom right, respectively. The displayed simulations were conducted on an 801×801 grid with domain spanning $\{x: -20 < x < 20\}$ and $\{y: -20 < y < 20\}$. The sequence shows approximate right-angle scattering of the charged vortices, the generation of an at-rest bubble at the system center of mass, and the generation of four bubbles being ejected outward from the collision. The color map for the plot at $n = 2500$ was rescaled to more clearly see the outgoing bubbles.

The final example of collisions to be discussed here can be seen in Fig. 11, which shows time-elapsd pictures of a collision with velocity slightly above where the vortices are observed to backscatter. The vortices are observed to touch, coalesce, interact, and eventually disperse. When the scalar vortices remain in close proximity, they create an effective trap for the fermion bound states that is significantly larger than the individual harmonic traps that originally confined each of the stationary vortices. The fermion bound states are observed to stably interact in the larger trap for a time $T \approx 100$, and then the dynamics become unstable and the fermion bound states (and the scalar vortices that form the trap they are contained within) eventually disperse. To better understand the transition from stable rotation to instability, it is helpful to consider the relative contributions of ψ_1^2 and ψ_4^2 over time and the implications to linear momentum, orbital angular momentum, and spin (Fig. 12). In the Dirac representation, the four-component spinor representing spin-up particles at rest is given by $\psi \propto (1, 0, 0, 0)$. In a frame that is boosted in the x - y plane, the new spinor is given by

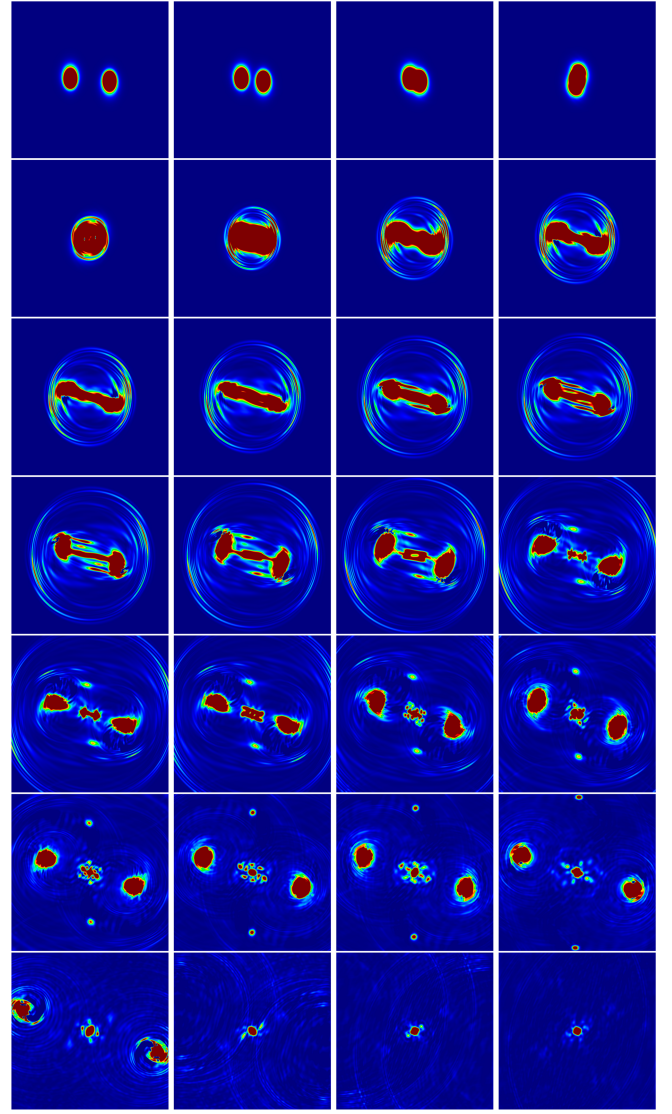


FIG. 10. Time slices of the energy density for the head-on collision of two charged vortices with $v_x(0) = \pm 0.75$ and $\psi_{1,0} = 1.5$. The plots represent times for $n = 0, 120, 200, 290, 400, 500, 600, 640, 680, 750, 800, 825, 875, 950, 1000, 1100, 1150, 1200, 1300, 1400, 1526, 1720, 1800, 2000, 2500, 3200, 3600$, and 4000 , where $t = n\Delta t$ and $\Delta t = 0.025$, displayed from top left to bottom right, respectively. The displayed simulations were conducted on an 801×801 grid with domain spanning $\{x: -20 < x < 20\}$ and $\{y: -20 < y < 20\}$. The sequence shows rich inelastic interactions that eject the vortices at off angles (not back or right-angle scattering), the creation and ejection of two bubbles, and the creation of a stationary bubble at the system center of mass.

$\psi \propto (1, 0, 0, (p_x + ip_y)/(E + m))$. As such, for a given fermion field strength ($\psi^\dagger \psi = \psi_1^2 + \psi_4^2$) the relative increase in ψ_4^2 can be interpreted as an increase in the average momentum $|p|^2 = p_x^2 + p_y^2$. For $T \gtrsim 100$, the momentum in the fermion field continues to increase until the vortices begin to radiate energy and eventually disperse.

