

Toward quantum simulations of sphaleron dynamics at colliders

Min Huang^{1,2,*} Ying-Ying Li^{1,†} Yandong Liu^{3,4,‡} and Hao Zhang^{1,2,5,§}

¹Theoretical Physics Division, *Institute of High Energy Physics*,
Chinese Academy of Sciences, Beijing 100049, China

²School of Physics, *University of Chinese Academy of Sciences*, Beijing 100049, China

³School of Physics and Astronomy, *Beijing Normal University*, Beijing 100875, China

⁴Key Laboratory of Multiscale Spin Physics (Ministry of Education), *Beijing Normal University*,
Beijing 100875, China

⁵Center for High Energy Physics, *Peking University*, Beijing 100871, China



(Received 20 October 2025; accepted 17 February 2026; published 25 March 2026)

Sphaleron dynamics in the Standard Model at high-energy particle collisions remains experimentally unobserved, with theoretical predictions hindered by its nonperturbative real-time nature. In this work, we investigate a quantum simulation approach to this challenge. Taking the 1 + 1D $O(3)$ nonlinear σ -model as a protocol towards studying dynamics of the sphaleron in electroweak theory, we identify the sphaleron configuration and establish lattice parameters that reproduce continuum sphaleron energies with controlled precision. We then develop quantum algorithms to simulate sphaleron evolutions where quantum effects can be included. This work lays the ground to establish quantum simulations for studying the interaction between classical topological objects and particles in the quantum field theory that are usually inaccessible to classical methods and computations.

DOI: [10.1103/wdf7-zt4c](https://doi.org/10.1103/wdf7-zt4c)

I. INTRODUCTION

As a saddle point of finite energy in the field space, sphaleron is a static physical solution to the electroweak (EW) field equations in Minkowski spacetime [1–5]. In the Standard Model (SM), the sphaleron process leads to baryon and lepton number violations, which at high temperature, can be a key ingredient for baryogenesis that aims to explain the origin of the observed baryon asymmetry [6,7], see recent reviews [8]. The sphaleron rates at finite temperature can be calculated using large-scale lattice simulations [9–12]. Additionally, there have been longstanding efforts in investigating whether such processes could be detected in high-energy collisions, such as those predicting the sphaleron production rates at colliders [13–20], and phenomenological studies of baryon number-violating processes at LHC [21] or ultrahigh-energy neutrino collisions [22]. Yet, the production rate of sphalerons in high-energy collisions has been calculated using various methods, resulting in predictions that

differ by orders of magnitude [19]. Such large discrepancies might be due to the real-time nature of multisphaleron processes that calculations in Euclidean spacetime fail to capture [20]. Thus, it is warranted to perform a renewed study of baryon number-violating processes in high-energy collisions based on real-time simulations achieved using the Hamiltonian descriptions.

In such collisions, incoming and outgoing particles correspond to excitations of quantum fields. Hamiltonians describing dynamics of these fields require reducing their infinite degrees of freedom to finite ones. By discretizing continuous spacetime into lattices and truncating local field amplitudes to finite values, the resulting Hilbert space, although finite in size, still grows exponentially with increasing spatial volume. With the advances in quantum technology and the emergence of a new computing paradigm, namely quantum computing, the number of storage units (qubits) is expected to grow polynomially with the system size, bringing new opportunities for performing such calculations nonperturbatively. Simulating dynamics in quantum field theories with quantum computer has also seen rapid advances in recent years, see reviews [23–26].

However, to achieve the goal of simulating a sphaleron process in particle collisions requires investigations into a line of questions, e.g., what are instanton solutions like on the lattice, how to manifest the sphaleron configurations based on a Hamiltonian with second quantizations. In this work, we initiate a first step towards this goal, which is investigating the sphaleron configuration on lattices.

*Contact author: huangmin@ihep.ac.cn

†Contact author: liyingying@ihep.ac.cn

‡Contact author: ydliau@bnu.edu.cn

§Contact author: zhanghao@ihep.ac.cn

We will focus on the $O(3)$ nonlinear σ -model (NLSM) in $1 + 1$ dimensional ($1 + 1D$) spacetime for its remarkable similarities to non-Abelian gauge theories [27]. With a $O(3)$ symmetry breaking term, this model admits the existence of sphaleron configuration, which can potentially serve as a toy model to investigate sphaleron productions at high-energy collisions in the EW theory. The paper is organized as follows. In Sec. II, we give a quick review of the sphaleron in an $O(3)$ NLSM. In Sec. III, we discuss the sphaleron configuration on lattices. Quantum simulation of sphaleron evolution will be outlined in Sec. IV. We then conclude in Sec. V.

II. THE NONLINEAR SIGMA MODEL

Simulating the sphaleron process in Weinberg-Salam theory is beyond the capability of quantum devices in the noisy intermediate-scale quantum (NISQ) era, where $O(100)$ qubits without error corrections are available. Fortunately, the sphaleron process exists widely in a lot of field theory models when there is uncontractible loop in the configuration space of the classical field [4]. The $O(3)$ NLSM in $1 + 1D$, has been studied before for its remarkable similarities to non-Abelian gauge theories [27]. When modified by an explicit symmetry-breaking term that breaks the $O(3)$ symmetry down to an $O(2)$ subgroup [28], it could be presented as a model for baryon-number and lepton-number violation which are essential ingredient to explain the matter-antimatter asymmetry observed in our Universe. Here we will investigate the sphaleron dynamics in this model with quantum computer as a first effort towards quantum simulating the sphaleron process in the Weinberg-Salam theory.

The classical action of the $O(3)$ NLSM with explicit symmetry-breaking term in $1 + 1D$ spacetime is (with $\omega > 0$)

$$S = \frac{1}{g^2} \int dt dx \left[\frac{1}{2} \partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n} - \omega^2 (1 + \mathbf{n} \cdot \mathbf{e}_z) \right], \quad (1)$$

where $\mathbf{n}(x) \cdot \mathbf{n}(x) \equiv 1$ is the normalized classical field from $\mathbb{R}^2$ to S^2 , $\mathbf{e}_z$ is the unit vector along the third axis in the target space. The second term in the action explicitly breaks the global $O(3)$ symmetry to its $O(2)$ subgroup of the rotations around the third axis in the target space.

The field $\mathbf{n}$ can be parametrized as¹

$$\mathbf{n}(\xi, \eta) = \begin{pmatrix} \sin \eta \sin \xi \\ \sin \eta \cos \eta (\cos \xi - 1) \\ -\sin^2 \eta \cos \xi - \cos^2 \eta \end{pmatrix}, \quad (2)$$

¹Due to the nontrivial topology of S^2 , this coordinate system has some singularities. So strictly speaking, it can be only safely used in the region of $\eta \in (0, \pi)$, $\xi \in (0, 2\pi)$.

and the action as a functional of the fields $\eta(t, x)$ and $\xi(t, x)$ can be written as

$$S = \frac{1}{g^2} \int dt dx \left\{ \frac{1}{2} [(1 - \cos \xi)^2 + \sin^2 \xi \cos^2 \eta] \partial_\mu \eta \partial^\mu \eta + \sin \xi \sin \eta \cos \eta \partial_\mu \eta \partial^\mu \xi + \frac{1}{2} \sin^2 \eta \partial_\mu \xi \partial^\mu \xi - \omega^2 \sin^2 \eta (1 - \cos \xi) \right\}. \quad (3)$$

The Hamiltonian for static configurations by neglecting the kinetic term as we derived in the Appendix is

$$H = \frac{1}{g^2} \int dx \left[\frac{1}{2} (1 - \cos \xi)^2 \left(\frac{\partial \eta}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial \eta}{\partial x} \sin \xi \cos \eta + \frac{\partial \xi}{\partial x} \sin \eta \right)^2 + \omega^2 \sin^2 \eta (1 - \cos \xi) \right]. \quad (4)$$

