

Radial scalar product for Kerr quasinormal modes

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A scalar product for quasinormal mode solutions to Teukolsky's homogeneous radial equation is presented. Evaluation of this scalar product can be performed either by direct integration, or by evaluation of a confluent hypergeometric functions. The related scalar product will be useful for better understanding analytic solutions to Teukolsky's radial equation, particularly the quasinormal modes, their potential spatial completeness, and whether the quasinormal mode overtone excitations may be estimated by spectral decomposition rather than fitting. With that motivation, the scalar product is applied to confluent Heun polynomials where it is used to derive their peculiar orthogonality and eigenvalue properties. A potentially new relationship is derived between the confluent Heun polynomials' scalar products and eigenvalues. Using these results, it is shown for the first time that Teukolsky's radial equation (and perhaps similar confluent Heun equations) are, in principle, exactly tridiagonalizable. To this end, "canonical" confluent Heun polynomials are conjectured.

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I. INTRODUCTION

The aftermath of binary black hole (BBH) merger is thought to be a spinning and perturbed black hole (BH) that *rings down* like a struck, and shimmering gong [1–6]. The corresponding gravitational radiation encodes information about the progenitor BBH system (i.e., masses, spins, momenta, and environment), and about the true nature of gravity [7–16]. Ongoing and planned experiments endeavor to detect and learn from that radiation with increasing precision [10,15,17–20]. At all stages, those efforts are guided by our mathematical understanding of BHs, their gravitational waves, and the special functions best suited to them [21–28]. Recently, it has become clear that there are limitations in our mathematical understanding of *ringdown* radiation, and that these limitations may actively confound the interpretation of numerical and experimental data [29–40].

The purpose of this article is to introduce a framework via a radial scalar-product that may offer a deeper, practical understanding of ringdown “overtones,” and potentially of gravitational waves from compact binaries more broadly. The primary technical focus of this work is the vector space structure of the eigensolutions to Teukolsky's equation for

quasinormal modes (QNMs), considered *after* the imposition of physical boundary conditions.

To date, the LIGO-Virgo-Kagra collaboration (LVK) has confidently observed gravitational radiation from 83 BBH mergers; among them, the ringdown radiation of 31 have been compared against gravitational wave signal models (theory predictions) to test for physics beyond general relativity (GR) [4–6,41–44]. Thus far, all BBH remnants have been found to be consistent with spinning (i.e., Kerr) BHs [42–44].

Planned gravitational wave detectors such as the Einstein Telescope (ET), Cosmic Explorer (CE), and the Laser Interferometer Space Antenna (LISA) are, for example, expected to detect signal to noise ratios of $\sim 10^3$ (ET and CE) and $\sim 10^5$ (LISA) from GW150914-like¹ BBHs of $10^2 M_\odot$ and $10^6 M_\odot$, respectively [45–47]. Such detections are expected to dramatically increase the potential for ringdown data analyses to verify proposed environmental, dark matter and quantum effects [10,37,48–51]. However, the correct interpretation of analyses involving ringdown is contingent upon our understanding of ringdown's underlying mathematical structure [8,9,52–64].

Here, *ringdown* will refer to linear combinations of BH QNMs [65–67]. While the gravitational waves from BBH *postmerger* will refer to all linear and nonlinear

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¹This example refers to BBHs that have similar configurations to that of the first gravitational wave detection: approximately nonspinning progenitors and having a luminosity distance of ~ 410 Mpc [2,22].

contributions [8], the mathematical structure of ringdown is that of the QNMs. Due to their astrophysical relevance, the QNMs of Kerr BHs are perhaps the most studied [8,9,52–64,68–71].

The Kerr QNMs emerge within the study of Einstein's equations linearized around the Kerr metric [1,66,67]. Within that topic, a central result is that each single QNM solution is a function of time (a damped sinusoid), times a function of spherical polar angles (a spheroidal harmonic), and times a function of radial distance (a confluent Heun function) [67,72–74].

The time domain's sinusoidal damping is determined by discrete QNM (or ringdown) frequencies [66]. For gravitational perturbations of Kerr, QNM frequencies are labeled by three integers: ℓ , m and n , where $\ell \geq 2$, $|m| \leq \ell$ and $n \geq 0$ [10]. Typically, ℓ is referred to as a polar index, m an azimuthal index, and n an “overtone” index. Respectively, ℓ , m and n originate from the polar, azimuthal and radial dimensions, as defined in a BH centered frame [66].

For physically relevant scenarios, i.e., where the QNM frequencies depend on ℓ , the spheroidal harmonics have recently been shown to not only be biorthogonal² with the so-called adjoint-spheroidal harmonics, but also complete [56]. This conclusion is based on the fact that ℓ is directly related to the eigenvalues of, e.g., Jacobi polynomials [77]. As a result, the spheroidal harmonics may be thought of as a basis of special functions, capable of representing gravitational waves from arbitrary sources.

This is well known to be true of the spherical harmonics, which are a special case of the spheroidal harmonics. Consequently, spherical harmonics play a variety of key roles in gravitational wave theory and data analysis, from the development of perturbative approximations (e.g., post-Newtonian theory and gravitational wave self force), and the storage of simulated gravitational waves (numerical relativity), to the evaluation of gravitational wave signal models for use in parameter inference [25,78–83]. Analogous applications for the spheroidal and adjoint-spheroidal harmonics have been proposed [84]. *Perhaps surprisingly, there has been comparatively little research into whether the QNM's radial functions are special functions of potentially broad use in gravitational wave theory* [64,69,85,86].

In 1973, Teukolsky derived the differential equation that the radial functions must satisfy, and at the time noted that his radial equation has two regular singular points and one irregular one, meaning that it cannot be transformed into a hypergeometric-type equation for which there are many well known analytic methods of solutions [87]. This appears to have informed a branch of research which seeks to solve Teukolsky's radial equation using numerical methods, e.g., [9,81,88–94].

²Two sequences, $\{a_i\}_{i=0}^{\infty}$ and $\{b_j\}_{j=0}^{\infty}$, are biorthogonal if $\langle a_i | b_j \rangle = \delta_{ij}$ [75,76].

However, in 1985 Leaver presented what was perhaps the first Frobenius-type series solution for the QNM's radial functions [66,95]. Leaver's work on QNMs intersects with topics in gravitational wave self-force, in which asymptotic solutions to Teukolsky's radial equation play a key role [96–99].

In parallel, there has been broad and growing interest in a generalization of the hypergeometric equation called the Heun equation [72,100–105]. Unlike the hypergeometric equation, which has three regular singular points, the Heun equation has four [100]. It was perhaps first pointed out by Leaver and then Fiziev that Teukolsky's radial equation is an instance of the *confluent Heun equation*, also known as a generalized spheroidal equation [102]. Fiziev's work appears to coincide with what is perhaps a recent and multidisciplinary interest in Heun equations [103]. For example, it has very recently been observed that a careful consideration of the confluent Heun function's asymptotic behavior allow them to be applied directly to astrophysically relevant scenarios in particle perturbation theory (e.g., gravitational waves from extreme mass ratio inspirals) [101].

Together, methods for evaluating the QNM's radial functions that are derived from Teukolsky, MST, and Heun underlay a vast (and growing) body of research. However, thus far, this work appears to have not fully addressed the following questions:

(Q1) The properties of many special functions in physics may be framed using ideas from Sturm-Liouville theory and functional analysis [106–109]. There, solutions to differential equations are conceptualized as eigenvectors, where the vector scalar products (i.e., symmetric nondegenerate bilinear forms [109]) are defined by an integral whose integrand is *weighted* such that the differential equations is formally *self-adjoint*. For example, the scalar product for the QNM's angular functions can be framed as that of the Jacobi polynomials (i.e., with the Jacobi *weight function*), and the Jacobi differential equation is self-adjoint with respect to that scalar product [77]. Further, the Jacobi polynomials are compatible with the spheroidal harmonic's analytic (i.e., regular and differentiable) structure [77]. In analogy:

Is there an appropriate scalar product for the QNM's radial functions?

(Q2) Classical polynomials (i.e., those of Hermite, Laguerre, and Jacobi) are known to be solutions to special cases of the hypergeometric equation [110]. It is well known that the QNM angular functions are underpinned by spin-weighted spherical harmonics, which are themselves underpinned by Jacobi polynomials [77]. Therefore, there is good reason to wonder whether an analogous conclusion be developed for the QNM's radial functions:

Are the QNM's radial functions underpinned by classical polynomials, or some generalization thereof? Equivalently, is the overtone label, i.e., n , directly related to the order of a complete and orthonormal polynomial sequence?

Regarding (Q1), many studies have encountered the need to perform integrals over solutions to Teukolsky's radial equation, e.g., [85,86,88,111–113]. Perhaps most relevant to the present article, Leaver (Ref. [86]) studied Schwarzschild (i.e., nonspinning) BHs, and used a Green's function approach to derive various spectral-type expansions, including those for QNMs. Similar approaches have recently been used to study nonlinear (particularly quadratic) QNMs of Schwarzschild, e.g., [64]. This branch of research intersects with (Q1), however it does not directly draw from Sturm-Liouville theory due to apparent divergences in QNM-type problems that are not present in classical resonance problems³ [86]. Interestingly, Ref. [86] observes that exactly such divergence may be overcome by allowing for *a radial integration path that is complex valued*. In this article, the same idea is encountered for Kerr, and is motivated by coordinate dependence.

In an adjacent branch of research, many studies have made use of what is called the “formal adjoint” of operators related to QNMs where, in effect, a weight function of 1 is considered, e.g., [85,102,113,115–117]. From that perspective, Teukolsky's radial equation is not self-adjoint due to the weight function or scalar product used. Consequently, it appears that an investigation of the QNMs' radial functions in the context of Sturm-Liouville theory with a physically motivated weight has not yet been pursued.

Regarding (Q2), a number studies have attempted to use classical polynomials to represent solutions to the QNMs' radial functions, e.g., [118–120]. These efforts encounter the following mathematical difficulty: while Teukolsky's radial equation may be similarity transformed to be similar to the equation that governs, e.g., Jacobi polynomials, the related transformations differ from those required by the physical boundary conditions [67,87,115]. In turn, this means that unlike Teukolsky's angular eigenvalue problem for QNMs, Teukolsky's radial eigenvalue problem cannot be exactly represented in terms of a standard spectral problem involving classical polynomials. Instead, one encounters a “generalized spectral problem” [118,120]. Although it has been recognized that Teukolsky's radial equation is a confluent Heun equation, it appears that *confluent Heun polynomials* have not been used to frame the QNM radial problem as purely spectral until the present work.

³Similarly, Ref. [114] implies that the QNM problem may not be compatible with Sturm-Liouville theory *because* QNMs damp in the time domain. In Sec. IV of this article, this is shown to not be the case. In particular, the QNM problem is non-Hermitian; however, this does not preclude the self-adjoint perspective taken in this work.

In 2014, Cook and Zalutskiy studied the connection between confluent Heun polynomials, “algebraically special” QNMs of nonspinning BH's, and “total transmission” QNMs in the presence of BH spin [121]. There, it was found that for special QNM frequencies, the associated radial functions can be directly identified with select confluent Heun polynomials. With (Q2) in mind, there is good reason to ask whether the confluent Heun polynomials are only relevant to the total transmission QNMs, or whether there exists a single formulation that connects confluent Heun polynomials to all QNM solutions?

This article provides a first-principles development of a radial scalar product for QNMs, thereby affirmatively answering (Q1). Toward (Q2), this article applies the scalar product to confluent Heun polynomials, where it is shown that the polynomials may be defined for *any* wave frequency, including the complex valued QNM frequencies. While a full treatment of (Q2) is determined to be outside of this article's scope, significant progress is provided toward an affirmative answer. In particular, this article concludes by showing that Teukolsky's radial equation is, at least in principle, exactly tridiagonalizable. To this end, the existence of potentially new “canonical confluent Heun polynomials” is conjectured.

II. SCOPE AND ORGANIZATION

A. Scope

This article discusses QNM solutions to Teukolsky radial equation in the context of isolated Kerr BHs, absent any effects beyond GR. Some aspects of the discussion make minimal physical assumptions, while others are specific to gravitational wave ringdown. Broadly, this work studies the discrete spectra of Teukolsky's radial equation. By construction, all versions of this equation are predicated on a single QNM frequency (i.e., the radial equation for QNMs may only be defined for a single frequency at a time).