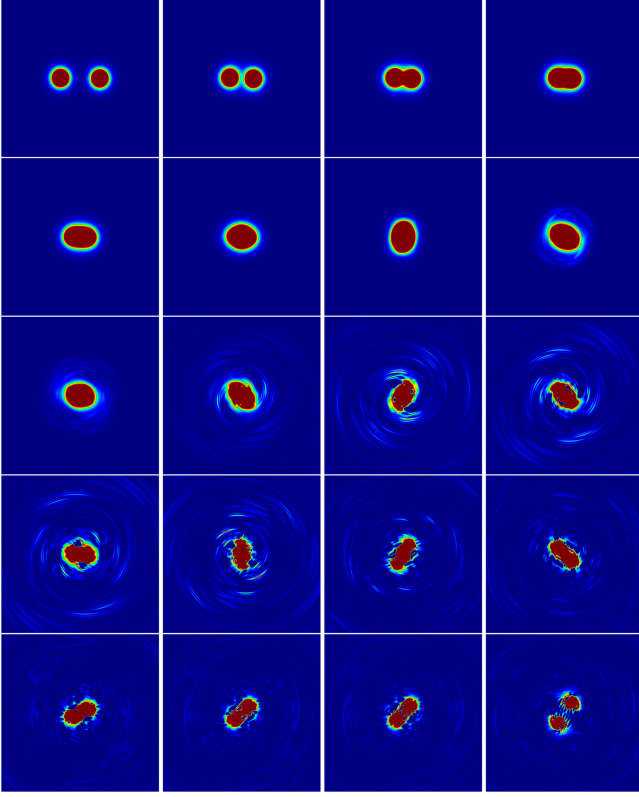


FIG. 11. Time slices of the energy density for the head-on collision of two charged vortices with $v_x(0) = \pm 0.25$ and $\psi_{1,0} = 1.0$. The plots represent times for $n = 0, 350, 550, 682, 725, 797, 972, 1132, 1222, 5467, 5930, 6173, 6337, 6754, 7981, 8262, 8579, 8665, 8693$, and 8810 , where $t = n\Delta t$ and $\Delta t = 0.025$, displayed from top left to bottom right, respectively. The displayed simulations were conducted on an 801×801 grid with domain spanning $\{x: -20 < x < 20\}$ and $\{y: -20 < y < 20\}$. The sequence shows coalescence around $n \approx 600$, stable interaction through $n \approx 4000$, and unstable dynamics through $n \approx 8600$, followed by eventual dispersal.

It is also helpful to remember that in isolation, prior to the interaction of the collision, each fermion bound state is an eigenstate of total angular momentum:

$$\hat{J}_z = \hat{L}_z + \hat{S}_z \quad (107)$$

$$= -i\partial_\theta + \hat{\Sigma}_z, \quad (108)$$

where $\Sigma_z = \frac{1}{2}\text{diag}(1, -1, 1, -1)$ is the spin projection matrix and all ψ are eigenstates with eigenvalue $j_z = +\frac{1}{2}$. While interacting, however, since total angular momentum is conserved while the intrinsic spin $\psi_1^2 - \psi_4^2$ decreases, Fig. 12 also implies an increase in orbital angular momentum during the mixing of the fermion states. This increase in both linear momentum and orbital angular momentum appears to contribute to the eventual instability and dispersal of the vortices.

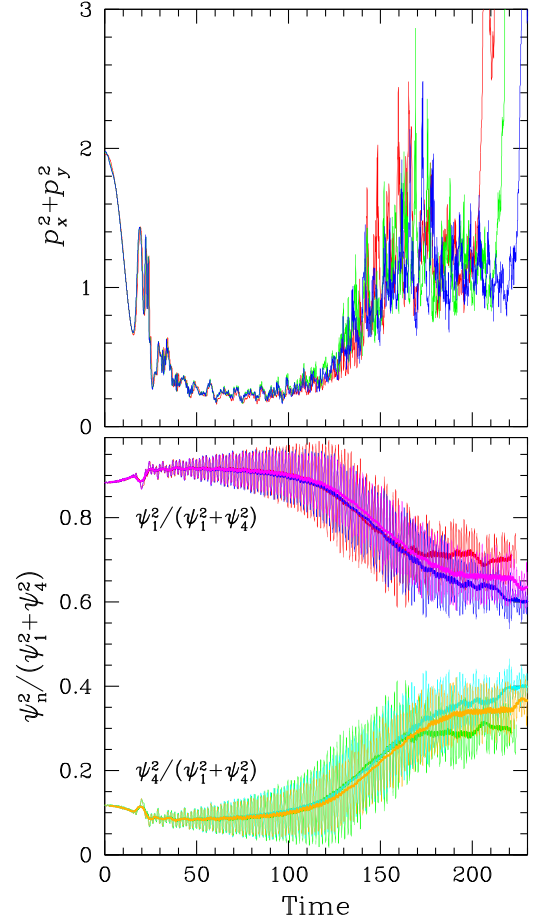


FIG. 12. Plots of the spatially integrated (linear) momentum squared as a function of time (top), for $l = 1, 2, 3$ in red, green, and blue, respectively. The bottom plot shows the instantaneous and time-averaged relative values of $\psi_l^2 / (\psi_1^2 + \psi_4^2)$ for levels $l = 1, 2, 3$ in red, blue, and magenta, respectively, and of $\psi_n^2 / (\psi_1^2 + \psi_4^2)$ for levels $l = 1, 2, 3$ in green, cyan, and orange, respectively. The simulations were conducted on $N \times N$ grids where $N = 200 \times 2^l + 1$.

Lastly, it is noteworthy to acknowledge the numerical challenges posed by this low-energy, deeply inelastic scattering class of solutions. The momentum gained by the fermion field increases the wave vector and creates gradients in the field that increase the solution error. A helpful diagnostic to monitor is the relative error in the total conserved energy:

$$E_{\text{error}}(t) = \frac{E(t) + E_{\text{rad}} - E_0}{E_0}, \quad (109)$$

where $E(t)$ is current energy in the grid at time t , E_{rad} is the radiated energy calculated by integrating the power through the outer boundary surface from $t = 0$ to t , and E_0 is the energy in the grid at $t = 0$. For most evolutions discussed in this section, $E_{\text{error}}(t)$ remained less than or on the order of 10^{-4} . However, for the evolution described in Fig. 11, where the fermion field underwent a significant

acceleration and spatial contraction, the error became significantly higher at late stages of the evolution; the error was roughly 10^{-4} at $t = 150$, but as the momentum (wave vector) of the field increased, the solution error increased to almost 10^{-2} by the end of the evolution when dissipative effects required for stability began to have a significant effect (Appendix A).

Throughout this work, $E_{\text{error}}(t)$ and the conservation of $\psi^\dagger\psi$ were monitored and grid resolutions were used that kept the error within acceptable tolerances. The methods employed maintain code stability, are convergent, and have errors that disappear in the limit that the lattice spacing goes to zero. However, if in future work one desires to study solutions that dynamically increase in wave vector for longer periods of time, one may want to employ adaptive mesh refinement or other techniques to more efficiently handle fine-resolution features. For additional insight and comparison, Appendix B discusses a toy model of a self-interacting fermion pulse that demonstrates similar underlying phenomenology in a simpler example.

