# Geometric origin of the energy-momentum tensor improvement terms

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In a flat background, the canonical energy momentum tensor of Lorentz and conformally invariant matter field theories can be improved to a symmetric and traceless tensor that gives the same conserved charges. We argue that the geometric origin of this improvement process is unveiled when the matter theory is coupled to metric-affine gravity. In particular, we show that the Belinfante-Rosenfeld improvement terms correspond to the matter theory's hypermomentum. The improvement terms in conformally invariant matter theories are also related to the hypermomentum; however, a general proof would require an extended investigation. We demonstrate our results through various examples, such as the free massless scalar, the Maxwell field, Abelian $p$-forms, the Dirac field, and a nonunitary massless scalar field. Possible applications of our method for theories that break Lorentz or special conformal invariance are briefly discussed.

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## I. INTRODUCTION

The energy momentum (e.m.) tensor is a fundamental quantity in any field theory as it is associated with its symmetries and the corresponding conserved charges. Nevertheless, even on a fixed flat background the e.m. of a matter action is not uniquely defined although its different forms give rise to the same conserved charges. For example, given a translation invariant action, one constructs by the standard Noether's procedure the so-called *canonical* e.m. tensor $t^{\mu\nu}$ that yields the conserved energy and spatial momentum of the theory. However, if the action is also Lorentz invariant the canonical e.m. tensor satisfies a certain property which enables one to define an "improved" e.m. tensor, the so-called Belinfante-Rosenfeld (BR) e.m. tensor $T^{\mu\nu}_{\rm BR}$, which is symmetric in $\mu, \nu$ [1,2]. Both $t^{\mu\nu}$ and $T^{\mu\nu}_{\rm BR}$ give rise to the same conserved charges, e.g., [3], i.e., the relativistic angular momenta; however, the symmetry property of the latter generically makes it preferable as the latter is explicitly symmetric under Lorentz transformations.

Similarly, for Lorentz invariant theories that are also invariant under scale transformations the Noether's procedure yields a conserved dilatation current which is given by a combination of the canonical e.m. tensor and a "virial" current [4]. If the theory is also invariant under special conformal transformations, the canonical e.m. tensor satisfies a certain property which enables one to "improve" it and define a conserved, symmetric, and traceless e.m. tensor $\Theta^{\mu\nu}$ [5]. Both $t^{\mu\nu}$ and $\Theta^{\mu\nu}$ give rise to the same conserved charges; however, the tracelessness of the latter generically makes it preferable as the latter is explicitly symmetric under scale and special conformal transformations.

Perhaps the most important property of the improved e.m. tensors $T^{\mu\nu}_{\rm BR}$ and $\Theta^{\mu\nu}$ is that these quantities are somehow "chosen" by Einstein gravity when the latter couples to matter. Indeed, the variation with respect to the metric of any diffeomorphic invariant extension of a Lorentz invariant matter action defines a symmetric and covariantly conserved e.m. tensor, the Hilbert or *metrical* e.m. tensor $T^{\mu\nu}$. This tensor generically coincides with the improved BR e.m. tensor $T^{\mu\nu}_{\rm BR}$ in the usual flat limit. However, if the flat action is conformally invariant, one might want to couple it to Einstein gravity in a Weyl invariant manner [6]. It turns out that the metrical e.m. tensor of a Weyl invariant action is also traceless, and it

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coincides with the improved conformal one $\Theta^{\mu\nu}$ in the usual flat limit.

Hence, coupling matter to gravity appears to circumvent the improvement process for the canonical e.m. tensor of matter theories in flat backgrounds. This is expected since coupling a flat matter theory to gravity is equivalent to making it diffeomorphic (and Weyl) invariant and hence to the gauging of all of its global flat spacetime symmetries. Nevertheless, it is interesting to ask whether there is a gravitational/geometrical analog of the improvement process that leads from the canonical flat e.m. tensor $t^{\mu\nu}$ to $T^{\mu\nu}_{\rm BR}$ and $\Theta^{\mu\nu}$. Motivated by this question, we study in this work the coupling of flat matter actions to metric-affine gravity (MAG) [7]; see also [8] for a review. Such a wider geometrization gives rise not only to a metrical e.m. tensor $T^{\mu\nu}$ of the curved action but also to the *hypermomentum* $\Delta_\lambda{}^{\mu\nu}$ defined as the variation of the action with respect to the affine connection [9,10]. In this case diffeomorphism invariance leads to a more complicated conservation equation that involves both $T^{\mu\nu}$ and $\Delta_\lambda{}^{\mu\nu}$. Furthermore, MAG extensions of Weyl invariance lead to trace identities that also involve both $T^{\mu\nu}$ and $\Delta_\lambda{}^{\mu\nu}$ [11].[1] Our observation is that the usual flat limits of the above conservation and trace identities differ from the corresponding ones of Einstein gravity due to the presence of the hypermomentum. This means that the flat limit of the MAG metrical e.m. tensor does not generically coincide neither with the BR e.m. tensor $T^{\mu\nu}_{\rm BR}$ nor with the improved conformal one $\Theta^{\mu\nu}$ [13]. This in turn implies that the hypermomentum is intimately related to the improvement terms of the canonical flat space e.m. tensor $t^{\mu\nu}$. The explicit relationship of the hypermomentum $\Delta_\lambda{}^{\mu\nu}$ to the improvement terms is theory dependent as we demonstrate in a number of examples that include scalars, fermions, and vector fields.

The paper is organized as follows. In Sec. II we briefly review the improvement procedure for the canonical e.m. tensor of matter theories in flat space. In Sec. III we study matter fields in MAG backgrounds and write down the conservation and trace conditions that follow from diffeomorphism and MAG-Weyl invariance. In Sec. IV we present examples of Lorentz invariant flat matter theories where the improvement process of their canonical e.m. tensor is captured by the hypermomentum. These include the Maxwell field, Dirac fermions and higher-form Abelian matter fields. In Sec. V we present examples of conformally invariant flat matter theories whose improvement process for their canonical e.m. tensor is also captured by the hypermomentum. These include canonical scalars as well as higher-derivative massless scalars. We summarize our results and discuss their possible generalization in Sec. VI. In Appendix A, we summarize the metric-affine framework. Then, in Appendixes B and C we revisit the canonical stress tensor as a variation with respect to the vielbein and derive its expression with the metric-affine theories. In Appendix D, we revisit the improvement terms in the various examples considered in the main text. Finally, in Appendix E we revisit improvements of the conserved current, and we apply it in the various cases studied in the main text.

[1]Trace anomaly in MAG has been recently studied in [12].

## II. THE IMPROVED ENERGY MOMENTUM TENSORS OF MATTER FIELDS IN A FLAT BACKGROUND

We briefly review here the well-known improvement process for matter fields in a flat background. This discussion can be found in many places in the literature, and we follow [5].[2] We consider matter fields $\phi^A(x)$, where $A$ collectively denotes spacetime or internal symmetry indices, described by the second order action

$$S[\phi^A] = \int d^dx \mathcal{L}(\phi^A, \partial_\mu \phi^A). \tag{2.1}$$

The metric is Minkowski with a mostly plus signature. A straightforward way to study the global spacetime symmetries of the action (2.1) is to consider general *active* coordinate transformations of the form $x^\mu \mapsto x^\mu - \xi^\mu(x)$ that change the action by a total derivative as[3]

$$\begin{aligned} S[\phi^A] &\mapsto S[\phi^A] + \delta^{(1)}_\xi S[\phi^A] \\ &= S[\phi^A] + \int d^dx \partial_\mu(\xi^\mu \mathcal{L}(\phi^A, \partial\phi^A)). \end{aligned} \tag{2.2}$$

Next, the fields transform under the active transformation as

$$\phi^A \mapsto \phi^A + \delta_\xi \phi^A = \phi^A + \xi^\mu \partial_\mu \phi^A + \Delta_\xi \phi^A. \tag{2.3}$$

We split $\delta_\xi \phi^A$ into a first term that it is always there and a second term, $\Delta_\xi \phi^A$, that depends on the tensorial properties of $\phi^A$ as well as on the form of the transformation. We further require that $\Delta_\xi \phi^A$ is at most linear in whatever small parameter is inside $\xi^\mu$. Under (2.3) the action (2.1) transforms as

$$\begin{aligned} &S[\phi^A] \mapsto S[\phi^A] + \delta^{(2)}_\xi S[\phi^A] \\ &\quad = S[\phi^A] + \int d^dx \left( \frac{\partial \mathcal{L}}{\partial \phi^A} \delta_\xi \phi^A + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} \partial_\mu (\delta_\xi \phi^A) \right). \end{aligned} \tag{2.4}$$

Requiring that $[\delta^{(1)}_\xi - \delta^{(2)}_\xi] S[\phi^A] = 0$ gives

[2]Generalizations of the Noether method have been considered in [14–19].

[3]This follows from the fact that the Lagrangian $\mathcal{L}(x)$ is just a scalar field.

$$\int d^d x[\partial_\mu(\xi^\nu t^\mu{}_\nu + \pi^{\mu,A}\Delta_\xi\phi^A) + \delta_\xi\phi^A \times \text{e.o.m.}] = 0, \tag{2.5}$$

where the canonical e.m. tensor $t^\mu{}_\nu$ is defined as

$$t^\mu{}_\nu = \pi^{\mu,A}\partial_\nu\phi^A - \delta^\mu{}_\nu\mathcal{L}, \qquad \pi^{\mu,A} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}, \tag{2.6}$$

and the equations of motion (e.o.m.) are as usual

$$\text{e.o.m.} = \frac{\partial\mathcal{L}}{\partial\phi^A} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\right). \tag{2.7}$$

Hence, on-shell we obtain the well-known (i.e., see [5]) relation that yields the conserved currents for a spacetime symmetry in flat space

$$\partial_\mu(\xi^\nu t^\mu{}_\nu + \pi^{\mu,A}\Delta_\xi\phi^A) = 0. \tag{2.8}$$

From (2.8) we can read the conditions satisfied by the canonical e.m. tensor $t^\mu{}_\nu$ which lead to the improvement process.

(i) Translations correspond to $\xi^\mu = a^\mu$ with $a^\mu$ constant. This gives

$$\Delta_a\phi^A = 0 \Rightarrow \partial_\mu t^\mu{}_\nu = 0. \tag{2.9}$$

(ii) Lorentz rotations correspond to

$$\xi^\mu = \omega^\mu{}_\nu x^\nu, \qquad \omega^{\mu\nu} = -\omega^{\nu\mu} = \text{constants}. \tag{2.10}$$

This gives

$$\Delta_\omega\phi^A = \omega_{\mu\nu}\Sigma^{\mu\nu,A}, \tag{2.11}$$

where $\Sigma^{\mu\nu,A} = -\Sigma^{\nu\mu,A}$ is a function of the field $\phi^A$ that depends on their Lorentz properties. Substituting (2.11) into (2.8) we obtain[4]

$$t^{\mu\nu} - t^{\nu\mu} = 2\partial_\rho\Psi^{\rho[\mu\nu]}, \tag{2.12}$$

where the quantity $\Psi^{\rho[\mu\nu]} = \pi^{\rho,A}\Sigma^{[\mu\nu],A}$ is referred to as the *spin tensor*. The condition (2.12) satisfied by the antisymmetric part of the canonical e.m. tensor is the imprint of Lorentz invariance of the action (2.1). It immediately leads to the well-known BR improvement procedure as [1,2]

$$t^{\mu\nu} = T^{\mu\nu}_{\text{BR}} + \partial_\rho(\Psi^{\mu[\nu\rho]} + \Psi^{\nu[\mu\rho]} + \Psi^{\rho[\mu\nu]}), \tag{2.13}$$

where the BR e.m. tensor $T^{\mu\nu}_{\text{BR}}$ is symmetric and on-shell conserved

$$\partial_\mu T^{\mu\nu}_{\text{BR}} = 0, \qquad T^{\mu\nu}_{\text{BR}} = T^{\nu\mu}_{\text{BR}}. \tag{2.14}$$

(iii) Dilatations correspond to $\xi^\mu = \lambda x^\mu$ with $\lambda$ constant, and hence

$$\Delta_\lambda\phi^A = \lambda\Sigma^A \tag{2.15}$$

for a function $\Sigma^A$ of the fields. Then from (2.8) we obtain

$$t^\mu{}_\mu + \partial_\mu N^\mu = 0, \tag{2.16}$$

with $N^\mu = \pi^{\mu,A}\Sigma^A$ referred to as the *virial current*. If the theory is Lorentz invariant (2.16) implies a condition for the trace of the Bellinfante-Rosenfeld e.m. tensor as

$$T^\mu_{\text{BR}\mu} + \partial_\mu(N^\mu + 2\Psi_\nu^{[\nu\mu]}) = 0. \tag{2.17}$$

(iv) Special conformal transformations correspond to $\xi^\mu = x^2 b^\mu - 2b^\nu x_\nu x^\mu$ with $b^\mu$ constants and hence

$$\Delta_b\phi^A = b_\mu\Sigma^{\mu,A} \tag{2.18}$$

for a function $\Sigma^{\mu,A}$ of the fields. Substituting (2.18) into (2.8) we obtain

$$2x_\mu t^{[\mu\nu]} - 2x^\nu t^\mu_\mu + \partial_\mu N^{\mu\nu} = 0, \tag{2.19}$$

with $N^{\mu\nu} = \pi^{\mu,A}\Sigma^{\nu,A}$. Using (2.12) and (2.16) we then find

$$N^\mu + 2\Psi_\nu{}^{[\nu\mu]} = \partial_\nu\Lambda^{\mu\nu}, \tag{2.20}$$

with $2\Lambda^{\mu\nu} = N^{\mu\nu} + 2x^\nu N^\mu + 2x_\rho\Psi^{\mu[\rho\nu]}$. Finally, assuming Lorentz invariance we can use (2.17) to obtain the condition satisfied by the BR e.m. tensor of a conformally invariant theory as [13,20]

$$T^\mu_{\text{BR}\mu} + \partial_\mu\partial_\nu\Lambda^{\mu\nu} = 0. \tag{2.21}$$

Using $\Lambda^{\mu\nu}$ it is then possible to construct a conserved, symmetric, and traceless e.m. tensor $\Theta^{\mu\nu}$ as in [13]

$$\begin{aligned}\Theta^{\mu\nu} = T^{\mu\nu}_{\text{BR}} &- \frac{1}{d-2}(\partial^\mu\partial_\rho\Lambda^{\rho\nu} + \partial^\nu\partial_\rho\Lambda^{\rho\mu} \\ &- \partial^2\Lambda^{\mu\nu} - \eta^{\mu\nu}\partial_\rho\partial_\sigma\Lambda^{\rho\sigma}) \\ &- \frac{1}{(d-1)(d-2)}(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\Lambda^\rho{}_\rho,\end{aligned} \tag{2.22}$$

where the additional term in (2.22) is identically conserved and

[4]Where we introduce our conventions for symmetrizing and antisymmetrizing $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$ and $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$.

$$\partial_\mu \Theta^{\mu\nu} = \Theta^\mu_\mu = 0, \qquad \Theta^{\mu\nu} = \Theta^{\nu\mu}. \tag{2.23}$$

The dilation and special conformal conserved currents are now nicely written as [20]

$$\begin{aligned} j^\mu_D &= x_\nu \Theta^{\mu\nu} \Rightarrow \partial_\mu j^\mu_D = 0, \\ K^{\mu\nu} &= x^2 \Theta^{\mu\nu} - 2x^\nu x_\kappa \Theta^{\mu\kappa} \Rightarrow \partial_\mu K^{\mu\nu} = 0. \end{aligned} \tag{2.24}$$

To recap, calculating directly from the Lagrangian the canonical e.m. tensor $t^{\mu\nu}$ allows us to deduce whether the matter theory is Lorentz, dilatation, and conformal invariant by checking the conditions (2.12), (2.16), and (2.20). Then, we can construct accordingly its corresponding improved versions.

