

Gluon condensate via Dirac spectral density: IR phase, scale anomaly, and IR decoupling

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Quark and gluon scalar densities, $\langle\bar{\psi}\psi\rangle$ and $\langle F^2\rangle$, reflect the degree of scale-invariance violations in $SU(N)$ gauge theories with fundamental quarks. It is known that $\langle\bar{\psi}\psi\rangle$ can be usefully scale-decomposed via the spectral density $\rho(\lambda)$ of Dirac modes. Here, we give such formula for $\langle F^2\rangle$, which reveals that the gluon condensate is a strictly UV quantity. For the recently found IR phase [1,2], where the infrared (IR) degrees of freedom separate out and become independent of the system's bulk, it implies that $\langle F^2\rangle$ due to this IR part vanishes. Its glue, thus, does not contribute to the scale anomaly of the entire system and is, in this sense, scale invariant consistently with the original claim. Associated formulas are used to define the IR decoupling of glue, which may serve as an alternative indicator of IR phase transition. Using the simplest form of coherent lattice QCD, we express the effective action of full QCD entirely via Dirac spectral density.

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I. INTRODUCTION

Quark and gluon scalar densities—namely, the expectation values $\langle\bar{\psi}\psi\rangle$ and $\langle F^2\rangle$ —play an important role in the dynamics of QCD. For example, in hadronic physics, they are well known as leading “condensates” entering the QCD sum rules [3]. At a more basic level, they are connected to fundamental symmetries, which makes them objects of prime interest for understanding the phases of nature's strong interactions and, more generally, the phases in a large class of theories where quarks and gluons interact in the same manner. For purposes of the present discussion, this class $\mathcal{T}$ will consist of asymptotically free vectorlike $SU(3)$ gauge theories with fundamental quarks. Thus, $\mathcal{T}$ involves theories with an arbitrary number $N_f < 16.5$ of fundamental quark flavors with arbitrary masses m_q , $q = 1, 2, \dots, N_f$ and at any temperature, including $T = 0$.

Viewed through the lens of scalar densities, each theory from $\mathcal{T}$ is characterized by $N_f + 1$ values $\langle F^2\rangle$, $\langle\bar{\psi}\psi_q\rangle$. But even their hypothetical full knowledge would not provide us with an understanding and classification of the phases in the entire $\mathcal{T}$. Indeed, the conventional symmetry-based

apparatus for such analyses simply does not have a full reach within $\mathcal{T}$. One useful but limited classification is usually made within the subclass of theories containing multiple massless flavors, where $\langle\bar{\psi}\psi\rangle$ indicates the spontaneous breakdown of the associated flavored chiral symmetry. Similar goes for considerations concerning the anomalous nature of flavor-singlet chiral symmetry. While these special circumstances and their implications are important for understanding certain aspects of low-energy “real-world” QCD, they are in themselves unlikely to reveal the phase structure of the entire $\mathcal{T}$. The situation is similar (but also different; see below) in the case of dilations where full-scale invariance requires quarks to be massless and temperature to be zero.

A dramatic departure from the traditional and purely symmetry-based considerations appear in Refs. [1,2] which, together with refinements in Refs. [4,5] and an additional state-of-the-art numerical evidence from Refs. [6–8], have led to the classification of phases in the entire $\mathcal{T}$. This primarily stemmed from the finding that systems in $\mathcal{T}$ can partition their degrees of freedom and become multicomponent. More precisely, there is a dynamical regime within $\mathcal{T}$ where deep infrared (IR) field fluctuations proliferate, separate out, and decouple from the rest of finite-energy fundamental fields (bulk). Such IR-bulk separation, or the property of being multicomponent in general, does not depend on symmetries and can occur anywhere in $\mathcal{T}$. The new regime became known as the *IR phase* of $\mathcal{T}$ and was associated with the restored scale invariance of glue in the IR component [2].

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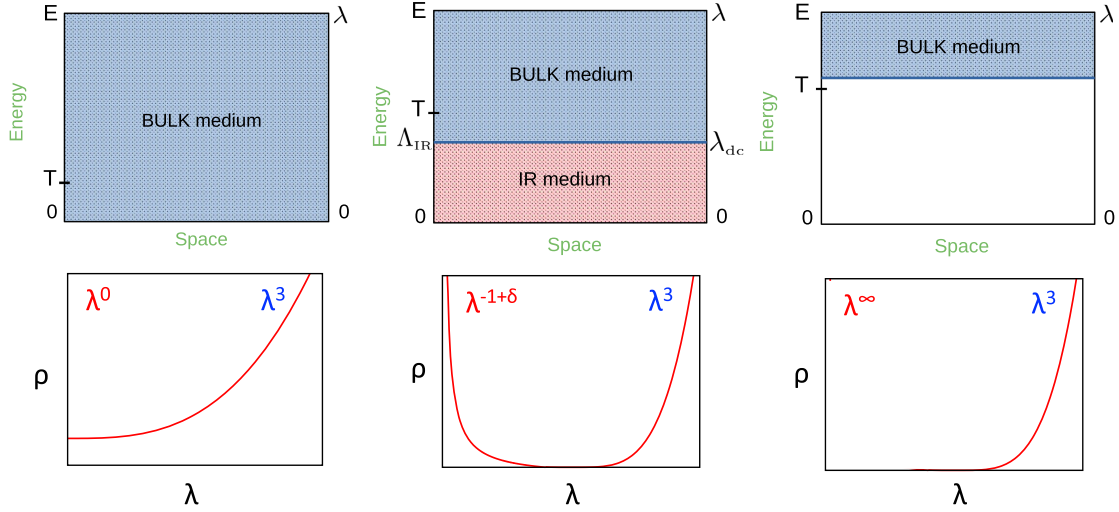