V. CONCLUSIONS

Results have been presented from numerical simulations of the flat-space nonlinear Maxwell-Klein-Gordon-Dirac equations with a spontaneously broken symmetry and repulsive boson-fermion interaction. The findings build upon historical vortex research [1–14] and complement other work that has explored fermionic bound states within vortices [5,15–17] by including massive Dirac fermions similar to those explored widely in the context of condensed matter physics [23–25,27,30–39].

Massive fermion bound state solutions have been shown to exist in the cores of scalar vortices when a repulsive boson-fermion interaction is present. Closed-form solutions were obtained that determined the dispersion relation and relationship between the interaction strength κ_m , effective fermion mass κ_d^{-1} , and width of bound state σ , for approximate solutions when sufficiently contained within the scalar vortex and when the fermion field strength did not significantly perturb the spontaneously broken vacuum of the condensate. The effective mass $\kappa_d^{-1} \propto m_F$ determines the mass gap, while the interaction strength κ_m determines the energy band of allowed states. When considered in the context of typical bulk parameters, the predicted mass gap and energies of bound states were shown to be comparable to those observed in gapped Dirac materials [40,41]. Numerical solutions to the stationary equations of motion were then obtained that confirmed the existence of bound states and demonstrated where the closed-form approximations ceased to be valid. The stationary bound state solutions were then time evolved and demonstrated stability until the fermion field was strong enough to significantly change the vacuum expectation value of the condensate, $\bar{\psi}\psi \rightarrow \kappa_m^{-1}$.

Simulations of head-on collisions of fermion bound states in vortices were conducted for a range of fermion field strengths and boost velocities. For low fermion field strength, head-on scattering closely resembled traditional right-angle scattering [7–9], but solutions still deviated from perfect right-angle scattering due to Coulombic interactions between the charged fermion field and the magnetic field of the scalar vortex, resulting in the impact parameter dynamically moving away from zero. As one increases fermion field strength, there are distinctly different behaviors for low- and high-velocity scattering.

For lower-velocity deep inelastic scattering, charged vortices were observed to coalesce, interact, become unstable, and disperse. The instability arises when the fermion bound states are confined within the large trap formed by the close proximity of the two scalar vortices. The larger trap is flatter and no longer spatially harmonic, and there is a dynamic mix of time-varying radial and axial electromagnetic fields that drive the dynamics of the fermion field. The mixing of bound states drives a decrease in total spin, which demands an increase in orbital angular momentum (in order to conserve total angular momentum), leading to instability and eventual dispersal. Additional understanding of the fermion field dynamics as instability arises could be a productive area for future study.

For high-velocity scattering and low fermion field strength, the vortices underwent near-right-angle scattering, and the creation of a new type of nontopological scalar-fermion bound state was observed. The bound states are bubblelike in that the scalar field interpolates between the spontaneously broken vacuum of the bulk, ϕ_0 , and the vacuum of (65) that becomes dynamically shifted away from ϕ_0 for large $\bar{\psi}\psi$. The solutions are reminiscent of the oscillons created by the vortex-antivortex collisions of Gleiser and Thorarinson [6], but the scalar bound states observed here do not have a harmonic time dependence and are, therefore, not oscillons; they arise from a balance between the outward self-repulsion of the fermion field (that does have a harmonic time dependence and nonzero orbital angular momentum) and the inward surface tension of the scalar bubble. The solutions also bring to mind a time-reversed version of the solutions of Srivastava [44], who discussed the creation of vortices by the interaction of both critical and subcritical bubbles in the context of cosmological phase transitions.

The realization of this work that vortex-vortex interactions can lead to the production of boson-fermion bubbles complements the findings of [44] and similarly could have impact on the understanding of gauged Abelian-Higgs phase transitions in models with a boson-fermion interaction. While the consequences of the boson-fermion interaction introduced here to the electronic structure of the bulk material were not discussed, this work provides insight into the existence, stability, and dynamics of bound states that could arise in condensed matter systems that

describe quasiparticles in Bose-Fermi models similar to the model (1). Finally, a dedicated and more detailed analysis of the boson-fermion bubbles discovered here, particularly their stability, may also be an interesting area for future study.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the author upon reasonable request.

APPENDIX A: NUMERICAL METHODS

This appendix describes the numerical methods for addressing potential numerical instabilities (including fermion doubler modes), the time evolution and boosting of charged vortices, and a number of self-consistency and monitoring techniques to ensure solution accuracy.

The time-evolution methods employed throughout this work implement second-order finite difference approximations to continuous (not lattice) equations of motion. While all finite difference approximations of partial differential equations may be susceptible to numerical instabilities, the modeling of Dirac fields using discretized approximations can be particularly challenging. Adding a “Wilson term” [45,46] to the Dirac equations of motion is a common approach to directly address *fermion doublers* on a lattice; the approach focuses on suppressing the $k \approx \pi/a$ mode for a lattice spacing a , which is the Nyquist limit of the numerical discretization. The approach is physically motivated and introduces an ultraviolet cutoff (low-pass filter) by adding a term proportional to $\partial_\mu^2 \psi$ to the equations of motion to specifically address the doubler modes. Figure 13 compares the result of using a Wilson term in the equations of motion to the numerically motivated approach of Kreiss-Oliger dissipation (the technique employed in this work). Kreiss-Oliger dissipation similarly adds a higher-order derivative term ($\partial_\mu^4 \psi$) that also suppresses high-wave-vector modes. Using von Neumann stability analysis yields the following amplification factors for second-order Crank-Nicolson time-centered difference schemes for the addition of a Wilson term and Kreiss-Oliger dissipation, respectively:

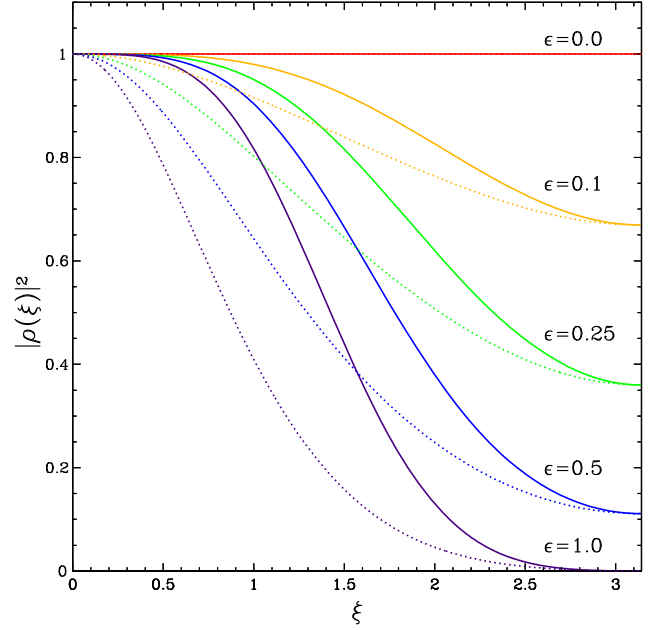


FIG. 13. Plots of amplification factors of modes for a physically motivated Wilson term (dotted lines) as it compares to a numerically motivated Kreiss-Oliger dissipation approach (solid lines). Plots for $\epsilon = \{0, 0.1, 0.25, 0.5, 1.0\}$ are shown in red, orange, green, blue and indigo, respectively. For a given ϵ , the Kreiss-Oliger approach has equal suppression at $\xi = \pi$ while having less attenuation for lower wave vectors.

$$|\rho(\xi)|_{\text{W}}^2 = \frac{1 + i\frac{\epsilon}{2}\sin(\xi) - \frac{\epsilon}{2}(\cos(\xi) - 1)}{1 - i\frac{\epsilon}{2}\sin(\xi) + \frac{\epsilon}{2}(\cos(\xi) - 1)}, \quad (\text{A1})$$

$$|\rho(\xi)|_{\text{KO}}^2 = \frac{1 + i\frac{\epsilon}{2}\sin(\xi) - \frac{\epsilon}{8}(3 - 4\cos(\xi) + \cos(2\xi))}{1 - i\frac{\epsilon}{2}\sin(\xi) + \frac{\epsilon}{8}(3 - 4\cos(\xi) + \cos(2\xi))}, \quad (\text{A2})$$

where $\xi = ka$ for lattice spacing a . These amplification factors can be seen in Fig. 13 for multiple values of the tunable parameter ϵ . In the Wilson term, one typically uses $\epsilon = ra$ with a tunable parameter r . The Courant factor $\lambda = dt/dx$ was set to $\lambda = 0.5$ in all simulations used in this work and in Fig. 13. Both approaches have tunable scaling parameters and vanish in the $a \rightarrow 0$ limit. While the Wilson term may be more appealing from a physical (vice numerical) perspective, the Kreiss-Oliger approach is a higher-order technique with a sharper frequency response that suppresses the doubler mode equally while having less negative effect on low-frequency modes. Most importantly, if doubler modes or high-wave-vector instabilities become prevalent in a solution, the dissipative approach employed here leads to solution errors that are quantified with (109) and can be monitored.

The following equations describe the transformation of the scalar, fermion, and electromagnetic fields from at-rest solutions in cylindrical coordinates (R, θ) to boosted solutions in Cartesian coordinates $(\tilde{t}, \tilde{x}, \tilde{y})$; the boosted

scalar and electromagnetic fields are given by

$$\phi_1(\tilde{t}, \tilde{x}, \tilde{y}) = \phi(R) \cos \theta, \quad (\text{A3})$$

$$\phi_2(\tilde{t}, \tilde{x}, \tilde{y}) = \phi(R) \sin \theta, \quad (\text{A4})$$

$$\Pi_1(\tilde{t}, \tilde{x}, \tilde{y}) \approx \frac{\phi_1(\tilde{t} + \Delta\tilde{t}, \tilde{x}, \tilde{y}) - \phi_1(\tilde{t} - \Delta\tilde{t}, \tilde{x}, \tilde{y})}{2\Delta\tilde{t}}, \quad (\text{A5})$$

$$\Pi_2(\tilde{t}, \tilde{x}, \tilde{y}) \approx \frac{\phi_2(\tilde{t} + \Delta\tilde{t}, \tilde{x}, \tilde{y}) - \phi_2(\tilde{t} - \Delta\tilde{t}, \tilde{x}, \tilde{y})}{2\Delta\tilde{t}}, \quad (\text{A6})$$

$$A_{\tilde{t}}(\tilde{t}, \tilde{x}, \tilde{y}) = \gamma(A_t(R) - v_0 \tilde{A}_\theta(R) \sin \theta), \quad (\text{A7})$$

$$A_{\tilde{x}}(\tilde{t}, \tilde{x}, \tilde{y}) = \gamma(v_0 A_t(R) - \tilde{A}_\theta(R) \sin \theta), \quad (\text{A8})$$

$$A_{\tilde{y}}(\tilde{t}, \tilde{x}, \tilde{y}) = \tilde{A}_\theta(R) \cos \theta, \quad (\text{A9})$$

$$E_{\tilde{x}}(\tilde{t}, \tilde{x}, \tilde{y}) = E_R(R) \cos \theta, \quad (\text{A10})$$

$$E_{\tilde{y}}(\tilde{t}, \tilde{x}, \tilde{y}) = \gamma E_R(R) \sin \theta - \gamma v_0 B_z(R), \quad (\text{A11})$$

$$B_{\tilde{z}}(\tilde{t}, \tilde{x}, \tilde{y}) = \gamma B_z(R) - \gamma v_0 E_R(R) \sin \theta, \quad (\text{A12})$$

$$E_{\tilde{z}}(\tilde{t}, \tilde{x}, \tilde{y}) = B_{\tilde{x}}(\tilde{t}, \tilde{x}, \tilde{y}) = B_{\tilde{y}}(\tilde{t}, \tilde{x}, \tilde{y}) = 0, \quad \text{and} \quad (\text{A13})$$

$$A_{\tilde{z}}(\tilde{t}, \tilde{x}, \tilde{y}) = 0, \quad (\text{A14})$$

while the boosted fermion field components are given by

$$\begin{aligned} a_1(\tilde{t}, \tilde{x}, \tilde{y}) &= \cosh(\lambda/2) \Psi_1(R) \cos(\omega\gamma(\tilde{t} + v_0\tilde{x})) \\ &\quad \pm \sinh(\lambda/2) \Psi_4(R) \sin(\theta - \omega\gamma(\tilde{t} + v_0\tilde{x})), \end{aligned} \quad (\text{A15})$$