## III. MATTER FIELDS IN CURVED BACKGROUNDS

### A. Matter fields coupled to Einstein gravity

Coupling the matter theory (2.1) to Einstein gravity is usually done by promoting the partial derivatives to covariant ones $\partial_\mu \mapsto \nabla_\mu$. Although this procedure is generally ambiguous due to the fact that the flat action (2.1) can be modified by total derivatives terms, it yields an unambiguous result regarding the e.m. tensor. Namely, suppose that we extend (2.1) to curved space as

$$S[\phi^A] \mapsto S[\phi^A; g] = \int d^d x \sqrt{-g}\, \mathcal{L}(\phi^A, \nabla_\mu \phi^A; g). \tag{3.1}$$

Under general diffeomorphisms $x^\mu \to x^\mu - \xi^\mu$ this transforms as

$$\begin{aligned} \delta_\xi S[\phi^A; g] &= -\frac{1}{2} \int d^d x \sqrt{-g} T^{\mu\nu} \mathcal{L}_\xi g_{\mu\nu} \\ &= \int d^d x \sqrt{-g} \xi_\nu \nabla_\mu T^{\mu\nu} - \int d^d x \partial_\mu (\sqrt{-g} \xi_\nu T^{\mu\nu}). \end{aligned} \tag{3.2}$$

As we are interested in the on-shell variation of the matter action, we have not indicated the additional terms that are proportional to the e.o.m. of the matter fields. As usual the Lie derivative of the metric is $\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\nu$ and the symmetric Hilbert metrical e.m. tensor $T^{\mu\nu}$ is defined as

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}, \qquad g := \det g_{\mu\nu}. \tag{3.3}$$

If we require that (3.1) is invariant under diffeomorphisms, this means that its variation vanishes up to boundary terms, and hence we obtain the on-shell conservation equation

$$\nabla_\mu T^{\mu\nu} = 0. \tag{3.4}$$

Restricting then back to flat space we see that $T^{\mu\nu}$ correspond to a symmetric and conserved e.m. tensor and might only differ from the improved BR e.m. tensor $T^{\mu\nu}_{\mathrm{BR}}$ by terms that do not alter the Poincaré charges.

If the flat matter theory on top of being Lorentz invariant is conformally invariant as well, then we may wish to couple it to Einstein gravity in a Weyl invariant manner [6]. Namely we might want to find a diffeomorphic invariant action that is also invariant under simultaneous Weyl rescaling of the metric $\delta_\Omega g_{\mu\nu} = 2\Omega g_{\mu\nu}$ and of the matter fields[5] $\delta_\Omega \phi^A = \Delta_\phi \Omega \phi^A$ with $\Delta_\phi$ the scaling dimension. The resulting curved action $S_w[\phi^A; g_{\mu\nu}]$ is not the same as (3.1); nevertheless, it gives rise to a metrical e.m. tensor for the $S_w$ action that is symmetric and covariantly conserved. However, Weyl invariance of the action gives

$$\begin{aligned} \delta_\Omega S_w[\phi^A; g] &= -\frac{1}{2} \int d^d x \sqrt{-g} T^{\mu\nu}_{(w)} \delta_\Omega g_{\mu\nu} \\ &= -\frac{1}{2} \int d^d x \sqrt{-g} T^{\mu\nu}_{(w)} 2\Omega g_{\mu\nu} = 0. \end{aligned} \tag{3.5}$$

This yields $g_{\mu\nu} T^{\mu\nu}_{(w)} = 0$ for the metrical e.m. tensor of the Weyl invariant action. Restricting finally to flat space we see that $T^{\mu\nu}_{(w)}$ correspond to a conserved, symmetric, and traceless e.m. tensor of a conformally invariant matter theory; hence, either it coincides with the conformally improved one $\Theta^{\mu\nu}$ or it differs from the latter by terms that do not alter the conformal charges. This Weyl-covariantization procedure is the standard method of obtaining the tensor $\Theta^{\mu\nu}$ in the case of complicated matter theories, i.e., [14].

### B. Matter fields in metric-affine backgrounds

It is not too surprising that Einstein and Weyl gravity couple to matter through the improved e.m. tensors $T^{\mu\nu}_{\mathrm{BR}}$ and $\Theta^{\mu\nu}$, respectively. After all, general covariance and Weyl covariance include Poincaré and conformal symmetry. Nevertheless, this leaves the flat space improvement procedure without a clear geometric interpretation. Our aim here is to provide a geometric framework for the e.m. improvement process in flat space, which we will argue is unveiled through the coupling of the matter theory to MAG.[6]

### C. Conservation and trace identities

The matter action (2.1) coupled to MAG in a holonomic (i.e., coordinate based) takes the form

[5]The Weyl rescaling of matter fields does not depend on their spin.

[6]For a reminder of metric-affine gravity and our notations see Appendix A.

$$S[\phi^A] \mapsto S[\phi^A; g, \Gamma] = \int d^d x \sqrt{-g}\, \mathcal{L}(\phi^A; g, \Gamma). \tag{3.6}$$

The fields $\phi^A(x)$ couple both to the metric $g_{\mu\nu}$ and to the independent affine connection $\Gamma^\lambda{}_{\mu\nu}$. Now, apart from the metrical e.m. tensor defined in (3.3) the action (3.6) gives rise to an additional quantity, the hypermomentum [9,10] defined as the variation of the matter part of the action with respect to the affine connection

$$\Delta_\lambda{}^{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta \Gamma^\lambda{}_{\mu\nu}}, \qquad \Delta^{\lambda\mu\nu} := g^{\lambda\rho} \Delta_\rho{}^{\mu\nu}. \tag{3.7}$$

Physically, the hypermomentum (3.7) encodes the microproperties of matter such as spin, dilation, and shear. It can be split into three pieces as (see [8])

$$\Delta^{\mu\nu\rho} = \Sigma^{\mu\nu\rho} + \frac{g^{\mu\nu}}{d} \Delta^\rho + \hat{\Delta}^{\mu\nu\rho}, \tag{3.8}$$

with

$$\Sigma^{\mu\nu\rho} := \Delta^{[\mu\nu]\rho} \quad \text{Spin}, \tag{3.9}$$

$$\Delta^\nu := \Delta^{\alpha\mu\nu} g_{\alpha\mu} \quad \text{Dilation}, \tag{3.10}$$

$$\hat{\Delta}^{\mu\nu\rho} := \Delta^{(\mu\nu)\rho} - \frac{1}{d} g^{\mu\nu} \Delta^\rho \quad \text{Shear}. \tag{3.11}$$

As in (3.2), the on-shell variation of the action (3.6) under the diffeomorphisms $x^\mu = x'^\mu + \xi^\mu$ is given by

$$\delta_\xi S = -\frac{1}{2} \int d^d x \sqrt{-g} (T_{\mu\nu} \mathcal{L}_\xi g^{\mu\nu} + \Delta_\lambda{}^{\mu\nu} \mathcal{L}_\xi \Gamma^\lambda{}_{\mu\nu}). \tag{3.12}$$

The Lie derivatives of the metric and the affine connection are

$$\begin{aligned} \mathcal{L}_\xi g_{\mu\nu} &= g_{\rho\nu} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho + \xi^\rho \partial_\rho g_{\mu\nu} \\ &= g_{\nu\rho} \nabla_\mu \xi^\rho + g_{\mu\rho} \nabla_\nu \xi^\rho - 2\xi^\rho (S_{\rho\mu\nu} + S_{\rho\nu\mu}) - \xi^\rho Q_{\rho\mu\nu}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \mathcal{L}_\xi \Gamma^\mu{}_{\kappa\lambda} &= -\partial_\rho \xi^\mu \Gamma^\rho{}_{\kappa\lambda} + \partial_\kappa \xi^\rho \Gamma^\mu{}_{\rho\lambda} \\ &\quad + \partial_\lambda \xi^\rho \Gamma^\mu{}_{\kappa\rho} + \xi^\rho \partial_\rho \Gamma^\mu{}_{\kappa\lambda} + \partial_\kappa \partial_\lambda \xi^\mu \\ &= \nabla_\kappa (2\xi^\rho S_{\rho\lambda}{}^\mu + \nabla_\lambda \xi^\mu) - R^\mu{}_{\lambda\kappa\rho} \xi^\rho. \end{aligned} \tag{3.14}$$

The last equality conforms that well-known fact that the Lie derivative of the affine-connection is a tensor. Plugging the above into (3.12), we find [8,21]

$$\delta_\xi S = \int d^d x \xi^\alpha \Xi_\alpha + \int d^d x \partial_\mu (\sqrt{-g} j^\mu), \tag{3.15}$$

with

$$\begin{aligned} \Xi_\alpha &= \sqrt{-g} (2\tilde{\nabla}_\mu T^\mu{}_\alpha - \Delta^{\lambda\mu\nu} R_{\lambda\mu\nu\alpha}) \\ &\quad + \hat{\nabla}_\mu \hat{\nabla}_\nu (\sqrt{-g} \Delta_\alpha{}^{\mu\nu}) - 2 S_{\mu\alpha}{}^\lambda \hat{\nabla}_\nu (\sqrt{-g} \Delta_\lambda^{\mu\nu}), \end{aligned} \tag{3.16}$$

where we introduce $\hat{\nabla}_\lambda = \nabla_\lambda - 2S_\lambda$[7] and

$$\begin{aligned} j^\mu &= -\xi_\nu \left( T^{\mu\nu} + \frac{g^{\nu\rho}}{2\sqrt{-g}} \hat{\nabla}_\lambda (\sqrt{-g} \Delta_\rho{}^{\mu\lambda}) \right) \\ &\quad + \frac{1}{2} \Delta_\lambda{}^{\nu\mu} \nabla_\nu \xi^\lambda + S_{\nu\rho}{}^\lambda \xi^\rho \Delta_\lambda{}^{\nu\mu}. \end{aligned} \tag{3.17}$$

Requiring that (3.15) vanishes up to boundary terms gives the *generalized conservation* of metric-affine gravity [8,21]

$$\Xi^\mu = 0. \tag{3.18}$$

In the MAG framework one defines generalized Killing vectors $\xi^\mu$ that satisfy [22][8]

$$\mathcal{L}_\xi g_{\mu\nu} = 0, \qquad \mathcal{L}_\xi \Gamma^\lambda{}_{\mu\nu} = 0. \tag{3.19}$$

In this case along with (3.18) we obtain the conservation law [22]

$$\partial_\mu (\sqrt{-g} j^\mu) = 0, \tag{3.20}$$

where the presence of the hypermomentum in (3.17) generalizes the usual definition of conserved currents in a matter theory coupled to Einstein gravity, i.e., $\partial_\mu (\sqrt{-g} \xi_\nu T^{\mu\nu}) = 0$.

The generalization of Weyl invariance [23–25] in the context of MAG in a MAG framework is not unique as one has the possibility to consider different transformations of the affine connection in conjunction with the Weyl rescaling of the metric [11]. Here we will consider the transformations

$$g_{\mu\nu} \mapsto e^{-2\Omega} g_{\mu\nu}, \qquad \Gamma^\rho{}_{\mu\nu} \mapsto \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Omega \delta^\rho_\mu, \tag{3.21}$$

for some arbitrary $\Omega = \Omega(x)$ which were first considered in [26–28]. These transformations correspond to a *local frame rescaling* of the form [11]

$$e_\mu{}^a \mapsto e^{-\Omega} e_\mu{}^a, \tag{3.22}$$

and the transformation of the generic affine connection in (3.21), with the spin connection kept fixed, can be read from (3.27) below. Under frame rescaling the action (3.6) transforms on-shell as

[7]The $\nabla$ covariant derivative acts on the scalar density as $\nabla_\lambda \sqrt{-g} = -\frac{1}{2} Q_\lambda \sqrt{-g}$ and trivially follows $\hat{\nabla}_\lambda \sqrt{-g} = -(\frac{1}{2} Q_\lambda + 2S_\lambda)\sqrt{-g}$ [see also Eq. (C5)].

[8]Equivalently the generalized Killing vectors can be defined as $\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi S_{\mu\nu\rho} = \mathcal{L}_\xi Q_{\mu\nu\rho} = 0$.

$$\delta_\Omega S = -\frac{1}{2}\int d^dx\sqrt{-g}(T_{\mu\nu}\delta_\Omega g^{\mu\nu} + \Delta_\lambda{}^{\mu\nu}\delta_\Omega\Gamma^\lambda{}_{\mu\nu})$$
$$= -\int d^dx\sqrt{-g}\Omega\left(T^\mu{}_\mu + \frac{1}{2\sqrt{-g}}\hat{\nabla}_\lambda(\sqrt{-g}\Delta_\mu{}^{\mu\lambda})\right)$$
$$+\frac{1}{2}\int d^dx\partial_\nu(\sqrt{-g}\Delta_\lambda{}^{\lambda\nu}\Omega). \tag{3.23}$$

Demanding that (3.23) vanishes up to boundary terms we obtain

$$T^\mu{}_\mu + \frac{1}{2\sqrt{-g}}\hat{\nabla}_\lambda(\sqrt{-g}\Delta_\mu{}^{\mu\lambda}) = 0, \tag{3.24}$$

which generalizes the trace condition of the metrical e.m. tensor in the case of Weyl gravity.