The underlying perspective of this work is that the structure QNM frequencies is influenced by the underlying structure of the discrete spectra of the radial equation for single frequencies. This concept is outlined in Sec. III A, and it is encapsulated by the overtone number, n . It is known that many references refer to the overtone number as being heuristically related to an “apparent” order in the imaginary part of QNM frequencies [10,122,123]. That apparent order is equally present in the QNM separation constants, and in the context of this work, each QNM separation constant is one of many radial eigenvalues.

Most mathematical arguments make no assumption about spin weight, s . However, $s \in \{-2, +2\}$ are given particular attention due to their correspondence with gravitational radiation. Further, Teukolsky's radial equation, even when formatted for the QNMs, may be considered for *any* frequency. Therefore, much of this article's content is not specific to the QNM's frequencies, but rather

the structure of Teukolsky’s radial equation when formatted with QNM boundary conditions. The reader is cautioned that the author does not aspire to the language of formal mathematical rigor (e.g., the text will not include theorems and lemmas).

This article’s primary focus is a frequency dependent scalar product that is uniquely determined by a choice of radial coordinate and the QNM boundary conditions. While it is noted in Sec. IV B that this product is related to a range of ostensibly frequency independent products, a key feature of all such products is that they are conditioned on the frequency parameter used to separate Teukolsky’s equation. In that sense, all such products are frequency dependent: some place the frequency dependence in a prefactor in front of the QNM radial functions, while others, including the one studies here, place the frequency dependence within the scalar product’s weight function. This concept is expanded upon in Sec. IV B.

Throughout this article, QNM frequencies are used to evaluate numerical examples due to their relevance to gravitational wave theory and experiment. QNM frequencies for this article have been generated using Leaver’s method via the publicly available POSITIVE and QNM Python packages [124,125]. All numerical results presented have been computed with 64 significant figures of precision via the MPMATH package [126].

B. Organization

Section III provides an overview of Kerr QNMs and Teukolsky’s radial equation. Section IV discusses a simplified version of Teukolsky’s radial equation in the context of Sturm-Liouville theory, and uses ideas therefrom to derive a scalar product for the Kerr QNMs. Section V focuses on two complementary numerical methods (direct integration and analytic continuation) for evaluating the scalar product. Section VI presents the limitations of direct integration and analytic continuation, and then compares and contrasts them.

Section VII presents two pedagogical applications of the scalar product. There, a pedagogical construction of the confluent Heun polynomials is provided, along with a derivation their most relevant properties. Section VIII discusses the implications of the scalar product and confluent Heun polynomials on Teukolsky’s radial equation. There, a class of “canonical confluent Heun polynomials” is proposed, and used to argue that Teukolsky’s radial equation may be exactly tridiagonalizable.

Finally, in Sec. IX, further implications of the scalar product and related results are discussed in the context of future work.

III. TEUKOLSKY’S EQUATION WITH QNM BOUNDARY CONDITIONS

In this section we connect the observable gravitational wave polarizations with this article’s topical starting point,

Teukolsky’s radial equation with QNM boundary conditions. Along the way, we will encounter spin weighted fields [25,26], the definition of QNMs [45,122], the importance of a scalar product and polynomials to Teukolsky’s angular equation, and how QNM solutions to Teukolsky’s radial equation may be framed in close analogy with the angular case [66]. These ideas converge on Teukolsky’s radial equation in a simplified form [73,101,127]. That equation sets the stage for our application of Sturm-Liouville theory [106,108,110,128], and the development of a radial scalar product in Sec. IV.

A. From gravitational wave strain to the perturbative wave quantity

Any gravitational wave detector will be sensitive to the two *strain* polarizations predicted by GR, h_+ and $h_\times$ [82,129]. The study of gravitational wave plane waves motivates an extremely useful complex valued strain given by

$$h = h_+ - ih_\times \quad (1a)$$

$$= \int_{-\infty}^t \int_{-\infty}^{t'} \psi_4(t'', r, \theta, \phi) dt'' dt'. \quad (1b)$$

In Eq. (1a), h is the complex valued gravitational wave strain [82]. We will consider it to depend on Boyer-Lindquist coordinates, where time, radius, polar angle, and azimuthal angle are denoted by t , r , θ and ϕ respectively [130]. In Eq. (1b), ψ_4 is the 5th Weyl scalar in Newman-Penrose notation [26,82]. In that formalism, ψ_4 is a complex valued scalar field of *spin weight* $s = -2$ [25,87]. Henceforth, complex valued quantities encode or modify the relative importance of h_+ and $h_\times$ within h . Within the study of gravitationally perturbed isolated BHs, two Weyl scalars are of particular interest: ψ_4 ($s = -2$), and the closely related ψ_0 ($s = +2$). Both encode information about gravitational waves [87].

For linear perturbations of Kerr, ψ_4 is treated in terms of a generic wave quantity, ψ

$$\psi = -(r - ia \cos \theta)^{-4} \psi_4. \quad (2)$$

In Eq. (2), the quantity scaling ψ is a spin coefficient [87], where $a = |J|/M$ is the BH spin magnitude. We will hold that $G = c = 1$, leaving a to have units of the black hole mass, M .

The use of ψ is that it satisfies exactly the same partial differential equation as other physically relevant wave quantities. In the case of the first Weyl scalar, ψ_0 , we simply have that $\psi = \psi_0$ [26]. Whether ψ refers to ψ_0 or ψ_4 is toggled by the spin weight of the field, s . Thus for ψ_0 ($s = +2$) or ψ_4 ($s = -2$), ψ satisfies Teukolsky’s equation, which is one way of representing Einstein’s equations linearized about the Kerr solution [27,67,87,115,131,132].

B. Teukolsky's equation and its QNM solutions

If we denote the partial differential operator for Teukolsky's equation as $\mathcal{L}_{tr\theta\phi}$, with

$$\mathcal{L}_{tr\theta\phi} = \mathcal{L}_{tr\theta\phi}(s, r, \theta, \partial_t, \partial_r, \partial_\theta, \partial_\phi), \quad (3)$$

then the schematic form of Teukolsky's equation is

$$\mathcal{L}_{tr\theta\phi}\psi(t, r, \theta, \phi) = 4\pi\Sigma T. \quad (4)$$

In Eq. (3), $\{\partial_t, \partial_r, \partial_\theta, \partial_\phi\}$ denote differentiation with respect to the Boyer-Lindquist coordinates [130]. An explicit definition of $\mathcal{L}_{tr\theta\phi}$ is provided in Appendix A. In Eq. (4), Σ is the usual quantity that appears in the Kerr metric,

$$\Sigma \equiv r^2 + a^2 \cos^2\theta, \quad (5)$$

and T is a source term derived from the stress-energy tensor. Since we are interested in isolated BHs, we have that

$$T(t, r, \theta, \phi) = 0. \quad (6)$$

The structure of $\mathcal{L}_{tr\theta\phi}$ [see Eq. (3)] is known to be such that ψ of the form

$$\psi \propto R(r)S(\theta)e^{-i\tilde{\omega}t}e^{-im\phi}, \quad (7)$$

separates⁴ $\mathcal{L}_{tr\theta\phi}$ into two coupled equations: one for $R(r)$ and another for $S(\theta)$. In Eq. (7), $\tilde{\omega}$ is generally a complex valued frequency

$$\tilde{\omega} = \omega - i/\tau. \quad (8)$$

In Eq. (8), one should think of any physical situation in which an isolated BH experiences a momentary and small perturbation. The postmerger of two sufficiently compact objects is an example of particular importance to astrophysics [4,5,133,134]. For the simplest scenarios, the resulting gravitational radiation will, qualitatively, ring down [13,65,114,122,135]. The central frequency of that ringing will be ω , and the exponential decay rate of that ringing will be $1/\tau$, where τ is strictly *positive*; otherwise, Eq. (7) would diverge at late times, and the BH would be linearly unstable [1,28,65,67].

In general, gravitational radiation will be a sum of such modelike terms (Eq. (7) weighted by constant factors that depend on spacetime geometry and the perturber [7,8,55,136]. The net spatial dependence of each term in the sum defines a single QNM [13,114,135]. There will

always be a period in time for which QNM radiation dominates over other solutions to the homogeneous radial equation, such as power-law tails, prompt emission, and nonlinear QNMs [64,86,112]. This inherently poorly defined region of QNM dominance is colloquially referred to as ringdown [65,122].

Properties of QNM solutions can be determined by applying the solution ansatz defined in Eq. (7), and then studying the resulting coupled eigenvalue problems. These are separate radial and angular differential equations, connected by the separation constant, A

$$\mathcal{L}_u S(u) = -AS(u) \quad (9a)$$

$$\mathcal{L}_r R(r) = +AR(r). \quad (9b)$$

Only the schematic structure of Eqs. (9a) and (9b) are important here, but reader may find full expressions for $\mathcal{L}_u$ and $\mathcal{L}_r$ in Appendix A. Henceforth, we will refer to Eq. (9a) as *the angular equation*, and Eq. (9b) as *the radial equation*. Respectively, *angular problem* and *radial problem* will refer to the solving of each eigenvalue equation.

In Eq. (9), $u = \cos(\theta)$, $\mathcal{L}_u$ will be referred to as the angular differential operator, and $\mathcal{L}_r$ will be referred to as the radial differential operator. The angular functions, $S(u)$, are called spheroidal harmonics, and we will refer to $R(r)$ as Teukolsky's radial functions. Both are known to be closely related to the confluent Heun functions⁵ [73,101,121,137].

C. Lessons from Teukolsky's angular problem: The roles of the scalar product and Jacobi polynomials

Although both angular and radial equations are known to be similar in form (e.g., Ref. [95]), there are good reasons to first study the angular equation, and then apply lessons therefrom to the radial equation. Relative to the radial equation, the angular equation is well known to be simpler; as a result, there are many well known and highly accurate methods for computing its global⁶ analytic solutions [54,66,69,81,121]. Pertinent to this article is the spectral method presented in Ref. [77], and *whether two key concepts therein may be applied to the radial problem*.

For the convenience of the reader, a modern and self-contained presentation this method is provided in Appendix B. There, the two key concepts are

- (i) The structure of the differential equation allows the development of a scalar product that is natural for the solution space. This concept motivates (Q1).
- (ii) The structure of the differential equation points the way to problem specific special functions, namely the Jacobi polynomials. In turn, the Jacobi

⁴The ansatz provided in Eq. (7) obscures the fact that our net solution will depend on the tortoise coordinate via the retarded time [67,87]. The radial part of the retarded time is ultimately encoded within $R(r)$.

⁵As noted by Leaver, they are also called generalized spheroidal wave functions [95].

⁶i.e., series solutions that apply across the physical domain.

polynomials may be used to exactly represent analytic solutions. This concept motivates (Q2).

If applicable to the radial problem, concept (i) may enable greater understanding of QNM (spatial) orthogonality, bi-orthogonality, and completeness [56]. Similarly, concept (ii) may yield a representation of the radial functions that may not only be computationally efficient, but may also lend clear insight into the mathematical meaning of the so-called overtone label, n . For these reasons, concepts (i) and (ii) will guide forthcoming discussion of the radial equation.

D. QNM solutions of Teukolsky's radial equation

The study of BH QNMs is traditionally contained within the study of spacetime exterior to the event horizon [1,67,138]. In this context, QNM boundary conditions are defined on two asymptotic surfaces: the event horizon and spatial infinity [67,87,115]. This makes it computationally and algebraically useful to consider a compactified radial coordinate [66,95],

$$\xi = (r - r_+)/ (r - r_-). \quad (10)$$

In Eq. (10), $r_+ = M + \sqrt{M^2 - a^2}$ is the outer Kerr horizon (i.e., the event horizon), and $r_- = M - \sqrt{M^2 - a^2}$ is the inner Kerr horizon. The compactified coordinate, ξ , has the effect of mapping $r = r_+$ to $\xi = 0$, $r = r_-$ to $\xi \rightarrow -\infty$, and $r \rightarrow \infty$ to $\xi = 1$.

The QNM boundary conditions are that the gravitational radiation is purely ingoing (toward the BH) at $\xi = 0$ and purely outgoing near $\xi = 1$. A derivation of these conditions is provided in, e.g., Refs. [67,121]. Since these conditions are imposed between $r = r_+$ and $r \rightarrow \infty$, we will only consider $r \in [r_+, \infty)$, and so $\xi \in [0, 1)$.