$$\begin{aligned} a_2(\tilde{t}, \tilde{x}, \tilde{y}) &= -\cosh(\lambda/2) \Psi_1(R) \sin(\omega\gamma(\tilde{t} + v_0\tilde{x})) \\ &\quad \mp \sinh(\lambda/2) \Psi_4(R) \cos(\theta - \omega\gamma(\tilde{t} + v_0\tilde{x})), \end{aligned} \quad (\text{A16})$$

$$\begin{aligned} b_1(\tilde{t}, \tilde{x}, \tilde{y}) &= \mp \cosh(\lambda/2) \Psi_4(R) \sin(\theta - \omega\gamma(\tilde{t} + v_0\tilde{x})) \\ &\quad - \sinh(\lambda/2) \Psi_1(R) \cos(\omega\gamma(\tilde{t} + v_0\tilde{x})), \quad \text{and} \end{aligned} \quad (\text{A17})$$

$$\begin{aligned} b_2(\tilde{t}, \tilde{x}, \tilde{y}) &= \pm \cosh(\lambda/2) \Psi_4(R) \cos(\theta - \omega\gamma(\tilde{t} + v_0\tilde{x})) \\ &\quad + \sinh(\lambda/2) \Psi_1(R) \sin(\omega\gamma(\tilde{t} + v_0\tilde{x})), \end{aligned} \quad (\text{A18})$$

where

$$\lambda = \tanh^{-1}(v_0), \quad (\text{A19})$$

$$R = \sqrt{\gamma^2(\tilde{x} + v_0\tilde{t})^2 + \tilde{y}^2}, \quad (\text{A20})$$

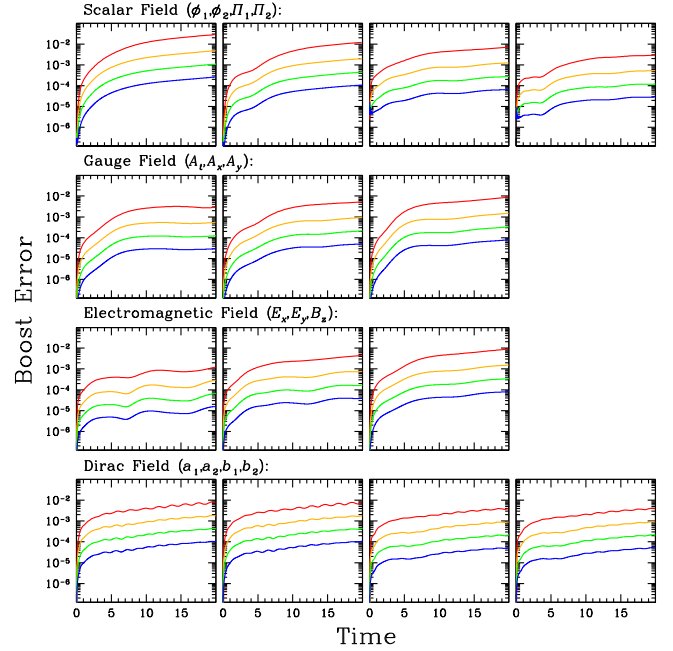


FIG. 14. Plots of the boost error over time for all fields when a single vortex is boosted with $v_x = 0.5$. The error is obtained by calculating the L2 norm of the difference between the time-evolved boosted initial data and the analytically boosted stationary solutions. Each field is evolved on an $N \times N$ grid, where $N = 200 \times 2^l + 1$ for levels $l = 0, 1, 2, 3$ in red, orange, green, and blue, respectively, with domain spanning $\{x: -20 < x < 20\}$ and $\{y: -20 < y < 20\}$. The error in all fields is observed to converge quadratically to zero.

$$\sin \theta = \tilde{y}/R, \quad (\text{A21})$$

$$\cos \theta = \gamma(\tilde{x} + v_0\tilde{t})/R, \quad \text{and} \quad (\text{A22})$$

$$\theta = \tan^{-1} \left(\frac{\tilde{y}}{\gamma(\tilde{x} + v_0\tilde{t})} \right). \quad (\text{A23})$$

Equations (A3)–(A18) were used in Sec. IV to generate initial data for head-on (zero impact parameter) collisions by setting $\tilde{t} = 0$ and superimposing solutions of the stationary equations of motion that were translated to $(\tilde{x}, \tilde{y}) = (\pm 5, 0)$ with boost velocities $\mp v_0$ and then time-evolved using (86)–(106). In this section, a similar approach is used, but with a *single* vortex boosted in the positive x direction. Those time-evolved solutions are then compared to directly transforming a stationary charged vortex solution of (42)–(49) with (A3)–(A18) to obtain boosted solutions for an arbitrary $\tilde{t}$. Figure 14 shows plots for each field of the boost error, which is defined to be the L2 norm of the difference between these two approaches and serves as a very strong method of verification of the boost methodology. It can be seen that every field component has boost error that converges to zero in accordance

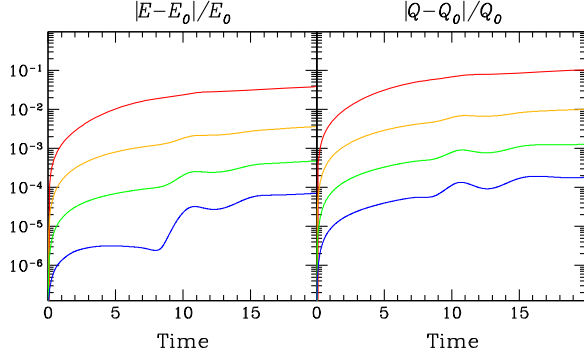


FIG. 15. Plots demonstrating conservation of energy (left) and charge (right) for the time evolution of the $(2+1)$ equations of motion for a head-on collision of two vortices. The relative change in energy (charge) is shown for levels $l = 0, 1, 2, 3$ in red, orange, green, and blue, respectively, for an $N \times N$ grid, where $N = 200 \times 2^l + 1$. The error in all fields is observed to converge quadratically to zero.

with the second-order finite difference scheme being employed.