FIG. 1. Types of thermal states for theories in T and schematics of their Dirac spectral densities $\rho(\lambda)$. Here, λ is the Dirac eigenvalue (scale) in the continuum-like notation where $D\psi_\lambda = i\lambda\psi_\lambda$. Left: the B phase (standard confined phase) involves a single-component system with correlated parts. Its leading IR power behavior λ^0 includes cases when density is logarithmically divergent. Middle: the IR phase involves a multicomponent system with IR separated and decoupled from the bulk. Λ_{IR} is the energy scale of IR-bulk separation, and λ_{dc} is the associated Dirac scale. Right: the hypothetical UV phase describes a single-component system of weakly interacting quarks and gluons.

IR-bulk separation occurs in the Dirac space of theories in T , as shown schematically by Fig. 1 in the setting of thermal transitions [2]. The theory that “confines” at $T = 0$ (left), such as pure-gluon QCD or real-world QCD, describes a single-component system at low T (blue bulk) and features field fluctuations correlated over any pair of Dirac (λ) and physical energy (E) scales. At a certain temperature T_{IR} , thermal fluctuations become strong enough to induce the creation of an independent IR component (red medium) whose degrees of freedom do not participate in the makeup of hadrons (“partial deconfinement”), and the IR phase ensues. Yet stronger disorder at $T > T_{\text{UV}} > T_{\text{IR}}$ may inhibit deep-IR ($\lambda \neq 0$) degrees of freedom, with the system becoming single-component again, possibly as a “deconfined” perturbative bulk. It is worth noting that, unlike the first two cases, this classic scenario for weakly coupled quark-gluon plasma has not yet been clearly identified in QCD, and its existence is uncertain. Schematics of Dirac spectral density $\rho(\lambda)$ in Fig. 1 convey how quark degrees of freedom are distributed across scales in the three regimes.

Since the separation of components is scale based, it is desirable—if at all possible—to think of scalar densities as composed of scale-dependent parts. Note that spectral density already partitions degrees of freedom by scale. This aspect was thus innately present in the arguments of the original works on IR phase [1,2], including arguments for the scale invariance of the IR component. However, a more formal treatment does need a scale decomposition of scalar densities, since departures from scale invariance due to the glue field and the quark field of mass m are quantified by (scale anomaly) [9–11]

$$T_{\mu\mu} = \frac{\beta(g)}{2g} \langle F^2 \rangle + (1 + \gamma_m(g)) m \langle \bar{\psi}\psi \rangle, \quad (1)$$

and only scale decompositions of $\langle F^2 \rangle$ and $m \langle \bar{\psi}\psi \rangle$ can isolate the contribution of a component.¹ To that end, we point out here the usefulness of the following expressions:

$$\begin{aligned} -m \langle \bar{\psi}\psi \rangle &= m \int_{\mathbb{R}^2[\mathbb{C}]} dS \frac{1}{\lambda + m} \rho_s^{\text{ef}}(\lambda), \\ \langle F^2 \rangle &= \frac{a}{c_S} \int_{\mathbb{R}^2[\mathbb{C}]} dS \lambda \rho_s^{\text{ef}}(\lambda), \end{aligned} \quad (2)$$

with $\lambda = \lambda_R + i\lambda_I$ being a complex eigenvalue of the lattice Dirac operator D defining quark dynamics in a regularized theory (UV cutoff $1/a$; IR cutoff $1/L$). This D is chosen such that the associated dimensionless constant c_S is nonzero (see below).² The quantity $\rho_s(\lambda) = \langle n(\lambda, dS) \rangle / (V_4 dS)$ is the *surface spectral density* of D , with $n(\lambda, dS)$ being the number of eigenvalues in the area $dS = d\lambda_R d\lambda_I$ around λ , and $V_4 = L^3/T$. The dimension of $\rho_s(\lambda)$ is a^{-2} instead of a^{-3} in the standard definition. The effective density $\rho_s^{\text{ef}} \equiv \rho_s - \rho_{s0}$ subtracts that of the free field.³ Note that $\rho_s^{\text{ef}} = \rho_s^{\text{ef}}(\lambda, a, L)$, and $\int_{\mathbb{R}^2[\mathbb{C}]} dS \rho_s^{\text{ef}}(\lambda) = 0$.

¹Notation here is standard, with T being the energy-momentum tensor and β, γ_m the conventional RG functions.