To find the sphaleron solution to the theory in the continuum, one considers static configurations $\hat{n}$ with fixed η , the energy obtained can be written as [28]

$$E = \frac{\sin^2 \eta}{g^2} \int dx \left[\frac{1}{2} \left(\frac{\partial \xi}{\partial x} \right)^2 + \omega^2 (1 - \cos \xi) \right]. \quad (5)$$

As seen, $\eta = \pi/2$ gives the maximal of the energy functional. At this maximal, one solves the configuration of ξ which minimizes the energy E to find a saddle point. Notice that $\xi \equiv 2k\pi$ ($k \in \mathbb{Z}$) corresponds to the ground state satisfying $\mathbf{n}(x) \equiv (0, 0, -1)^T$, independent of the η values, which should not be considered as the sphaleron solutions. With the boundary condition,²

$$\lim_{x \rightarrow -\infty} \xi(x) = 0, \quad \lim_{x \rightarrow +\infty} \xi(x) = 2\pi, \quad (7)$$

the configuration of ξ satisfying the Euler-Lagrange equation is solved as (since the arcsin function is not continued at $[0, 2\pi]$)

$$\xi^{\text{sph}}(x) = \begin{cases} 2 \arcsin(\text{sech}(\omega x)), & x < 0 \\ 2\pi - 2 \arcsin(\text{sech}(\omega x)), & x \geq 0 \end{cases}, \quad (8)$$

²The sphaleron configuration is homotopically equivalent to the ground state configuration since $\pi_1(S^2)$ is trivial. One can verify this conclusion with the solution (8) directly by transforming the field configuration along the path

$$\mathbf{n}(\xi(x, s), \eta(x, s)) = \mathbf{n}\left(\xi^{\text{sph}}(x), \frac{\pi}{2}(1 - s)\right). \quad (6)$$

It is easy to verify that this continuous map connects the sphaleron configuration ($s = 0$) and the ground-state configuration ($s = 1$). The reason for such a boundary condition is because that the variation along the η -direction has been frozen.

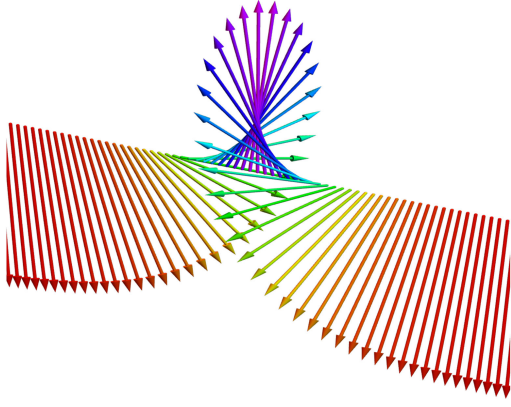


FIG. 1. $\mathbf{n}(x)$ configurations in the $x-z$ plane at different x for the sphaleron solution.

or

$$\mathbf{n}(x) = \begin{pmatrix} -2 \operatorname{sech}(\omega x) \tanh(\omega x) \\ 0 \\ -1 + 2 \operatorname{sech}^2(\omega x) \end{pmatrix}. \quad (9)$$

Substituting into the energy equation with $\eta = \pi/2$, one can find out the saddle point (see Fig. 1) and get the sphaleron energy,

$$E_{\text{sph}} \Big|_{\eta=\frac{\pi}{2}} = \frac{8\omega}{g^2}. \quad (10)$$

To prove that Eq. (8) is a saddle point in the configuration space mathematically rigorously, we consider the infinitesimal variation,

$$\xi(x) \rightarrow \xi^{\text{sph}}(x) + \varepsilon \delta \xi(x), \quad \eta(x) \rightarrow \frac{\pi}{2} + \varepsilon \delta \eta(x), \quad \varepsilon \rightarrow 0^+. \quad (11)$$

Then the variation of the Hamiltonian can be expanded to $\mathcal{O}(\varepsilon^2)$ as

$$H \rightarrow H^{(0)} + \varepsilon H^{(1)} + \varepsilon^2 H^{(2)} + \mathcal{O}(\varepsilon^3). \quad (12)$$

The zeroth-order term is just the Hamiltonian of Eq. (8),

$$H^{(0)} = \frac{1}{g^2} \int dx \left[\frac{1}{2} \left(\frac{\partial \xi^{\text{sph}}}{\partial x} \right)^2 + \omega^2 (1 - \cos \xi^{\text{sph}}) \right] = \frac{8\omega}{g^2}. \quad (13)$$

The linear term,

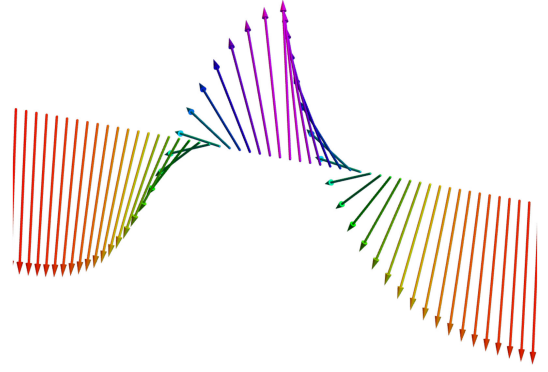


FIG. 2. Same as Fig. 1 but for the solution in Eq. (15).

$$\begin{aligned} H^{(1)} &= \frac{1}{g^2} \int dx \left[\left(\frac{\partial \xi^{\text{sph}}}{\partial x} \right) \left(\frac{\partial \delta \xi}{\partial x} \right) + \omega^2 \delta \xi \sin \xi^{\text{sph}} \right] \\ &= \frac{1}{g^2} \int dx \left[\left(\frac{\partial \xi^{\text{sph}}}{\partial x} \right) \left(\frac{\partial \delta \xi}{\partial x} \right) + \delta \xi \frac{\partial^2 \xi^{\text{sph}}}{\partial x^2} \right] \\ &= \frac{1}{g^2} \int dx \frac{\partial}{\partial x} \left[\delta \xi \frac{\partial \xi^{\text{sph}}}{\partial x} \right] \\ &= \frac{1}{g^2} \delta \xi \frac{\partial \xi^{\text{sph}}}{\partial x} \Big|_{x \rightarrow -\infty}^{x \rightarrow +\infty} \\ &= \frac{2\omega}{g^2} \delta \xi \operatorname{sech}(\omega x) \Big|_{-\infty}^{+\infty}, \end{aligned} \quad (14)$$

since $\operatorname{sech}(\omega x)$ goes to 0 faster than any polynomial of x when x goes to infinity, we have $H^{(1)} = 0$ if $\delta \xi$ is at most polynomially divergent when x goes to infinity.³

³If one defines the “sphaleron” solution as (see Fig. 2)

$$\xi^{\text{sph}}(x) = 2 \arcsin(\operatorname{sech}(\omega x)), \quad (15)$$

without carefully treating of the discontinuity of the arcsin function, the linear term will become

$$\begin{aligned} H^{(1)} &= \frac{2\omega}{g^2} \int dx \sqrt{\tanh^2(\omega x)} \left[-\left(\frac{\partial \delta \xi}{\partial x} \right) \operatorname{csch}(\omega x) + \omega \delta \xi \operatorname{sech}(\omega x) \right] \\ &= \frac{2\omega}{g^2} \int dx \left[-\left(\frac{\partial \delta \xi}{\partial x} \right) \operatorname{sech}(\omega x) + \delta \xi \frac{\partial}{\partial x} (-\operatorname{sech}(\omega x)) \right] \operatorname{sign}(x) \\ &= \frac{2\omega}{g^2} \left\{ \int_0^{+\infty} dx \frac{\partial}{\partial x} [-\delta \xi \operatorname{sech}(\omega x)] - \int_{-\infty}^0 dx \frac{\partial}{\partial x} [-\delta \xi \operatorname{sech}(\omega x)] \right\} \\ &= -\frac{2\omega}{g^2} [\delta \xi \operatorname{sech}(\omega x)_{x \rightarrow +\infty} + \delta \xi \operatorname{sech}(\omega x)_{x \rightarrow -\infty} - 2\delta \xi(0)], \end{aligned} \quad (16)$$

which is no longer a pure boundary term and not zero. Such a nonvanished first order variation is clearly a proof that Eq. (15) is not a static solution of the equation of motion, and could not be a sphaleron.