Next, Fig. 15 demonstrates energy and charge conservation of two charged vortices boosted at one another that undergo near-right-angle scattering. The error in both

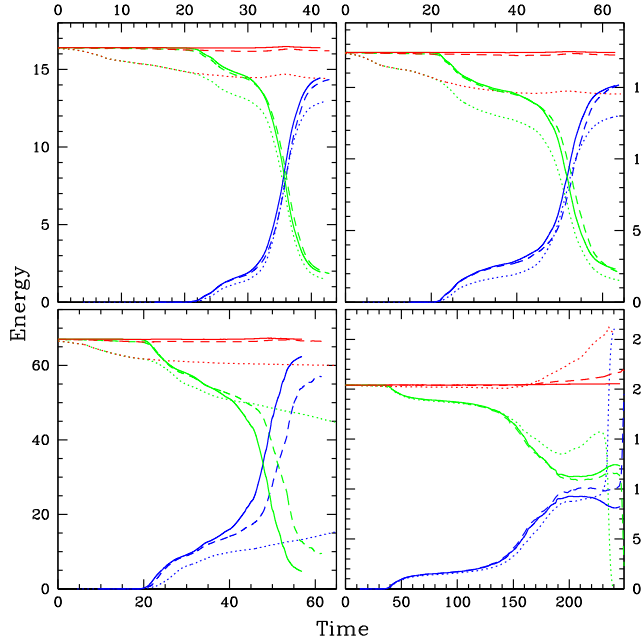


FIG. 16. Plots of total energy (red), energy currently in the computational domain (green), and cumulative radiated energy (blue). The upper left figure corresponds to the evolution from Fig. 8; the upper right figure corresponds to the evolution from Fig. 9; the lower left figure corresponds to the evolution from Fig. 10; and the lower right figure corresponds to the evolution from Fig. 11. Simulations were performed on an $N \times N$ grid, where $N = 200 \times 2^l + 1$ for $l = 1, 2, 3$, with dotted, dashed, and solid lines, respectively. Total energy is observed to be conserved in a convergent manner.

energy and charge conservation can also be seen to converge to zero with second-order accuracy as desired.

Lastly, Fig. 16 displays the energy conservation for each of the simulations discussed in Figs. 8–11. For each simulation, the total energy, energy in the grid, and cumulative radiated energy are displayed for multiple grid resolutions. The cumulative radiated energy at time t is calculated by integrating the power radiated through a Gaussian surface around the computational domain from the beginning of the simulation to time t . The cumulative radiated energy is then added to the energy in the grid at t to obtain the total energy at t . For the $l = 3$ grids, the simulations displayed by Figs. 8–10 demonstrate total energy conservation to a few parts in 10^4 . The simulation displayed by Fig. 11 demonstrates similar performance until the $m = 2$ vortex becomes unstable and disperses. By the end of the simulation, conservation of total energy is demonstrated to slightly better than one part in 10^2 . Whether the higher error in these solutions arises from doubler modes related to the discretization of the Dirac equation or simply from accelerating fields that increase the wave vector, the resultant error is monitored and understood.

APPENDIX B: SELF-INTERACTION IN THE 1D MAXWELL-DIRAC EQUATION

While an interesting and related in-depth analysis of the self-interaction properties of the classical and quantum Maxwell-Dirac equations can be found in [29], this appendix provides a simple and intuitive example that more simply demonstrates the effects seen by the interacting fermion fields confined to vortices described in this work, particularly the last scattering example in Sec. IV that demonstrated unstable behavior and is visualized in Fig. 11.

A toy model is desired that is as close as possible to (1) to remain most relevant to this work while also removing boson-fermion interactions and focusing solely on the properties of the fermion field. Setting $\phi = 0$ everywhere, the model essentially becomes

$$\mathcal{L} = \sqrt{-g}(L_M + L_D). \quad (\text{B1})$$

Slightly different dimensionless variables are required, since the ones used in the body of this work are in terms of condensate parameters that are now being ignored. However, one can use ξ as an arbitrary parameter to set the relative length scale of the fermion bound state and use $\kappa_d = \lambda/\xi$ as the dimensionless model parameter. This leads to 1D equations of motion:

$$\partial_t a_1 = -\partial_x b_1 - \frac{1}{2} A_t a_2 - \frac{1}{2} A_x b_2 + \frac{1}{2} A_y b_1 + \kappa_d^{-1} a_2, \quad (\text{B2})$$

$$\partial_t a_2 = -\partial_x b_2 + \frac{1}{2} A_t a_1 + \frac{1}{2} A_x b_1 + \frac{1}{2} A_y b_2 - \kappa_d^{-1} a_1, \quad (\text{B3})$$

$$\partial_t b_1 = -\partial_x a_1 - \frac{1}{2} A_t b_2 - \frac{1}{2} A_x a_2 - \frac{1}{2} A_y a_1 - \kappa_d^{-1} b_2, \quad (\text{B4})$$

$$\partial_t b_2 = -\partial_x a_2 + \frac{1}{2} A_t b_1 + \frac{1}{2} A_x a_1 - \frac{1}{2} A_y a_2 + \kappa_d^{-1} b_1, \quad (\text{B5})$$

$$\partial_t E^x = -\alpha_M^{-1} \kappa_d^{-2} (a_1 b_1 + a_2 b_2), \quad (\text{B6})$$

$$\partial_t E^y = -\partial_x B^z - \alpha_M^{-1} \kappa_d^{-2} (a_1 b_2 - a_2 b_1), \quad (\text{B7})$$

$$\partial_t B_z = -\partial_x E_y, \quad (\text{B8})$$

$$\partial_t A_x = -E_x + \partial_x A_t, \quad (\text{B9})$$

$$\partial_t A_y = -E_y, \quad \text{and} \quad (\text{B10})$$

$$\partial_t A_t = \partial_x A_x, \quad (\text{B11})$$

where $\alpha_M = 16\pi\alpha'$ for the fine structure constant α' . Figure 17 shows time evolution of these equations for the self-interacting fermion field and the free fermion field ($A_\mu = 0$). The initial data are taken to be

$$a_1 = a_{1,0} \exp(-x^2/\sigma^2), \quad (\text{B12})$$

$$a_2 = 0, \quad (\text{B13})$$

$$b_1 = 0, \quad \text{and} \quad (\text{B14})$$

$$b_2 = \kappa_d x a_1, \quad (\text{B15})$$

and the appropriate Maxwell equations are used to solve for the electromagnetic fields. The fields are time evolved using Eqs. (B2)–(B11), and discrete Fourier transforms are taken at each time to obtain $a(k)$ and $b(k)$ for both the free and interacting fields. The Gaussian initial data $a_1(t)$ naturally give rise to Gaussian $a(k)$. For the free field, the solutions are the well-known 1D solutions and disperse in accordance with the dispersion relation $\omega^2 = \sigma^{-2} + \kappa_d^{-2}$; the field disperses in the spatial domain, and the spectral content $a(k)$ and $b(k)$ does not change over time. For the