²Denoting the integration domain $\mathbb{R}^2[\mathbb{C}]$ conveys that components of $\mathbb{R}^2$ have meaning in the complex plane $\mathbb{C}$.

³We will see that in the glue case, the free-field subtraction is necessary to merely define the operator. In the quark case, it removes the leading UV divergence at nonzero m , which keeps the two expressions symmetric in this sense.

Albeit not in this form, the quark relation in Eq. (2) appeared in considerations leading to the Banks-Casher relation [12]. The glue expression is new and exploits the ideas put forward in Refs. [13–15]. Together, they expose, in an explicit way, the key difference between the two scalar densities in terms of their scale makeup: while the kinematic factor $1/(\lambda + m)$ enhances IR and suppresses UV in the quark case, its counterpart λ does the exact opposite in the glue case. In fact, the glue expression invokes a surprising possibility that the gluon condensate is an entirely UV quantity. We will show how Eq. (2) clarify the proposed relation of the IR phase to scale invariance [1,2], and how to use the associated formulas to characterize its IR-bulk decoupling. We make all these results solid by writing the corresponding expressions [Eqs. (7) and (8)] for the important case of the overlap Dirac operator.

II. GLUON CONDENSATE

We now derive the expression for gluon condensate in Eq. (2). To that end, consider the Euclidean lattice setup for theories in $\mathcal{T}$, consisting of $N_s^3 \times N_\tau$ sites of a hypercubic lattice with UV cutoff $1/a$, IR cutoff $1/L$ ($L = N_s a$), and temperature T ($1/T = N_\tau a$). Lattice spacing a is controlled by the gauge coupling g , but this relationship can be kept

$$F^2(x, U) \equiv \frac{1}{c_s a^3} \text{tr}_{cs} [D_{x,x}(U) - D_{x,x}(\mathbb{I})]$$

where “Tr” denotes the full trace of the Dirac matrix. When performing the QCD average, we replaced $F^2(x)$ with $(\sum_y F^2(y))/(N_s^3 N_\tau)$, permitted by virtue of hypercubic translation invariance. For general Ginsparg-Wilson operators [16], nonzero c_s is expected due to their nonultra-locality [17–19]. In the case of overlap Dirac operators [20], this was shown via explicit computation in Ref. [15].

Unlike in the continuum, eigenvalues λ_j of lattice D are not purely imaginary and details of the spectrum vary in different formulations. To cover all possibilities, we introduced in Sec. I the surface spectral density $\rho_s(\lambda)$ of eigenvalues in the complex plane. This density is usefully represented as $\rho_s(\lambda, U) = \sum_j \delta(\lambda_R - \lambda_R^j) \delta(\lambda_I - \lambda_I^j) / V_4$, with $\lambda = \lambda_R + i\lambda_I$, $\lambda^j = \lambda^j(U)$, and $V_4 = L^3/T$. Then,

$$\langle \text{Tr} D \rangle = V_4 \int_{\mathbb{R}^2[\mathbb{C}]} d\mathcal{S} \rho_s(\lambda), \quad \rho_s(\lambda) \equiv \langle \rho_s(\lambda, U) \rangle, \quad (5)$$

which consequently turns the expression for gluon condensate in Eq. (4) into one in Eq. (2), as desired.

implicit here. Let $U \equiv \{U_\mu(x)\}$ be a configuration of the SU(3) gauge field and $D = D(U)$ a lattice Dirac operator. The latter is explicitly assumed to be local, gauge covariant, and to respect hypercubic symmetries. Local *gauge operators* of definite space-time transformation properties can then be constructed from D via suitable tracing operations [13,14]. In the case of $F^2(x) \equiv \text{tr}_c F_{\mu\nu} F_{\mu\nu}(x)$, one starts from [13–15]

$$\text{tr}_{cs} \hat{D}_{x,x}(U) - \text{tr}_{cs} \hat{D}_{x,x}(\mathbb{I}) = c_s a^4 \text{tr}_c F_{\mu\nu} F_{\mu\nu}(x, A) + \mathcal{O}(a^6) \quad (3)$$

for transcription U of a classical continuum gauge field A onto the hypercubic lattice with spacing a , and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$. Here, $\mathbb{I}$ denotes the free field $U_\mu(x) \equiv \text{diag}\{1, 1, 1\}$, and tr_c is the trace in color while tr_{cs} is the trace in color-spin, and c_s is a constant. Note that $\hat{D} = aD$ is dimensionless, but $F_{\mu\nu}$ has its physical dimension. (In Refs. [13–15], D denotes the operator in lattice units.) Equation (3) implies that, if $c_s \neq 0$, quantum operator $F^2(x)$ can be defined via indicated Dirac matrix elements, namely

$$\longrightarrow \langle F^2 \rangle = \frac{a}{c_s L^3} \langle \text{Tr}[D(U) - D(\mathbb{I})] \rangle, \quad (4)$$

III. GLUON CONDENSATE WITH OVERLAP OPERATORS

To analyze the general implications of Eq. (2), it is desirable to examine them for operators D that mimic continuum features to the largest extent possible. To that end, we first note that neither quark nor gluon lattice condensate is guaranteed to be real-valued *a priori*. A natural way to ensure this is to impose γ_5 -Hermiticity—namely, $\gamma_5 D \gamma_5 = D^\dagger$ —which forces eigenvalues to appear in complex-conjugate pairs, and physically amounts to continuum-like treatment of anti-particles. We will thus assume that D is γ_5 -Hermitian from now on.