The quadratic term can be expanded as

$$H^{(2)} = \frac{1}{2g^2} \int dx \left[\omega^2 (1 - 2\text{sech}^2(\omega x)) \delta\xi^2 + \left(\frac{\partial\delta\xi}{\partial x} \right)^2 - 8\omega^2 \text{sech}^2(\omega x) \delta\eta^2 + \frac{8\omega \sinh(\omega x)}{\cosh^3(\omega x)} \delta\eta \left(\frac{\partial\delta\eta}{\partial x} \right) + 4\text{sech}^4(\omega x) \left(\frac{\partial\delta\eta}{\partial x} \right)^2 \right]. \quad (17)$$

It is easy to verify that for the variation,

$$\delta\xi = 0, \quad \delta\eta = \theta(x_0 - x)\theta(x_0 + x), \quad x_0 > 0, \quad (18)$$

where $\theta(\cdot)$ is the Heaviside- θ function, the $H^{(2)} = -8\omega \tanh(\omega x_0)/g^2$ is negative. On the other hand, for the variation

$$\delta\xi = \theta(x_0 - x)\theta(x_0 + x), \quad \delta\eta = 0, \quad x_0 > 0, \quad (19)$$

we have

$$H^{(2)} = \frac{\omega}{g^2} [\omega x_0 - 2 \tanh(\omega x_0)], \quad (20)$$

which is positive if $x_0 > 2/\omega$. Thus, the second order variation of the energy of the static solution [Eq. (8)] is indefinite which means that the solution is a saddle point in the configuration space.

III. CLASSICAL SPHALERON SOLUTIONS ON A LATTICE

To simulate the classical 1 + 1D $O(3)$ NLSM with quantum algorithms, one first discretizes the space on lattices. The classical lattice Hamiltonian describing the $O(3)$ nonlinear σ -model for a system with $2N + 1$ sites and lattice spacing $a \equiv L/2N$, as we also derive in the Appendix, is given by [29–33]

$$H_{\text{tot}}^{\text{latt}} = \frac{g^2}{2a} \sum_{k=-N}^N \mathbf{L}_k^2 + \frac{1}{ag^2} \sum_{k=-N}^{N-1} (1 - \mathbf{n}_k \cdot \mathbf{n}_{k+1}) + \frac{a\omega^2}{g^2} \sum_{k=-N}^N (1 + \mathbf{n}_k \cdot \mathbf{e}_z), \quad (21)$$

where the integer k labels the spatial lattice and a is the lattice spacing, respectively. The first term $\mathbf{L}_k^2$ is recognized as the angular momentum and the term $\mathbf{n}_k \cdot \mathbf{n}_{k+1}$ as the interaction term between each lattice site and its nearest neighbors. As the sphaleron solution is a static solution where the field is time-independent, the angular momentum $\mathbf{L}_k^2$ could be neglected, which leads to the Hamiltonian,

$$H^{\text{latt}} = \frac{1}{ag^2} \sum_{k=-N}^{N-1} (1 - \mathbf{n}_k \cdot \mathbf{n}_{k+1}) + \frac{a\omega^2}{g^2} \sum_{k=-N}^N (1 + \mathbf{n}_k \cdot \mathbf{e}_z), \quad (22)$$

for solving the sphaleron configurations on the lattice.

To study the sphaleron solution after discretization, it is necessary to revisit the coordinate system on the lattice at site k ,

$$\mathbf{n}_k = \begin{pmatrix} \sin \eta_k \sin \xi_k \\ \sin \eta_k \cos \eta_k (\cos \xi_k - 1) \\ -\sin^2 \eta_k \cos \xi_k - \cos^2 \eta_k \end{pmatrix}. \quad (23)$$

The energy for fixed value of $\eta_k \equiv \eta_0$ can be written as

$$H^{\text{latt}} = \frac{\sin^2 \eta_0}{ag^2} \left\{ \sum_{k=-N}^{N-1} [1 - \cos(\xi_k - \xi_{k+1})] + a^2 \omega^2 \sum_{k=-N}^N (1 - \cos \xi_k) \right\}. \quad (24)$$

To find the sphaleron configuration in the framework of the lattice field theory, we add the following boundary conditions along with $\eta_0 = \pi/2$, as inspired by the continuous case:

$$\xi_{-N} = 0, \quad \xi_N = 2\pi, \quad (25)$$

and find the sequence $\{\xi_k\}$ which minimizes H^{latt} . However, there is a fundamental difference between the continuous case and the discrete case since the coordinate space is no longer path connected, and it is meaningless to distinguish 0 from 2π (for more discussions, see [34] for example). This subtlety can be manifested by noticing that

$$\xi_k = \begin{cases} 0, & k < m \\ 2\pi, & k \geq m \end{cases}, \quad (26)$$

for any integer m in the range $-N < m \leq N$ is always a (trivial) solution which minimizes the H^{latt} and satisfies the boundary condition (25).

To resolve this subtlety and avoid the above trivial solution, we require at least one site i with ξ_i different from 0 and 2π . In the continuous case, when x varies from $-\infty$ to $+\infty$, $\xi(x)$ takes on every value in the interval $[0, 2\pi]$. So we can assign ξ_i at site i to α which is not integer times to 2π . In principle, the different choice of i and α corresponds the translation of the position of the sphaleron along the space direction. For simplicity and to minimize the boundary effect, we assign the site $i = 0$ to have $\xi_0 = \alpha$, which leads to the condition,

$$\xi_{-N} = 0, \quad \xi_0 = \alpha, \quad \xi_N = 2\pi. \quad (27)$$

It is easy to see that the sequence which minimizes H^{latt} must have the form,

$$0, \xi_{N-1}, \dots, \xi_1, \alpha, 2\pi - \xi_1, \dots, 2\pi - \xi_{N-1}, 2\pi. \quad (28)$$

Again, as we have pointed out, it is meaningless to distinguish this configuration from

$$0, \xi_{N-1}, \dots, \xi_1, \alpha, \xi_1, \dots, \xi_{N-1}, 0, \quad (29)$$

on a lattice, so we usually work on the second one for simplicity.

We now want to show that the configuration (28) goes to the classical sphaleron solution in the continuum limit. The proof will be separated to two steps. First, we show that the consequence is monotonic. To show that, let us focus on the left-half segment; $\{0, \xi_{N-1}, \dots, \alpha\}$. If there exists a three-site $\{\xi_{k-1}, \xi_k, \xi_{k+1}\}$ which is not monotonic, then either

$$\text{Case I: } \xi_{k-1} > \xi_k, \quad \xi_{k+1} > \xi_k, \quad (30)$$

or

$$\text{Case II: } \xi_{k-1} < \xi_k, \quad \xi_{k+1} < \xi_k. \quad (31)$$

The variation of the Hamiltonian to the variable ξ_k is

$$\begin{aligned} \delta H^{\text{latt}} &= \frac{\sin^2 \eta_0}{ag^2} [\sin(\xi_k - \xi_{k+1}) + \sin(\xi_k - \xi_{k-1})] \\ &\quad + a^2 \omega^2 \sin \xi_k \delta \xi_k. \end{aligned} \quad (32)$$

In the continuum limit ($N \rightarrow +\infty, a = L/2N \rightarrow 0$ with ω and g fixed), the third term vanishes as $\mathcal{O}(a^2)$, decaying faster than the first two terms which scale as $\mathcal{O}(a)$. So for Case I, δH^{latt} is negative when $\delta \xi_k > 0$, and for Case II, δH^{latt} is negative when $\delta \xi_k < 0$. Both these variations pull $\{\xi_{k-1}, \xi_k, \xi_{k+1}\}$ back to a monotonic consequence.