The QNM boundary conditions are asymptotic in the sense that they are restrictions on the functional form of ψ near the event horizon and spatial infinity. As used by Leaver, these boundary conditions amount to a *similarity transformation* on the radial problem [66,95,100]. This transformation proceeds by seeking solutions to the radial problem of the form

$$R(r(\xi)) = \mu(\xi)f(\xi). \quad (11)$$

In Eq. (11), $\mu(\xi)$ is a product of functional forms required by the asymptotic boundary conditions,

$$\begin{aligned} \mu(\xi) = & e^{\frac{2i\delta\tilde{\omega}}{1-\xi}}(1-\xi)^{1+2(s-iM\tilde{\omega})} \\ & \times \xi^{-(iM\tilde{\omega}+s)+\frac{i(am-2M^2\tilde{\omega})}{2\delta}}. \end{aligned} \quad (12)$$

The factors that make up μ are determined by structure of Frobenius (series) solutions to Eq. (9b). In Eq. (12), the argument of the exponential, as well as the powers of ξ and $1-\xi$ are selected by the physical boundary

conditions [73,95,121]. In turn, Eq. (11) strictly requires that $f(\xi)$ be nonzero at $\xi = 0$ and $\xi = 1$; otherwise the scalings imparted by μ (i.e., the QNM boundary conditions) would be spoiled. This requirement makes $f(\xi)$ the natural quantity for which to seek a power series solution.

The application of Eq. (11) to radial eigenproblem [Eq. (9b)] has the effect of transforming $\mathcal{L}_r$ into a new differential operator, $\mathcal{L}_\xi$,

$$[\mu(\xi)^{-1}\mathcal{L}_r\mu(\xi(r))f](\xi) = [\mu(\xi)^{-1}A\mu(\xi)]f(\xi) \quad (13a)$$

$$\mathcal{L}_\xi f(\xi) = Af(\xi). \quad (13b)$$

In Eq. (13a), we have applied the solution ansatz given by Eq. (11) to the radial eigenproblem, Eq. (9b). We have then divided by μ . Since μ is closely related to the singular exponents used in Frobenius' method, the division is exact [66]. The result, Eq. (13b), is a transformed eigenproblem that has the same eigenvalues, A . It is in this sense that $\mathcal{L}_\xi$ is formally *similar* to $\mathcal{L}_r$ [76,139,140],

$$\mathcal{L}_\xi = \mu(\xi)^{-1}\mathcal{L}_r\mu(\xi(r)). \quad (14)$$

While we refer to the effect of Eqs. (13) and (14) as a similarity transformation, it is sometimes referred to as an elementary power transformation, or a homotopic transformation [100,101,141]. The reader should note that Eqs. (13a) and (14) contain a coordinate transformation (r to ξ), as well as a similarity transformation.

In Eq. (14), the new differential operator, $\mathcal{L}_\xi$, has the form given by Eq. (15).

$$\begin{aligned} \mathcal{L}_\xi = & (C_0 + C_1(1-\xi)) + (C_2 + C_3(1-\xi) \\ & + C_4(1-\xi)^2)\partial_\xi + \xi(\xi-1)^2\partial_\xi^2 \end{aligned} \quad (15)$$

There, the constants, C_0 to C_4 , are defined in Eq. (16) using the physical parameters discussed previously.

$$\delta = \sqrt{M^2 - a^2} \quad (16a)$$

$$\begin{aligned} C_0 = & -2am\tilde{\omega} - 2i\tilde{\omega}(-\delta + M(2s+1)) \\ & + \tilde{\omega}^2(\delta + M)(\delta + 7M) \end{aligned} \quad (16b)$$

$$\begin{aligned} C_1 = & 8M^2\tilde{\omega}^2 + s(4iM\tilde{\omega} - 1) + 6iM\tilde{\omega} - 1 \\ & - (4M\tilde{\omega} + i)\frac{(am - 2M^2\tilde{\omega})}{\delta} \end{aligned} \quad (16c)$$

$$C_2 = 4i\delta\tilde{\omega} \quad (16d)$$

$$C_3 = -2(s+1) + 4i\tilde{\omega}(M - \delta) \quad (16e)$$

$$C_4 = s + 3 - 6iM\tilde{\omega} + \frac{i(am - 2M^2\tilde{\omega})}{\delta} \quad (16f)$$

Equation (15) has a few special properties. We have started with Boyer-Lindquist coordinates, but one also arrives at Eq. (15) if one instead begins with Kerr ingoing or Kerr outgoing coordinates [115]. One also arrives at Eq. (15) (albeit with $s \rightarrow -s$ and $A \rightarrow A - 2s$) if one uses the alternative tetrad choice discussed in Ref. [115], which amounts to $t \rightarrow -t$ and $\phi \rightarrow -\phi$. As noted in, e.g., Refs. [28,115], the use of either $s = 2$ or $s = -2$ results in mathematically different descriptions of the same gravitational radiation.

Equation (15) has regular singular points at $\xi \in \{-\infty, 0\}$ and one irregular singular point at $\xi = 1$. Thus Eq. (15) is an instance of the confluent Heun differential operator [100]. This is also true in the Schwarzschild limit. While it is sometimes said that Ref. [95] treats Eq. (15) as what is called a generalized spheroidal operator, we note here that this is a slight misnomer [100,103]. Equation (15) may only be transformed into a generalized spheroidal operator if one applies an asymptotic boundary condition at $r = r_-$ that is *inconsistent* with QNMs [66,121]. Appendix C provides an expanded disambiguation between different forms of the confluent Heun equation and the radial equation.

Equations (13b)–(16) are commonly used to define series solutions for $f(\xi)$. The requirement that these series solutions converge on $\xi \in [0, 1)$ happens to be equivalent to finding eigenvalues, A , such that certain continued fractions are zero [66,95,142]. In particular, see Ref. [95] for a proof of convergence for series solutions, example applications of continued fractions to the radial and angular eigenproblems, and how the eigenvalues may be determined by searching for roots of continued fractions.

From the use of continued fractions, values of A are well known to be discrete. However, the use of continued fractions has arguably obscured deeper understanding of the distribution of eigenvalues, A , and their related radial (labeled in n) and angular functions (labeled in ℓ and m). In search of that understanding, it appears that there has been relatively little application of Sturm-Liouville theory to Eq. (15) in the BH context [65,66,100,111,122,136,143–145]. Nevertheless, since the Kerr QNMs' eigenvalues are well known to be unique for most physical scenarios⁷ (e.g., [57,69,127,146]), the QNM's themselves are necessarily linearly independent,⁸ and thus well suited to Sturm-Liouville theory [108].

⁷There is some ambiguity in how one defines the eigenvalues. In particular, the fact that C_0 is nonzero means that the eigenvalue that one starts with, A , need not strictly define the eigenvalue that is used in Eq. (9a). Thus in, e.g., the limit that $a = 0$, where $A = \ell(\ell + 1) - s(s + 1)$ for every radial eigenfunction, the effective eigenvalue, $A - C_0$, may be a unique function of $\tilde{\omega}$.

⁸More precisely, finite sets of QNMs will be linearly independent when their eigenvalues are unique. Infinite sets of QNMs will be *minimal* if their eigenvalues remain distinct as some order label on the set goes to infinity [56,76].

IV. DEFINITION OF A RADIAL SCALAR PRODUCT

The aim of Sturm-Liouville theory is to understand the existence and asymptotic behavior of eigenvalues and eigenfunctions [106,108,128]. Here, we will focus on a facet of the theory that pertains to whether there exists a *scalar product* and an associated *weight function*, for which $\mathcal{L}_\xi$ is *self-adjoint*. While a scalar product may be developed under which the problem is self-adjoint (symmetric under scalar products), it is still the case the QNM radial problem is non-Hermitian; e.g., it has nontrivially complex eigenvalues.

The immediate aim of what follows is to derive a scalar product and related weight function such that Eq. (15) is *formally* self-adjoint. In other words, a function, $W(\xi)$, is determined such that $W(\xi)\mathcal{L}_\xi$ is in Sturm-Liouville form. As the scalar product will ultimately be an integral over ξ involving $W(\xi)$, the behavior $W(\xi)$ at the domain boundaries is addressed in Sec. V.

A. The scalar product and the weight function

By scalar product, we mean a symmetric bilinear form that takes two nonsingular functions,⁹ $a(\xi)$ and $b(\xi)$, and performs an operation on them that results in a number [109,147]. In less technical parlance, we simply mean a bra-ket operation that is similar to what one encounters in quantum mechanics

$$\langle a | b \rangle = \int_0^1 a(\xi) b(\xi) W(\xi) d\xi. \quad (17)$$

In Eq. (17), $W(\xi)$ is called a measure, or weight function. It will soon be the focus of our discussion.

Since there is no conjugation involved in Eq. (17), the scalar-product shown is not conjugate-symmetric. This feature distinguishes it from an inner-product, and allows analysis of $\mathcal{L}_\xi$, irrespective of its complex valued coefficients. The analysis of immediate interest pertains whether $W(\xi)$ may be defined such that $\mathcal{L}_\xi$ is formally self-adjoint [108,128].

The adjoint of $\mathcal{L}_\xi$ is defined to be an operator, $\mathcal{L}_\xi^\dagger$, such that

$$\langle a | \mathcal{L}_\xi b \rangle = \langle \mathcal{L}_\xi^\dagger a | b \rangle. \quad (18)$$

Chapter 5, Sec. 3 of Ref. [108] provides a detailed overview of how $\mathcal{L}_\xi^\dagger$ may be explicitly defined using integration by parts. In that derivation, the fact that the coefficient of the second derivative term in Eq. (15) vanishes at $\xi = 0$ and $\xi = 1$ is key to Sturm-Liouville boundary conditions being satisfied [128]. There, it is also shown that $\mathcal{L}_\xi$ is formally

⁹The reader should not confuse the nonsingular but otherwise arbitrary function, $a(\xi)$, with the BH spin, a .

self-adjoint, i.e.

$$\mathcal{L}_\xi = \mathcal{L}_\xi^\dagger, \quad (19)$$

if it also holds that

$$\mathcal{L}_\xi = p_0(\xi) + W(\xi)^{-1} \partial_\xi [(W(\xi) p_2(\xi)) \times \partial_\xi] \quad (20a)$$

$$= p_0(\xi) + \frac{\partial_\xi (W(\xi) p_2(\xi))}{W(\xi)} \partial_\xi + p_2(\xi) \partial_\xi^2. \quad (20b)$$

In Eq. (20), $p_0(\xi)$ and $p_2(\xi)$ are the coefficient functions seen in zeroth and second derivative terms of Eq. (15), respectively. If we define p_1 to be the coefficient function of ∂_ξ in Eq. (15), then

$$p_0 = C_0 + C_1(1 - \xi), \quad (21a)$$

$$p_1 = C_2 + C_3(1 - \xi) + C_4(1 - \xi)^2 \quad (21b)$$

$$p_2 = \xi(\xi - 1)^2. \quad (21c)$$

In Eq. (20a), the appearance of the weight function is a direct result of Eqs. (18) and (19). To go from Eqs. (20a) to (20b), we have applied the product rule (also known as Leibniz's rule), to $\partial_\xi [(W p_2) \times \partial_\xi]$. For $\mathcal{L}_\xi$ in Eq. (15) to be equivalent to that in Eq. (20b), it must be the case that coefficients of ∂_ξ match,

$$p_1 = W^{-1} \partial_\xi (W p_2) \quad (22a)$$

$$W^{-1} \partial_\xi W = p_2^{-1} (p_1 - \partial_\xi p_2). \quad (22b)$$

In Eq. (22a) we have simply equated the first derivative coefficient of Eq. (20b) with that of Eq. (15). In Eq. (22b), we have used Leibniz's rule to expand the right-hand side of Eq. (22a), and then we have rearranged terms to emphasize that Eq. (22b) is a linear differential equation whose left-hand side is equal to $\partial_\xi \ln W$.

Equation (22b) can be solved for W using the coefficient functions defined in Eq. (21),

$$W(\xi) = \frac{1}{p_2(\xi)} \exp \left(\int^\xi \frac{p_1(\xi')}{p_2(\xi')} d\xi' \right) \quad (23a)$$

$$= e^{\frac{C_2}{1-\xi}} (1 - \xi)^{-C_2-C_3-2} \xi^{C_2+C_3+C_4-1}. \quad (23b)$$

In Eq. (23a), the integration constant has been chosen to be zero, and in Eq. (23b) we explicitly evaluate the integral using Eqs. (21b) and (21c).