It is also desirable to ensure continuum-like chiral properties on the lattice. At present, this amounts to using Ginsparg-Wilson (GW) Dirac operators [16].⁴ Among them, the one-parameter family of *overlap operators* D based on the Wilson-Dirac matrix [20] has been studied most extensively. They are given by

⁴See, e.g., Refs. [18,19] for full specification of the class.

$$\frac{a}{\Delta}D = 1 + \frac{\hat{D}_W - \Delta}{\sqrt{(\hat{D}_W - \Delta)^\dagger(\hat{D}_W - \Delta)}}, \quad \Delta \in (0, 2), \quad (6)$$

where $\hat{D}_W$ is the dimensionless, massless Wilson-Dirac operator. In Ref. [15], it was shown that $c_S = c_S(\Delta) \neq 0$ for the above $D = D(\Delta)$. They can thus be used to express the gluon condensate via Eq. (2).

To that end, note that the γ_5 -Hermitian spectrum of D traces the circle of radius Δ centered at $(\Delta, 0)$, so that the eigenvalues $\lambda_R + i\lambda_I$ satisfy $a(\lambda_R^2 + \lambda_I^2) = 2\lambda_R\Delta$. The expression involving spectral density along the circle is obtained via substituting $\rho_s(\sigma \cos \varphi, \sigma \sin \varphi) = \rho(\sigma)\delta(\varphi - \cos^{-1}(a\sigma/2\Delta))/\sigma$ upon transfer to polar coordinates in Eq. (2). We obtain

$$\langle F^2 \rangle_{a,L} = \frac{a^2}{c_S \Delta} \int_0^{(\frac{2\Delta}{a})^-} d\sigma \sigma^2 \rho^{\text{ef}}(\sigma, a, L) + T \frac{\langle n_0 \rangle_{a,L} 2\Delta}{L^3 c_S}, \quad (7)$$

where $\rho^{\text{ef}} = \rho - \rho_0$ has been defined previously and $\langle n_0 \rangle$ is the average number of exact zero modes equal to the number of modes with real eigenvalue $2\Delta/a$. While the former do not contribute to $\langle F^2 \rangle$, the discrete contribution of the latter was separated out. In the continuum limit, the energy-like variable σ (magnitude of lattice eigenvalue) coincides with λ of the continuum Euclidean formulation, where eigenvalues are parametrized by $i\lambda$. Thus, $\rho(\sigma)$ is associated with the upper branch ($\lambda \geq 0$) of the continuum density. The analogous derivation for the quark expression in Eq. (2) leads to

$$\begin{aligned} -m \langle \bar{\psi} \psi \rangle_{a,L} &= \int_0^{(\frac{2\Delta}{a})^-} d\sigma \frac{m^2 + m^2 + a\sigma^2 m/\Delta}{m^2 + \sigma^2 + a\sigma^2 m/\Delta} \rho^{\text{ef}}(\sigma, a, L) \\ &+ T \frac{\langle n_0 \rangle_{a,L}}{L^3} \left(1 + \frac{am}{2\Delta + am} \right). \end{aligned} \quad (8)$$

Here, all real modes contribute and are separated out in the second term of the expression. We emphasize that Eqs. (7) and (8) are fully regularized scale decompositions of these quantities.

IV. THE USES: GLUON CONDENSATE AS A UV QUANTITY

The above implies that, in a well-defined sense, the gluon condensate in QCD is a UV quantity. The underlying logic is that the contribution of Dirac scales, up to any finite renormalized value λ_R , vanishes in the continuum limit. Indeed, writing the associated integral in Eq. (7) via renormalized quantities $\sigma_R = \sigma/Z_S$, $m_R = m/Z_S$, $\rho_R(\sigma_R) = Z_S \rho(Z_S \sigma_R)$ [21,22], we obtain

$$\begin{aligned} \langle F^2 \rangle_L[\lambda_R] &= \lim_{a \rightarrow 0} \frac{a^2 Z_S^2}{c_S \Delta} \int_0^{\lambda_R} d\sigma_R \sigma_R^2 \rho_R^{\text{ef}}(\sigma_R, a, L) = 0, \\ \forall \quad 0 < \lambda_R < \infty, \end{aligned} \quad (9)$$

where the notation $\langle F^2 \rangle[\lambda_R]$ means discarding all bare Dirac scales whose renormalized counterpart exceeds λ_R . The limit is zero, since ρ_R^{ef} , as well as the associated integral, has a well-defined continuum limit at fixed L , and since the a -dependence of Z_S is at most logarithmic. Note also that the real modes, separated in Eq. (7), contribute at the scale of UV cutoff and thus do not enter Eq. (9).