Second, we show that the value of α should be π . Let us consider the variation to the variable α ,

$$\delta H^{\text{latt}} = \frac{\sin^2 \eta_0}{ag^2} [2 \sin(\alpha - \xi_1) + a^2 \omega^2 \sin \alpha] \delta \alpha. \quad (33)$$

Once again, one may neglect the second term in the continuum limit since it is of higher order of a . Then if $\alpha > 0$, the variation is greater than 0 when $\delta \alpha > 0$ since the sequence is increased in this case. If $\alpha < 0$, the variation is greater than 0 when $\delta \alpha < 0$ since the sequence is increased. So to get the extreme value of the H^{latt} , one has to push α to the boundary of its domain, which means that one should choose $\alpha = \pm\pi$. We would like to point it out that even with the constraints, the lattice model could not distinguish the sphaleron solution [Eq. (8)] from the “wrong” sphaleron solution [Eq. (15)]. One need additional conditions such as requiring $\xi(x)$ to increase from 0 to 2π after taking the continuum limit. This is again because that the discrete model cannot faithfully reflect the topological properties of the configuration [34]. In this work, we add the constraints,

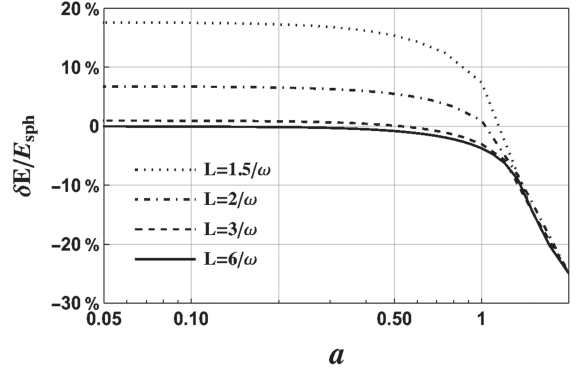


FIG. 3. Relative errors of the sphaleron energy on the lattice to its continuum value E_{sph} , with $\delta E \equiv E_{\text{sph}}(L, a) - E_{\text{sph}}$ for different lattice volume L and lattice spacing a with $\omega = 1/2$.

$$\begin{aligned} \xi_{-N} &= 0, & \xi_0 &= \pi, & \xi_N &= 2\pi, \\ \text{for } \forall k \in [-N, N], & & \xi_k &< \xi_{k+1}. \end{aligned} \quad (34)$$

The last constraint will guarantee the continuum limit of the configuration from the lattice model is Eq. (8) instead of Eq. (15).

Substituting the constraints Eq. (34) into Eq. (24), we numerically solve for the minimum values and corresponding solutions for different values of N within the domain $\xi_k \in [0, 2\pi], k \in [-N, N]$. These solutions $\{\xi_k^{\text{sph}}\}$, together with $\eta_k = \pi/2$, gives the saddle point, i.e., the sphaleron configuration $\{\mathbf{n}_k^{\text{sph}}\}$ of the O(3) field on lattice, whose energy is denoted by $E_{\text{sph}}(L, a)$. To reliably study sphaleron configurations on the lattice, we need to systematically control the distortions introduced to the sphaleron configurations from the finite volume L and finite lattice spacing a . We systematically compare the sphaleron energy on the lattice to its continuum value E_{sph} across different lattice volumes L and spacings a . The numerical result is shown in Fig. 3.

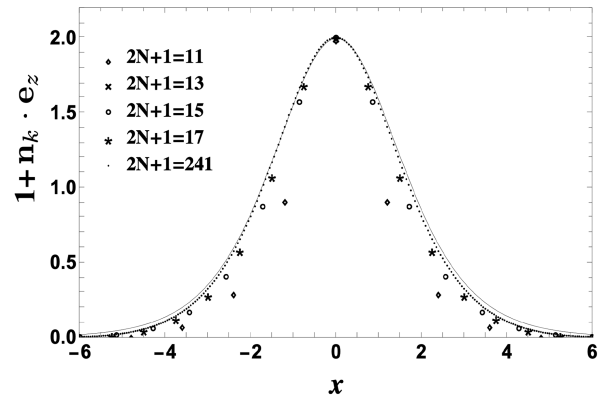


FIG. 4. The sphaleron configuration on the lattice across different number of lattice sites. The parameters are chosen to be $L = 12$, $\omega = 1/2$, and $g = 1$. The thin black line is the sphaleron solution in the continuum.

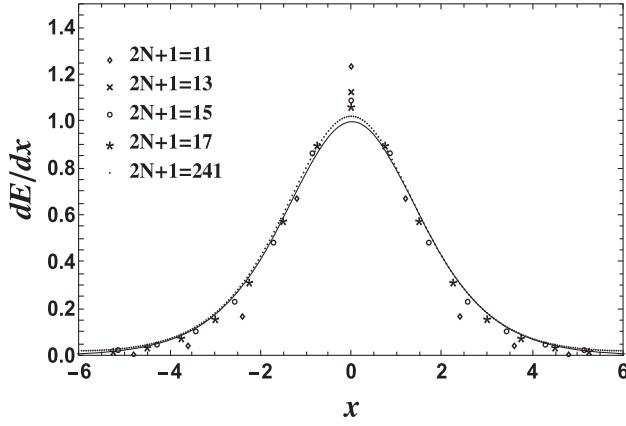


FIG. 5. The energy density of the sphaleron configuration on the lattice across different number of lattice sites $2N + 1$. The parameters are chosen to be $L = 12$, $\omega = 1/2$, and $g = 1$. The thin black line is the sphaleron solution in the continuous limit. To get a symmetric distribution, we modify the kinetic energy part of the k th site in Eq. (24) to be $[1 - \frac{1}{2}\cos(\xi_k - \xi_{k+1}) - \frac{1}{2}\cos(\xi_{k-1} - \xi_k)]$.

To further investigate the convergence to the continuous limit $a \rightarrow 0$, we compare the distribution of $1 + \mathbf{n}_k \cdot \mathbf{e}_z$ and the energy density dE/dx of the sphaleron configuration on the lattice with the continuous one in Figs. 4 and 5. We observe that as the number of lattice sites $2N + 1$ increases, i.e., the lattice spacing a decreases, the lattice sphaleron configuration approaches the continuum solution. Additionally, a rapid convergence of the lattice sphaleron configuration can be observed as the system size increases to $2N + 1 \geq 15$, with errors becoming well-controlled.

IV. THE QUANTUM SIMULATIONS

To initiate our study of sphaleron production and decay in high-energy collisions, we develop formulations to simulate the process of sphaleron decay, which require not only defining the classical sphaleron configuration in the framework of quantum field, but also accounting for the quantum effects of the $\mathbf{n}$ fields to incorporate sphaleron decaying to particles. At quantum level, the Hamiltonian describing the dynamics of $\mathbf{n}$ field can be written as

$$\hat{H}_{\text{tot}} = \frac{g^2}{2a} \sum_{k=-N}^N \hat{\mathbf{L}}_k^2 + \frac{1}{ag^2} \sum_{k=-N}^{N-1} (\mathbb{1} - \hat{\mathbf{n}}_k \cdot \hat{\mathbf{n}}_{k+1}) + \frac{a\omega^2}{g^2} \sum_{k=-N}^N (\mathbb{1} + \hat{\mathbf{n}}_k \cdot \mathbf{e}_z), \quad (35)$$

which is lifting the field variable $\mathbf{L}_k$ and $\mathbf{n}_k$ in $H_{\text{tot}}^{\text{latt}}$ as operator $\hat{\mathbf{L}}_k$ and $\hat{\mathbf{n}}_k$ acting on local fields at site k . The commutation relations follow those of the quantum rotor models, as given by [35]

$$[\hat{L}^\alpha, \hat{L}^\beta] = i\epsilon^{\alpha\beta\gamma} \hat{L}^\gamma, \quad [\hat{L}^\alpha, \hat{n}^\beta] = i\epsilon^{\alpha\beta\gamma} \hat{n}^\gamma$$