It will soon be useful to simplify our notation for $W(\xi)$ by defining a shorthand for combinations of parameters,

$$B_0 = C_2 + C_3 + C_4 - 1 \quad (24a)$$

$$B_1 = -C_2 - C_3 - 2 \quad (24b)$$

$$B_2 = C_2. \quad (24c)$$

Consequently, the weight function takes a less cumbersome form,

$$W(\xi) = \xi^{B_0} (1 - \xi)^{B_1} e^{\frac{B_2}{1-\xi}}, \quad (25)$$

and the scalar product between two any two arbitrary functions, $a(\xi)$ and $b(\xi)$, may now be written as,

$$\langle a | b \rangle = \int_0^1 a(\xi) b(\xi) \xi^{B_0} (1 - \xi)^{B_1} e^{\frac{B_2}{1-\xi}} d\xi. \quad (26)$$

B. Properties and nonuniqueness of the weight function

With Eq. (26), we arrive at one of this article's key results. Equation (26) along with Eqs. (16) and (24) define a scalar product for which Teukolsky's radial equation, Eq. (15), is self-adjoint. In turn, self-adjointness, in principle, implies that as long as the eigenvalues may be defined, and that they are unique, then the eigenfunctions are orthogonal, and very likely complete [100]. However, there are good reasons why the explicit demonstration of orthogonality and completeness are beyond the scope of this article.¹⁰ The most pressing reason has to do with the fact that $W(\xi)$, *naively*, appears to diverge on the event horizon, and at spatial infinity. Understanding that *apparent* divergence and its consequences is preliminary to further investigation.

Visual inspection of Eq. (23) points to the origin of these divergences. In Eq. (23a), the factor

$$p_2(\xi)^{-1} = \xi^{-1} (1 - \xi)^{-2} \quad (27)$$

clearly creates a potential for the weight function to become singular at $\xi = 0$ and $\xi = 1$. For this reason, the radial eigen-problem may be said to be of the *singular*¹¹ Sturm-Liouville type [128]. In Eq. (23b), we see that the net exponents of ξ and $1 - \xi$ may still prevent divergence if their total *real part* is positive. Similarly, the exponential factor may diverge if the real part of $C_2 = 4i\delta\tilde{\omega}$ is greater than zero.

Thus, to determine the physical conditions in which $W(\xi)$ diverges, one might embark on an analysis of the sign of parameters within Eq. (23b) or, equivalently, Eq. (25). While this is a somewhat cumbersome exercise, it is easy to

¹⁰In particular, one route to demonstrating completeness of the single-frequency radial functions would be to construct an isomorphism between them, and a known basis [56,76].

¹¹Note that, e.g., the Jacobi and Legendre polynomial problems are also singular in this way [106].

see that the exponential factor,

$$e^{C_2/(1-\xi)} = e^{4i\delta\tilde{\omega}/(1-\xi)}, \quad (28)$$

will generally diverge at spatial infinity. For that we need to recall the discussion below Eq. (8): Since the imaginary part of $\tilde{\omega}$ must be negative for time-domain stability, the real part of $4i\delta\tilde{\omega}/(1-\xi)$ is generally *positive* for the QNMs,

$$\text{Re} \frac{4i\delta\tilde{\omega}}{1-\xi} = \frac{4\delta}{\tau(1-\xi)} > 0. \quad (29)$$

The inequality in Eq. (29) holds because $\delta = \sqrt{M^2 - a^2} > 0$ and $\tau > 0$. Thus it would appear that $W(\xi)$ will always diverge on at least one boundary. This appearance happens to be just that.

Above, we have implicitly assumed that $\xi = 0$ and $\xi = 1$ are approached from the real line. While that is a physically meaningful assumption, previous studies have found it to be an unnecessary practical impediment [85,95,111,142]. In particular, when evaluating the scalar product, ξ need not be a real valued quantity. This point is explored in the two sections that follow. A kind of “physicist’s analytic continuation” is detailed in Sec. VA, and then a complementary analytic continuation technique is presented in Sec. VB.

We conclude this section by pointing out two alternative perspectives on the weight function. The first hinges on the fact that Teukolsky’s equation allows for the definition of a weight function prior to applying the QNM boundary conditions. The second pertains to whether the weight function is unique, once the QNM boundary conditions have been applied.

The first perspective is that the form of the weight function is closely linked to the scaling function, $\mu(\xi)$, defined in Eq. (12). In particular, it is easy show [see, e.g., Eq. (B25)] that

$$W(\xi) = \mu(\xi)^2 W_0(\xi). \quad (30)$$

In Eq. (31), W_0 is the radial equation’s weight function *before* applying the similarity transformation shown in Eq. (15),

$$W_0(\xi) = \xi^s (1-\xi)^{-2(s+1)}. \quad (31)$$

This weight function is independent of $\tilde{\omega}$, m and a , and it is directly relevant for the original radial functions, $R(\xi)$ [see Eq. (9b)]. In essence, Eq. (30) defines how the weight function changes under similarity transformations on the main problem. One way to arrive at Eq. (30) is to note that such transformations preserve the value of scalar products. While W_0 has already been used in operator perturbation theory (e.g., [145]), it is not directly considered here because the corresponding differential operator does not

support polynomial solutions that are compatible with the QNM boundary conditions.

The second perspective is that $W(\xi)$ is not the only weight function for which $\mathcal{L}_\xi$ is formally self-adjoint. It is left as an exercise for the reader to show that one can actually enforce a strictly positive and real valued weight function,

$$\mathcal{W}(\xi) = |\xi^{B_0} (1-\xi)^{B_1} e^{\frac{B_2}{1-\xi}}|. \quad (32)$$

This exercise follows exactly the steps discussed in Eqs. (20)–(23), but with the requirement that the weight function is wrapped within an absolute value. Further, one must use the fact that $\partial_\xi \mathcal{W} = \text{sign}(W) \partial_\xi W$. While $\mathcal{W}(\xi)$ is amenable to the direct integration approach that we will discuss shortly, it is not at all clear to the author how it may be used with the analytic continuation approach that will be discussed in Sec. VB. Further, there do not appear to be significant gains in using $\mathcal{W}(\xi)$ rather than $W(\xi)$. Since complexification of the radial coordinate is also needed for $\mathcal{W}(\xi)$, scalar products will in general still be complex valued. $\mathcal{W}(\xi)$ is noted for potential future interest. Henceforth, this article will focus on the evaluation of scalar products with the comparatively simpler weight function, $W(\xi)$.

V. EVALUATION OF SCALAR PRODUCTS

The scalar product developed in the previous section has been copied below,

$$\langle a \rangle b = \int_0^1 a(\xi) b(\xi) \xi^{B_0} (1-\xi)^{B_1} e^{\frac{B_2}{1-\xi}} d\xi. \quad (33)$$

One of the simplest ways to evaluate Eq. (33) is direct integration (i.e., explicit numerical integration via discretization of the domain). However, divergence of the weight function at spatial infinity or the BH horizon means that naive integration along real values of ξ is bound to fail. So other means must be considered.

In Sec. VA, we make use of coordinate dependence to develop a method to *directly integrate* Eq. (33). In Sec. VB, we note that Eq. (33) coincides with the integral representation of the Tricomi confluent hypergeometric function, also known as the Kummer function of the second kind [106,148]. That fact, along with the well known *analytic continuation* of e.g., the Gamma function to complex inputs, is used to analytically evaluate Eq. (33). This section concludes with a brief comparison of the pros and cons of direct integration and analytic continuation.

A. Direct integration

An extremely useful lesson from calculus is that, given an integral, we may change the coordinate representation of its integrand without changing the value of the integral. So

while the integrand of Eq. (33) is coordinate dependent, its value, $\langle a \rangle b$, is coordinate independent. This concept leads us to seek a new real valued coordinate, z , such that the integrand of Eq. (33) does not diverge. Henceforth, we will use $\xi(z)$ to refer to a complex valued integration path, and ξ to refer to the quantity defined by Eq. (10).

The following definition of $\xi(z)$ has the desired effect of augmenting singular behavior at the horizon and spatial infinity,

$$\xi(z) = \frac{x_2 z^{x_0}}{(x_2 + 1)z^{x_0} - 1}. \quad (34)$$

In Eq. (34), x_0 and x_2 are coordinate parameters that modify B_0 and B_2 , respectively. We will consider $\text{Re}(x_0) > 0$, such that $\xi(0) = 0$. The dependence of Eq. (34) on x_2 is motivated by the complex radial coordinate used in Ref. [142]. It may also be developed by placing the Laguerre polynomial equation into a ξ -like compactified coordinate, and then studying that system's weight function. The dependence of Eq. (34) on x_0 is motivated by the observation that z^{x_0} is the simplest apparent coordinate modification that impacts singular behavior arising from B_0 .

The effect of this coordinate choice is found by applying Eq. (34) to the scalar product's integrand,

$$\langle a \rangle b = \int_0^1 a(\xi(z)) b(\xi(z)) [W(\xi(z)) \partial_z \xi(z)] dz. \quad (35)$$

In Eq. (35), we have used the fact that $d\xi = \partial_z \xi(z) dz$ to write the integral explicitly in z . According to Eq. (34), there may exist coordinate parameters, x_0 and x_2 , such that $\xi(z=0) = 0$, and $\xi(z=1) = 1$, which would mean that the limits of integration do not change.

Within Eq. (34), the quantity in square brackets is of particular practical use. It is an *effective weight function*,

$$W_{\text{eff}}(z) = W(\xi(z)) \partial_z \xi(z) \quad (36a)$$

$$= e^{\frac{B_2 x_2}{z^{x_0} - 1} z^{x_0} (B_0 + 1) - 1} (1 - z^{x_0})^{B_1} \times (1 - (x_2 + 1)z^{x_0})^{-B_0 - B_1 - 2}. \quad (36b)$$

While Eq. (34) provides a coordinate that modifies the boundary behavior of $W(\xi)$, it is Eq. (36) that must be analyzed to determine the special values of x_0 and x_2 that prevent the effective weight function from diverging at the domain boundaries. In essence, we have two independent conditions: (i) $W_{\text{eff}}(0) = 0$, and (ii) $W_{\text{eff}}(1) = 0$. We also have two unknowns: x_0 and x_1 . Thus, x_0 and x_1 may be determined (up to positive constants) by careful analysis of W_{eff} .

This analysis is somewhat tedious, but tractable. First, one should focus on the factor of $z^{x_0(B_0+1)-1}$ within W_{eff} . If x_0 is chosen appropriately, then the real part of

$x_0(B_0 + 1) - 1$ should equal some positive constant, K_0 , thereby allowing W_{eff} to go to zero as z goes to zero. Since, B_0 is complex valued, x_0 must be also. These ideas result in the following constraints,

$$K_0 = -1 + [1 + \text{Re}(B_0)]\text{Re}(x_0) - \text{Im}(B_0)\text{Im}(x_0), \quad (37a)$$

$$K_0 > 0. \quad (37b)$$

If we assume that $\text{Re}(x_0)$ only depends on $\text{Re}(B_0)$, then the result for x_0 follows,

$$x_0 = K_0 \frac{\text{sgn}(\text{Re}(B_0) + 1)}{\text{Re}(B_0) + 1} + i \frac{-1 + K_0[\text{sgn}(\text{Re}(B_0) + 1) - 1]}{\text{Im}(B_0)}. \quad (38)$$

Note that, in Eq. (38), $\text{Re}(x_0) > 0$, as is needed for $\xi|_{z=0}$ to be zero.

Given, x_0 , one may then determine x_2 . Returning to W_{eff} , if its exponential factor has a negative argument as $z \rightarrow 1$, then W_{eff} is assured to be regular in that limit. To that end, one should consider $e^{\frac{B_2 x_2}{z^{x_0} - 1}}$, and seek x_2 such that

$$\lim_{z \rightarrow 1} \frac{B_2 x_2}{z^{x_0} - 1} \sim -\frac{K_2}{1 - z}, \quad (39a)$$

$$K_2 > 0. \quad (39b)$$

To evaluate the left-hand side of Eq. (39a), one can use the binomial theorem¹²; this results in a quantity with the same form as the right-hand-side of Eq. (39a). Upgrading the asymptotic equivalence in Eq. (39a) to an equality yields the following solution for x_2 ,

$$x_2 = K_2 x_0 / B_2. \quad (40)$$

Note that, since x_0 is complex valued, the factor of $(1 - (x_2 + 1)z^{x_0})^{-B_0 - B_1 - 2}$ within W_{eff} will never encounter a root. Thus it is of little practical concern.