Two additional remarks are worth making. First, the above argument was made in the context of a finite system of size L . Regarding the infinite system, the same reasoning and conclusion applies for the order of limits $\lim_{L \rightarrow \infty} \lim_{a \rightarrow 0}$. For the reversed order, the first limit $\lim_{L \rightarrow \infty}$ is finite but a dependent, and it could in principle lead to such a UV divergence of the integral in Eq. (9) that would produce a nonzero contribution to gluon condensate. However, such UV divergence is unlikely to occur, since it would have to be generated by IR Dirac modes. We thus propose that the vanishing of any IR contribution to the gluon condensate holds for both ways of removing cutoffs.

Second, note that the situation is very different for the quark part of the trace anomaly. At finite, fixed L , we have from Eq. (8), upon taking the continuum limit,

$$\begin{aligned} -m \langle \bar{\psi} \psi \rangle_L[\lambda_R] &= 2 \int_0^{\lambda_R} d\sigma_R \frac{1}{1 + \sigma_R^2/m_R^2} \rho^{\text{ef}}(\sigma_R, L) \\ &+ T \frac{\langle n_0 \rangle_L}{L^3}, \end{aligned} \quad (10)$$

where the second term is due to zero modes. The IR contribution is clearly nonzero, with the second term vanishing in the $L \rightarrow \infty$ limit. The above form also shows that the quark contribution to the trace anomaly becomes a strictly IR quantity in the chiral limit, which is familiar from considerations leading to the Banks-Casher relation [12].

V. THE USES: IR SCALE INVARIANCE IN THE IR PHASE

In the original work [2], the IR phase was defined by the negative power in IR behavior of Dirac spectral density, and thus by a power-law enhancement of deep IR degrees of freedom. The associated classification of phases in set $\mathcal{T}$ is

$$\begin{aligned} \rho(\lambda) &\propto \lambda^p, \quad \lambda \rightarrow 0 \Rightarrow \\ \text{phase} &= \text{B if } p=0, \quad \text{IR if } p<0, \quad \text{UV if } p>0, \end{aligned} \quad (11)$$

where continuum-like notation ($\sigma \rightarrow \lambda$) was used, but the classification is already well defined on the lattice. In the IR phase, p is usefully written as $p = -1 + \delta$ because, at least in thermal cases, $\delta > 0$ is small and may vanish in the

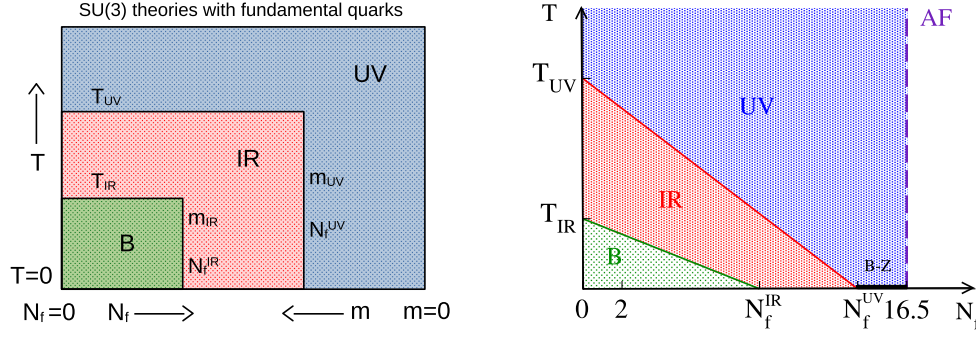


FIG. 2. Left: schematic view of phases in set $\mathcal{T}$ based on a degree of deep-IR degrees of freedom proliferation and their scale invariance. The direction of arrows for parameters indicates the direction of possible phase changes along the chain $B \rightarrow IR \rightarrow UV$ [2]. Right: the case of near-massless quarks with the Banks-Zaks (B-Z) regime and the asymptotic freedom (AF) boundary indicated.

continuum limit [2]. The “B” in “B phase” refers to the broken IR scale invariance of glue: the standard IR characteristic of “confined phase.” Here, spectral density is expected to logarithmically diverge away from the chiral limit [2,23], and thus $p = 0$. In the not-yet-observed UV phase, IR degrees of freedom are power-law suppressed with the power possibly infinite if spectral density is zero in deep IR. The associated phase structure of $\mathcal{T}$ is shown in Fig. 2 (left).

Two aspects [2] of the new IR phase are especially relevant for the present discussion.⁵ (i) Due to $p < 0$ in IR and $p = 3$ in UV (asymptotic freedom), with an intervening regime of severe mode depletion, the $\rho(\lambda)$ of theories in the IR phase exhibits a bimodal structure with the IR part separated from the bulk in scale/energy. This invoked the conjecture of IR-bulk decoupling, namely that the IR part is independent from the bulk and acts as an autonomous subsystem. (ii) Due to the near-pure negative-power behavior of ρ and the connection to the conformal window [2,25,26] [see Fig. 2 (right)], it was conjectured that the *glue* of the IR part is scale invariant, at least asymptotically. Thus, although defined via Dirac spectral properties, the physics behind the above structure of $\mathcal{T}$ was proposed to be driven by glue and interpreted in glue terms [1,2]. In that vein, the new Eq. (7) is perhaps the purest expression of the implied connection and allows us to express the key properties of IR phase, such as (i) and (ii), in more standard field-theoretic terms.