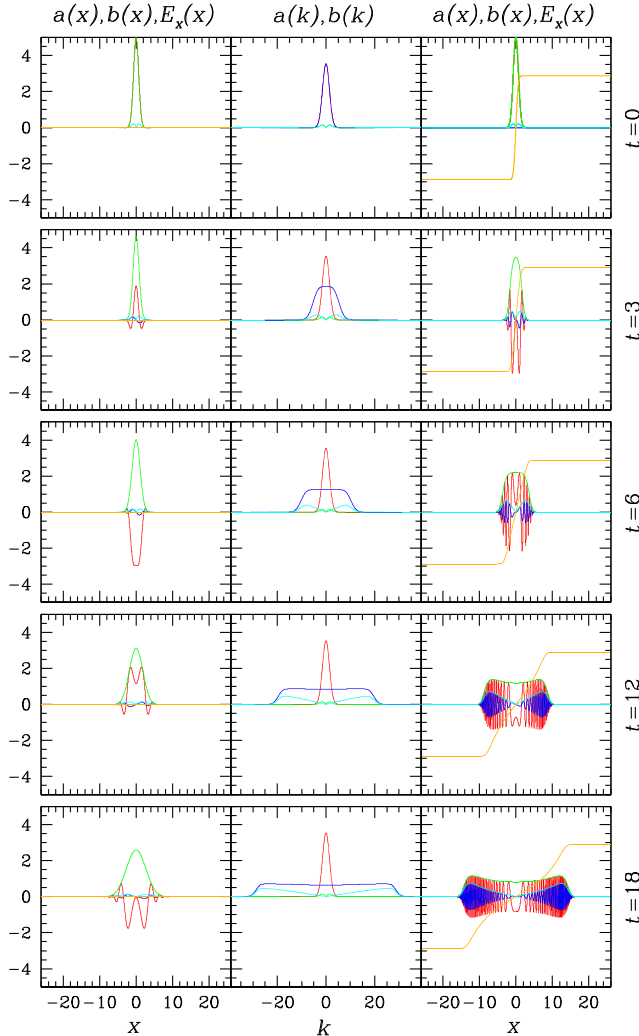


FIG. 17. Plots of $a(x)$, $|a(x)|$, $b(x)$, $|b(x)|$, and $E_x(x)$ for the 1D plane-symmetric free fermion field (left column) and self-interacting fermion and Maxwell fields (right column) and $a(k)$ and $b(k)$ for both fields (center column). The $a(x)$, $|a(x)|$, $b(x)$, $|b(x)|$, and $E_x(x)$ fields are plotted in red, green, blue, cyan, and orange, respectively. $a(k)$ and $b(k)$ for the free field are plotted in red and green, respectively, while $a(k)$ and $b(k)$ for the self-interacting field are in blue and cyan, respectively. The free field is observed to disperse with constant spectral content $a(k)$ and $b(k)$, while the self-interacting field disperses with an accelerating wave front and increasing wave vector over time.

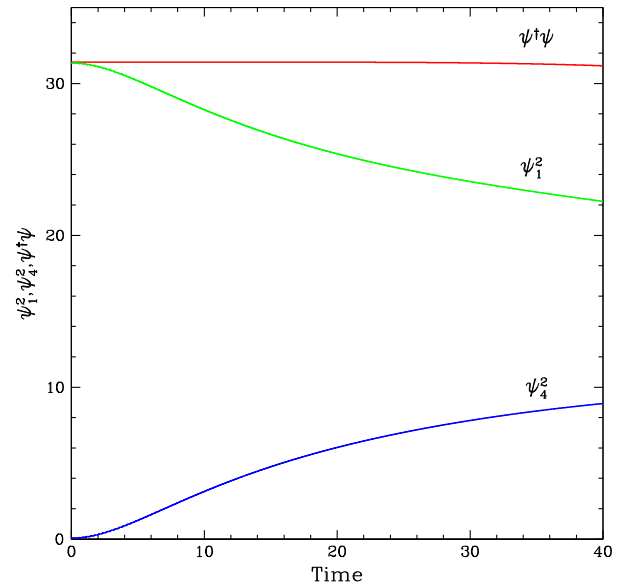


FIG. 18. Plots of ψ^2 , ψ_1^2 , and ψ_4^2 (red, green, and blue, respectively) for the 1D plane-symmetric self-interacting field. The total fermion field strength $\psi^\dagger\psi$ is conserved, while ψ_1^2 decreases, and ψ_4^2 (which is proportional to p_x^2) increases.

self-interacting field, however, the simple dispersion relation does not hold and the wave front of the dispersing fermion field accelerates based on strong Coulombic self-repulsion. As the field accelerates, p_x increases, which increases the wave vector k_x in the field components observed in Fig. 17. Similar to the analysis in Sec. IV, the increase in momentum (p_x^2) can also be observed by considering the relative proportion of ψ_1 and ψ_4 in the overall field strength. Figure 18 clearly shows a relative increase in ψ_4^2 , which is expected since $\psi_4^2 \propto p_x^2 \propto k_x^2$.

Despite its simplicity, this toy model helps demonstrate how a self-interacting fermion field can evolve from a simple smoothly varying Gaussian distribution to a distribution with high-frequency components when in the presence of persistent strong electromagnetic fields. Again, it is important to emphasize that even if high-wave-vector components do emerge, such components are manageable and evolutions can be terminated when the conservation of energy or $\psi^\dagger\psi$ deviates beyond an acceptable tolerance.