The simple flat limits of the conservation (3.18) and trace (3.24) MAG equations do not coincide with the ones coming from coupling matter to Einstein and Weyl gravity due to the presence of the hypermomentum. For example, the only property of the flat limit of the metrical MAG e.m. tensor $T^{\mu\nu}$ is that it is symmetric—it is neither conserved nor traceless in general. It is only conserved when the hypermomentum is antisymmetric in the last pair of indices. Hence, it does not generically give the improved BR $T^{\mu\nu}_{\rm BR}$ or the improved conformal $\Theta^{\mu\nu}$ e.m. tensors. Nevertheless, there is important information that we have gained from the above endeavor. We can formulate the above matter system using the vielbein formalism and consider as independent variables the tangent space metric $g_{ab}$,[9] the coframe 1-form $\vartheta^a = e_\mu{}^a dx^\mu$, and the connection (or spin connection) 1-form $\omega_{ab} = \omega_{ab\mu}dx^\mu$. Then the matter action functional now reads

$$S[\phi^A] \mapsto S_e[\phi^A; e, \omega, g] = \int d^dx\sqrt{-g}\,\mathcal{L}_e(\phi^A; e, \omega, g), \tag{3.25}$$

and the additional matter source is the *MAG canonical* e.m. tensor given by

$$\tau^a{}_\mu := -\frac{1}{\sqrt{-g}}\frac{\delta S_e}{\delta e^\mu{}_a}, \tag{3.26}$$

which is the fundamental e.m. tensor in MAG. We do not consider the variation with respect to the spin connection as it coincides with that of the connection, namely (3.7), due to the identity[10]

$$\nabla_\mu e_\nu{}^a := \partial_\mu e_\nu{}^a + \omega^a{}_{b\mu}e_\nu{}^b - \Gamma^\rho{}_{\nu\mu}e_\rho{}^a = 0, \tag{3.27}$$

relating the holonomic affine connection to the (anholonomic) connection and the vielbeins.[11] The crucial observation is that the MAG canonical e.m. tensor is related to the standard canonical e.m. tensor that one obtains from the Lagrangian $\mathcal{L}(\phi^A, \nabla_\mu\phi^A; g)$ when it is coupled to Einstein gravity. This follows from

$$\tau^\mu{}_\nu = \tau^\mu{}_a e_\nu{}^a = -e^\mu{}_c\left\{\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial e^\nu{}_c}\mathcal{L}_e + \frac{\partial\mathcal{L}_e}{\partial e^\nu{}_c}\right\}$$
$$= \delta^\mu_\nu\mathcal{L}_e - e^\mu{}_c\frac{\partial\mathcal{L}_e}{\partial e^\nu{}_c}, \tag{3.28}$$

where we have used the identities

$$\frac{\partial e^\lambda{}_c}{\partial e_\mu{}^a} = -e^\mu{}_c e^\lambda{}_a, \qquad \frac{\partial\sqrt{-g}}{\partial e^\nu{}_c} = -\sqrt{-g}e_\nu{}^c. \tag{3.29}$$

To take care of the last term on the right-hand side of (3.28) we must distinguish between the ordinary affine covariant derivatives and the MAG Weyl covariant derivatives. For ordinary affine covariant derivatives the vielbein appears only through the covariant derivative of the matter fields

$$\nabla_a\phi^A = e^\rho{}_a\nabla_\rho\phi^A \Rightarrow \frac{\partial(\nabla_a\phi^A)}{\partial e^\lambda{}_c} = \delta^c_a\nabla_\lambda\phi^A, \tag{3.30}$$

since the spin connection is an independent variable. Then assuming that the matter action depends only algebraically on the vielbein we obtain

$$\frac{\partial\mathcal{L}_e}{\partial e^\nu{}_c} = \frac{\partial\mathcal{L}_e}{\partial(\nabla_a\phi^A)}\frac{\partial(\nabla_a\phi^A)}{\partial e^\nu{}_c} = \frac{\partial\mathcal{L}_e}{\partial(\nabla_c\phi^A)}\nabla_\nu\phi^A. \tag{3.31}$$

Inserting the above into (3.28) we obtain

$$\tau^\mu{}_\nu = -e^\mu{}_c\frac{\partial\mathcal{L}_e}{\partial(\nabla_c\phi^A)}\nabla_\nu\phi^A + \delta^\mu{}_\nu\mathcal{L}_e$$
$$= -\frac{\partial\mathcal{L}_e}{\partial(\nabla_\mu\phi^A)}\nabla_\nu\phi^A + \delta^\mu{}_\nu\mathcal{L}_e \equiv t^\mu{}_\nu. \tag{3.32}$$

For Weyl covariant derivatives the vielbein appears also in the variation of the Weyl connection (i.e., torsion vector) [11] (see also [29,30]). For example, consider the action of the Weyl covariant derivative on a tensor weight scalar

$$D_a\phi_A = e^\rho{}_a D_\rho\phi_A, \quad D_\rho\phi_A = \nabla_\rho\phi_A - \alpha S_\rho\phi_A, \quad \alpha = \frac{2w_A}{d-1}, \tag{3.33}$$

where $w_A$ is the weight of the field(s) $\phi_A$. The end result is

[9]This can be chosen to be the Minkowski metric by an appropriate choice of the vielbeins (i.e., for orthonormal). We adopt the mostly plus convention for the latter, that is, $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$.

[10]Sometimes mistakenly referred to as *the vielbein postulate*.

[11]The variation with respect to the tangent space metric gives an important relation that is a combination of the variations with respect to vielbein and spin connection [8] (see below).

$$\tau^{\mu}_{(w)^{\nu}} = -\frac{\partial \mathcal{L}_e}{\partial(D_\mu \phi^A)} D_\nu \phi^A + \delta^\mu{}_\nu \mathcal{L}_e + \alpha \hat{\nabla}_\lambda \left( \phi^A \frac{\partial \mathcal{L}_e}{\partial(D_\rho \phi^A)} \delta^{[\mu}_\rho \delta^{\lambda]}_\nu \right) = t^\mu{}_\nu + \alpha \hat{\nabla}_\lambda \left( \phi^A \frac{\partial \mathcal{L}_e}{\partial(D_\rho \phi^A)} \delta^{[\mu}_\rho \delta^{\lambda]}_\nu \right), \tag{3.34}$$

where we recall that $\hat{\nabla}_\mu = \nabla_\mu - 2S_\mu$ and the subscript $w$ in $\tau^{\mu}_{(w)^{\nu}}$ denotes the inclusion of the Weyl covariant derivative. By construction, $\tau^{\mu}_{(w)^{\nu}}$ is traceless and the right-hand side of (3.34) is the covariantized form of (2.16).

Nevertheless, independently of the nature of the covariant derivatives, we can show that the tensors (3.26), (3.3), and (3.7) are not independent. Using the chain rule and the identity (3.27), we find[12]

$$\tau^\mu{}_\nu = T^\mu{}_\nu + \frac{1}{2\sqrt{-g}} \hat{\nabla}_\lambda (\sqrt{-g} \Delta_\nu{}^{\mu\lambda}). \tag{3.35}$$

Equation (3.35) tells us that the knowledge of $T^{\mu\nu}$ and $\Delta_\lambda{}^{\mu\nu}$ gives us access to the canonical e.m. tensor of the matter theory: in the case of ordinary affine covariant derivatives to the standard canonical e.m. tensor and in the case of a Weyl covariant derivatives to the *traceless part* of the standard canonical e.m. tensor (3.24). This is a useful result in cases where the only information one has regarding the matter theory is the variation of its effective action with respect to the metric or other background fields, and hence there is no information regarding its canonical e.m. tensor. Such a case is, notably, AdS/CFT holography. Using (3.35) we further find

$$\frac{1}{\sqrt{-g}} \hat{\nabla}_\mu (\sqrt{-g} \tau^\mu{}_\alpha) = \frac{1}{2} \Delta^{\lambda\mu\nu} R_{\lambda\mu\nu\alpha} - \frac{1}{2} Q_{\alpha\mu\nu} T^{\mu\nu} - 2 S_{\alpha\mu\nu} \tau^{\mu\nu}, \tag{3.36}$$

expressed in a holonomic frame [8,32]; see also [33,34] for conservation laws in metric-torsion theories.

### D. The trivial flat limit and the improvement terms

The simplest flat limit of the above results corresponds to setting the metric to be flat and the affine connection to be zero. We then obtain from (3.35) and (3.36)

$$\tau^{\mu\nu} = T^{\mu\nu} + \frac{1}{2} \partial_\lambda \Delta^{\nu\mu\lambda}, \qquad \partial_\mu \tau^{\mu\nu} = 0. \tag{3.37}$$

Also, the flat limit of (3.24) reads

$$T^\mu{}_\mu + \frac{1}{2} \partial_\lambda \Delta_\mu{}^{\mu\lambda} = 0. \tag{3.38}$$

The results (3.37) and (3.38) correspond to the improvement process of the flat canonical e.m. tensor, and we see that the improvement terms are related to the flat limit of the hypermomentum.

The flat limit of (3.17) yields the current

$$j^\mu = -\xi_\lambda t^{\mu\lambda} + \frac{1}{2} \Delta^{\lambda\nu\mu} \partial_\nu \xi_\lambda, \tag{3.39}$$

where $\xi^\mu$ is an arbitrary vector field. This coincides with the corresponding result (2.8) that was found from the Noether procedure in flat space, which clarifies further the role of the hypermomentum in the improvement process.

Restricting then $\xi^\mu$ to the Killing vectors of Minkowski, i.e., the translations and Lorentz generators, we will be able to show that the flat limit of the hypermomentum correspond to the BR improvement terms. From (3.39) we obtain

$$\partial_\mu j^\mu = \frac{1}{2} (-2 \partial_\mu \xi_\nu T^{\mu\nu} + \partial_\lambda \partial_\mu \xi_\nu \Delta^{\nu\lambda\mu}), \tag{3.40}$$

upon employing (3.37). For translations $\xi^\mu = a^\mu$ the conservation of the current (3.39) yields the on-shell conservation of the canonical energy-momentum tensor

$$\partial_\mu t^\mu{}_\nu = 0. \tag{3.41}$$

For Lorentz transformations $\xi^\mu = \omega^\mu{}_\nu x^\nu$ the current (3.39) becomes

$$(J^\mu)^{\nu\rho} = t^{\mu\nu} x^\rho - t^{\mu\rho} x^\nu - \Sigma^{\nu\rho\mu}, \tag{3.42}$$

and its conservation gives

$$\partial_\mu (J^\mu)^{\nu\rho} = 0 \Rightarrow t^{\mu\nu} - t^{\nu\mu} + \partial_\lambda \Sigma^{\mu\nu\lambda} = 0, \tag{3.43}$$

where $\Sigma^{\mu\nu\lambda}$ is the spin part of the hypermomentum defined in (3.9). We see that (3.43) is equivalent to the condition (2.12) if we identify

$$\Sigma^{\mu\nu\rho} = -2\Psi^{\rho[\mu\nu]}, \tag{3.44}$$

and as promised the improvement terms used in (2.13) correspond to the flat limit of the hypermomentum. This provides a geometric interpretation of the BR improvement procedure.

There is no improvement process for dilatations with $\xi^\mu = \lambda x^\mu$. In flat space they just lead to the condition (2.16) for the trace of the canonical e.m. tensor; therefore, we do not expect that the *virial current* $N^\mu$ in (2.16) is related to the hypermomentum. From (3.37) we obtain the corresponding trace condition for the symmetric e.m. tensor $T^{\mu\nu}$ as

[12]The result (3.35) was originally found in Eq. (5.2.16) of [8] expressed in exterior forms language and in a coordinate basis in Eq. (55) of [31] for $F = 1$. See also [32]. We review its derivation in Appendix C.

$$T^{\mu}{}_{\mu} = \partial_\mu \left( N^\mu + \frac{1}{2}\Delta^\mu \right). \tag{3.45}$$

The improvement procedure comes about when we consider special conformal transformations $\xi^\mu = b^\mu x^2 - 2x^\mu b_\nu x^\nu$, and the on-shell conserved tensor $\partial_\mu K^\mu{}_\nu = 0$ reads

$$\begin{aligned} K^\mu{}_\nu &= x^2 t^\mu{}_\nu - 2x_\nu x^\rho t^\mu{}_\rho - 2(x^\rho \Sigma_{\nu\rho}{}^\mu - x_\nu \Sigma^{\mu\rho}{}_\rho) \\ &\quad + 2(x_\nu \partial_\rho Z^{\mu\rho} - Z^\mu{}_\nu) \\ &= x^2 T_{\mathrm{BR}}{}^\mu{}_\nu - 2x_\nu x^\rho T_{\mathrm{BR}}{}^\mu{}_\rho + 2(x_\nu \partial_\rho Z^{\mu\rho} - Z^\mu{}_\nu), \end{aligned} \tag{3.46}$$

provided that

$$T_{\mathrm{BR}\mu}{}^\mu = \partial_\mu \partial_\nu Z^{\mu\nu}, \qquad Z^{\mu\nu} = Z^{\nu\mu}, \tag{3.47}$$

which translates to the condition

$$\partial_\nu Z^{\mu\nu} = \Lambda^\mu + \frac{1}{2}\Delta^\mu - \Sigma^{\mu\nu}{}_\nu. \tag{3.48}$$

As we will see in detail in the examples that follow in Secs. IV and V, the role of the hypermomentum is the restoration of Lorentz or scale invariance in the MAG setup. The results will match those found using the BR or the virial current procedures.

## IV. LORENTZ INVARIANCE IMPROVEMENT TERMS

We will present below explicit field theory examples where coupling to MAG corresponds to the BR improvement process.

### A. The electromagnetic field

The action for the free Maxwell field is

$$S_{\mathrm{EM}} = \int d^d x \mathcal{L}_{\mathrm{EM}}, \qquad \mathcal{L}_{\mathrm{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{4.1}$$

where, as usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the electromagnetic field. The action is invariant under gauge transformations $A_\mu \mapsto A_\mu + \partial_\mu f$, and its equations of motion are given by

$$\partial_\lambda F^{\mu\lambda} = 0. \tag{4.2}$$

The corresponding canonical energy-momentum tensor is found to be

$$t^{\mu\nu} = F^{\mu\lambda} \partial^\nu A_\lambda - \frac{1}{4} \eta^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}. \tag{4.3}$$

This is neither symmetric nor gauge invariant. To remedy the situation and obtain an improved tensor that meets these requirements one uses the BR method, revisited in Appendix D 2, which amounts to adding the term [35]

$$-F^{\mu\lambda} \partial_\lambda A^\nu, \tag{4.4}$$

giving the symmetric and also gauge invariant electromagnetic energy-momentum tensor

$$T^{\mu\nu} = F^{\mu\lambda} F^\nu{}_\lambda - \frac{1}{4} \eta^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}. \tag{4.5}$$

The e.m. tensor (4.5) can be alternatively obtained if we minimally couple the action (4.1) to Einstein gravity as

$$S_{\mathrm{EM}} = -\frac{1}{4} \int d^d x \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{4.6}$$

and then use (3.3). The resulting Hilbert metrical e.m. tensor coincides in the flat limit with (4.5). However, as we have mentioned above the minimal coupling of the theory to Einstein gravity obscures the access to the canonical e.m. tensor. This is rectified by coupling the theory to MAG as follows. We consider a curved metric and replace

$$\partial_\mu \mapsto \nabla_\mu, \tag{4.7}$$

where $\nabla_\mu$ is now the covariant derivative with respect to a generic affine connection, and then the field strength generalizes to

$$\hat{F}_{\mu\nu} \coloneqq \nabla_\mu A_\nu - \nabla_\nu A_\mu = F_{\mu\nu} + 2S_{\mu\nu}{}^\lambda A_\lambda, \tag{4.8}$$

which is not gauge invariant,[13] and the corresponding Lagrangian becomes

$$\begin{aligned} \hat{\mathcal{L}}_{\mathrm{EM}} &= -\frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \\ &= -\frac{1}{4} (F_{\mu\nu} F^{\mu\nu} + 4S_{\mu\nu}{}^\rho F^{\mu\nu} A_\rho + 4S_{\mu\nu}{}^\rho S^{\mu\nu\lambda} A_\rho A_\lambda). \end{aligned} \tag{4.9}$$

The associated hypermomentum to this matter sector is easily computed using (3.7), yielding in the flat spacetime limit

$$\Delta_\lambda{}^{\mu\nu} = 2A_\lambda F^{\mu\nu}. \tag{4.10}$$

Then, using the equations of motion (4.2) we obtain

$$\frac{1}{2} \partial_\lambda \Delta^{\nu\mu\lambda} = F^{\mu\lambda} \partial_\lambda A^\nu, \tag{4.11}$$

[13]The breaking of gauge invariance

$$\delta_\Lambda \hat{\mathcal{L}}_{\mathrm{EM}} = -S_{\mu\nu}{}^\rho \hat{F}^{\mu\nu} \partial_\rho \Lambda, \qquad \delta_\Lambda A_\mu = \partial_\mu \Lambda,$$

is not fatal at this point as this is only an intermediate step as at the end we will go back to the flat background (i.e., we set $g = \eta$ and $\Gamma^\lambda{}_{\mu\nu} = 0$) and the gauge invariance will then be restored.

which demonstrates that the hypermomentum provides a geometric origin for the BR improvement term (4.4). The above discussion can be trivially extended for the case of a Proca field.