To that end, it is important to realize that (i) and (ii) are in fact connected [2,4]. For example, while the IR-bulk separation is very suggestive of IR-bulk decoupling, it is not sufficient for the claim. But (ii) offers a dynamical reason, which is easy to see in the thermal IR phase at $N_f = 0$. Indeed, assume that, upon entering the IR phase, this theory of glue becomes scale invariant below some energy $\Lambda_{IR} < T$. The gauge coupling then stops running at that scale, which requires nonanalyticities in the internal structure of the theory. These nonanalyticities can then also facilitate the

IR-bulk decoupling. Conversely, nonanalyticities generated by breaking off the IR into a physically distinct component can make scale invariance in IR possible.⁶

There are at least two aspects of scale invariance to study in these IR phase circumstances. The first one isolates the IR component as a field system, bringing $\Lambda_{IR} \rightarrow \infty$ by virtue of an overall rescaling, and it aims to study scale-invariant field theory so defined—e.g., in the strongly coupled part of the conformal window. The second one views the multicomponent system as a whole and seeks to understand the role its IR part plays in violations of scale invariance. Focusing on this second aspect of (ii), let $\lambda_{dc} = \lambda_{dc}(a)$ be the scale marking the IR-bulk boundary in the Dirac spectrum (see Fig. 1). The contribution $\langle F^2 \rangle_{IR}$ of the IR component (IR medium) to full $\langle F^2 \rangle$ is

$$\langle F^2 \rangle_{IR} \equiv \frac{a^2}{c_S \Delta} \int_0^{\lambda_{dc}(a)} d\sigma \sigma^2 \rho^{ef}(\sigma, a) \rightarrow 0 \text{ for } a \rightarrow 0. \quad (12)$$

Its approach to zero in the continuum limit ensues due to obvious integrability at any finite a , and because $\lambda_{dc}(a)$ varies at most logarithmically in the vicinity of $a = 0$. Hence, the IR component of the system in the IR phase does not contribute to scale anomaly and is, from this point of view, scale invariant. The above argument is generic for the IR phase of theories in $\mathcal{T}$, and it does not depend on whether L is kept fixed (finite) or taken to infinity at each a .

VI. THE USES: IR-BULK DECOUPLING IN THE IR PHASE

Aspects of the present analysis, and those of Refs. [13–15], can also be used to study IR-bulk decoupling in IR phase [point (i) in Sec. V]. The chief idea is that “decoupling” is identified with “decorrelation,” which turns such parts into

⁵See Ref. [24] for more detailed discussion of features that make the $p < 0$ regime a truly distinct phase.

⁶Thermal onset of these internal nonanalyticities is predicted to occur at $T = T_{IR}$ of IR phase transition [4,5], because they induce nonanalytic T dependence of physical observables at that point.

independent subsystems. Here, we analyze the glue part of the system which is our focus in this work. The full account of spectral correlations and decoupling will be given elsewhere.

Since the IR-bulk separation is based on Dirac scales, we need the notion of correlation among such scale-based parts. This requires analogues of expressions in previous sections, but for a given gauge background U . To that end, let us consider the action-like dimensionless construct $\mathcal{F}^2(U)$:

$$\begin{aligned}\mathcal{F}^2(U) &\equiv a^4 \sum_x F^2(x, U) = \frac{a}{c_S} \text{Tr}[D(U) - D(\mathbb{I})] \\ &= V_4 \frac{a}{c_S} \int_{\mathbb{R}^2[C]} dS \lambda \rho_s^{\text{ef}}(\lambda, U),\end{aligned}\quad (13)$$

written here for general $D(U)$ with $c_S \neq 0$, and where Eqs. (4) and (5) and their notation are used. For overlap Dirac operators, we then have a specific expression [analogous to (7)], which allows us to quantify the contribution to $\mathcal{F}^2(U)$ from the range of lattice Dirac scales σ ($0 \leq \eta_1 \leq \sigma \leq \eta_2 < 2\Delta/a$) as

$$\begin{aligned}\mathcal{F}^2(U, \eta_1, \eta_2) &\equiv \frac{V_4 a^2}{c_S \Delta} \int_{\eta_1}^{\eta_2} d\sigma \sigma^2 \rho^{\text{ef}}(\sigma, U) \\ &= \frac{a^2}{c_S \Delta} \sum_{\eta_1 < \sigma_j \leq \eta_2} [\sigma_j^2(U) - \sigma_j^2(\mathbb{I})],\end{aligned}\quad (14)$$

where the filter only allows values $\sigma_j(U)$ and (separately) $\sigma_j(\mathbb{I})$ in the specified range. Note that the formula for given non-negative η_1, η_2 in fact combines (equal) contributions from both the upper and lower branches of the Dirac spectrum. It is fully regularized and allows for defining correlations among different “parts” of the system with “partitions” based on Dirac spectra.