In Eqs. (38) and (40) we define the values of the x_0 and x_2 are allowed to take such that W_{eff} is regular (actually zero) at the event horizon and spatial infinity. In doing so, we have transferred the coordinate dependence to two new, strictly positive, parameters, K_0 and K_2 . What remains is the possibility that $|\xi(z)| \gg 1$.

If K_0 and K_2 are not chosen such that $|\xi(z)| \lesssim 1$, then numerical evaluation of Eq. (35) can encounter extremely large numbers. This numerically problematic possibility can be avoided if one determines K_0 and K_2 by requiring that $\xi(z)$ is contained as much as possible within a unit circle centered about $\xi = 1/2$. This may be accomplished

¹²For this it is useful to work with $z' = 1 - z$.

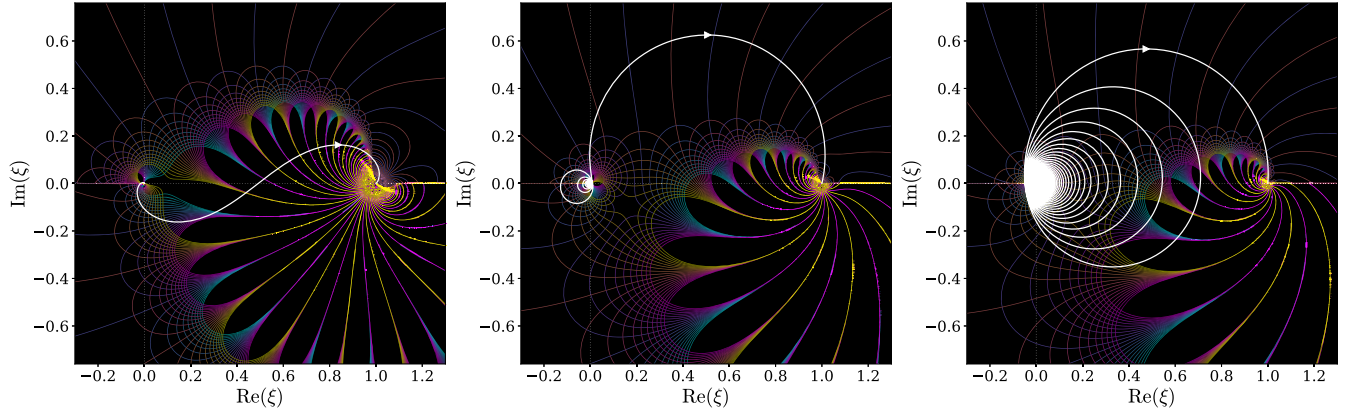


FIG. 1. Examples integration paths (white curves) against weight function contours (colored contours) for the $(s, \ell, m, n) = (-2, 2, 2, 3)$ QNM at three BH spins: (left) $a/M = 0.7$, $M\tilde{\omega} = 0.4713 - 0.5843i$, (center) $a/M = 0.99$, $M\tilde{\omega} = 0.8695 - 0.2058i$, and (right) $a/M = 0.999$, $M\tilde{\omega} = 0.9558 - 0.07299i$. Contours for the real and imaginary parts of the weight function, $W(\xi) = \text{Re}(W) + i\text{Im}(W)$, are shown. For $\text{Re}(W)$, colors between cyan and magenta represent dimensionless values between -50 and 50 respectively. For $\text{Im}(W)$, colors between magenta and yellow represent values between -50 and 50 respectively. Convergence of contours at spatial infinity ($\text{Re}(\xi) = 1$) corresponds to the divergence of the weight function. Horizontal contours along $\text{Re}(\xi) < 0$ and $\text{Re}(\xi) > 0$ are branch cuts. For left, center and right panels, coordinate parameters, (K_0, K_2) , are $(0.8530, 23.4621)$, $(0.7105, 1.4010 \times 10^{-4})$ and $(0.1585, 2.5088 \times 10^{-4})$, respectively. Dotted horizontal and vertical lines mark the respective locations of the real and imaginary axes.

via the following minimization,

$$\{K_0, K_2\} = \text{argmin}_{K_0, K_2} S(K_0, K_2, z) \quad (41a)$$

where

$$S(K_0, K_2, z) = |1/2 - \text{mean}_z(|\xi(z)|)| + \max_z |\text{Im} \xi(z)| + |\min_z \text{Re} \xi(z)|. \quad (41b)$$

In Eq. (41), argmin outputs the values for which S is minimized in K_0 and K_2 , mean_z computes the unweighted mean over values of z , $\min_z$ returns the smallest value of a quantity in z , and $\max_z$ returns the largest value of a quantity in z .

In practice, one has a number of remaining options for actually evaluating Eq. (35). For the evaluation of Eq. (41), it is found to be useful to densely sample, on log-scale, z near 0 and 1. The coordinate freedom associated with x_0 introduces oscillations into $\xi(z)$ which must be well resolved for the accurate determination of K_0 and K_2 . Similarly, for the evaluation of Eq. (35), one will want to densely sample near the boundaries; it is found that $z = \frac{1}{2}(1 + \sin(t\pi/2))$, with t uniformly spaced on $(-1, 1)$, is sufficient for this purpose. For the determination of K_0 and K_2 , and for the evaluation of Eq. (35), it is found that at least 10^4 points in z are needed for robust results. With these details in hand, it is now possible to compute the integration path, $\xi(z)$, and related scalar-products, Eq. (35), using, e.g., 5th order spline integration [149].

Figure 1 shows three example integration paths, $\xi(z)$, set against contours of the weight function, $W(\xi)$, in the

complex plane. In focus is the effect of BH spin a . The result for $a = 0$ is qualitatively similar to that for $a = 0.7$. Branch cuts are visible as overlapping horizontal contours along the real line for $\xi < 0$ and $\xi > 1$. The shape of integration paths are seen to vary significantly with increasing BH spin, changing most drastically near the extremal limit, $a = M$.

For $a = 0.99$ (the central panel), the integration path appears to cross the branch cut¹³ many times. The top panel of Fig. 2 shows this in detail. There, the apparent crossing of branch cuts is an artifact of presentation: $\xi(z)$ actually runs along all branches of the Riemann surface defined by $W(z)$, but in Fig. 1, only a single branch of $W(\xi)$ is shown [150]. This can be understood by noting that the transformed scalar-product, Eq. (35), uses an integration path that follows the real line, where there are no branch cuts. This is shown explicitly in the bottom panel of Fig. 2.

B. Analytic continuation

The scalar product developed in Sec. IV is,

$$\langle a \rangle b = \int_0^1 a(\xi) b(\xi) \xi^{B_0} (1 - \xi)^{B_1} e^{\frac{B_2}{1-\xi}} d\xi. \quad (42)$$

¹³For complex numbers y , We define the principle branch of $\log(y)$ by taking $\arg(y) \in (-\pi, \pi)$ and place the branch cut on the line defined by $\Re(y) < 0$. As seen in Fig. 1, this convention results in a branch cut along $\text{Re} \xi < 0$ (for powers of ξ), and another along $\text{Re} \xi > 1$ (for powers of $1 - \xi$).

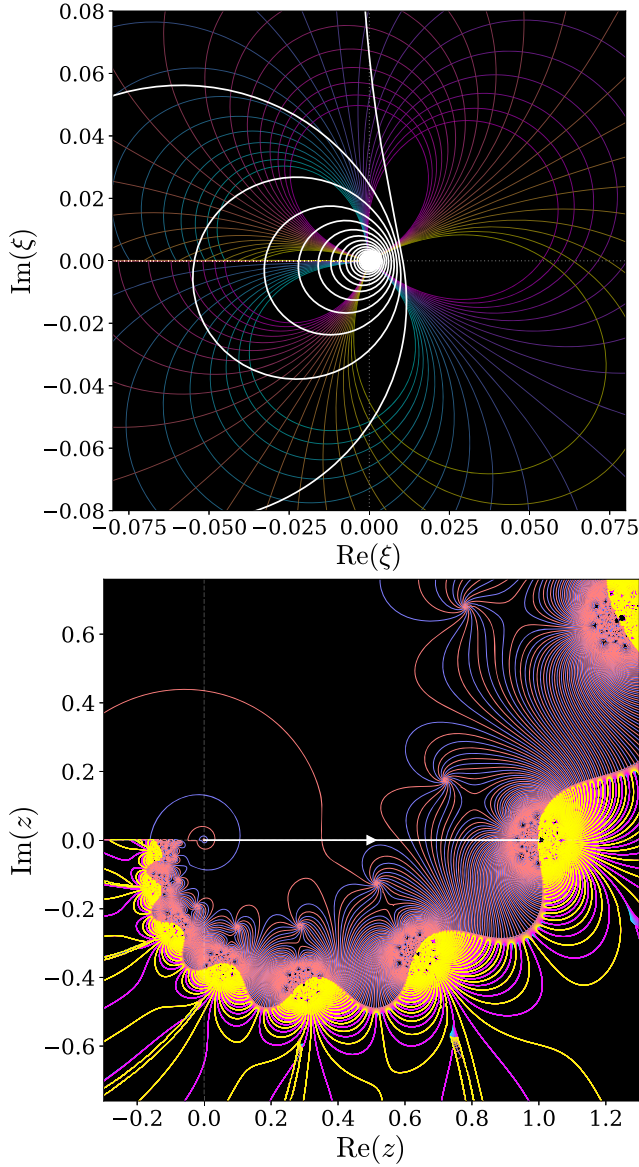


FIG. 2. Examples integration paths (white curves) against weight function contours (colored contours) for the $(s, \ell, m, n) = (-2, 2, 2, 3)$ QNM at BH spin $a/M = 0.99$. (top) A zoom-in of the central panel of Fig. 1 with axes centered around the BH horizon at $\xi = 0$. (bottom) Visualization of the coordinate transformed integration path and effective weight function for the same case. Contour formatting and coordinate parameters are identical to those in Fig. 1.

The method described in the previous section uses coordinate freedom to extend the domain upon which Eq. (42) can be evaluated. This is the basic purpose of the set of techniques known as *analytic continuation* [100,151]. It happens that there is a more direct, and ultimately more robust, route to evaluating Eq. (42) with analytic continuation.

This alternative route begins with the observation that if $a(\xi) = b(\xi) = 1$, then the left-hand side of

Eq. (42) coincides exactly with the integral representation of the Tricomi¹⁴ confluent hypergeometric function [95,106,110,148,152],

$$\int_0^1 \xi^{B_0} (1 - \xi)^{B_1} e^{\frac{B_2}{1-\xi}} d\xi = e^{B_2} \Gamma(B_0 + 1) U(1 + B_0, -B_1, -B_2). \quad (43)$$

In Eq. (43), $\Gamma(x)$ is the usual Euler-Bernoulli Gamma function, and $U(x, y, z)$ is the Tricomi confluent hypergeometric function,

$$U(x, y, z) = \frac{\Gamma(1 - y)}{\Gamma(x - y + 1)} M(x, y, z) + \frac{\Gamma(y - 1)}{\Gamma(x)} z^{1-y} M(x - y + 1, 2 - y, z). \quad (44)$$

Both may be evaluated for complex inputs by considering the appropriate integration contour [153]. Finally, in Eq. (44), $M(x, y, z)$ is the confluent hypergeometric function of the first kind (See, e.g., Eq. (13.2.42) of Ref. [152]).

The gamma function is constructed to extend the factorial function to the complex numbers. The confluent hypergeometric function is a product of gamma functions and a power series in z , and so it is straightforward to evaluate.