A comment is in order concerning the scale invariance of electromagnetism (4.1) but non conformal invariance in $d \neq 4$. Indeed by taking the trace of (4.5) we find that it takes on-shell the form of a total derivative

$$T^{\mu}{}_{\mu} = -\frac{d-4}{4} F_{\mu\nu}F^{\mu\nu} = \partial_\mu V^\mu, \qquad V^\mu = -\frac{d-4}{2} A_\nu F^{\mu\nu}. \tag{4.12}$$

From the above and Eq. (5.3) we can construct an on-shell conserved dilation current. Despite been nongauge invariant, the corresponding dilation charge $Q = \int d^{d-1}x j^0_D$ is on-shell gauge invariant up to boundary terms. However, in this case the scale does not implement conformal invariance. More concretely, $V^\mu$ in (4.12) cannot be expressed as $\partial_\nu Z^{\mu\nu}$, where $Z^{\mu\nu}$ is a symmetric rank-2 tensor, while keeping gauge invariance.[14]

### 1. Abelian p-forms

The above easily extends to the action of Abelian $p$-forms

$$\mathcal{L}_p = -\frac{1}{2p!} F_{\mu_1\mu_2\cdots\mu_p}F^{\mu_1\mu_2\cdots\mu_p}, \qquad F_{\mu_1\mu_2\cdots\mu_p} = p\partial_{[\mu_1}A_{\mu_2\cdots\mu_p]}, \tag{4.13}$$

whose equations of motion read

$$\partial_{\mu_1}F^{\mu_1\mu_2\cdots\mu_p} = 0, \tag{4.14}$$

where $A_{\mu_2\cdots\mu_p}$ is a massless $p-1$ form. The corresponding canonical e.m. tensor is then given by

$$t^{\mu\nu} = \frac{1}{(p-1)!} F^{\mu\mu_2\cdots\mu_p}\partial^\nu A_{\mu_2\cdots\mu_p} - \frac{1}{2p!}\eta^{\mu\nu}F_{\mu_1\mu_2\cdots\mu_p}F^{\mu_1\mu_2\cdots\mu_p}, \tag{4.15}$$

which is neither symmetric nor gauge invariant under the following transformation:

$$A_{\mu_1\cdots\mu_{p-1}} \mapsto A_{\mu_1\cdots\mu_{p-1}} + (p-1)\partial_{[\mu_1}\Lambda_{\mu_2\cdots\mu_{p-1}]}, \tag{4.16}$$

with $\Lambda_{\mu_2\cdots\mu_{p-1}}$ a $p-2$ form. Similar to electromagnetism the BR improvement of (4.15) comes from the hypermomentum. For that we just need the minimal coupling of (4.13) to MAG according to (4.7), and we find

$$\hat{F}_{\mu_1\mu_2\cdots\mu_p} = p\nabla_{[\mu_1}A_{\mu_2\cdots\mu_p]} = F_{\mu_1\mu_2\cdots\mu_p} + pS_{[\mu_1\mu_2}{}^\rho A_{|\rho|\mu_3\cdots\mu_p]}, \tag{4.17}$$

where $\rho$ is excluded from the antisymmetry. The corresponding action is given by

$$\hat{\mathcal{L}}_p = -\frac{1}{2p!}\hat{F}_{\mu_1\mu_2\cdots\mu_p}\hat{F}^{\mu_1\mu_2\cdots\mu_p}, \tag{4.18}$$

whereas the associated hypermomentum can be computed through (3.7)

$$\Delta_\lambda{}^{\mu\nu} = \frac{2}{(p-2)!}A_{\lambda\mu_3\cdots\mu_p}F^{\mu\nu\mu_3\cdots\mu_p}. \tag{4.19}$$

Inserting the latter and (4.15) into (3.37) we find on-shell

$$T^{\mu\nu} = \frac{1}{(p-1)!}\left(F^{\mu\mu_2\cdots\mu_p}F^\nu{}_{\mu_2\cdots\mu_p} - \frac{1}{2p}\eta^{\mu\nu}F_{\mu_1\mu_2\cdots\mu_p}F^{\mu_1\mu_2\cdots\mu_p}\right), \tag{4.20}$$

which is symmetric and gauge invariant. Its trace takes on-shell the form of a total derivative

$$T^{\mu}{}_{\mu} = -\frac{d-2p}{2p!}F_{\mu_1\mu_2\cdots\mu_p}F^{\mu_1\mu_2\cdots\mu_p} = \partial_\mu\Lambda^\mu,$$
$$\Lambda^\mu = -\frac{d-2p}{2(p-1)!}A_{\mu_2\cdots\mu_p}F^{\mu\mu_2\cdots\mu_p}. \tag{4.21}$$

Extending the discussion after (4.12), we find that the theory is scale but nonconformal invariant for $d \neq 2p$.

## B. The Dirac field

Let us consider the massive Dirac field in a curved $d$-dimensional spacetime with a mostly plus metric described in terms of the Hermitian action[15]

$$S_{\mathrm{D}} = \int d^dx\sqrt{-g}\mathcal{L}_{\mathrm{D}},$$
$$\mathcal{L}_{\mathrm{D}} = -\frac{i}{2}(\bar{\psi}\gamma^c\nabla_c\psi - \nabla_c\bar{\psi}\gamma^c\psi) - im\bar{\psi}\psi, \tag{4.22}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$. The $\gamma$-matrices obey the Clifford algebra

[14] In detail, there exists such a rank-2 tensor, namely

$$Z^{\mu\nu} = -\frac{d-4}{2}\left(A^\mu A^\nu - \frac{1}{2}\eta^{\mu\nu}A_\kappa A^\kappa\right),$$

upon enforcing the Lorentz gauge $\partial_\mu A^\mu = 0$. However, the corresponding symmetric, conserved, and traceless tensor $\Theta^{\mu\nu}$ read through (2.22) and (4.5) as well as the corresponding conserved charges defined through (2.24), e.g., $Q_\nu = \int d^{d-1}xK^0{}_\nu$, explicitly depend on the gauge field. Hence, scale does not extend to conformal invariance without breaking the gauge symmetry of the theory.

[15] The gamma matrices in the mostly minus signature metrics are related to those of the mostly plus (we adopt here) as follows: $\gamma^\mu|_{\text{mostly minus}} = i\gamma^\mu|_{\text{mostly plus}}$.

$$\gamma^a\gamma^b+\gamma^b\gamma^a=2\eta^{ab}, \tag{4.23}$$

where $(\gamma^0)^\dagger=-\gamma^0$ and $(\gamma^i)^\dagger=\gamma^i$ or simply $(\gamma^a)^\dagger=\gamma^0\gamma^a\gamma^0$. In addition, the $\gamma$'s obey the useful identity

$$[\gamma^a,\Sigma^{bc}]=\eta^{ab}\gamma^c-\eta^{ac}\gamma^b, \tag{4.24}$$

where $\Sigma^{ab}=\frac{1}{4}[\gamma^a,\gamma^b]$. In (4.22) the torsionful covariant derivatives act on the spinor fields $\psi$ and $\bar\psi$ on the tangent frame metric as (see 8.35 of [36])

$$\nabla_c\psi=\partial_c\psi+\frac{1}{2}\omega_{abc}\Sigma^{ab}\psi,\quad \nabla_c\bar\psi=\partial_c\bar\psi-\frac{1}{2}\omega_{abc}\bar\psi\Sigma^{ab}, \tag{4.25}$$

where $\partial_c=e^\mu{}_c\partial_\mu$ and $\omega_{abc}$ are the spin connection coefficients, being antisymmetric in $a$, $b$ indices.[16] The spinor field transforms under infinitesimal Lorentz transformations (2.10) where

$$\delta_\omega\psi=-\frac{1}{2}\omega_{ab}\Sigma^{ab}\psi,\qquad \delta_\omega\bar\psi=\frac{1}{2}\omega_{ab}\bar\psi\Sigma^{ab}, \tag{4.26}$$

where $\omega_{ab}$ are the infinitesimal parameters not to be confused with the spin-connection coefficients $\omega_{abc}$. We can also derive the equations of motion of (4.22). It is a rather long but straightforward calculation to show that

$$\gamma^a(\nabla_a-S_a)\psi+m\psi=0,\qquad (\nabla_a-S_a)\bar\psi\gamma^a-m\bar\psi=0, \tag{4.27}$$

which is in agreement with [37].

Starting from the action (4.22) we can compute the hypermomentum (3.7) which in the flat spacetime takes the form

$$\Delta^{\nu\mu\lambda}=-\frac{i}{2}\bar\psi(\gamma^\lambda\Sigma^{\mu\nu}+\Sigma^{\mu\nu}\gamma^\lambda)\psi, \tag{4.28}$$

where we have used (3.27). In addition, using the Clifford algebra (4.23) we find that the hypermomentum tensor turns out to be totally antisymmetric (in agreement with [37]) since

$$\Delta^{\nu\mu\lambda}=-i\bar\psi\gamma^{[\lambda}\Sigma^{\mu\nu]}\psi=\frac{i}{2}\bar\psi\gamma^{[\nu}\gamma^\mu\gamma^{\lambda]}\psi. \tag{4.29}$$

We can easily find the divergence of (3.7)

$$\partial_\lambda\Delta^{\nu\mu\lambda}=i\bar\psi(\gamma^\mu\partial^\nu-\gamma^\nu\partial^\mu)\psi+\frac{i}{2}\partial^\mu(\bar\psi\gamma^\nu\psi)-\frac{i}{2}\partial^\nu(\bar\psi\gamma^\mu\psi), \tag{4.30}$$

where we used the equations of motion (4.27) in flat spacetime and the third of (4.24). Now, the corresponding canonical on-shell energy-momentum tensor of the Dirac field is derived in Appendix D 3 in Eq. (D16), and it is restated here for the reader's convenience:

$$t^{\mu\nu}=\frac{i}{2}(\bar\psi\gamma^\mu\partial^\nu\psi-\partial^\nu\bar\psi\gamma^\mu\psi). \tag{4.31}$$

Substituting (4.30) and (4.31) into (3.37) we find the conserved improved tensor (also derived in [38])

$$T^{\mu\nu}=\frac{i}{4}(\bar\psi\gamma^\mu\partial^\nu\psi-\partial^\nu\bar\psi\gamma^\mu\psi)+\frac{i}{4}(\bar\psi\gamma^\nu\partial^\mu\psi-\partial^\mu\bar\psi\gamma^\nu\psi), \tag{4.32}$$

where we note $T^\mu{}_\mu=-im\bar\psi\psi$. This coincides with the stress energy tensor obtained using the BR method in Appendix D 3 and Eq. (D21). From (3.37) and (4.29) we also see explicitly why the usual treatment of taking the symmetric part of $t_{\mu\nu}$ in (4.31) for the Dirac field, in order to derive the symmetric energy-momentum tensor, works in this case. The reason is that the hypermomentum is totally antisymmetric (4.29), and thus the symmetric part of $t_{\mu\nu}$ in (3.37) equal to

$$t_{(\mu\nu)}=T_{\mu\nu}, \tag{4.33}$$

and also $T_{\mu\nu}$ is conserved. In our formulation, therefore, it becomes clear why this method of simply symmetrizing the canonical stress energy tensor (4.31) works out in the case of the Dirac field. Finally, the gravitational definition of the energy-momentum tensor (3.3) fails here since spinors do not couple to the metric but rather to the vielbeins. This also proves the superiority of our method over the standard procedure, since in our approach no metric variations are performed.

### C. Higher derivative scalar theories

We now move to a nontrivial example of the Belinfante-Rosenfeld improvement process which demonstrates the power of our approach. Consider higher-derivative scalar theories as those considered, e.g., in [39] with action

$$S_4=-\frac{1}{2}\int d^dx(\partial^2\phi)^2,\qquad \partial^2\phi=\eta^{\mu\nu}\partial_\mu\partial_\nu\phi, \tag{4.34}$$

whose equations of motion are given by

$$\partial^2\partial^2\phi=0. \tag{4.35}$$

[16]This is so by demanding metricity $Q_{\rho\mu\nu}=-\nabla_\rho g_{\mu\nu}=0$ in the case at hand. For spaces with nonmetricity, the generalization to the Dirac equation is not known. We thank Friedrich Hehl for related discussions on this point.

Since the Lagrangian in (4.34) is higher derivative, the canonical e.m. tensor is given by

$$t^{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi + \eta^{\mu\nu}\mathcal{L} - 2\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\lambda\phi)}\partial^\nu\partial_\lambda\phi + \partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\lambda\phi)}\partial^\nu\phi\right) = 2\partial^\mu\partial^\nu\phi\partial^2\phi - \frac{1}{2}\eta^{\mu\nu}(\partial^2\phi)^2 - \partial^\mu(\partial^2\phi\partial^\nu\phi). \tag{4.36}$$

This can be improved (see Appendix D 4 for details) to the following symmetric BR e.m. tensor:

$$T^{\mu\nu}_{\mathrm{BR}} = 2\partial^\mu\partial^\nu\phi\partial^2\phi - \frac{1}{2}\eta^{\mu\nu}(\partial^2\phi)^2 - \partial^\mu(\partial^2\phi\partial^\nu\phi) - \partial^\nu(\partial^2\phi\partial^\mu\phi) + \eta^{\mu\nu}\partial_\lambda(\partial^2\phi\partial^\lambda\phi), \tag{4.37}$$

where the last two terms correspond to the BR improvement terms.

We can then show that the above BR e.m. tensor coincides with the flat limit of the following Hilbert metrical e.m. tensor:

$$T^{\mu\nu}_{\mathrm{BR}} = 2\tilde{\nabla}^\mu\partial^\nu\phi\tilde{\nabla}^2\phi - \frac{1}{2}\eta^{\mu\nu}(\tilde{\nabla}^2\phi)^2 - \tilde{\nabla}^\mu(\tilde{\nabla}^2\phi\tilde{\partial}^\nu\phi) - \tilde{\nabla}^\nu(\tilde{\nabla}^2\phi\partial^\mu\phi) + \eta^{\mu\nu}\tilde{\nabla}_\lambda(\partial^2\phi\partial^\lambda\phi), \tag{4.38}$$

which is in turn obtained using (3.3) when we minimally couple the action (4.34) to Einstein gravity. As before, the latter procedure does not give us a geometric understanding of the BR improvement terms. For that we would need access to the canonical e.m. tensor, and this can be achieved if we couple the theory to MAG using the following:

$$S_4 = -\frac{1}{2}\int d^dx\sqrt{-g}(\nabla^2\phi)^2, \qquad \nabla^2\phi = g^{\mu\nu}\nabla_\mu\partial_\nu\phi, \tag{4.39}$$

where now $\nabla_\mu$ is the affine connection. The crucial point is that when we use (3.3) to compute the Hilbert metrical e.m. tensor from (4.39), we need to take into account the nontrivial connection and the result is not the same as (4.38). In fact, the corresponding e.m. tensor is not even covariantly conserved. Restricting to flat space we obtain

$$T^{\mu\nu} = 2\partial^\mu\partial^\nu\phi\partial^2\phi - \frac{1}{2}\eta^{\mu\nu}\partial^2\phi\partial^2\phi. \tag{4.40}$$

The reason that (4.40) is not conserved is the presence of hypermomentum, which we can calculate using (3.7) and the definition of the covariant Laplacian

$$\nabla^2\phi = g^{\mu\nu}(\partial_\mu\partial_\nu\phi - \Gamma_{\nu\mu}{}^\rho\partial_\rho\phi), \tag{4.41}$$

as

$$\Delta^{\nu\mu\lambda} = -2\eta^{\mu\lambda}\partial^2\phi\partial^\nu\phi. \tag{4.42}$$

We can verify that using (4.40) and (4.42) into (3.37) we find the canonical e.m. tensor (4.36).