$$[\hat{n}^\alpha, \hat{n}^\beta] = 0, \quad \alpha, \beta, \gamma \in \{x, y, z\}, \quad (36)$$

where we have used unbolded notation with the subscript of the lattice index suppressed to denote the individual components of the vector operators $\hat{\mathbf{L}}_k$ and $\hat{\mathbf{n}}_k$. The local field can be represented the state $|l, m\rangle_k$ which are eigenstates of the operators $\hat{\mathbf{L}}_k^2$ and $\hat{\mathbf{L}}_k \cdot \mathbf{e}_z$. The state $|l, m\rangle_k$ form the angular momentum basis at local sites. After imposing the cutoff at l_{max} , $\hat{\mathbf{L}}_k^2$ becomes a finite-dimensional matrix with eigenvalues $l(l+1)$ for $l = 0, 1, \dots, l_{\text{max}}$, where each eigenvalue has multiplicity $2l+1$ corresponding to the degeneracy in the quantum number $m \in \{-l, -l+1, \dots, l\}$. Note that for a finite truncation l_{max} , the commutation relation in Eq. (36) is violated. The dimension of the local Hilbert space at each site becomes $\text{dim} = (l_{\text{max}} + 1)^2$, requiring $n_q = \lceil \log_2(\text{dim}) \rceil$ qubits to represent the local field at a single site when implementing the system on a quantum device. The field degrees of freedom at site k can be mapped to qubits by encoding the states $|l, m\rangle_k$ into the computational basis states of the qubit system. The matrix representation of operators $\hat{\mathbf{n}}_k \cdot \mathbf{e}_i$ ($i = x, y, z$) in the computational basis $|l, m\rangle_k$ can be obtained by noting that $\hat{\mathbf{n}}_k \cdot (\mathbf{e}_x \pm i\mathbf{e}_y) = \mp\sqrt{2}\hat{X}_{\pm 1}$ and $\hat{\mathbf{n}}_k \cdot \mathbf{e}_z = \hat{X}_0$ with the matrix representation of $\hat{X}_M$ given by [33]

$$\langle l, m | \hat{X}_M | l', m' \rangle = (-1)^m \sqrt{(2l+1)(2l'+1)} \times \begin{pmatrix} l & l' & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & 1 & l' \\ -m & M & m' \end{pmatrix}. \quad (37)$$

We expect to study the evolution of sphaleron configuration within the above framework and extract physical observables, such as sphaleron lifetime, sphaleron decay width to particles. In the following, we layout two possible strategies. One strategy is to consider the classical sphaleron configuration as a background field. This strategy has been used in [36] to study the classical evolution of sphalerons in the SM and calculate the averaged number of Higgs and W -bosons from sphaleron decays. To allow interpretations of particles produced exclusively, quantizations are necessary. Yet, such classical evolution of sphalerons might be taken as background fields added to the quantized Hamiltonian to allow studies of partial decay width, particle number distributions etc. For our case in the $O(3)$ model, it would result in the following Hamiltonian by neglecting the constant terms in Eq. (38):

$$\hat{H}_{\text{tot}}^{\text{bg}} = -\frac{1}{ag^2} \sum_{k=-N}^{N-1} (\hat{\mathbf{n}}_k + \mathbf{n}_k(t)) \cdot (\hat{\mathbf{n}}_{k+1} + \mathbf{n}_{k+1}(t)) + \frac{a\omega^2}{g^2} \sum_{k=-N}^N (\hat{\mathbf{n}}_k + \mathbf{n}_k(t)) \cdot \mathbf{e}_z + \frac{g^2}{2a} \sum_{k=-N}^N \hat{\mathbf{L}}_k^2. \quad (38)$$

In the above, we have introduced $\mathbf{n}_k(t)$ as the time-dependent classical background field with $\mathbf{n}_k(t=0) \equiv \mathbf{n}_k^{\text{sph}}$.

This method, with the classically computed $\mathbf{n}_k(t)$ driving $\hat{H}_{\text{tot}}^{\text{bg}}$, should offer a clear path to study the exclusive decay of sphalerons into particles.

Another strategy would be constructing the sphaleron initial states as quantum coherent state, which can interpolate between quantum and classical descriptions [37]. Quantum coherent states corresponding to the classical field configurations of a sphaleron have been used to study the decay products of sphalerons [38] in the framework of SM. However, within the framework of the $O(3)$ NLSM, the standard construction of coherent states, i.e., Glauber-type coherent states of quantum harmonic oscillator [39] is not applicable. As Perelomov emphasized [40,41], the Glauber-type coherent states rely essentially on the nilpotent and noncompact group structure of the Heisenberg-Weyl group, and cannot be straightforwardly extended to compact groups such as $O(3)$. For this reason, one may lead to consider the generalized coherent states introduced by Perelomov [40,41], which are defined by acting with the group transformations on a chosen reference state in the Hilbert space. In general, neither the choice of the reference state nor the associated group action is unique. The possible influence of this ambiguity on the $O(3)$ sphaleron process is not yet fully understood and deserves further investigation.

In the following, we construct the sphaleron coherent state by choosing the lowest-weight state as the reference state [37]. The construction is carried out under the requirement that, in the infinite-dimensional Hilbert space limit, i.e., as $l_{\text{max}} \rightarrow \infty$, the spatial field configuration of the sphaleron coherent state approaches the discretized classical sphaleron configuration obtained in Sec. III. With the local state $\{|l, m\rangle_k\}$ on site k as described previously, the lowest-weight reference state of the full chain can then be written as

$$|\psi_0\rangle = \bigotimes_{k=-N}^N \sum_{l=0}^{l_{\text{max}}} \frac{1}{\sqrt{l_{\text{max}} + 1}} |l, -l\rangle_k. \quad (39)$$

The group transformation $\hat{R}$ acting on the full chain is given by the tensor product of $O(3)$ group rotations $\hat{R}_k(\theta_k, \phi_k)$ acting on each site, i.e., $\hat{R} = \bigotimes_{k=-N}^N \hat{R}_k(\theta_k, \phi_k)$.

Here $\hat{R}_k(\theta_k, \phi_k)$ denotes the action that rotates the angular momentum of the state from the z -axis to the direction $\mathbf{n}_k(\theta_k, \phi_k)$, where $\mathbf{n}_k(\theta_k, \phi_k) = (\sin \theta_k \cos \phi_k, \sin \theta_k \sin \phi_k, \cos \theta_k)$ with $\theta_k \in [0, \pi]$ and $\phi_k \in [0, 2\pi]$. Acting with $\hat{R}$ on the reference state $|\psi_0\rangle$, we obtain the initial state as $|\psi_{\text{init}}\rangle$,

$$\begin{aligned} |\psi_{\text{init}}\rangle &= \left(\bigotimes_{k=-N}^N \hat{R}_k(\theta_k, \phi_k) \right) |\psi_0\rangle \\ &\equiv \bigotimes_{k=-N}^N |\mathbf{n}(\theta_k, \phi_k)\rangle_k, \end{aligned} \quad (40)$$

with

$$|\mathbf{n}(\theta_k, \phi_k)\rangle_k = \hat{R}_k(\theta_k, \phi_k) \sum_{l=0}^{l_{\text{max}}} \frac{1}{\sqrt{l_{\text{max}} + 1}} |l, -l\rangle_k. \quad (41)$$

The explicit action of the rotation operator on a lowest-weight state can be written as [42]

$$\begin{aligned} \hat{R}_k(\theta_k, \phi_k) \sum_{l=0}^{l_{\text{max}}} |l, -l\rangle_k &= e^{-i\phi_k \hat{L}_z} e^{-i\theta_k \hat{L}_y} \sum_{l=0}^{l_{\text{max}}} |l, -l\rangle_k \\ &= e^{\zeta(\theta_k, \phi_k) \hat{L}_+ - \zeta^*(\theta_k, \phi_k) \hat{L}_-} \sum_{l=0}^{l_{\text{max}}} |l, -l\rangle_k, \end{aligned}$$

in the above equation, we have defined $\zeta(\theta, \phi) \equiv -\frac{\theta}{2} e^{-i\phi}$.