To formulate IR-bulk decoupling of glue in the IR phase, it is practical to make two preparatory steps. First, it is convenient to replace the variable σ with $t = \sigma/m$, where m may be, e.g., the smallest nonzero quark mass in the theory, because it does not get renormalized [21,22]. Second, rather than the usual connected correlation $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle_c \equiv \langle \mathcal{O}_1 \mathcal{O}_2 \rangle - \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle$, we consider the *normalized* connected Pearson correlation $\langle \dots \rangle_{\text{nc}}$ (covariance divided by standard deviations), namely

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle_{\text{nc}} \equiv \frac{\langle \mathcal{O}_1 \mathcal{O}_2 \rangle_c}{\sqrt{\langle \mathcal{O}_1 \mathcal{O}_1 \rangle_c \langle \mathcal{O}_2 \mathcal{O}_2 \rangle_c}}, \quad (15)$$

which scales out magnitudes of correlated observables and thus expresses pure correlation.

Let t_{ref} be an arbitrary but fixed Dirac scale inside the bulk of the target continuum theory in the IR phase defined by Eq. (11). We note in passing that the boundary between

IR and bulk in the IR phase may be defined by the existence of a point $t_3 = \sigma_3/m > 0$ in the Dirac spectrum of a UV-regularized theory, such that the spatial IR dimension of Dirac eigenmodes at $t > t_3$ is the ordinary $d_{\text{IR}} = 3$, while $d_{\text{IR}} = 0$ for modes at $0 < t < t_3$ [4,5,7]. With that, $t_{\text{ref}} > t_3(a)$ for all sufficiently small a . IR-bulk decoupling in the IR phase then may be formulated as follows: In the IR phase, there exists at least one $0 < \tilde{t} < t_{\text{ref}}$ such that the corresponding partitions of $\mathcal{F}^2(0, t_{\text{ref}})$ decorrelate, and the decoupling scale t_{dc} is the largest of such scales—namely,

$$\lim_{a \rightarrow 0} \lim_{L \rightarrow \infty} \langle \mathcal{F}^2(0, \tilde{t}) \mathcal{F}^2(\tilde{t}, t_{\text{ref}}) \rangle_{\text{nc}} = 0 \quad \text{and} \quad t_{\text{dc}} \equiv \sup\{\tilde{t}\}. \quad (16)$$

Note that segment $(\tilde{t}, t_{\text{ref}})$ contains part of the bulk for each $\tilde{t}$.

A few remarks are important here. (i) It is implicitly understood that the above definition does not depend on the choice of t_{ref} . (ii) Regarding t_{dc} , the available evidence [4–7] embodied in the metal-to-critical scenario [5] favors the simplest possibility that $t_{\text{dc}} = t_3 = t_A$ (see also Ref. [24]), where $t_A > 0$ is the Anderson-like point of Refs. [27–29]. However, studies directly confirming the above form of IR-bulk decoupling, as well as direct computation of t_{dc} , are yet to be performed. (iii) It is consistent with the current knowledge that decoupling Eq. (16) may occur already before taking the continuum limit—i.e., even at nonzero a . (iv) Detection of the IR phase (or even its definition) via the presence of the above IR-bulk decoupling may be a fruitful approach to its numerical investigation. (v) Normalized Pearson correlations of $\mathcal{F}^2$ segments, such as those entering Eq. (16), can be expressed via spectral values $\sigma_i(U)$ alone. The general expression is

$$\langle \mathcal{F}^2(\eta_1, \eta_2) \mathcal{F}^2(\gamma_1, \gamma_2) \rangle_{\text{nc}} = \left\langle \sum_{\eta_1 < \sigma_i \leq \eta_2} \sigma_i^2(U) \sum_{\gamma_1 < \sigma_j \leq \gamma_2} \sigma_j^2(U) \right\rangle_{\text{nc}}, \quad (17)$$

as follows from Eqs. (14) and (15), or in the same way via spectral t -values.

VII. THE USES: COHERENT LATTICE QCD

The term *coherent lattice QCD* refers to formulations of lattice-regularized QCD constructed from a single object—namely, a suitable lattice Dirac operator. Such theories were first proposed in Refs. [13,14], and their construction utilized the same ideas as those employed here. The formulas involving Dirac spectral density, suggested in this work, provide an additional insight in their construction that we now make explicit.

Let D be any lattice Dirac operator with $c_S \neq 0$. The basic coherent lattice QCD has the form

$$S = \frac{1}{2g^2 c_S} \text{Tr}[D(U) - D(\mathbb{I})] + a^4 \sum_{q=1}^{N_f} \bar{\psi}_q(D(U) + m_q)\psi_q, \quad (18)$$

where N_f is the number of quark flavors with masses m_q . In original Refs. [13,14], this action was written in terms of dimensionless lattice objects, and factors involving powers of a were thus absent. Note that the gauge part is in fact $\mathcal{F}^2(U)/2g^2$, with $\mathcal{F}^2$ given in Eq. (13).