Although the MAG metrical e.m. tensor (4.40) does not coincide with the BR one, we can still show that the improvement terms in (4.37) come from the hypermomentum. For that we can use (2.12) and (3.37) to establish that

$$\Psi^{\rho[\mu\nu]} = \frac{1}{2}\Delta^{[\nu\mu]\rho}. \tag{4.43}$$

Then, from (2.13) we find that the BR improvement terms are generally given by

$$\frac{1}{2}\partial_\lambda(\Delta^{[\lambda\nu]\mu} + \Delta^{[\lambda\mu]\nu} + \Delta^{[\nu\mu]\lambda}) = \partial^\nu(\partial^2\phi\partial^\mu\phi) - \eta^{\mu\nu}\partial_\lambda(\partial^2\phi\partial^\lambda\phi), \tag{4.44}$$

which are exactly minus the improvement terms of (4.37), verifying our claim.

## V. SPECIAL CONFORMAL INVARIANCE IMPROVEMENT TERMS

We present below examples where coupling to MAG reveals the relevance of hypermomentum to the improvement process of the e.m. tensor in a conformal field theory.

### A. The massless scalar

The simplest example is that of a massless free scalar field in $d$ dimensions with action

$$S_{\mathrm{scalar}} = -\frac{1}{2}\int d^dx(\partial\phi)^2, \qquad (\partial\phi)^2 = \eta^{\kappa\lambda}\partial_\kappa\phi\partial_\lambda\phi. \tag{5.1}$$

The canonical e.m. tensor of (5.1) is symmetric and coincides with the improved BR one as

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}(\partial\phi)^2. \tag{5.2}$$

This is the e.m. tensor that we would have obtained as the flat limit of the metrical e.m. tensor when we minimally couple the action (5.1) to Einstein gravity. However, the action (5.1) is scale invariant, and this gives rise to an on-shell conserved dilatation current as

$$j^\mu_D = x_\nu T^{\mu\nu} - V^\mu. \tag{5.3}$$

The e.m. tensor (5.2) has a nonvanishing trace that takes the form of a total derivative

$$T^\mu{}_\mu = -\frac{d-2}{2}(\partial\phi)^2 = \partial_\mu V^\mu, \qquad V^\mu = -\frac{d-2}{4}\partial^\mu\phi^2. \tag{5.4}$$

Moreover, since

$$V^\mu = \partial_\nu Z^{\mu\nu}, \qquad Z^{\mu\nu} = -\frac{d-2}{4}\eta^{\mu\nu}\phi^2, \tag{5.5}$$

condition (2.21) holds and using (2.22) we can build an improved traceless e.m. tensor [5]

$$\Theta_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 + \frac{1}{4}\frac{d-2}{d-1}(\eta_{\mu\nu}\partial^2\phi^2 - \partial_\mu\partial_\nu\phi^2), \tag{5.6}$$

which is conserved, symmetric, and also traceless. The corresponding scale and special conformal currents can be read through (2.24). Equivalently we find the same expression using the improved energy-momentum tensor

$$T^{\mu\nu}_{\text{imp}} = t^{\mu\nu} - \frac{1}{d-1}(\eta^{\mu\nu}\partial_\kappa V^\kappa - \partial^\nu V^\mu). \tag{5.7}$$

For $d = 4$ it coincides with the improved stress energy tensor in [6], which was also derived in Appendix A of [40] by adding the total divergence term $\partial^2\phi$ with an appropriate coefficient in the action (5.1).

The usual way to find the improved conformal e.m. tensor $\Theta^{\mu\nu}$ is to couple the matter action (5.1) to Einstein gravity in a Weyl invariant way, and this corresponds to finding the Weyl invariant extension of the flat Laplacian [6]. That process is well-known but it does not provide a geometric origin for the improvement terms in (5.6). Here we will show that coupling the action (5.1) to MAG and requiring an extended Weyl invariance, namely local frame invariance, are also straightforward and give a geometric meaning to the improvement terms through hypermomentum. The frame invariant extension of (5.1) can be achieved if we used the extended covariant derivative that takes the form of a minimal coupling with a Weyl connection [23–25] (see also [11,29,30] in the context of MAG). Following the latter reference we define

$$\partial_\mu \to \nabla_\mu \to D_\mu = \nabla_\mu + \alpha S_\mu. \tag{5.8}$$

The coefficient $\alpha$ will be found requiring that the resulting action

$$S = -\frac{1}{2}\int d^dx\sqrt{-g}\,(D\phi)^2 \tag{5.9}$$

is invariant under the local rescaling

$$\phi \mapsto e^{w\Omega}\phi, \qquad w = \frac{d-2}{2}, \tag{5.10}$$

accompanied by the frame rescaling (3.21). From the latter we find that the torsion tensor transforms as

$$S_{\mu\nu}{}^\rho \mapsto S_{\mu\nu}{}^\rho + \frac{1}{2}(\partial_\mu\Omega\delta^\rho_\nu - \partial_\nu\Omega\delta^\rho_\mu) \tag{5.11}$$

and the torsion vector (A9) as

$$S_\mu \mapsto S_\mu + \frac{d-1}{2}\partial_\mu\Omega. \tag{5.12}$$

We then find that for

$$\alpha = \frac{d-2}{d-1}, \tag{5.13}$$

the Weyl covariant derivative (5.8) transforms simply as

$$D_\mu\phi \mapsto e^{\frac{d-2}{2}\Omega}D_\mu\phi, \tag{5.14}$$

yielding an invariant action (5.9). Note also that the Ricci scalar $\tilde{R}$ transforms as

$$\tilde{R} \mapsto e^{2\Omega}(\tilde{R} + 2(d-1)\tilde{\nabla}^2\Omega - (d-1)(d-2)(\partial\Omega)^2). \tag{5.15}$$

We can now compute the hypermomentum (3.7), which in the flat limit becomes

$$\Delta_\nu{}^{\mu\lambda} = \frac{d-2}{d-1}\delta^{[\mu}_\nu\partial^{\lambda]}\phi^2. \tag{5.16}$$

This can be decomposed as in (3.8)

$$\Delta^{\nu\mu\lambda} = \Sigma^{\nu\mu\lambda} + \frac{\eta^{\mu\nu}}{d}\Delta^\lambda + \hat{\Delta}^{\nu\mu\lambda}, \quad \Sigma^{\nu\mu\lambda} = \frac{1}{2}\frac{d-2}{d-1}\eta^{\lambda[\mu}\partial^{\nu]}\phi^2,$$
$$\Delta^\lambda = \frac{d-2}{2}\partial^\lambda\phi^2, \quad \hat{\Delta}^{\nu\mu\lambda} = \frac{1}{2}\frac{d-2}{d-1}\left(\frac{\eta^{\mu\nu}}{d}\partial^\lambda - \eta^{\lambda(\mu}\partial^{\nu)}\right)\phi^2. \tag{5.17}$$

Then, starting from (5.16) we find

$$\partial_\lambda\Delta^{\nu\mu\lambda} = \frac{1}{d}\eta^{\mu\nu}\partial_\lambda\Delta^\lambda + \partial_\lambda\hat{\Delta}^{\nu\mu\lambda} = \frac{d-2}{2(d-1)}(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\phi^2, \tag{5.18}$$

where $\partial_\lambda\Sigma^{\nu\mu\lambda} = 0$. Substituting (5.18) into (3.37), we finally find (also derived in [38])

$$t^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}(\partial\phi)^2 + \frac{1}{4}\frac{d-2}{d-1}(\eta^{\mu\nu}\partial^2\phi^2 - \partial^\mu\partial^\nu\phi^2), \tag{5.19}$$

which coincides with (5.6).

In fact, we will show that in teleparallel gravity (see [41] for a review), where we consider connections with vanishing curvature for the connection $\Gamma^\lambda{}_{\mu\nu}$, the action (5.9) is

equivalent with that of the conformally coupled scalar given in Eq. (5.24) of Appendix D 1, up to a boundary term. To prove this we consider the Ricci scalar $R$ of the MAG theory, e.g., Eq. (13) in [42], specialized to vanishing nonmetricity[17]

$$R = \tilde{R} + S_{\mu\nu\rho}S^{\mu\nu\rho} - 2S_{\mu\nu\rho}S^{\rho\mu\nu} - 4S_\mu S^\mu - 4\tilde{\nabla}_\mu S^\mu = \tilde{R} - 4\frac{d-2}{d-1}S_\mu S^\mu - 4\tilde{\nabla}_\mu S^\mu, \tag{5.20}$$

where in the second equality we have assumed a vectorial type torsion of the form

$$S_{\mu\nu}{}^\rho = \frac{1}{d-1}(S_\mu \delta^\rho_\nu - S_\nu \delta^\rho_\mu). \tag{5.21}$$

Then, we consider (5.9), which can be rewritten as

$$S = \int d^dx\sqrt{-g}\left\{-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\frac{d-2}{d-1}\left(\frac{d-2}{d-1}S_\mu S^\mu\phi^2 - S^\mu\partial_\mu\phi^2\right)\right\}, \tag{5.22}$$

or upon using Eq. (5.20) we find

$$S = \int d^dx\sqrt{-g}\left\{-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\frac{d-2}{d-1}\left(\frac{1}{4}(\tilde{R}-R)\phi^2 - \tilde{\nabla}_\mu(S^\mu\phi^2)\right)\right\}. \tag{5.23}$$

Ignoring the last term (as it is a boundary term) and imposing flatness of the connection, i.e., $R^\lambda{}_{\alpha\beta\gamma} = 0$, we have $R = 0$ and we find exact agreement with the action of the well-known action of the conformally coupled scalar [6]

$$S_{\rm cs} = \int d^dx\sqrt{-g}\left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{8}\frac{d-2}{d-1}\tilde{R}\phi^2\right), \tag{5.24}$$

which is revisited in Appendix D 1.

### B. Higher derivative scalars

The action (4.34) is scale invariant. The trace of the canonical and the BR e.m. tensors (4.36) and (4.37), respectively, take on-shell, respectively, the forms

$$t^\mu{}_\mu = \partial_\mu\tilde{V}^\mu, \qquad T^\mu_{{\rm BR}\mu} = \partial_\mu V^\mu, \tag{5.25}$$

where the virial currents are given by

$$\tilde{V}^\mu = \frac{1}{2}((2-d)\partial^\mu\phi\partial^2\phi + (d-4)\phi\partial^\mu\partial^2\phi), \tag{5.26}$$

$$V^\mu = \frac{1}{2}(d\partial^\mu\phi\partial^2\phi + (d-4)\phi\partial^\mu\partial^2\phi). \tag{5.27}$$

The corresponding conserved dilatation current takes the form of Eq. (5.3). Moreover, the theory is also conformal invariant [13,20] since the virial currents take the form since

$$\tilde{V}^\mu = \partial_\nu\tilde{Z}^{\mu\nu}, \qquad V^\mu = \partial_\nu Z^{\mu\nu}, \tag{5.28}$$

with

$$\tilde{Z}^{\mu\nu} = -\frac{d-3}{2}(2\partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}(\partial\phi)^2) + \frac{1}{2}(d-4)\eta^{\mu\nu}\phi\partial^2\phi,$$
$$Z^{\mu\nu} = 2\partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}(\partial\phi)^2 + \frac{1}{2}(d-4)\eta^{\mu\nu}\phi\partial^2\phi. \tag{5.29}$$

From the latter we can construct the conserved Noether tensor (3.46)

$$\partial_\mu K^{\mu\nu} = 0, \quad K^{\mu\nu} = x^2 T^{\mu\nu}_{\rm BR} - 2x^\nu x_\kappa T^{\mu\kappa}_{\rm BR} + 2(\partial_\kappa Z^{\mu\kappa}x^\nu - Z^{\mu\nu}). \tag{5.30}$$

Having at hand the BR improved e.m. tensor, as well as $Z^{\mu\nu}$ we can use formula (2.22) to construct the conformally improved, conserved symmetric and traceless e.m. tensor $\Theta^{\mu\nu}$ as [see also Eqs. (4.4) and (4.5) in [14]]

$$\begin{aligned}\Theta_{\mu\nu} = {} & 2\partial_\mu\partial_\nu\phi\partial^2\phi - \frac{1}{2}\eta_{\mu\nu}(\partial^2\phi)^2 - \partial_\mu(\partial^2\phi\partial_\nu\phi) \\ & - \partial_\nu(\partial^2\phi\partial_\mu\phi) + \eta_{\mu\nu}\partial_\lambda(\partial^2\phi\partial^\lambda\phi) \\ & + \frac{1}{2}\frac{d-4}{d-1}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)(\phi\partial^2\phi) \\ & + \frac{2}{d-2}(2\partial_\rho(\partial_\mu\partial_\nu\phi\partial^\rho\phi) - \eta_{\mu\nu}\partial_\rho\partial_\sigma(\partial^\rho\phi\partial^\sigma\phi)) \\ & - \frac{d}{(d-1)(d-2)}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)(\partial\phi)^2,\end{aligned} \tag{5.31}$$

obeying

$$\partial^\mu\Theta_{\mu\nu} = -\partial_\nu\phi\partial^2\partial^2\phi, \qquad \eta^{\mu\nu}\Theta_{\mu\nu} = -\frac{1}{2}(d-4)\phi\partial^2\partial^2\phi, \tag{5.32}$$

which vanish upon using the equations of motion (4.35). The corresponding scale and special conformal currents can be read through (2.24).

To obtain $\Theta^{\mu\nu}$ one would have had to couple the action (4.34) to Einstein gravity in a Weyl invariant way. This is equivalent to having to obtain the Weyl extension of the square of the Laplacian $\nabla^2$ (see, e.g., [43]). This procedure,

[17]We note that under a frame rescaling the Ricci scalar simply transforms as $R \mapsto e^{2\Omega}R$, where the additional pieces in the transformations of the torsion, the torsion vector, and the $\tilde{R}$, that is, Eqs. (5.11), (5.12), and (5.15), drop out.

however, does not provide a geometric interpretation for the improvement terms. We show below that coupling (4.34) to MAG reveals the geometric origin of the improvement terms, and, in fact, it is rather straightforward.