By properties of $O(3)$ rotation operators and their relation to spherical harmonics [43], the local state can be given by

$$|\mathbf{n}(\theta_k, \phi_k)\rangle_k = \frac{2\sqrt{\pi}}{\sqrt{l_{\text{max}} + 1}} \sum_{l=0}^{l_{\text{max}}} \sum_{m=-l}^l Y_{lm}^*(\theta_k, \phi_k) |l, m\rangle_k. \quad (42)$$

Recalled the purpose of this construction is to ensure that, in the limit $l_{\text{max}} \rightarrow \infty$, the expectation values of the position operators reproduce the corresponding classical field configuration. Applying the local state defined in (42) to the direction operators $\hat{\mathbf{n}}_k$, one finds that for each component $\mathbf{e}_i$ ($i = x, y, z$),

$$\langle \mathbf{n}(\theta_k, \phi_k) | \hat{\mathbf{n}}_k \cdot \mathbf{e}_i | \mathbf{n}(\theta_k, \phi_k) \rangle_k = \frac{l_{\text{max}}}{l_{\text{max}} + 1} \mathbf{n}_k(\theta_k, \phi_k). \quad (43)$$

with the prefactor approaching unity as $l_{\text{max}} \rightarrow \infty$. Note that with such construction, the expectation value of $\hat{\mathbf{n}}_k \cdot \mathbf{e}_i$ aligns with $\mathbf{n}_k(\theta_k, \phi_k)$ even for finite l_{max} .

To ensure that the initial state $|\psi_{\text{init}}\rangle$ represents the sphaleron state, the rotation angles (θ_k, ϕ_k) at each site are chosen to match the local orientation of the classical sphaleron field $\{\mathbf{n}_k^{\text{sph}}\}$, specified by $(\theta_k^{\text{sph}}, \phi_k^{\text{sph}})$. This leads to the sphaleron coherent state as

$$|\psi_{\text{init}}^{\text{sph}}\rangle = \bigotimes_{k=-N}^N |\mathbf{n}(\theta_k^{\text{sph}}, \phi_k^{\text{sph}})\rangle_k. \quad (44)$$

While the potential term of the Hamiltonian in (35) reduces to its classical counterpart, the kinetic term grows with l_{max} and diverges as $l_{\text{max}} \rightarrow \infty$. This divergence follows from the uncertainty principle, since a precise specification of the field orientation $\hat{\mathbf{n}}_k$ is imposed. As a result, a finite cutoff l_{max} should be retained not only for the technical reasons discussed previously, namely to render the Hilbert space finite-dimensional and thus amenable to quantum

simulation, but also to restrict the description of sphaleron dynamics to a finite energy range.

The real-time evolution for $|\psi_{\text{init}}^{\text{sph}}\rangle$ under $\hat{H}_{\text{tot}}$ can be realized using the Trotterization scheme [44]. Consider the three noncommuting terms in $\hat{H}_{\text{tot}}$ as

$$\begin{aligned}\hat{H}_1 &= \frac{g^2}{2a} \sum_k \hat{\mathbf{L}}_k^2, \\ \hat{H}_2 &= \frac{1}{ag^2} \sum_k (\mathbb{1} - \hat{\mathbf{n}}_k \cdot \hat{\mathbf{n}}_{k+1}), \\ \hat{H}_3 &= \frac{a\omega^2}{g^2} \sum_k (\mathbb{1} + \hat{\mathbf{n}}_k \cdot \mathbf{e}_z),\end{aligned}\quad (45)$$

the second-order Trotterization for evolving $\hat{H}_{\text{tot}}$ for a total time t with Trotter step Δt can therefore be written as $[U(\Delta t)]^s$, with $s = t/\Delta t$ and

$$U(\Delta t) = \prod_{l=1}^{l=3} U_l(\Delta t) \prod_{l=3}^{l=0} U_l(\Delta t), \quad (46)$$

with $U_l(\Delta t) = e^{-i\hat{H}_l\Delta t/2}$.

In the following, we describe how to implement the time evolution operator in Eq. (46) using quantum circuits. When an estimation of the circuit depth is necessary, we focus on the universal quantum gate set $\{R_x(\theta), R_y(\theta), R_z(\theta), \text{CNOT}\}$ where the gates included are widely supported by near-term devices. Given that operators acting on different lattice sites commute and $\hat{H}_1$ is diagonal in the computational basis, $e^{-i\hat{H}_1\Delta t/2}$ can be implemented with $R_z(\theta)$ gates and CNOT gates with a circuit depth of $\mathcal{O}(2^{n_q})$ for general cases. Define $U_{k,k+1}(\Delta t) = \exp(-i\theta_1\Delta t\hat{\mathbf{n}}_k \cdot \hat{\mathbf{n}}_{k+1})$ and $U_{k,z}(\Delta t) = \exp(-i\theta_2\Delta t\hat{\mathbf{n}}_k \cdot \mathbf{e}_z)$ with $\theta_1 = -1/2ag^2$ and $\theta_2 = -a\omega^2/2g^2$, we have

$$\begin{aligned}U_2(\Delta t) &= e^{i\theta_1\Delta t} \prod_k U_{k,k+1}(\Delta t), \\ U_3(\Delta t) &= e^{i\theta_2\Delta t} \prod_k U_{k,z}(\Delta t).\end{aligned}\quad (47)$$

Given the matrix representation of $\hat{\mathbf{n}}_k \cdot \mathbf{e}_i$ in the computational basis, $\hat{\mathbf{n}}_k \cdot \hat{\mathbf{n}}_{k+1}$ and $\hat{\mathbf{n}}_k \cdot \mathbf{e}_z$ can be systematically decomposed into linear combinations of Pauli strings with operator-Schmidt techniques [45], allowing the time evolution operator $U_{k,k+1}(\Delta t)$ and $U_{k,z}(\Delta t)$ to be implemented with quantum circuits using Trotterizations or qDRIFT algorithm [46]. For the case of $n_q < 12$ where matrix exponentiation is classically tractable, these evolution operators can also be exactly decomposed into quantum gates. The time-evolution operators $U_2(\Delta t)$ and $U_3(\Delta t)$ can thus be constructed subsequently following Eq. (47).

We consider the quantum algorithm based on a lattice with $2N + 1 = 5$ sites and the simplest nontrivial

truncation $l_{\text{max}} = 1$ which leads to $n_q = 2$. In this case, the mapping of the local states to the computational basis can be chosen as $|l=0, m=0\rangle_k \rightarrow |00\rangle_k$, $|l=1, m=-1\rangle_k \rightarrow |01\rangle_k$, $|l=1, m=0\rangle_k \rightarrow |10\rangle_k$ and $|l=1, m=1\rangle_k \rightarrow |11\rangle_k$ at site k . Here, $|00\rangle_k$, $|01\rangle_k$, $|10\rangle_k$, and $|11\rangle_k$ represent the computational basis states of the two qubits at lattice site k . An exact decomposition of $U_{k,k+1}(\Delta t)$ using the QISKIT transpiler yields a CNOT circuit depth of order $\mathcal{O}(3 \times 10^4)$. Given the substantial circuit depths involved, implementing the corresponding time evolution on a NISQ device would require dedicated quantum error mitigation or correction techniques in order to obtain reliable results.

As a test, we investigate the time evolution of the initial state $|\psi_{\text{init}}^{\text{sph}}\rangle$ with noiseless simulators, focusing on quantifying Trotterization errors in Eq. (46) through their impact on the energy of the system. With the exact construction of both $U_{k,k+1}(\Delta t)$ and $U_{k,z}(\Delta t)$, our primary observable is defined as the relative deviation of the total energy from its initial value,

$$\varepsilon_E(t) = \left| \frac{\langle \hat{H}_{\text{tot}}(t) \rangle - \langle \hat{H}_{\text{tot}}(0) \rangle}{\langle \hat{H}_{\text{tot}}(0) \rangle} \right|, \quad (48)$$

where $\langle \hat{H}_{\text{tot}}(t) \rangle$ is the expectation value of $\hat{H}_{\text{tot}}$ in Eq. (35) for the state evolved to time t .