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- [1] H. Nielsen and P. Olesen, *Nucl. Phys.* **B61**, 45 (1973).
 - [2] M. B. Hindmarsh and T. W. B. Kibble, *Rep. Prog. Phys.* **58**, 477 (1995).
 - [3] H. J. de Vega, *Phys. Rev. D* **18**, 2932 (1978).
 - [4] H. J. de Vega and F. A. Schaposnik, *Phys. Rev. D* **14**, 1100 (1976).
 - [5] K. Kleidis, A. Kuiroukidis, P. Nerantzi, and D. Papadopoulos, *Gen. Relativ. Gravit.* **42**, 31 (2011).
 - [6] M. Gleiser and J. Thorarinson, *Phys. Rev. D* **76**, 041701 (2007).
 - [7] E. Shellard and P. Ruback, *Phys. Lett. B* **209**, 262 (1988).
 - [8] P. Ruback, *Nucl. Phys.* **B296**, 669 (1988).
 - [9] K. Moriarty, E. Myers, and C. Rebbi, *Phys. Lett. B* **207**, 411 (1988).
 - [10] E. Myers, C. Rebbi, and R. Strilka, *Phys. Rev. D* **45**, 1355 (1992).
 - [11] J. Dziarmaga, *Phys. Rev. D* **49**, 5609 (1994).
 - [12] B. P. Abbott (LIGO Scientific and Virgo Collaborations), *Phys. Rev. D* **97**, 102002 (2018).
 - [13] T. Helfer, J. C. Aurrekoetxea, and E. A. Lim, *Phys. Rev. D* **99**, 104028 (2019).
 - [14] J. J. Blanco-Pillado, K. D. Olum, and J. M. Wachter, *Phys. Rev. D* **100**, 023535 (2019).
 - [15] C. R. Nohl, *Phys. Rev. D* **12**, 1840 (1975).
 - [16] R. Jackiw and P. Rossi, *Nucl. Phys.* **B190**, 681 (1981).
 - [17] G. Lozano, M. V. Manias, and F. A. Schaposnik, *Phys. Rev. D* **38**, 601 (1988).
 - [18] H. Weigel, M. Quandt, and N. Graham, *Phys. Rev. Lett.* **106**, 101601 (2011).
 - [19] A. A. Abrikosov, *Sov. Phys. JETP* **5**, 1174 (1957).
 - [20] A. A. Abrikosov, *Rev. Mod. Phys.* **76**, 975 (2004).
 - [21] D. Cribier, B. Jacrot, L. M. Rao, and B. Farnoux, *Phys. Lett.* **9**, 106 (1964).
 - [22] R. Roth and K. Burnett, *Phys. Rev. A* **69**, 021601 (2004).
 - [23] V. Krasnow and I. Stasyuk, *Condens. Matter Phys.* **22**, 1 (2019).
 - [24] A. Albus, F. Illuminati, and J. Eisert, *Phys. Rev. A* **68**, 023606 (2003).
 - [25] M. Lewenstein, L. Santos, M. A. Baranov, and H. Fehrmann, *Phys. Rev. Lett.* **92**, 050401 (2004).
 - [26] F. Illuminati and A. Albus, *Phys. Rev. Lett.* **93**, 090406 (2004).
 - [27] M. Bukov and L. Pollet, *Phys. Rev. B* **89**, 094502 (2014).
 - [28] N. I. Chott, Q. Su, and R. Grobe, *Phys. Rev. A* **76**, 010101 (2007).
 - [29] Q. Z. Lv, S. Norris, Q. Su, and R. Grobe, *Phys. Rev. A* **90**, 034101 (2014).
 - [30] J. von Milczewski, F. Rose, and R. Schmidt, *Phys. Rev. A* **105**, 013317 (2022).
 - [31] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature (London)* **415**, 39 (2002).
 - [32] M. Greiner, I. Bloch, O. Mandel, T. W. Hänsch, and T. Esslinger, *Phys. Rev. Lett.* **87**, 160405 (2001).
 - [33] H. Fehrmann, M. Baranov, B. Damski, M. Lewenstein, and L. Santos, *Opt. Commun.* **243**, 23 (2004).
 - [34] M. Cramer, J. Eisert, and F. Illuminati, *Phys. Rev. Lett.* **93**, 190405 (2004).
 - [35] T. Wehling, A. Black-Schaffer, and A. Balatsky, *Adv. Phys.* **63**, 1 (2014).
 - [36] L. Classen, I. F. Herbut, L. Janssen, and M. M. Scherer, *Phys. Rev. B* **93**, 125119 (2016).
 - [37] L. Ye, M. Kang, J. Liu, F. von Cube, C. R. Wicker, T. Suzuki, C. Jozwiak, A. Bostwick, E. Rotenberg, D. C. Bell *et al.*, *Nature (London)* **555**, 638 (2018).
 - [38] M. Yang, W. Zhao, D. Mu, Z. Shi, J. Zhong, Y. Li, Y. Liu, J. Zhong, N. Cheng, W. Zhou *et al.*, *Phys. Rev. Lett.* **133**, 256601 (2024).
 - [39] Z. Lin, C. Wang, P. Wang, S. Yi, L. Li, Q. Zhang, Y. Wang, Z. Wang, H. Huang, Y. Sun *et al.*, *Phys. Rev. B* **102**, 155103 (2020).
 - [40] P. Kumar, R. Skomski, P. Manchanda, A. Kashyap, and P. Dowben, *Curr. Appl. Phys.* **14**, S136 (2014).
 - [41] M. Fischer, L. Sousa, L. Castro, L. Ribeiro, R. de Sousa Junior, G. Silva, and P. Neto, *Sci. Rep.* **9**, 17990 (2019).
 - [42] N. S. Manton and P. Sutcliffe, *Topological Solitons*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2004).
 - [43] E. P. Honda, *Phys. Rev. D* **102**, 056011 (2020).
 - [44] A. M. Srivastava, *Phys. Rev. D* **46**, 1353 (1992).
 - [45] K. G. Wilson, *Phys. Rev. D* **10**, 2445 (1974).
 - [46] B. Messias de Resende, F. C. de Lima, R. H. Miwa, E. Vernek, and G. J. Ferreira, *Phys. Rev. B* **96**, 161113 (2017).