As before the main idea is to couple (4.34) to MAG in a Weyl invariant manner. For that we need to construct MAG Weyl-covariant derivatives on scalars $\Phi$ and mixed tensors $X_{\mu_1\cdots\mu_p}{}^{\nu_1\cdots\nu_q}$ of corresponding weights $w$. We do that with the help of the torsion vector $S_\mu$ as

$$D_\mu\Phi=\left(\partial_\mu-\frac{2w}{d-1}S_\mu\right)\Phi,$$
$$D_\mu X_{\mu_1\cdots\mu_p}{}^{\nu_1\cdots\nu_q}=\nabla_\mu X_{\mu_1\cdots\mu_p}{}^{\nu_1\cdots\nu_q}-2\frac{w+p-q}{d-1}S_\mu X_{\mu_1\cdots\mu_p}{}^{\nu_1\cdots\nu_q}. \tag{5.33}$$

These covariant derivatives transform under the following Weyl transformations:

$$g_{\mu\nu}\mapsto e^{-2\Omega}g_{\mu\nu},\quad \Phi\mapsto e^{w\Omega}\Phi,$$
$$X_{\mu_1\cdots\mu_p}{}^{\nu_1\cdots\nu_q}\mapsto e^{w\Omega}X_{\mu_1\cdots\mu_p}{}^{\nu_1\cdots\nu_q},\quad w=\frac{d-4}{2}, \tag{5.34}$$

and the frame transformations of $S_\mu$ in (5.12) as

$$D_\mu\Phi\mapsto e^{w\Omega}D_\mu\Phi,\quad D_\mu X_{\mu_1\cdots\mu_p}{}^{\nu_1\cdots\nu_q}\mapsto e^{w\Omega}D_\mu X_{\mu_1\cdots\mu_p}{}^{\nu_1\cdots\nu_q}. \tag{5.35}$$

Then, the following action is manifestly Weyl and frame rescaling invariant:

$$S_4=-\frac{1}{2}\int d^dx\sqrt{-g}(D^2\phi)^2,\quad D^2\phi=g^{\mu\nu}D_\mu D_\nu\phi, \tag{5.36}$$

where

$$D^2\phi=\nabla^2\phi-\frac{2w}{d-1}\nabla_\lambda S^\lambda\phi-\frac{4w+2}{d-1}S^\lambda\partial_\lambda\phi+\frac{4w(w+1)}{(d-1)^2}S_\lambda S^\lambda\phi. \tag{5.37}$$

From the above, using (3.7) and (5.34) we find that the hypermomentum tensor in the flat limit equals

$$\Delta^{\nu\mu\lambda}=-2\eta^{\mu\lambda}\partial^2\phi\partial^\nu\phi+\frac{d-4}{d-1}(\partial^\mu(\phi\partial^2\phi)\eta^{\lambda\nu}-\partial^\lambda(\phi\partial^2\phi)\eta^{\mu\nu})-2\frac{d-3}{d-1}\partial^2\phi(\eta^{\lambda\nu}\partial^\mu\phi-\eta^{\mu\nu}\partial^\lambda\phi), \tag{5.38}$$

which, of course, differs from (4.42). Inserting (4.40) and (5.38) into (3.37), we find

$$t_{\mu\nu}=2\partial_\mu\partial_\nu\phi\partial^2\phi-\frac{1}{2}\eta_{\mu\nu}(\partial^2\phi)^2-\partial_\mu(\partial^2\phi\partial_\nu\phi)-\partial_\nu(\partial^2\phi\partial_\mu\phi)+\eta_{\mu\nu}\partial_\lambda(\partial^2\phi\partial^\lambda\phi)+\frac{1}{2}\frac{d-4}{d-1}(\partial_\mu\partial_\nu-\eta_{\mu\nu}\partial^2)(\phi\partial^2\phi)+\frac{2}{d-1}(\partial_\nu(\partial_\mu\phi\partial^2\phi)-\eta_{\mu\nu}\partial_\lambda(\partial^\lambda\phi\partial^2\phi)), \tag{5.39}$$

where the terms in the second to fourth lines are identically conserved line-by-line. In addition, starting from (5.39) we find

$$\partial^\mu t_{\mu\nu}=-\partial_\nu\phi\partial^2\partial^2\phi,\quad \eta^{\mu\nu}t_{\mu\nu}=-\frac{1}{2}(d-4)\phi\partial^2\partial^2\phi; \tag{5.40}$$

hence, upon using (4.35) we find that $t_{\mu\nu}$ is on-shell divergenceless and traceless. Comparing (5.39) with (5.31) we find that their difference corresponds to an off-shell improvement term regarding the conservation but also the trace. Thus, the two stress tensors are equivalent, yielding the same Noether charges up to boundary terms.

Although the action (4.34) is Lorentz invariant, the stress tensor (5.39) is nonsymmetric

$$t_{\mu\nu}-t_{\nu\mu}+\frac{4}{d-1}\partial_\lambda(\delta^\lambda_{[\mu}\partial_{\nu]}\phi\partial^2\phi)=0, \tag{5.41}$$

and using (3.43) we read

$$\Sigma_{\mu\nu}{}^\lambda=\frac{4}{d-1}\delta^\lambda_{[\mu}\partial_{\nu]}\phi\partial^2\phi, \tag{5.42}$$

whose trace is given by

$$\Sigma_{\lambda\nu}{}^\lambda=2\partial_\nu\phi\partial^2\phi, \tag{5.43}$$

which satisfies

$$\Sigma_{\lambda\nu}{}^\lambda=\partial^\mu Z_{\mu\nu},\qquad Z_{\mu\nu}=2\partial_\mu\phi\partial_\nu\phi-\eta_{\mu\nu}(\partial\phi)^2. \tag{5.44}$$

Then, using (5.42) we find

$$\Sigma_{\nu\lambda\mu}+\Sigma_{\mu\nu\lambda}+\Sigma_{\mu\lambda\nu}=\frac{4}{d-1}(\eta_{\mu\nu}\partial_\lambda\phi\partial^2\phi-\eta_{\lambda\nu}\partial_\mu\phi\partial^2\phi), \tag{5.45}$$

so the corresponding Belinfante-Rosenfeld scale (BRS) energy-momentum tensor (2.13), using (3.44), reads

$$T^{\rm BRS}_{\mu\nu} = 2\partial_\mu\partial_\nu\phi\partial^2\phi - \frac{1}{2}\eta_{\mu\nu}(\partial^2\phi)^2 - \partial_\mu(\partial^2\phi\partial_\nu\phi) - \partial_\nu(\partial^2\phi\partial_\mu\phi) + \eta_{\mu\nu}\partial_\lambda(\partial^2\phi\partial^\lambda\phi) + \frac{1}{2}\frac{d-4}{d-1}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)(\phi\partial^2\phi) = T^{\rm BR}_{\mu\nu} + \frac{1}{2}\frac{d-4}{d-1}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)(\phi\partial^2\phi), \tag{5.46}$$

where in the last step we have employed (4.37). The tensor (5.46) is on-shell conserved, symmetric, and its trace assumes the form

$$\eta^{\mu\nu}T^{\rm BRS}_{\mu\nu} = \eta^{\mu\nu}(t_{\mu\nu} + \partial_\lambda\Sigma_\mu{}^\lambda{}_\nu) = -\frac{1}{2}(d-4)\phi\partial^2\partial^2\phi + \partial^\mu\partial^\nu Z_{\mu\nu}, \tag{5.47}$$

where we have used the second of (5.40) and (5.44). The latter formula ensures the on-shell conformal invariance of (4.34) as we can construct the corresponding conserved, symmetric, and traceless tensor $\Theta^{\mu\nu}$ using (2.22), where $(T^{\rm BR}_{\mu\nu}, \Lambda_{\mu\nu}) \mapsto (T^{\rm BRS}_{\mu\nu}, -Z_{\mu\nu})$. The corresponding scale and special conformal currents can be read through (2.24).

### C. Six-derivative case

Let us now consider the six-derivative free field also considered, e.g., in [39] with action given by

$$S_6 = -\frac{1}{2}\int d^dx(\partial_\mu\partial^2\phi)^2, \qquad \partial^2\phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi, \tag{5.48}$$

and the equations of motion are given by

$$\partial^2\partial^2\partial^2\phi = 0. \tag{5.49}$$

Generalizing it to curved spacetimes we consider for metric connections the action

$$S_6 = -\frac{1}{2}\int d^dx\sqrt{-g}(\nabla_\mu\nabla^2\phi)^2, \qquad \nabla^2\phi = \eta^{\mu\nu}\nabla_\mu\nabla_\nu\phi, \tag{5.50}$$

leading to the metric stress tensor (3.3) and the hypermomentum tensor (3.7)

$$T_{\mu\nu} = \nabla_\mu\nabla^2\phi\nabla_\nu\nabla^2\phi - 2\nabla_\mu\nabla_\nu\phi\nabla^2\nabla^2\phi - \frac{1}{2}g_{\mu\nu}(\nabla_\lambda\nabla^2\phi)^2,$$
$$\Delta_\nu{}^{\mu\lambda} = 2g^{\lambda\mu}\nabla_\nu\phi\nabla^2\nabla^2\phi, \tag{5.51}$$

where $\phi$ is a scalar field, that is, $\nabla_\mu\phi = \partial_\mu\phi$. Using (3.37) for the trivial vacuum we easily find the nonsymmetric stress tensor

$$t_{\mu\nu} = \partial_\mu\partial^2\phi\partial_\nu\partial^2\phi - 2\partial_\mu\partial_\nu\phi\partial^2\partial^2\phi - \frac{1}{2}g_{\mu\nu}(\partial_\lambda\partial^2\phi)^2 + \partial_\mu(\partial_\nu\phi\partial^2\partial^2\phi) \tag{5.52}$$

that coincides with the canonical stress tensor for the action (5.48). The tensor (5.52) is on-shell conserved since

$$\partial^\mu t_{\mu\nu} = \partial_\nu\phi\partial^2\partial^2\partial^2\phi. \tag{5.53}$$

Using the Belinfante-Rosenfeld procedure (2.13) or considering the action (5.50) for a Levi-Civita connection we find

$$T^{\rm BR}_{\mu\nu} = \partial_\mu\partial^2\phi\partial_\nu\partial^2\phi - 2\partial_\mu\partial_\nu\phi\partial^2\partial^2\phi - \frac{1}{2}g_{\mu\nu}(\partial_\lambda\partial^2\phi)^2 + \partial_\mu(\partial_\nu\phi\partial^2\partial^2\phi) + \partial_\nu(\partial_\mu\phi\partial^2\partial^2\phi) - g_{\mu\nu}\partial_\lambda(\partial^\lambda\phi\partial^2\partial^2\phi), \tag{5.54}$$

where the second line is identically conserved.

The action (5.50) is invariant under the global transformations

$$g_{\mu\nu} \mapsto e^{-2\Omega}g_{\mu\nu}, \qquad \phi \mapsto e^{w\Omega}\phi, \qquad w = \frac{d-6}{2}. \tag{5.55}$$

As in the four-derivative case we make the symmetry local using as a Weyl connection the torsion vector reaching the expression for the hypermomentum tensor

$$\Delta_\nu{}^{\mu\lambda} = 2\eta^{\lambda\mu}\partial_\nu\phi\partial^2\partial^2\phi - \frac{d-6}{d-1}(\delta^\lambda_\nu\partial^\mu - \delta^\mu_\nu\partial^\lambda)(\phi\partial^2\partial^2\phi) + 2\frac{d-5}{d-1}(\delta^\lambda_\nu\partial^\mu\phi - \delta^\mu_\nu\partial^\lambda\phi)\partial^2\partial^2\phi - \frac{d-2}{d-1}(\delta^\lambda_\nu\partial^\mu\partial^2\phi - \delta^\mu_\nu\partial^\lambda\partial^2\phi)\partial^2\phi. \tag{5.56}$$

Using the latter and the metric tensor in (5.51) we find scale canonical stress tensor

$$t_{\mu\nu} = \partial_\mu\partial^2\phi\partial_\nu\partial^2\phi - 2\partial_\mu\partial_\nu\phi\partial^2\partial^2\phi - \frac{1}{2}g_{\mu\nu}(\partial_\lambda\partial^2\phi)^2 + \partial_\mu(\partial_\nu\phi\partial^2\partial^2\phi) - \frac{1}{2}\frac{d-6}{d-1}(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2)(\phi\partial^2\partial^2\phi) + \frac{d-5}{d-1}(\partial_\nu(\partial_\mu\phi\partial^2\partial^2\phi) - \eta_{\mu\nu}\partial_\lambda(\partial^\lambda\phi\partial^2\partial^2\phi)) - \frac{1}{2}\frac{d-2}{d-1}(\partial_\nu(\partial_\mu\partial^2\phi\partial^2\phi) - \eta_{\mu\nu}\partial_\lambda(\partial^\lambda\partial^2\phi\partial^2\phi)), \tag{5.57}$$

satisfying

$$\partial^\mu t_{\mu\nu} = \partial_\nu \phi \partial^2\partial^2\partial^2\phi, \qquad \eta^{\mu\nu} t_{\mu\nu} = \frac{1}{2}(d-6)\phi\partial^2\partial^2\partial^2\phi. \tag{5.58}$$

The latter stress tensor is not a symmetric tensor satisfying (3.43), where

$$\Sigma_{\mu\nu\lambda} = -\frac{8}{d-1}\eta_{\lambda[\mu}\partial_{\nu]}\phi\partial^2\partial^2\phi - \frac{d-2}{d-1}\eta_{\lambda[\mu}\partial_{\nu]}\partial^2\phi\partial^2\phi. \tag{5.59}$$

Its trace reads

$$\Sigma_{\lambda\nu}{}^{\lambda} = -4\partial_\nu\phi\partial^2\partial^2\phi - \frac{1}{2}(d-2)\partial_\nu\partial^2\phi\partial^2\phi = \partial^\mu Z_{\mu\nu}, \tag{5.60}$$

where (see also [44])

$$\begin{aligned} Z_{\mu\nu} &= 2(\phi\partial_\mu\partial_\nu\partial^2\phi + \partial_\mu\partial_\nu\phi\partial^2\phi) \\ &\quad - 2(\eta_{\mu\nu}\phi\partial^2\partial^2\phi + 2\partial_{(\mu}\phi\partial_{\nu)}\partial^2\phi) - \frac{1}{2}(d-2)\eta_{\mu\nu}(\partial^2\phi)^2. \end{aligned} \tag{5.61}$$

We can now define the BRS stress tensor (2.13), using (3.44), which reads

$$T^{\mathrm{BRS}}_{\mu\nu} = T^{\mathrm{BR}}_{\mu\nu} - \frac{1}{2}\frac{d-6}{d-1}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)(\phi\partial^2\partial^2\phi), \tag{5.62}$$

where $T^{\mathrm{BR}}_{\mu\nu}$ was given in (5.54). The trace reads

$$\begin{aligned} \eta^{\mu\nu}T^{\mathrm{BRS}}_{\mu\nu} &= \eta^{\mu\nu}(t_{\mu\nu} + \partial_\lambda \Sigma_{\mu}{}^{\lambda}{}_{\nu}) \\ &= \frac{1}{2}(d-6)\phi\partial^2\partial^2\partial^2\phi + \partial^\mu\partial^\nu Z_{\mu\nu}, \end{aligned} \tag{5.63}$$

where we have used the second of (5.58) and (5.60). The latter formula ensures the on-shell conformal invariance of (5.48) as we can construct the corresponding conserved, symmetric, and traceless tensor $\Theta^{\mu\nu}$ using (2.22), where $(T^{\mathrm{BR}}_{\mu\nu}, \Lambda_{\mu\nu}) \mapsto (T^{\mathrm{BRS}}_{\mu\nu}, -Z_{\mu\nu})$. The corresponding scale and special conformal currents can be read through (2.24).

Similarly we can extend the above treatment for other higher-derivative free fields

$$\begin{aligned} S_{4n} &= -\frac{1}{2}\int d^dx(\partial^{2n}\phi)^2, \\ S_{4n+2} &= -\frac{1}{2}\int d^dx(\partial_\mu\partial^{2n}\phi)^2, \quad n \geq 1. \end{aligned} \tag{5.64}$$

## VI. CONCLUSION AND OUTLOOK

The Belinfante-Rosenfeld and conformal improvements of the canonical e.m. tensor of matter theories in flat spacetimes are complicated but straightforward procedures. In most cases they are circumvented by coupling the theory to Einstein or Weyl gravity, but this leaves open the question regarding the geometrical origin of the process. In this work we have provided a geometric origin of the improvement processes by extending the coupling of the matter system to metric-affine geometry, namely allowing for connection-matter couplings. We have explicitly shown in a number of examples that the improvement terms are directly related to the hypermomentum of the matter theory. Nevertheless, this relationship is theory dependent, and in the case of conformal improvement terms there does not appear to be a general formula although we have demonstrated it in some nontrivial cases.