We will use the same set of parameters in the classical lattice sphaleron solution and calculate $\varepsilon_E(t)$ for different Δt used. To rigorously bound Trotter errors for all physical processes in the system, we choose a benchmark evolution time $t = 35$, which is significantly larger than the longest characteristic timescale, $1/\Delta E \sim 5$, where ΔE denotes the energy gap between the ground state and the first excited state. The results are shown in Fig. 6. As can be seen, $\varepsilon_E(t)$, with its magnitude smaller than 1% for $\Delta t < 1$, display an approximately power-law dependence on Δt , indicating that the errors are dominated by the leading-order term, which for second-order Trotterization scales as $\mathcal{O}(\Delta t^2)$. Furthermore, it establishes the range of Δt values for reliable Trotterized simulations.

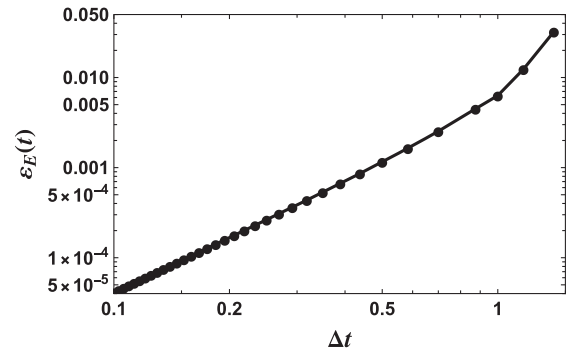


FIG. 6. Trotterization error $\varepsilon_E(t)$ as a function of the Trotterization step size Δt , evaluated over an evolution period $t = 35$.

The work presented above should be understood as an operational construction of quantum states whose spatial field configurations approximate those of the classical lattice sphaleron within a finite-dimensional Hilbert space. In this framework, such states can be prepared and evolved using quantum circuits, thereby providing a concrete platform for investigating sphaleron-related dynamics at the level of field configurations. At present, however, a rigorous and widely accepted definition of a quantum sphaleron state has yet to be established. Moreover, the simulations reported here are restricted to small system sizes and limited qubit numbers, which in turn constrain the range of accessible observables. Extending these studies to larger systems and exploring additional dynamical signatures therefore constitute natural directions for future work.

V. CONCLUSIONS

The production of sphaleron at high energy collisions in the SM is yet to be observed. Despite decades of theoretical investigation, reliable predictions of sphaleron production rates remain elusive. As a nonperturbative quantum process requiring real-time field theory simulations, this challenge may ultimately be resolved through quantum computation. In this work, we initiate this research program by studying the $1 + 1\text{D } O(3)$ NLSM, which, when coupled to fermions, provides a tractable framework for studying sphaleron production in high-energy collisions. With lattice formulation, we explicitly identified the sphaleron configuration as compared to that in the continuous spacetime theory. Furthermore, we established appropriate lattice volumes and spacings that reproduce the sphaleron energy of the continuum theory with controlled precision. To simulate dynamical processes involving sphaleron production from particle collisions and subsequent decay, we have provided one way of constructing the classical sphaleron field configuration from quantum states, developed the quantum algorithms for the sphaleron evolution using Trotterization. Future work should extend these efforts along three directions; developing quantum algorithms to extract sphaleron production rates in high-energy collisions as well as its decay branching ratio to particle, measuring Chern-Simons number evolution during sphaleron dynamics, and incorporating fermionic degrees of freedom into the framework.

ACKNOWLEDGMENTS

The work of M. H. and H. Z. is supported in part by the National Science Foundation of China under Grants No. 12575114, No. 12235001, and No. 12075257. Y.-Y. L. is supported in part by the National Science Foundation of China under Grant No. 12522509, the National Key R&D Program of China (Grant No. 2025YFA1614200), and IHEP under Grant No. E55153U1. M. H. would like to thank Dr. K. Hu at the Particle Astrophysics Center of the Institute of High Energy Physics (IHEP), Chinese Academy of Sciences, for his support and assistance in the programming

of quantum simulations. Y.-Y. L. and H. Z. thanks Prof. Andrew Cohen for useful discussion.

DATA AVAILABILITY

In this theoretical work, the data can be generated using the material provided in the manuscript.

APPENDIX: CONSTRUCTING THE DISCRETIZED HAMILTONIAN

Rewrite the phase space coordinates of $\mathbf{n}$ using polar coordinates,

$$\mathbf{n}(\theta, \varphi) = (\sin \theta(x) \cos \varphi(x), \sin \theta(x) \sin \varphi(x), \cos \theta(x)),$$

the Lagrangian can be written as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2g^2} (\partial_\mu \mathbf{n})^2 - \frac{\omega^2}{g^2} (1 + \mathbf{n} \cdot \vec{e}_z) \\ &= \frac{1}{2g^2} (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta - (\partial_x \mathbf{n})^2) - \frac{\omega^2}{g^2} (1 + \mathbf{n} \cdot \mathbf{e}_z) \end{aligned}$$

with

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= \frac{\dot{\theta}}{g^2} \\ \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} &= \frac{\dot{\varphi} \sin^2 \theta}{g^2} \end{aligned}$$

the classical Hamiltonian can be obtained via Legendre transformation as

$$H_{\text{tot}} = \frac{1}{2g^2} \int dx (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta + (\partial_x \mathbf{n})^2 + 2\omega^2 (1 + \mathbf{n} \cdot \mathbf{e}_z)), \quad (\text{A1})$$

where the first two terms are the kinetic energy, and the rest are taken as potential energies.

Next, we proceed with discretization in the x -direction while omitting the constant term,

$$\begin{aligned} H_{\text{tot}}^{\text{latt}} &= \frac{a}{2g^2} \sum_k \left(\dot{\theta}_k^2 + \dot{\varphi}_k^2 \sin^2 \theta_k - \frac{2}{a^2} \mathbf{n}_k \mathbf{n}_{k+1} \right) \quad (\text{A2}) \\ &\quad + \frac{a\omega^2}{g^2} \sum_k (1 + \mathbf{n}_k \cdot \mathbf{e}_z) \\ &= \sum_k \left(\frac{1}{2} I \omega_k^2 - \frac{1}{2g^2} \frac{2}{a} \mathbf{n}_k \mathbf{n}_{k+1} \right) \\ &\quad + \frac{a\omega^2}{g^2} \sum_k (1 + \mathbf{n}_k \cdot \mathbf{e}_z). \quad (\text{A3}) \end{aligned}$$

Since the phase space is a spherical surface, the kinetic energy part is expressed as rotational energies in terms with the moment of inertia $I = a/g^2$ and the angular velocity vector with the angular velocity vector,

$$\boldsymbol{\omega}_k = \begin{pmatrix} \cos \theta_k \cos \varphi_k \dot{\theta}_k - \sin \varphi_k \sin \theta_k \dot{\varphi}_k \\ \cos \theta_k \sin \varphi_k \dot{\theta}_k + \cos \varphi_k \sin \theta_k \dot{\varphi}_k \\ -\sin \theta_k \dot{\theta}_k \end{pmatrix}.$$

We thus define the corresponding angular momentum as

$$\mathbf{L}_k(\theta_k, \varphi_k) \equiv I \boldsymbol{\omega}_k \quad (\text{A4})$$

Rewriting the Hamiltonian in terms of angular momentum in vector form, we obtain,

$$H_{\text{tot}}^{\text{latt}} = \frac{g^2}{2a} \sum_k \mathbf{L}_k^2 - \frac{1}{ag^2} \sum_k \mathbf{n}_k \mathbf{n}_{k+1} + \frac{a\omega^2}{g^2} \sum_k (1 + \mathbf{n}_k \cdot \mathbf{e}_z) \quad (\text{A5})$$