This parade of special functions has the following use: Every $a(\xi)$ and $b(\xi)$ of interest to Eq. (42) will be smooth, and so possesses a series expansion,

$$a(\xi) = \sum_{j=0} a_j \xi^j \quad \text{and} \quad b(\xi) = \sum_{k=0} b_k \xi^k. \quad (45)$$

This allows every scalar product to be expressed as a sum over scalar products of monomials,

$$\langle a \rangle b = \sum_{j,k} a_j b_k \langle \xi^{j+k} \rangle. \quad (46)$$

In Eq. (46), $\langle \xi^p \rangle$ is the p th *monomial moment* defined as,

$$\langle \xi^p \rangle = \int_0^1 \xi^p W(\xi) d\xi \quad (47a)$$

$$= \int_0^1 \xi^{B_0+p} (1 - \xi)^{B_1} e^{\frac{B_2}{1-\xi}} d\xi \quad (47b)$$

$$= e^{B_2} \Gamma(B_0 + p + 1) U(1 + B_0 + p, -B_1, -B_2). \quad (47c)$$

In Eq. (47a), we have effectively defined $\langle \xi^p \rangle$ to be $\langle 1 \rangle \xi^p$. In Eq. (47b), we have applied the definition of the scalar

¹⁴The nomenclature for these functions appears to be particularly inconsistent across sources. Some readers may find Refs. [152] and [148] of particular use.

product, and in Eq. (47c), we have equated the integral over ξ with the product of special functions found in Eq. (43), but with $B_0 \rightarrow B_0 + p$. The result is that the scalar product of any two smooth functions may be equated with a sum over confluent hypergeometric terms,

$$\langle a \rangle b = e^{B_2} \sum_{j,k} a_j b_k \Gamma(B_0 + j + k + 1) \times U(1 + B_0 + j + k, -B_1, -B_2). \quad (48)$$

In practice, it is useful to precompute monomial moments using Eq. (47), and then compute scalar products using Eq. (46). The monomial moments are defined up to an overall constant. Henceforth, we chose to normalize all $\langle \xi^p \rangle$ such that $\langle \xi^1 \rangle = 1$,

$$\langle \xi^p \rangle \leftarrow \langle \xi^p \rangle / \langle \xi^1 \rangle. \quad (49)$$

VI. LIMITATIONS OF DIRECT INTEGRATION AND ANALYTIC CONTINUATION

Direct integration, analytic continuation, and their application to monomial moments and confluent Heun polynomials are all subject to various caveats and limitations. Here, these facets are reviewed. At times, technical details are omitted for brevity, but particular attention is given to practical workarounds. While analytic continuation and direct integration are equivalent when their respective caveats are satisfied, they are otherwise found to be complementary. Ultimately, analytic continuation is determined to be easier to use in practice.

A. Direct Integration: Disallowed BH spins, and the zero frequency limit

The effective weight function introduced in Sec. VA was

$$W_{\text{eff}}(z) = e^{\frac{B_2 x_2}{x_0 - 1}} z^{x_0(B_0 + 1) - 1} (1 - z^{x_0})^{B_1} \times (1 - (x_2 + 1)z^{x_0})^{-B_0 - B_1 - 2}. \quad (50)$$

In Sec. VA, a direct integration approach was introduced to evaluate scalar product using $W_{\text{eff}}(z)$. This approach has two limiting features: (i) for a given value of $\tilde{\omega}$, there are values of the BH spin, a , such that an integration path cannot be constructed, and (ii) when $\tilde{\omega} = 0$, the weight function may diverge at $\xi = 1$.

For any value of $\tilde{\omega} = \omega - i/\tau$, there may exist values of the BH spin,

$$a = a_*, \quad (51)$$

such that an integration path cannot be constructed as described in Sec. VA. This limitation results from the requirement that $\text{Re}(x_0) > 0$.

In Sec. VA, the coordinate parameter x_0 was introduced so that $W_{\text{eff}}(z)$ is zero at $z = 0$. The value found for x_0 to have that effect was,

$$x_0 = K_0 \frac{\text{sgn}(\text{Re}(B_0) + 1)}{\text{Re}(B_0) + 1} + i \frac{-1 + K_0[\text{sgn}(\text{Re}(B_0) + 1) - 1]}{\text{Im}(B_0)} \quad (52)$$

where $\text{sgn}(x)$ is defined by $x = \text{sgn}(x)|x|$, and

$$B_0 = -\left(\frac{2M(\delta + M)}{\tau\delta} + s\right) + i\left(\frac{am - 2M^2\omega}{\delta} - 2M\omega\right). \quad (53)$$

and $K_0 > 0$. As seen in Eq. (52), the real part of x_0 is generally regular, since $\text{sgn}(\text{Re}(B_0) + 1)$ is defined to be zero when $\text{Re}(B_0) + 1$ is zero. However, the imaginary part of x_0 may diverge if $\text{Im}(B_0) = 0$. If we had constructed x_0 differently, e.g., by allowing $\text{Re}(x_0)$ to depend on both $\text{Re}(B_0)$ and $\text{Im}(B_0)$, then the requirement that $\text{Re}(x_0) > 0$ would still have resulted in a $1/\text{Im}(B_0)$ divergence.

Taking ω as known, there may exist values of the BH spin, namely a_* , such that

$$\text{Im}(B_0)|_{a=a_*} = 0 \quad (54a)$$

Applying the definition of B_0 [see Eq. (53)], and recalling that $\delta = \sqrt{M^2 - a^2}$ allows Eq. (54a) to be solved for a_* ,

$$a_* = \frac{4mM^2\omega}{m^2 + 4M^2\omega^2}. \quad (54b)$$

For physical applications, it happens that for any QNM with (ℓ, m, n) , having frequency

$$\tilde{\omega}(a) = \omega(a) - i/\tau(a), \quad (55)$$

there is a discrete set BH spins where ω is such that $a = a_*$. For a given QNM ℓ, m and n , this set may be defined by first evaluating $\omega(a)$ using e.g., `positive.physics.qnmobj` [124], and then searching for values of a such that

$$|a - a_*(\tilde{\omega}(a))| = 0. \quad (56)$$

For $(\ell, m, n) = (2, 2, 0)$, there is only one such BH spin,

$$a_* \approx 0.996632. \quad (57)$$

It is instructive to contemplate the behavior of x_0 when a is above this critical value.

Toward extremal BH spin, $a \rightarrow 1$, x_0 typically remains regular, meaning that an integration path, $\xi(z)$, may still be constructed. This is a result of how x_0 depends on B_0 . Since $\lim_{a \rightarrow 1} |\text{Re}(B_0)| \rightarrow \infty$, $\text{Re}(x_0)$ tends to zero in that same limit. Similarly, it may be deduced that $\text{Im}(x_0)$ tends to zero in that limit. Recall from Eq. (34) that the integration path, $\xi(z)$, is not defined when $x_0 = 0$. Despite this, the reasoning above implies that $\xi(z)$ may be constructed in the near extremal limit, within the limits of numerical precision; i.e., when x_0 is small, but strictly nonzero.

Lastly, a second coordinate parameter, x_2 , was introduced so that exponential divergence of the weight function was avoided (see Sec. VA). The full expression for x_2 was

$$x_2 = \frac{K_2 x_0}{B_2} \quad (58a)$$

$$= \frac{K_2 x_0}{4i\delta\tilde{\omega}}. \quad (58b)$$

In Eq. (58b), we have used the fact that $B_2 = 4i\delta\tilde{\omega}$. Since x_2 is proportional to $\tilde{\omega}^{-1}$, it generally diverges in the zero frequency limit. However, since only $x_2 B_2 = K_2 x_0$ appears in the weight function (see, e.g., Eq. (50), $W_{\text{eff}}(z)$ may still be evaluated.

Further, in the zero frequency limit there is no exponential divergence of the weight function at $\xi = 1$, and so there is no formal need for x_2 *at all* in that limit. Seeing as

$$W_{\text{eff}}(z)|_{\tilde{\omega}=0} \propto (1 - z^{x_0})^{B_1}, \quad (59)$$

W_{eff} will have a pole at $z = 1$ if $\text{Re}(B_1) < 0$. This is of no concern when $\tilde{\omega} \neq 0$, since x_2 is designed to enforce exponential damping at $z = 1$; otherwise, when $\tilde{\omega} = 0$, it is possible to construct a different coordinate, z' , such that this pole is avoided. This work-around may *only* be viable when $\tilde{\omega} = 0$; otherwise, it is found that when $\tilde{\omega} \neq 0$, the required coordinate change necessarily introduces oscillations in $\xi(z')$ that grow exponentially as z increases. Thus, when $\tilde{\omega} = 0$, one can only use coordinate freedom to either avoid the essential singularity at $\xi(z) = 1$ (i.e., that resulting from the exponential), *or* the pole at $\xi(z) = 1$, but not both.

In this, direct integration's underlying limitation is one of mathematical perspective: it happens to be more robust to use analytic continuation via integration over closed contours, rather than singularity avoiding integration over finite paths as has been discussed above. For example, Ref. [100] details how single and double Pochhammer contours may be used in similar mathematical settings. This approach makes use of standard theorems in complex analysis, and is ultimately equivalent to analytic continuation as outlined in Sec. VB.

B. Analytic continuation and limitations of the gamma function

It was found in Sec. VB that scalar products may be evaluated analytically using monomial moments, $\langle \xi^p \rangle$,

$$\langle \xi^p \rangle = \Gamma(1 + B_0 + p)U(1 + B_0 + p, -B_1, -B_2). \quad (60a)$$

The qualitative behavior of the monomial moments can be described by

$$\langle \xi^p \rangle \approx \frac{\Gamma(1 + B_0 + p)\Gamma(1 + B_1)}{\Gamma(2 + B_0 + p + B_1)}, \quad (60b)$$

where the leading term in the power series definition for $U(1 + B_0 + p, -B_1, -B_2)$ has been used to approximate it with $\Gamma(1 + B_1)/\Gamma(2 + B_0 + p + B_1)$ [152]. Since $B_2 = 4i\delta\tilde{\omega}$, Eq. (60b) is exact in the zero frequency limit, $\tilde{\omega} = 0$, or in the extremal spin limit, $\delta = \sqrt{M^2 - a^2} = 0$.

For either Eq. (60a) or Eq. (60b), $\langle \xi^p \rangle$ will be regular when instances of the gamma function are regular. Since $\Gamma(z)$ is well known to have simple poles when z is a negative integer, it follows that $\langle \xi^p \rangle$ can diverge when,

$$1 + B_0 + p \in \{z \in \mathbb{Z} | z < 0\}, \quad (61a)$$

and

$$1 + B_1 \in \{z \in \mathbb{Z} | z < 0\}. \quad (61b)$$

In Eq. (61), $\{z \in \mathbb{Z} | z < 0\}$ denotes the set of all negative integers. The practical implications of Eq. (61) may be investigated by rewriting B_0 and B_1 in terms of physical parameters,

$$p - s + 1 - \frac{2M(\delta + M)}{\tau\delta} + \frac{i(am - 2M\omega(\delta + M))}{\delta} = -z_0, \quad (62a)$$

$$2s + 1 - \frac{4M}{\tau} - 4iM\omega = -z_1. \quad (62b)$$

In Eq. (62), z_0 and z_1 are natural numbers, and we have used Eq. (8), Eqs. (16) and (24) to expand in terms of basic physical quantities. Equation (62a) is equivalent to Eq. (61a), and Eq. (62b) is equivalent to Eq. (61b).

Inspection of Eqs. (62a) and (62b) reveals that they can only hold when $M\tilde{\omega}$ is *purely imaginary* (i.e., $\omega = 0$). This limits the physical relevance of these equations to the total transmission QNMs [121]. Furthermore, Eq. (62a) can only be true if $a = 0$, thus it is only relevant to the algebraically special modes of Schwarzschild [52,121].

Both Eqs. (62a) and (62b) have nontrivial implications for evaluation of the scalar product with analytic continuation. If Eq. (62a) holds, then there exist some monomials, ξ^p , such that the respective monomial moments are infinite when defined by Eq. (60a). If Eq. (62b) holds, then no monomial moments are finite according to Eq. (60a).

These limitations are, again, a result of mathematical perspective: evaluation of the scalar product with Gamma

and confluent hypergeometric functions may fail, depending on $\text{Re}(B_0)$ or $\text{Re}(B_1)$; however, we saw in the previous section that the viability of direct integration only depends on $\text{Im}(B_0)$. Thus the two perspectives have complementary limitations.

C. Direct integration vs analytic continuation

In practice, analytic continuation has the following relative attributes that may make it preferable in most circumstances. Since analytic continuation does not require the determination of optimal coordinate parameters (see Sec. VA), its implementation requires fewer steps. In addition, analytic continuation requires monomial moments to compute scalar products, and does not require the storage of arrays [discretizations of $\xi(z)$] of length 10^5 or greater (see Sec. VA). Instead, monomial moments with degrees up to twice that of the largest polynomial order of interest may be precomputed, ideally using arbitrary precision arithmetic [126], and then used for all scalar products (Sec. VB). Thus, while direct integration may serve as a valuable alternative, particularly in cases where the Gamma function cannot be evaluated, analytic continuation is expected to be structurally simpler and computationally more efficient.