Our main result is that by coupling the matter theory to MAG rather than just Einstein gravity and then taking the flat limit we arrive at a geometric interpretation of the improvement terms in flat space, both for Lorentz as well as for conformal invariance, in terms of the hypermomentum. Moreover, the coupling of the matter theory to MAG can be done in a minimal way, and this avoids the complicated procedure of Weyl extending differential operators in Einstein gravity. The point is that to obtain the flat limit of the hypermomentum we just need to know the linear terms in the affine connection in any MAG extension of the matter theory. This is a significant simplification; however, it is not without ambiguity in the case of a MAG Weyl extension. In this work, we have considered the simple case of local frame rescaling, and we have shown in explicit nontrivial examples how the hypermomentum determines the improvement term.

Extensions of our work may offer a new perspective on the geometric and gravitational description of matter systems invariant under subsets or limits of Poincaré or conformal symmetries. Such systems may arise in AdS/CFT holography or in fluid dynamics. In particular, since coupling to MAG gives us access to the canonical e.m. tensor of the matter theory, we can describe matter systems with less symmetries, e.g., systems without Lorentz symmetry, or systems with scalar but not conformal symmetry. Looking at our main equations (3.18) and (3.24) we note that had we considered nontrivial flat limits, i.e., situations where the connection does not vanish while the metric stays flat, then we could describe matter systems whose canonical e.m. tensor cannot be improved to be symmetric or traceless.[18] Other possibilities include the description of Carrollian, Galilean, or even Aristotelian matter theories, e.g., [46–50]. We plan to return in these questions in the near future.

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[18]The possibility that Lorentz symmetry breaking terms might have geometric origins was also discussed in [45].

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## DATA AVAILABILITY

No data were created or analyzed in this study.

## APPENDIX A: METRIC-AFFINE GRAVITY

We shall work in the metric-affine framework. To this end we consider a $d$-dimensional manifold of Lorentzian signature, which we endow with the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, $\mu,\nu = 0,1,2,\ldots,d-1$ and a generic affine connection $\nabla$ with components $\Gamma^\lambda{}_{\mu\nu}$. The covariant derivative acts on a mixed (1, 1) tensor, say $A^\mu{}_\nu$, as

$$\nabla_\kappa A^\mu{}_\nu = \partial_\kappa A^\mu{}_\nu + \Gamma^\mu{}_{\lambda\kappa}A^\lambda{}_\nu - \Gamma^\lambda{}_{\nu\kappa}A^\mu{}_\lambda. \tag{A1}$$

We also define the torsion and curvature of the connection as usual through

$$[\nabla_\kappa,\nabla_\lambda]A^\mu{}_\nu = R^\mu{}_{\rho\kappa\lambda}A^\rho{}_\nu - R^\rho{}_{\mu\kappa\lambda}A^\mu{}_\rho + 2S_{\kappa\lambda}{}^\rho\nabla_\rho A^\mu{}_\nu, \tag{A2}$$

where

$$S_{\mu\nu}{}^\lambda \coloneqq \Gamma^\lambda{}_{[\mu\nu]}, \qquad S_{\mu\nu\rho} = g_{\rho\lambda}S_{\mu\nu}{}^\lambda, \tag{A3}$$

and

$$R^\mu{}_{\nu\alpha\beta} \coloneqq 2\partial_{[\alpha}\Gamma^\mu{}_{|\nu|\beta]} + 2\Gamma^\mu{}_{\rho[\alpha}\Gamma^\rho{}_{|\nu|\beta]}, \tag{A4}$$

respectively, and $\nu$ is excluded from antisymmetry on the curvature tensor. From the curvature tensor we can define three kinds of contractions, namely

$$R_{\nu\beta} = R^\mu{}_{\nu\mu\beta}, \quad \hat{R}_{\alpha\beta} = R^\mu{}_{\mu\alpha\beta}, \quad \check{R}^\mu{}_\beta = R^\mu{}_{\nu\alpha\beta}g^{\nu\alpha} \tag{A5}$$

and their corresponding contractions

$$R = g^{\mu\nu}R_{\mu\nu} = -g^{\mu\nu}\check{R}_{\mu\nu}, \qquad g^{\mu\nu}\hat{R}_{\mu\nu} = 0. \tag{A6}$$

Obviously these objects need only a connection to be defined and do not require a metric. On the other hand, a metric (along with the connection) is essential to construct the nonmetricity tensor, which we define via

$$Q_{\rho\mu\nu} = -\nabla_\rho g_{\mu\nu} = -\partial_\rho g_{\mu\nu} + \Gamma^\lambda{}_{\mu\rho}g_{\lambda\nu} + \Gamma^\lambda{}_{\nu\rho}g_{\mu\lambda} \tag{A7}$$

that measures the failure of the metric to be covariantly constant. Out of nonmetricity we can construct two vectors by the following contractions:

$$Q_\mu \coloneqq Q_{\mu\alpha\beta}g^{\alpha\beta}, \qquad q_\mu \coloneqq Q_{\alpha\nu\mu}g^{\alpha\nu}. \tag{A8}$$

As for torsion we can define a vector as follows[19]:

$$S_\mu \coloneqq S_{\mu\alpha}{}^\alpha. \tag{A9}$$

Note that with the above definitions of torsion and nonmetricity the affine connection can be decomposed as (see, for instance, [51])

$$\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} + N^\lambda{}_{\mu\nu}, \tag{A10}$$

where

$$\tilde{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{A11}$$

is the usual Levi-Civita connection constructed from $g_{\mu\nu}$, and the additional tensor $N^\lambda{}_{\mu\nu}$ is defined through

$$N^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(Q_{\mu\nu\rho} + Q_{\nu\rho\mu} - Q_{\rho\mu\nu}) - g^{\rho\lambda}(S_{\rho\mu\nu} + S_{\rho\nu\mu} - S_{\mu\nu\rho}), \tag{A12}$$

the so-called distortion tensor, measuring the departure from the Riemannian geometry. Out of the distortion we can extract three traces as $N^{(1)}_\nu \coloneqq N^\alpha{}_{\alpha\nu}$, $N^{(2)}_\nu \coloneqq N^\alpha{}_{\nu\alpha}$, and $N^{(3)}_\nu \coloneqq N_{\alpha\mu\nu}g^{\mu\nu}$, which are linear combinations of the torsion and nonmetricity vectors as can be trivially verified.

## APPENDIX B: CANONICAL ENERGY-MOMENTUM TENSOR

Let us now prove here the known fact that the canonical energy-momentum tensor defined by the variation with respect to the vielbein as it is read from Eq. (3.26),

$$t^\mu{}_\nu = t^\mu{}_a e_\nu{}^a = \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta e_\mu{}^a}e_\nu{}^a, \tag{B1}$$

coincides with the tensorial canonical expression (3.32).

[19]Along with a pseudovector $t_\mu \coloneqq \epsilon_{\mu\nu\alpha\beta}S^{\nu\alpha\beta}$, defined only for $n = 4$.

We first employ the chain rule on (B1), and we find

$$\begin{aligned} t^{\mu}{}_{\nu} &= \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L}_M)}{\partial e^{\lambda}{}_{c}}\frac{\partial e^{\lambda}{}_{c}}{\partial e_{\mu}{}^{a}}e_{\nu}{}^{a} \\ &= -\frac{1}{\sqrt{-g}}\delta^{\lambda}_{\nu}e^{\mu}{}_{c}\frac{\partial(\sqrt{-g}\mathcal{L}_M)}{\partial e^{\lambda}{}_{c}} \\ &= -e^{\mu}{}_{c}\left\{\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial e^{\nu}{}_{c}}\mathcal{L}_M + \frac{\partial\mathcal{L}_M}{\partial e^{\nu}{}_{c}}\right\}, \end{aligned} \tag{B2}$$

where in the second equality we used the identity

$$\frac{\partial e^{\lambda}{}_{c}}{\partial e_{\mu}{}^{a}} = -e^{\mu}{}_{c}e^{\lambda}{}_{a}. \tag{B3}$$

Then, from the relation $g_{\mu\nu} = e_{\mu}{}^{a}e_{\nu}{}^{b}\eta_{ab}$ we easily obtain

$$\frac{\partial\sqrt{-g}}{\partial e^{\nu}{}_{c}} = -\sqrt{-g}e_{\nu}{}^{c}, \tag{B4}$$

and this with (B2) it follows that

$$t^{\mu}{}_{\nu} = \delta^{\mu}_{\nu}\mathcal{L}_M - e^{\mu}{}_{c}\frac{\partial\mathcal{L}_M}{\partial e^{\nu}{}_{c}}. \tag{B5}$$

We now need only to take care of the last term on the right-hand side of the latter. Since the vielbein appears only through the covariant differentiation of the matter fields, we have

$$\nabla_a\phi^A = e^{\rho}{}_{a}\nabla_{\rho}\phi^A, \tag{B6}$$

and therefore,

$$\frac{\partial(\nabla_a\phi^A)}{\partial e^{\lambda}{}_{c}} = \delta^{c}_{a}\nabla_{\lambda}\phi^A. \tag{B7}$$

Then on the premise that the matter action depends algebraically on the vielbein (i.e., not on its derivatives), employing the chain rule again and using (3.30), we find

$$\frac{\delta\mathcal{L}_M}{\delta e^{\nu}{}_{c}} = \frac{\partial\mathcal{L}_M}{\partial(\nabla_a\phi^A)}\frac{\partial(\nabla_a\phi^A)}{\partial e^{\nu}{}_{c}} = \frac{\partial\mathcal{L}_M}{\partial(\nabla_c\phi^A)}\nabla_{\nu}\phi^A. \tag{B8}$$

Upon inserting the above into (3.28) we establish the claimed equivalence of (B1) with (3.32)

$$\begin{aligned} t^{\mu}{}_{\nu} &= -e^{\mu}{}_{c}\frac{\partial\mathcal{L}_M}{\partial(\nabla_c\phi^A)}\nabla_{\nu}\phi^A + \delta^{\mu}_{\nu}\mathcal{L}_M \\ &= -\frac{\partial\mathcal{L}_M}{\partial(\nabla_{\mu}\phi^A)}\nabla_{\nu}\phi^A + \delta^{\mu}_{\nu}\mathcal{L}_M \equiv t^{\mu}_{c\,\nu}. \end{aligned} \tag{B9}$$

## APPENDIX C: PROOF OF EQ. (3.35)

Let us start with (3.26) and use the chain rule

$$t^{a}{}_{\mu}(x) := -\frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta e^{\mu}{}_{a}(x)} = -\frac{1}{\sqrt{-g}}\int d^dy\left\{\frac{\delta S_{\mathrm{M}}}{\delta g^{\kappa\lambda}(y)}\frac{\delta g^{\kappa\lambda}(y)}{\delta e^{\mu}{}_{a}(x)} + \frac{\delta S_{\mathrm{M}}}{\delta\Gamma^{\rho}{}_{\kappa\lambda}(y)}\frac{\delta\Gamma^{\rho}{}_{\kappa\lambda}(y)}{\delta e^{\mu}{}_{a}(x)}\right\}. \tag{C1}$$

To proceed we employ $g^{\kappa\lambda} = \eta^{ab}e^{\kappa}{}_{a}e^{\lambda}{}_{b}$ and (3.27),[20] yielding the expression for $t^{\nu}{}_{\mu} = t^{a}{}_{\mu}e^{\nu}{}_{a}$,

$$t^{\nu}{}_{\mu} = T^{\nu}{}_{\mu} + \frac{1}{2}\Delta_{\mu}{}^{\kappa\lambda}\Gamma^{\nu}{}_{\kappa\lambda} - \frac{1}{2}\Delta_{\rho}{}^{\nu\lambda}\Gamma^{\rho}{}_{\mu\lambda} - \frac{1}{2}\Delta_{\mu}{}^{\kappa\lambda}e^{\nu}{}_{a}\partial_{\lambda}e^{a}{}_{\kappa} + \frac{1}{2\sqrt{-g}}\partial_{\rho}(\sqrt{-g}\Delta_{\mu}{}^{\kappa\rho}e^{a}{}_{\kappa})e^{\nu}{}_{a}, \tag{C2}$$

where we have used the definitions (3.3) and (3.7). To proceed we use (A1) for $\Delta_{\mu}{}^{\nu\rho}$ yielding after some algebra

$$t^{\nu}{}_{\mu} = T^{\nu}{}_{\mu} + \frac{1}{2}\nabla_{\lambda}\Delta_{\mu}{}^{\nu\lambda} + \frac{1}{2}(\partial_{\rho}\ln\sqrt{-g} - \Gamma^{\lambda}{}_{\rho\lambda})\Delta_{\mu}{}^{\nu\rho}. \tag{C3}$$

Then, using (A3) and (A7), we obtain

$$\Gamma^{\lambda}{}_{\rho\lambda} = \frac{1}{2}g^{\lambda\sigma}\partial_{\rho}g_{\lambda\sigma} + \frac{1}{2}Q_{\rho} + 2S_{\rho} \Rightarrow \partial_{\rho}\ln\sqrt{-g} - \Gamma^{\lambda}{}_{\rho\lambda} = \nabla_{\rho}\ln\sqrt{-g} - 2S_{\rho}, \tag{C4}$$

where

$$\frac{1}{2}g^{\lambda\sigma}\partial_{\rho}g_{\lambda\sigma} = \partial_{\rho}\ln\sqrt{-g}, \qquad Q_{\rho} = -2\nabla_{\rho}\ln\sqrt{-g}, \tag{C5}$$

which can easily be found through (A7) and (A11). Employing (C4) into (C3) we find

[20]Assuming that the spin-connection $\omega_{ab\mu}$ is independent of $e^{\mu}{}_{a}$ and using the identity $x\delta'(x) = -\delta(x)$.

$$t^{\nu}{}_{\mu} = T^{\nu}{}_{\mu} + \frac{1}{2\sqrt{-g}}\hat{\nabla}_{\lambda}(\sqrt{-g}\Delta_{\mu}{}^{\nu\lambda}), \tag{C6}$$

where we recall $\hat{\nabla}_{\lambda} = \nabla_{\lambda} - 2S_{\lambda}$, which is in agreement with (3.35).

## APPENDIX D: REVISITING KNOWN EXAMPLES

In this appendix we revisit the derivation of the canonical energy-momentum tensor and the corresponding improvement terms in various known examples.

### 1. Conformally coupled scalar

Let us consider the action of the conformally coupled scalar [6], given in (5.24) and restated here for the reader's convenience,

$$S_{\rm cs} = \int d^d x\sqrt{-g}\left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{8}\frac{d-2}{d-1}\tilde{R}\phi^2\right), \tag{D1}$$

where $(\partial\phi)^2 = g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi$ and $\tilde{R}$ is the Ricci scalar of the metric $g_{\mu\nu}$. The action is invariant under the Weyl transformation (5.10)

$$g_{\mu\nu} \mapsto e^{-2\Omega}g_{\mu\nu}, \qquad \phi \mapsto e^{\frac{d-2}{2}\Omega}\phi, \tag{D2}$$

where the Ricci scalar transforms as (5.15). Working out the gravitational stress energy tensor (3.3) we find

$$T_{\mu\nu} = \partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 + \frac{1}{4}\frac{d-2}{d-1}(\tilde{G}_{\mu\nu}\phi^2 + g_{\mu\nu}\tilde{\nabla}^2\phi^2 - \tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\phi^2), \tag{D3}$$

where $\tilde{\nabla}$ is the covariant derivative constructed out of the Levi-Civita tensor for the metric $g_{\mu\nu}$ and $\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R}$ is the corresponding Einstein tensor. This tensor is conserved and traceless upon using the equations of motion of (5.24)

$$\Delta_2\phi = 0, \qquad \Delta_2 = \tilde{\nabla}^2 - \frac{1}{4}\frac{d-2}{d-1}\tilde{R}, \tag{D4}$$

where $\Delta_2$ is the conformal Laplacian or the Yamabe operator. Under the Weyl transformation (D2) the Yamabe operator transforms as

$$\Delta_2\phi \mapsto e^{\frac{d+2}{2}\Omega}\Delta_2\phi, \tag{D5}$$

yielding the Weyl invariant action

$$S_{\rm cs} = \frac{1}{2}\int d^d x\sqrt{-g}\phi\Delta_2\phi, \tag{D6}$$

which up to a boundary term is equivalent to (5.24). Finally, upon taking the flat limit in (D3) we readily find

$$T_{\mu\nu} = \partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 + \frac{1}{4}\frac{d-2}{d-1}(\eta_{\mu\nu}\partial^2\phi^2 - \partial_{\mu}\partial_{\nu}\phi^2), \tag{D7}$$

which coincides with (5.6).