-
- [1] J. Milnor, *Morse Theory*, Annals of Mathematics Studies (Princeton University Press, 1963).
- [2] C. H. Taubes, The existence of a nonminimal solution to the SU(2) Yang-Mills Higgs equations on R^3 , *Commun. Math. Phys.* **86**, 257 (1982).
- [3] N. S. Manton, Topology in the Weinberg-Salam theory, *Phys. Rev. D* **28**, 2019 (1983).
- [4] P. Forgacs and Z. Horvath, Topology and saddle points in field theories, *Phys. Lett.* **138B**, 397 (1984).
- [5] F. R. Klinkhamer and N. S. Manton, A saddle point solution in the Weinberg-Salam theory, *Phys. Rev. D* **30**, 2212 (1984).
- [6] A. D. Sakharov, Violation of CP invariance, C asymmetry, and baryon asymmetry of the universe, *Pis'ma Zh. Eksp. Teor. Fiz.* **5**, 32 (1967).
- [7] A. G. Cohen, D. B. Kaplan, and A. E. Nelson, Progress in electroweak baryogenesis, *Annu. Rev. Nucl. Part. Sci.* **43**, 27 (1993).
- [8] D. E. Morrissey and M. J. Ramsey-Musolf, Electroweak baryogenesis, *New J. Phys.* **14**, 125003 (2012).
- [9] J. Ambjorn, T. Askgaard, H. Porter, and M. E. Shaposhnikov, Sphaleron transitions and baryon asymmetry: A numerical real time analysis, *Nucl. Phys.* **B353**, 346 (1991).
- [10] M. D'Onofrio, K. Rummukainen, and A. Tranberg, Sphaleron rate in the minimal Standard Model, *Phys. Rev. Lett.* **113**, 141602 (2014).
- [11] C. Bonanno, F. D'Angelo, M. D'Elia, L. Maio, and M. Naviglio, Sphaleron rate from a modified Backus-Gilbert inversion method, *Phys. Rev. D* **108**, 074515 (2023).
- [12] C. Bonanno, F. D'Angelo, M. D'Elia, L. Maio, and M. Naviglio, Sphaleron rate of $N_f = 2 + 1$ QCD, *Phys. Rev. Lett.* **132**, 051903 (2024).
- [13] V. A. Rubakov and M. E. Shaposhnikov, Electroweak baryon number nonconservation in the early universe and in high-energy collisions, *Usp. Fiz. Nauk* **166**, 493 (1996).
- [14] M. J. Gibbs, A. Ringwald, B. R. Webber, and J. T. Zadzorny, Monte carlo simulation of baryon and lepton number violating processes at high energies, *Z. Phys. C* **66**, 285 (1995).
- [15] F. Bezrukov, D. Levkov, C. Rebbi, V. Rubakov, and P. Tinyakov, Suppression of baryon number violation in electroweak collisions: numerical results, *Phys. Lett. B* **574**, 75 (2003).
- [16] F. Bezrukov, D. Levkov, C. Rebbi, V. Rubakov, and P. Tinyakov, Semiclassical study of baryon and lepton number violation in high-energy electroweak collisions, *Phys. Rev. D* **68**, 036005 (2003).
- [17] A. Ringwald, An upper bound on the total cross-section for electroweak baryon number violation, *J. High Energy Phys.* **10** (2003) 008.
- [18] A. Ringwald, Electroweak instantons/sphalerons at VLHC?, *Phys. Lett. B* **555**, 227 (2003).
- [19] S. H. H. Tye and S. S. C. Wong, Bloch wave function for the periodic sphaleron potential and unsuppressed baryon and lepton number violating processes, *Phys. Rev. D* **92**, 045005 (2015).
- [20] S. H. H. Tye and S. S. C. Wong, Baryon number violating scatterings in laboratories, *Phys. Rev. D* **96**, 093004 (2017).
- [21] J. Ellis and K. Sakurai, Search for sphalerons in proton-proton collisions, *J. High Energy Phys.* **04** (2016) 086.
- [22] J. Ellis, K. Sakurai, and M. Spannowsky, Search for sphalerons: IceCube vs. LHC, *J. High Energy Phys.* **05** (2016) 085.
- [23] C. W. Bauer *et al.*, Quantum simulation for high-energy physics, *PRX Quantum* **4**, 027001 (2023).
- [24] A. Di Meglio *et al.*, Quantum computing for high-energy physics: State of the art and challenges. Summary of the QC4HEP Working Group, *PRX Quantum* **5**, 037001 (2024).
- [25] C. W. Bauer, Z. Davoudi, N. Klco, and M. J. Savage, Quantum simulation of fundamental particles and forces, *Nat. Rev. Phys.* **5**, 420 (2023).
- [26] Y. Fang, C. Gao, Y.-Y. Li, J. Shu, Y. Wu, H. Xing, B. Xu, L. Xu, and C. Zhou, Quantum frontiers in high energy physics, *Sci. China Phys. Mech. Astron.* **68**, 260301 (2025).
- [27] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Two-dimensional sigma models: Modeling non-perturbative effects of quantum chromodynamics, *Phys. Rep.* **116**, 103 (1984).
- [28] E. Mottola and A. Wipf, Unsuppressed fermion-number violation at high temperature: An O(3) model, *Phys. Rev. D* **39**, 588 (1989).
- [29] C. J. Hamer, J. B. Kogut, and L. Susskind, Strong-coupling expansions and phase diagrams for the O(2), O(3), and O(4) Heisenberg spin systems in two dimensions, *Phys. Rev. D* **19**, 3091 (1979).

- [30] A. Alexandru, P. F. Bedaque, H. Lamm, and S. Lawrence (NuQS Collaboration), σ models on quantum computers, *Phys. Rev. Lett.* **123**, 090501 (2019).
- [31] A. Alexandru, P. F. Bedaque, A. Carosso, M. J. Cervia, and A. Sheng, Qubitization strategies for bosonic field theories, *Phys. Rev. D* **107**, 034503 (2023).
- [32] J. Y. Araz, S. Schenk, and M. Spannowsky, Toward a quantum simulation of nonlinear sigma models with a topological term, *Phys. Rev. A* **107**, 032619 (2023).
- [33] F. Bruckmann, K. Jansen, and S. Kühn, $O(3)$ nonlinear sigma model in $1+1$ dimensions with matrix product states, *Phys. Rev. D* **99**, 074501 (2019).
- [34] J.-Y. Chen, Instanton density operator in Lattice QCD from higher category theory, *SciPost Phys.* **19**, 158 (2025).
- [35] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, Cambridge, England, 2011).
- [36] M. Hellmund and J. Kripfganz, The decay of the sphaleron, *Nucl. Phys.* **B373**, 749 (1992).
- [37] W.-M. Zhang, D. H. Feng, and R. Gilmore, Coherent states: Theory and some applications, *Rev. Mod. Phys.* **62**, 867 (1990).
- [38] J. Zadrozny, Sphaleron decay products: A coherent state analysis, *Phys. Lett. B* **284**, 88 (1992).
- [39] R. J. Glauber, Coherent and incoherent states of the radiation field, *Phys. Rev.* **131**, 2766 (1963).
- [40] A. M. Perelomov, Coherent states for arbitrary lie groups, *Commun. Math. Phys.* **26**, 222 (1972).
- [41] A. Perelomov, *Generalized Coherent States and Their Applications*, Theoretical and Mathematical Physics (Springer, New York, 1986).
- [42] A. M. Perelomov, Generalized coherent states and some of their applications, *Sov. Phys. Usp.* **20**, 703 (1977).
- [43] E. Wigner, Einige folgerungen aus der schrödingerschen theorie für die termstrukturen, *Z. Phys.* **43**, 624 (1927).
- [44] M. Suzuki, Fractal decomposition of exponential operators with applications to many-body theories and Monte Carlo simulations, *Phys. Lett. A* **146**, 319 (2002).
- [45] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, Cambridge, England, 2010).
- [46] E. Campbell, Random compiler for fast Hamiltonian simulation, *Phys. Rev. Lett.* **123**, 070503 (2019).