The free-field part of the glue action is a constant that can be omitted in the definition of the theory. The effective glue action after integrating out the quark variables reads

$$\begin{aligned} S_{\text{eff}}(U) &= \frac{1}{2g^2 c_S} \text{Tr} D(U) - \sum_{q=1}^{N_f} \text{Tr} \ln(aD(U) + am_q) \\ &= V_4 \int_{\mathbb{R}^2[\mathbb{C}]} d\mathcal{S}\rho_s(\lambda, U) \left[\frac{1}{2g^2 c_S} \lambda \right. \\ &\quad \left. - \sum_{q=1}^{N_f} \ln(a\lambda + am_q) \right], \end{aligned} \quad (19)$$

where λ is a complex variable, and the notation is introduced in connection with Eq. (2). Thus, the defining object of the theory, the action, is scale-decomposed and expressed in terms of Dirac spectral density. There is another hidden constant in the above expression, isolated by factoring am_q in log terms. Discarding this constant leads to the effective action in the form

$$\begin{aligned} S_{\text{eff}}(U) &= V_4 \int_{\mathbb{R}^2[\mathbb{C}]} d\mathcal{S}\rho_s(\lambda, U) \left[\frac{1}{2g^2 c_S} \lambda - \sum_{q=1}^{N_f} \ln(1 + \lambda/m_q) \right] \\ &\equiv S_{\text{eff}}^G(U) + S_{\text{eff}}^Q(U). \end{aligned} \quad (20)$$

For the family of overlap Dirac operators, we then have, in particular for glue and quark parts,

$$2g^2 S_{\text{eff}}^G(U) = \frac{2\Delta}{c_S} n_0(U) + \frac{a^2 V_4}{c_S \Delta} \int_0^{(\frac{2\Delta}{a})^-} d\sigma \rho^{\text{ef}}(\sigma, U) \sigma^2, \quad (21)$$

$$\begin{aligned} -S_{\text{eff}}^Q(U) &= n_0(U) \sum_{q=1}^{N_f} \ln \left(1 + \frac{2\Delta}{am_q} \right) \\ &\quad + V_4 \int_{0^+}^{(\frac{2\Delta}{a})^-} d\sigma \rho^{\text{ef}}(\sigma, U) \\ &\quad \times \sum_{q=1}^{N_f} \ln \left[1 + \frac{\sigma^2}{m_q^2} (1 + am_q/\Delta) \right], \end{aligned} \quad (22)$$

where the contribution of real modes is again separated. The construction of other coherent lattice QCD actions, such as the symmetric logarithmic case [13,14], proceeds along the same lines.

VIII. EPILOGUE

Dirac spectral density $\rho(\lambda)$ in QCD specifies the distribution of its quark degrees of freedom over Dirac scales. Its knowledge thus facilitates, among other things, our understanding of how these scales contribute to the composition of important quark observables such as $\langle \bar{\psi}\psi \rangle$. For example, we learned that $\langle \bar{\psi}\psi \rangle$ is a strictly IR quantity in the chiral limit, which is a particular way to interpret the approach of Banks and Casher [12].

Here, we showed, via Eqs. (2) and (7), that $\rho(\lambda)$ also determines how scalar *glue density* gets apportioned across these scales. The new formulas reveal that glue density, in a stark contrast to quark density, is a strictly UV quantity. As such, it also provides for the analogue of the Banks-Casher relation for gluon condensate. Details will be discussed in a dedicated account, but our reasoning makes it clear that this quantity is encoded in the $1/\lambda$ power term in UV asymptotics of $\rho(\lambda)$.

The outlook on $\rho(\lambda)$ as the distribution of degrees of freedom over scales was pervasive in works that led to the discovery of the IR phase [1,2,4,5]. The Definition of the phase is also expressed via IR asymptotics of $\rho(\lambda)$ [2]. But its key features, such as scale invariance and IR-bulk separation, were always chiefly attributed to gluonic rather than quark degrees of freedom. This was not contradictory with the explanation that matrix elements of the Dirac operator are in fact gauge-covariant glue operators, which makes $\rho(\lambda)$ the unusual (scale-dependent) *glue operator* as well. Here, we made the needed direct link between Dirac spectral density and the aforementioned driving glue effects.

Our formulas have several applications, particularly in relation to IR phase, where these considerations first arose. Here, we discussed the question of glue scale invariance of the IR part and the precise definition of IR-bulk decoupling. They are all based on our formulas featuring Dirac spectral density, which possibly also opens new ways of numerical evaluation. Our considerations are most potent in regularizations known as coherent lattice QCD [13,14], whose full effective actions can also be expressed via Dirac spectral density.

Finally, we wish to emphasize that although the use of the overlap Dirac operator for expressing gluon condensate [Eq. (7)] has many theoretical advantages, simpler operators may be preferable in numerical studies. In particular, the formula for all lattice Dirac operators with $c_S \neq 0$ is given in Eq. (2). For example, simple Wilson-like Dirac operators with $c_S \neq 0$ can be considered.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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