### 2. The electromagnetic field

Let us consider the Maxwell field whose action is given by (4.1) and the energy-momentum tensor $t^{\mu}{}_{\nu}$ is given by (4.3) that is not symmetric, although the action (4.1) is invariant under Lorentz transformation. To symmetrize (4.3) we follow BR procedure (see [52]), and we demand at first that the Maxwell action (4.1) is invariant under local Lorentz transformations (2.10) under which

$$\delta_{\omega}A^{\mu} = -\omega^{\mu}{}_{\nu}A^{\nu}. \tag{D8}$$

Doing so we find the spin-current (3.42) where

$$\Sigma^{\mu\nu\rho} = A^{\rho}F^{\mu\nu} - A^{\nu}F^{\mu\rho}, \tag{D9}$$

obeying (3.9) with the hypermomentum given in (4.10). Starting from (2.13) and using (3.44), (4.3), and (D9) we find

$$T^{\mu\nu}_{\rm BR} = t^{\mu\nu} - \partial_{\lambda}(A^{\nu}F^{\mu\lambda}) = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}, \tag{D10}$$

which coincides with (4.5).

Equivalently we could have considered the Maxwell field in a curved background, whose action is given by

$$S_{\rm EM} = \int d^d x\sqrt{-g}\mathcal{L}_{\rm EM}, \qquad \mathcal{L}_{\rm EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{D11}$$

and upon using (3.3) and then restricting to flat we find agreement with (D10).

### 3. Dirac field

Let us consider the massive Dirac field in a $d$-dimensional spacetime described by the Hermitian action, the flat limit of (4.22)

$$S_{\rm D} = \int d^d x\mathcal{L}_{\rm D},$$
$$\mathcal{L}_{\rm D} = -\frac{i}{2}(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\gamma^{\mu}\psi) - im\bar{\psi}\psi, \quad \bar{\psi} = \psi^{\dagger}\gamma^0, \tag{D12}$$

whose equations of motion read

$$(\gamma^{\mu}\partial_{\mu} + m)\psi = 0, \quad \partial_{\mu}\bar{\psi}\gamma^{\mu} - m\bar{\psi} = 0. \tag{D13}$$

Demanding invariance under a local translation $\delta_{\varepsilon}x^{\mu} = -\varepsilon^{\mu}(x)$, we easily find

$$\delta_\varepsilon S_{\rm D} = -\int d^d x \partial_\mu \varepsilon_\nu t^{\mu\nu}, \tag{D14}$$

where the off-shell conserved canonical energy-momentum tensor is given by

$$t^{\mu\nu} = \frac{i}{2}(\bar\psi\gamma^\mu\partial^\nu\psi - \partial^\nu\bar\psi\gamma^\mu\psi) + \eta^{\mu\nu}\mathcal{L}_D, \tag{D15}$$

also read through (2.6). Upon using the equations of motion (D13) it simplifies to

$$t^{\mu\nu} = \frac{i}{2}(\bar\psi\gamma^\mu\partial^\nu\psi - \partial^\nu\bar\psi\gamma^\mu\psi). \tag{D16}$$

This is not symmetric despite the action (D12) being invariant under Lorentz transformations. To obtain a symmetric stress energy tensor we follow BR procedure and we demand at first that the Dirac action (D12) is invariant under local Lorentz transformations (2.10) where the spinor transforms as (4.26). Demanding that the Dirac action (D12) is invariant under local Lorentz transformations (2.10)

$$\delta_\alpha S_{\rm D} = -\frac{1}{2}\int d^d x \partial_\mu \omega_{\nu\rho}(J^\mu)^{\nu\rho}, \tag{D17}$$

where the conserved spin currents $\partial_\mu(J^\mu)^{\nu\rho} = 0$ are given by

$$(J^\mu)^{\nu\rho} = t^{\mu\nu}x^\rho - t^{\mu\rho}x^\nu - \Sigma^{\mu\nu\rho}, \tag{D18}$$

where

$$\Sigma^{\mu\nu\rho} = \frac{i}{2}\bar\psi(\gamma^\mu\Sigma^{\nu\rho} + \Sigma^{\nu\rho}\gamma^\mu)\psi, \tag{D19}$$

which coincides with the hypermomentum (4.28)

$$\Sigma^{\mu\nu\rho} = \Delta^{\rho\mu\nu}, \tag{D20}$$

trivially satisfying (3.9), since it is totally antisymmetric due to (4.29).

Finally, it can easily be seen that the BR stress energy tensor $T^{\mu\nu}_{\rm BR}$ defined in (2.13) coincides with the tensor in (4.32). Indeed, starting from (2.13), (3.44), and (3.37) we easily find

$$\begin{aligned} T^{\mu\nu}_{\rm BR} &= t^{\mu\nu} + \frac{1}{2}\partial_\lambda\Sigma^{\lambda\mu\nu} = \tau^{\mu\nu} - \frac{1}{2}\partial_\lambda\Delta^{\nu\mu\lambda} = T^{\mu\nu} \\ &= \frac{i}{4}(\bar\psi\gamma^\mu\partial^\nu\psi - \partial^\nu\bar\psi\gamma^\mu\psi) + \frac{i}{4}(\bar\psi\gamma^\nu\partial^\mu\psi - \partial^\mu\bar\psi\gamma^\nu\psi), \end{aligned} \tag{D21}$$

where in the last step we have used (4.32). In passing we note that the derived BR stress energy tensor is given by the symmetrization of the canonical one (D16), where the additional term is on-shell conserved.

### 4. Nonunitary scalar CFTs

Let us consider the non-unitary scalar CFT (conformal field theory) described by the action (4.34), and the canonical energy-momentum tensor $t_{\mu\nu}$ is given by (4.36) that is not symmetric, despite the action being invariant under Lorentz transformations. To symmetrize (4.36) we follow BR procedure, and we demand at first that the action (4.34) is invariant under local Lorentz transformations. Doing so we find the on-shell conserved spin current (3.42) where

$$\Sigma^{\mu\nu\rho} = \partial^2\phi(\eta^{\mu\nu}\partial^\rho\phi - \eta^{\mu\rho}\partial^\nu\phi), \tag{D22}$$

obeying (3.9), where the hypermomentum was given in (4.42). Finally, starting from (2.13) and using (3.44) and (D22), we reach (4.37), which is restated here for the reader's convenience,

$$\begin{aligned} T^{\mu\nu}_{\rm BR} &= 2\partial^\mu\partial^\nu\phi\partial^2\phi - \frac{1}{2}\eta^{\mu\nu}\partial^2\phi\partial^2\phi - \partial^\mu(\partial^2\phi\partial^\nu\phi) \\ &\quad - \partial^\nu(\partial^2\phi\partial^\mu\phi) + \eta^{\mu\nu}\partial_\lambda(\partial^2\phi\partial^\lambda\phi). \end{aligned} \tag{D23}$$

## APPENDIX E: IMPROVED NOETHER CURRENT

In this appendix, we will consider improvements of the conserved currents where analog considerations have been considered in [5] and see also [14–19].

Let us consider the action in flat spacetime

$$S = \int d^d x \mathcal{L}(x), \tag{E1}$$

and varying $x^\mu$ as

$$x^\mu \mapsto x'^\mu = x^\mu - \xi^\mu(x), \tag{E2}$$

we find that the action transforms infinitesimally as

$$S \mapsto S + \int d^d x \partial_\mu(\xi^\mu\mathcal{L}). \tag{E3}$$

Alternatively, we may consider that the Lagrangian density $\mathcal{L}$ is a function of fields $\phi$ (not necessarily scalars) and their (first-order) derivatives $\mathcal{L} = \mathcal{L}(\phi, \partial\phi)$. The fields $\phi$ vary under the transformation (E2) infinitesimally as

$$\delta_\xi\phi = \phi'(x) - \phi(x) = \xi^\mu\partial_\mu\phi + \hat\delta_\xi\phi, \tag{E4}$$

where $\hat\delta_\xi$ includes any tensorial transformations of $\phi$.[21] Under the latter the action transforms as the Lie derivative

[21]Like as, a covariant or a contravariant tensor transforms as

$$\delta_\xi A_\mu = A'_\mu(x) - A_\mu(x) = \xi^\nu\partial_\nu A_\mu + \partial_\mu\xi^\nu A_\nu,$$
$$\delta_\xi B^\mu = B^{\mu'}(x) - B^\mu(x) = \xi^\nu\partial_\nu B^\mu - B^\nu\partial_\nu\xi^\mu,$$

as it can easily be proved by varying $A = A_\mu(x)dx^\mu$ and $B = B^\mu(x)\partial_\mu$.

$$S \mapsto S + \int d^d x\, \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\xi \phi \right), \tag{E5}$$

upon using the equations of motion. Matching the variations (E3) and (E5) and also using (E4), we find the generalized Noether current $S^\mu$

$$\partial_\mu S^\mu = 0, \qquad S^\mu = \xi^\nu t^\mu{}_\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \hat{\delta}_\xi \phi, \tag{E6}$$

where $t^\mu{}_\nu$ is the canonical conserved stress energy tensor given by

$$t^\mu{}_\nu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \delta^\mu{}_\nu \mathcal{L}. \tag{E7}$$

For translations $\xi^\mu = a^\mu$ and a scalar field $\phi$ Eq. (E6) leads to the conservation of the canonical stress energy tensor

$$\partial_\mu t^\mu{}_\nu = 0. \tag{E8}$$

In what follows we will work out three known cases where the additional term in (E6) contributes.

### 1. Scale

First, we consider the free scalar (5.1) for scale transformations

$$x'^\mu = \lambda^{-1} x^\mu, \qquad \xi^\mu = \varepsilon x^\mu,$$
$$\phi'(x') = \det\left(\frac{\partial x'}{\partial x}\right)^{-\frac{\Delta_\phi}{d}} \phi(x) = \lambda^{-\Delta_\phi} \phi(x), \tag{E9}$$

where $\lambda = 1 + \varepsilon$ and $\Delta_\phi = \frac{1}{2}(d-2)$. The additional term in (E4) can easily be found to be

$$\hat{\delta}_\xi \phi = -\varepsilon \Delta_\phi \phi \tag{E10}$$

and upon inserting into (E6) we obtain the on-shell conserved current

$$\partial_\mu S^\mu = 0, \qquad S^\mu = x^\nu t^\mu{}_\nu + \frac{1}{2} \Delta_\phi \partial^\mu \phi^2, \tag{E11}$$

where $t^{\mu\nu}$ was given in (5.2) and the current precisely matches (5.3) and (5.4).

### 2. Special conformal

Second, we consider the action of special conformal transformations on the free scalar (5.1),

$$x'^\mu = \Omega^{-1}(x^\mu + b^\mu x^2), \qquad ds'^2 = \eta_{\mu\nu} dx'^\mu dx'^\nu = \Omega^{-2} \eta_{\mu\nu} dx^\mu dx^\nu,$$
$$\phi'(x') = \det\left(\frac{\partial x'}{\partial x}\right)^{-\frac{\Delta_\phi}{d}} \phi(x) = \Omega^{-\Delta_\phi} \phi(x), \qquad \Omega = 1 + 2b \cdot x + x^2 b^2, \tag{E12}$$

where $\Delta_\phi = \frac{1}{2}(d-2)$. The additional term in (E4) can easily be found to be

$$\hat{\delta}_\xi \phi = -2\Delta_\phi x_\nu b^\nu \phi, \qquad \xi^\mu = b^\mu x^2 - 2 b^\nu x^\mu x_\nu. \tag{E13}$$

Upon using the latter in the condition (E6), we find that it is not on-shell conserved, satisfying

$$\partial_\mu S^\mu = -\Delta_\phi b^\mu \partial_\mu \phi^2 \quad \text{where } S^\mu = \xi^\rho t^\mu{}_\rho - \Delta_\phi \partial^\mu \phi^2 b^\rho x_\rho. \tag{E14}$$

Yet this divergence can be readily written as a conserved quantity

$$\partial_\mu (S^\mu + \Delta_\phi b^\mu \phi^2) = 0$$
$$\Rightarrow \partial_\mu (x^2 t^\mu{}_\rho - 2 x^\nu t^\mu{}_\nu x_\rho + 2(\partial_\nu Z^{\mu\nu} x_\rho - Z^\mu{}_\rho)) = 0, \tag{E15}$$

where

$$Z^{\mu\nu} = -\frac{1}{2} \Delta_\phi \eta^{\mu\nu} \phi^2, \tag{E16}$$

which is in agreement with (3.46) and (5.5).

The origin of the additional term can be understood upon transforming the free scalar (5.1) under the special conformal transformations (E12), which is invariant but up to a boundary term [4]

$$S_{\text{scalar}} = -\frac{1}{2} \int d^d x \partial_\mu \phi \partial^\mu \phi \Rightarrow$$
$$S'_{\text{scalar}} = -\frac{1}{2} \int d^d x \Omega^{2-d} \partial_\mu (\Omega^{\Delta_\phi} \phi) \partial^\mu (\Omega^{\Delta_\phi} \phi)$$
$$= -\frac{1}{2} \int d^d x \{ (\partial\phi)^2 + \Delta_\phi \partial_\mu \ln\Omega \partial^\mu \phi^2 + \Delta_\phi^2 (\partial \ln\Omega)^2 \phi^2 \}$$
$$= -\frac{1}{2} \int d^d x \{ (\partial\phi)^2 + \Delta_\phi \partial_\mu (\partial^\mu \ln\Omega \phi^2) \}$$
$$= S_{\text{scalar}} - \frac{\Delta_\phi}{2} \int d^d x \partial_\mu (\phi^2 \partial^\mu \ln\Omega), \tag{E17}$$

where in the second to last equation we have used the following identity that is valid for the special conformal transformations (E12):

$$\Delta_\phi(\partial \ln \Omega)^2 = \partial^2 \ln \Omega. \tag{E18}$$

### 3. Lorentz

Third, we consider the Maxwell field (4.1) which is invariant under Lorentz transformations

$$\xi^\nu = \omega^\nu{}_\rho x^\rho, \qquad \hat{\delta}_\xi A_\mu = -\omega_\mu{}^\nu A_\nu, \tag{E19}$$

where $\omega_{\mu\nu}$ is an antisymmetric tensor. Using the above and (4.3) in (E6) we find

$$\partial_\mu J^{\mu\nu\rho} = 0, \qquad J^{\mu\nu\rho} = x^\rho t^{\mu\nu} - x^\nu t^{\mu\rho} + F^{\mu\rho}A^\nu - F^{\mu\nu}A^\rho, \tag{E20}$$

which is in agreement with (3.42) and (D9).


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