All forthcoming numerical results will make use of analytic continuation with 64 digits of precision via the public MPMATH package [126].

VII. PEDAGOGICAL APPLICATIONS

Previous sections provide a foundation for applying the scalar product, Eq. (63), to physical scenarios in which the series expansion for radial functions is either known (analytic continuation), or unknown (direct integration). Direct integration is applicable in both scenarios.

$$\langle a \rangle b = \int_0^1 a(\xi) b(\xi) \xi^{B_0} (1 - \xi)^{B_1} e^{\frac{B_2}{1-\xi}} d\xi. \quad (63)$$

Before leaping into applications to Teukolsky's radial equation, there are good reasons to first look closely at two pedagogical cases.

In this section, we will first apply the scalar product to the monomial moments for different QNM values of the frequency $\tilde{\omega}$, paying particular attention to the effect of spin weight, s . For a broad range of scenarios, monomial moments with $s = 2$ decrease monotonically with monomial order, while this is less so the case for $s = -2$. We will then apply the scalar product to the confluent Heun polynomials.

Previously, Cook and Zalutskiy considered confluent Heun polynomials for BHs in Refs. [102,121]. There, the polynomials were defined such that a special choice $\tilde{\omega}$ resulted in $\mathcal{L}_\xi$ Eq. (15)] having polynomial eigenfunctions.

These are the well known algebraically special and total transmission QNMs [28,52,102,121,154,155].

Here, we take a different perspective: We will *a priori* define $\tilde{\omega}$ to be *any* QNM frequency.¹⁵ This requires that, in effect, we redefine the potential of $\mathcal{L}_\xi$ such that its eigenfunctions are polynomials. The consequences of this choice enable confluent Heun polynomials to be defined for arbitrary frequencies; in turn, this will have a significant impact on forthcoming results and discussion.

We will use the scalar product to derive the orthogonality between confluent Heun polynomials of *like order*. Lastly, we will use the scalar product to derive a relationship between the confluent Heun polynomials' eigenvalues, and a specific average of ξ . This relationship provides some insight into the structure of the eigenvalues.

A. Monomial moments

The scalar-product of any two smooth functions may be evaluated using a linear combination of monomial moments,

$$\langle a \rangle b = \sum_{j,k} a_j b_k \langle \xi^{j+k} \rangle. \quad (64)$$

For many common scalar products, monomial moments $\langle \xi^p \rangle$ decrease monotonically with p . This is easily demonstrated by considering a *hypothetical* constant weight function

$$W(\xi)_{\text{hypothetical}} = 1, \quad (65a)$$

$$\langle a \rangle b_{\text{hypothetical}} = \int_0^1 a(\xi) b(\xi) d\xi, \quad (65b)$$

$$\langle \xi^p \rangle_{\text{hypothetical}} = (p+1)^{-1} \sim p^{-1}. \quad (65c)$$

In Eq. (65a), a weight function of 1 results in a scalar product equivalent to that used for the shifted Legendre polynomials, Eq. (65b). In turn, this results in monomial moments that are asymptotically equivalent to one over the degree of the monomial, Eq. (65c). In practice, this means that if the sequences comprised of a_j and b_k are convergent, then so too is $a_j b_k \langle \xi^{j+k} \rangle$. We will now examine this argument in the context of the QNM's scalar product. Our goal is to gain a basic and qualitative understanding of how monomial moments depend on two physical parameters: s and $M\tilde{\omega}$.

We previously observed that monomial moments may be computed via analytic continuation of the confluent hypergeometric function, U , and the gamma function, Γ ,

¹⁵In the language of Ref. [100], we choose to focus on Class I of confluent Heun polynomials.

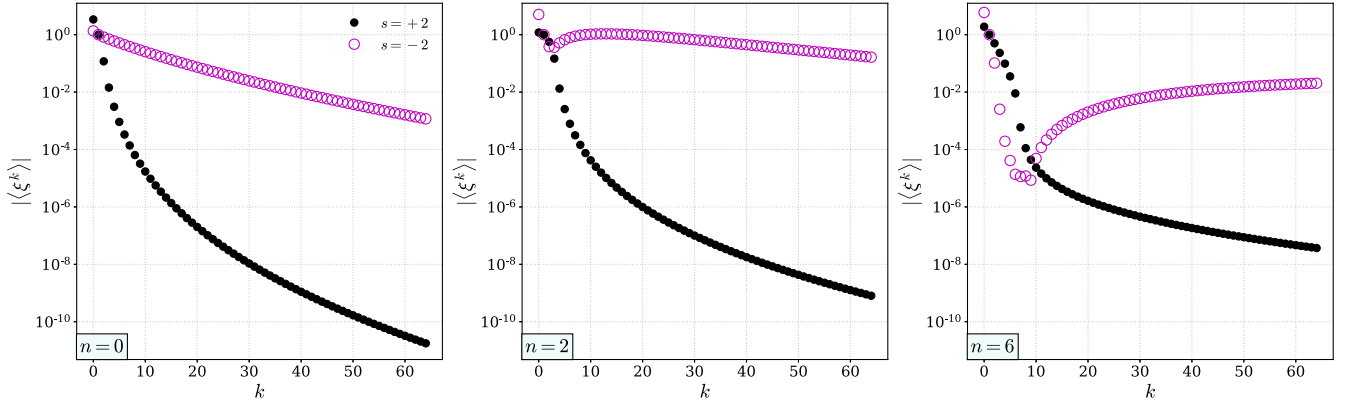


FIG. 3. Example distributions of absolute monomial moments, $|\langle \xi^k \rangle|$ v.s. monomial order, k , for the $(\ell, m) = (2, 2)$ QNM at BH dimensionless spin of $a/M = 0.7$. Left, central and right panels show results for n of 0, 2, and 6, respectively. The corresponding QNM frequencies are $M\tilde{\omega} = 0.5326 - 0.0808i$, $M\tilde{\omega} = 0.4999 - 0.4123i$, and $M\tilde{\omega} = 0.4239 - 1.0954i$, respectively. At focus is the effect of negating the spin weight, s , for increasing overtone label. For all cases shown, $|\langle \xi \rangle| = 1$ has been enforced. Closed circles show points for $s = +2$, and open circles show points for $s = -2$.

$$\langle \xi^p \rangle = \Gamma(1 + B_0 + p)U(1 + B_0 + p, -B_1, -B_2) \quad (66)$$

The confluent hypergeometric function, U , is defined as a series expansion in its third input [152]. Thus some understanding of $\langle \xi^p \rangle$ may be found by examining U expanded in powers of B_2 . Examination of the zeroth order term happens to be sufficient for *qualitative* understanding of how $\langle \xi^p \rangle$ behaves when p is large. For precise and predictive understanding, one would have to extend the forthcoming analysis to higher orders. This will not be done here so that the following discussion remains fairly simple.

To zeroth order in its third input, U takes the approximate form,

$$U(x, y, z) \approx \frac{\Gamma(1 - y)}{\Gamma(x - y + 1)} + \mathcal{O}(z). \quad (67)$$

When applied to Eq. (66), this yields

$$\langle \xi^p \rangle \approx \frac{\Gamma(1 + B_0 + p)\Gamma(1 + B_1)}{\Gamma(2 + B_0 + p + B_1)} \quad (68a)$$

$$\sim p^{-1-B_1}\Gamma(1 + B_1). \quad (68b)$$

To go from Eq. (68a) to Eq. (68b), we have used the fact that, as $p \rightarrow \infty$, $\Gamma(p + \alpha) \sim p^\alpha \Gamma(p)$ [110]. The physics of the situation is revealed by using Eqs. (24) and (16) to rewrite B_1 in terms of M , $\tilde{\omega}$, and s ,

$$\langle \xi^p \rangle \sim p^{-1-2s+4iM\tilde{\omega}} \quad (69a)$$

$$\sim p^{-1-2s+4M/\tau} e^{4iM\omega \ln(p)}. \quad (69b)$$

In Eq. (69a), we see that the previous result of p^{-1} is effectively generalized to depend on the spin weight, s , and the dimensionless frequency, $M\tilde{\omega}$. Since the net exponent

of p is now complex valued, it is the real part of the exponent that determines the large- p behavior of $|\langle \xi^p \rangle|$. In Eq. (69b), this is shown explicitly by using $\tilde{\omega} = w - i/\tau$. Recall that τ is strictly positive as is needed for time domain stability.

The presence of $e^{4iM\omega \ln(p)}$ in Eq. (69b) means that, in general, the real and imaginary parts of $\langle \xi^p \rangle$ will trace out a logarithmic spiral.

The presence of $p^{-1-2s+4M/\tau}$ within Eq. (69b) allows us to discern three additional features:

- (i) There exist some combination of s and M/τ such that monomial moments either decrease, asymptote to a constant, or increase as p tends toward infinity.
- (ii) Since Eq. (69b) depends on p^{-2s} , monomial moments for $s = +2$ will be much more inclined to decrease as $p \rightarrow \infty$

$$\langle \xi^p \rangle|_{s=+2} \sim p^{-8} \langle \xi^p \rangle|_{s=-2} \quad (70)$$

In Eq. (70), we have used the fact that if $\tilde{\omega}$ is a QNM frequency, then $\tilde{\omega}|_{s=+2} = \tilde{\omega}|_{s=-2}$.

- (iii) Lastly, the presence of $4M/|\tau|$ in Eq. (69b) means that, as τ decreases, the rate of change of $\langle \xi^p \rangle$ will typically increase. This is perhaps most relevant for the QNM overtones, for which $|\tau|$ decreases as the overtone label, n , increases [52,71].

These three qualitative features are visible in Fig. 3:

- (i) In the left panel ($n = 0$), and for $s = -2$, we see that the monomial moments $|\langle \xi^p \rangle|$ decrease monotonically with p . However, for $s = -2$ and $n \in \{2, 6\}$, moments are nonmonotonic, and at $n = 6$, they increase as p increases.
- (ii) In each panel, we see that monomial moments for spin weight $s = +2$ are significantly depressed relative to those for $s = -2$.

- (iii) The scale of each panel is identical so that they effect of increasing overtone label (i.e., decreasing τ) is easily visible for both spin weights.

The fact that absolute monomial moments, $|\langle \xi^p \rangle|$, are not strictly decreasing with p has implications for the numerical behavior of scalar products. Naively, it might seem that, if $|\langle \xi^p \rangle|$ increases with p , then scalar products might diverge. This happens to not be the case because the monomial moments are generally complex valued.

We have previously noted that the real and imaginary parts of $\langle \xi^p \rangle$ will trace out a logarithmic spiral in p . This implies that linear combinations of monomial moments will, in general, have a telescoping quality: as p increases, successive terms will interfere destructively, meaning that although $|\langle \xi^p \rangle|$ may increase, its net contribution must be considered along with adjacent terms. This telescoping quality behoves the use of arbitrary precision (e.g., MPMATH [126]) when evaluating moments by analytic continuation¹⁶ [95,100]. However, this quality also prompts a question.

Since the weight function, $W(\xi)$, allows the computation of arbitrary monomial moments, the monomials are a basis for the scalar product space defined by Eq. (64) [109,147]. Thus, it is fair to wonder whether there exists a related basis that is more suitable for numerical approximation? This is the basic question that links the study of monomial moments to applied polynomial theory [156]. A complete investigation of this question is beyond the current scope. Nevertheless, a first step toward an answer may be found by applying the scalar product to *confluent Heun polynomials* [28,102,104,121,152,157].

B. Confluent Heun polynomials

The differential operator that results from applying QNM boundary conditions to Teukolsky's radial equation was

$$\mathcal{L}_\xi = (C_0 + C_1(1 - \xi)) \quad (71a)$$

$$+ (C_2 + C_3(1 - \xi) + C_4(1 - \xi)^2) \partial_\xi \quad (71b)$$

$$+ \xi(\xi - 1)^2 \partial_\xi^2. \quad (71c)$$

Since the operator given in Eq. (71) is of the confluent Heun type, its eigensolutions are the aptly named confluent Heun functions [100,104].

The subset of these with terminating series solutions are, unsurprisingly, named confluent Heun polynomials [100,121,152,157].

A confluent Heun polynomial of order p is typically considered to result from special values of C_0 and C_1 [121,157]. In the context of Teukolsky's radial equation, these special values have been interpreted as a constraint

on $\tilde{\omega}$. This is the case for the algebraically special modes of Schwarzschild and the total transmission modes of Kerr [52,102,121]. Here, a different perspective on the confluent Heun polynomial conditions for BHs will be used. This perspective is based on two observations:

- (i) Classical polynomials are typically defined to be eigenfunctions of a purely differential operator (i.e., one with no ∂_ξ^0 term). For the confluent Heun operator, Eq. (71), this purely differential part is denoted by Eqs. (71b), and (71c) and repeated below in Eq. (72),

$$\mathcal{D}_\xi = (C_2 + C_3(1 - \xi) + C_4(1 - \xi)^2) \partial_\xi + \xi(\xi - 1)^2 \partial_\xi^2. \quad (72)$$

- (ii) Any attempt to construct order p polynomials that are eigenfunctions of $\mathcal{D}_\xi$ requires that the “eigenvalue” be a *linear* function of the domain variable,

$$\mathcal{D}_\xi \mathbf{y}_{pk}(\xi) = \sigma_{pk}(\xi) \mathbf{y}_{pk}(\xi), \quad (73a)$$

$$\sigma_{pk}(\xi) = \lambda_{pk} + \mu_p \xi. \quad (73b)$$

Plainly, $\sigma_{pk}(\xi)$ is not an eigenvalue in the traditional sense. Instead, it is an instance of a *two parameter* eigenvalue problem [100,158]. We will see that μ_p is uniquely determined by the order of the polynomial, meaning that it need not be conceptually separated from the traditional eigenvalue, λ_{pk} .

In what follows, we will apply these ideas to construct confluent Heun polynomials that are valid for *any* value of $\tilde{\omega}$ (i.e., *any* QNM frequency). We will see that the operator, $\mathcal{D}_\xi$, is self-adjoint under the same scalar product that we have discussed previously. We will then use the self-adjointness of this operator to prove and numerically demonstrate the orthogonality of confluent Heun polynomials for *fixed* polynomial order. Lastly, we will use self-adjointness to derive an expression for the confluent Heun polynomials' eigenvalues in terms of ratios of scalar products.

1. Construction

Let $\mathbf{y}_{pk}(\xi)$ be a confluent Heun polynomial of order p ,

$$\mathbf{y}_{pk}(\xi) = \sum_{q=0}^p a_{pkq} \xi^q. \quad (74)$$

We will soon see that the label, k , is useful because there are will generally be $p + 1$ order p polynomials.

The polynomial coefficients, a_{pkq} , are determined by applying the form above for $\mathbf{y}_{pk}(\xi)$ to the eigenrelationship given by Eq. (73a). One then asserts that the coefficients of each power in ξ must be equal on both sides of Eq. (73a).

¹⁶Leaver came to the same conclusion during his analysis of series expansions in ξ , which he refers to as Jaffé-type expansion [95].

The result is a three term recursion relationship or, equivalently, a discrete eigenvalue relationship. To simplify our notation, we will henceforth drop two indices, p and k , from a_{pkq} ,

$$a_q \leftarrow a_{pkq}. \quad (75)$$

Thus the discrete eigenvalue relationship takes the schematic form,

$$a_{q-1} \alpha_q + a_q \beta_q + a_{q+1} \gamma_q = \lambda_{pk} a_q, \quad (76)$$

where

$$\alpha_q = -(q-1)(C_4 + q - 2) - \mu_p \quad (77a)$$

$$\beta_q = q(C_3 + 2(C_4 + q - 1)) \quad (77b)$$

$$\gamma_q = -(q+1)(C_2 + C_3 + C_4 + q). \quad (77c)$$

An initial boundary condition,

$$a_0 \beta_0 + a_1 \gamma_0 = \lambda_{pk} a_0, \quad (78)$$

is imposed for consistency with Eq. (74). In Eq. (78), $a_0 = 1$ is a common initial normalization choice.

A necessary but insufficient condition for the three-term recursion to terminate at the p th term, is simply that $\alpha_{p+1} = 0$. This is the so-called $\Delta_{p+1} = 0$ condition [121,152]. In our notation, this takes the form,

$$\alpha_{p+1} = p(p + C_4 - 1) - \mu_p = 0. \quad (79)$$

In turn, this can be interpreted as a constraint that allows the determination of μ_p ,

$$\mu_p = p(p + C_4 - 1). \quad (80)$$

Thus μ_p must be nonzero for the existence of nontrivial polynomial solutions to Eq. (73a).

Furthermore, if μ_p is given by Eq. (80), then the discrete eigenvalue relationship given by Eq. (76) defines a finite dimensional matrix eigenvalue problem¹⁷ [100,121,152]. The matrix of interest, $\hat{\mathbf{A}}_p$, is asymmetric and tridiagonal,

$$\hat{\mathbf{A}}_p = \begin{pmatrix} \beta_0 & \gamma_0 & 0 & \cdots & 0 \\ \alpha_1 & \beta_1 & \gamma_1 & \ddots & 0 \\ 0 & \alpha_2 & \beta_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \gamma_{p-1} \\ 0 & 0 & \cdots & \alpha_p & \beta_p \end{pmatrix}. \quad (81)$$

By construction, the eigenvectors of $\hat{\mathbf{A}}_p$ are the polynomial coefficients, and the eigenvalues of $\hat{\mathbf{A}}_p$ are the polynomial eigenvalues,

$$\{a_{pkq}\}_{q=0}^p = k^{\text{th}} \text{ Eigenvector of } \hat{\mathbf{A}}_p, \quad (82a)$$

$$\lambda_{pk} = k^{\text{th}} \text{ Eigenvalue of } \hat{\mathbf{A}}_p. \quad (82b)$$

Since $\hat{\mathbf{A}}_p$ is a $(p+1) \times (p+1)$ matrix, there will be $p+1$ eigenvalues, λ_{pk} . If the eigenvalues are unique, then we may impose an order on $\mathbf{y}_{pk}$, whereby

$$k \in \{0, 1, 2, \dots, p\}. \quad (83)$$

In general, the $p+1$ values of λ_{pk} will correspond to the roots of $\hat{\mathbf{A}}_p$'s characteristic polynomial. Lacking a deeper understanding how the roots of all such polynomials are distributed, the following ordering will henceforth be used

$$\text{Im } \lambda_{p0} < \text{Im } \lambda_{p1} < \text{Im } \lambda_{p2} < \dots < \text{Im } \lambda_{pp}. \quad (84)$$

In this sense, for a given order p , there is a *multiplex* comprised of $p+1$ solutions, where each solution is labeled by the *multiplex index*, k .

2. Orthogonality between polynomials of like order

If each λ_{pk} is unique, then each corresponding polynomial, $\mathbf{y}_{pk}$, will be linearly independent [139]. Since $\mathcal{L}_\xi = \mathcal{D}_\xi + (C_0 + (1-\xi)C_1)$ is self-adjoint with respect to the scalar product, Eq. (64), $\mathcal{D}_\xi$ is also self-adjoint,

$$\mathcal{D}_\xi = \mathcal{D}_\xi^\dagger. \quad (85)$$

Here, we will use the self-adjointness of $\mathcal{D}_\xi$ to show that when λ_{pk} are unique, $\mathbf{y}_{pk}$ must be orthogonal. To that end we will use a standard argument that is based on the behavior of $\langle \mathbf{y}_{p'k'} | \mathcal{D}_\xi \mathbf{y}_{pk} \rangle$ when (p', k') swaps places with (p, k) . Along the way, we will encounter a twist that results from the fact that $\mathbf{y}_{pk}$ satisfy a nonstandard eigenvalue equation, Eq. (73a).

Let us begin by analyzing the matrix whose elements are $\langle \mathbf{y}_{p'k'} | \mathcal{D}_\xi \mathbf{y}_{pk} \rangle$. Since $\mathcal{D}_\xi$ is self-adjoint with respect to the scalar product Eq. (26)], we have that

¹⁷Note that, along with the series' boundary conditions (Eqs. (78)–(79)), it is the finite nature of the system of equations that justifies the eigenproblem interpretation. In contrast, non-polynomial solutions to the radial problem will not satisfy Eq. (79), meaning that, e.g., a continued fraction based method, and a related boundary condition at the infinite end of the series are required [66,95].

$$\langle \mathbf{y}_{p'k} \rangle \mathcal{D}_\xi \mathbf{y}_{pk} = \langle \mathcal{D}_\xi \mathbf{y}_{p'k} \rangle \mathbf{y}_{pk}. \quad (86)$$

Applying the eigenvalue relationship,

$$\mathcal{D}_\xi \mathbf{y}_{pk}(\xi) = (\lambda_{pk} + \mu_p \xi) \mathbf{y}_{pk}(\xi) \quad (87)$$

to both sides of Eq. (86) yields

$$\begin{aligned} \lambda_{pk} \langle \mathbf{y}_{p'k} \rangle \mathbf{y}_{pk} + \mu_p \langle \mathbf{y}_{p'k} | \xi | \mathbf{y}_{pk} \rangle \\ = \lambda_{p'k} \langle \mathbf{y}_{p'k} \rangle \mathbf{y}_{pk} + \mu_{p'} \langle \mathbf{y}_{p'k} | \xi | \mathbf{y}_{pk} \rangle. \end{aligned} \quad (88)$$

A useful rearrangement of Eq. (88) is given by Eq. (89),

$$(\lambda_{pk} - \lambda_{p'k}) \langle \mathbf{y}_{p'k} \rangle \mathbf{y}_{pk} = (\mu_{p'} - \mu_p) \langle \mathbf{y}_{p'k} | \xi | \mathbf{y}_{pk} \rangle. \quad (89)$$

The aforementioned *twist* is that, unlike in single parameter eigenvalue problems, the right-hand side of Eq. (89) is nonzero. Nevertheless, the standard orthogonality argument is recovered by restricting Eq. (89) to polynomials of like order,

$$p = p'. \quad (90)$$

The resulting version of Eq. (89) is,

$$(\lambda_{pk} - \lambda_{pk'}) \langle \mathbf{y}_{pk'} \rangle \mathbf{y}_{pk} = 0. \quad (91)$$

If λ_{pk} are distinct for different k , then Eq. (91) can only hold if

$$\langle \mathbf{y}_{pk'} \rangle \mathbf{y}_{pk} = \eta_{pk}^2 \delta_{k'k}. \quad (92)$$

In Eq. (92), η_{pk} is an inverse normalization constant. Henceforth, we will work with the *normalized confluent Heun polynomials*,

$$y_{pk} = \eta_{pk}^{-1} \mathbf{y}_{pk}, \quad (93)$$

where,

$$\langle y_{pk'} \rangle y_{pk} = \delta_{k'k}. \quad (94)$$

3. A scalar product expression for eigenvalues

For polynomials of unlike order, Eq. (89) may be recast as a relationship between polynomial eigenvalues. This is done by solving for λ_{pk} ,

$$\lambda_{pk} = \lambda_{p'k} + (\mu_{p'} - \mu_p) \frac{\langle y_{p'k} | \xi | y_{pk} \rangle}{\langle y_{p'k} \rangle y_{pk}}. \quad (95)$$

Equation (95) applies to all (p, k) and (p', k') , where $p \neq p'$. An instructive special case is that of

$$p' = 0. \quad (96)$$

This corresponds to y_{00} , which must be constant. If y_{00} is a constant, then for the eigenvalue relation Eq. (87) to hold, it must be true that

$$\lambda_{00} = \mu_0 = 0. \quad (97)$$

Applying Eq. (97) to Eq. (95) gives

$$\lambda_{pk} = \mu_p \frac{\langle y_{00} | \xi | y_{pk} \rangle}{\langle y_{00} \rangle y_{pk}}. \quad (98)$$

Recalling that $\mu_p = p(p + C_4 - 1)$, Eq. (98) is equivalent to

$$\lambda_{pk} = p(p + C_4 - 1) \frac{\langle y_{00} | \xi | y_{pk} \rangle}{\langle y_{00} \rangle y_{pk}}. \quad (99)$$

If y_{pk} are known, then Eq. (99) allows the computation of the related eigenvalues via the scalar product. In most instances this will not be needed to define the eigenvalues; however, Eqs. (95)–(99) are presented to provide a way to check consistency between evaluations of the scalar product, and the underlying confluent Heun system.

4. Numerical examples

Numerical computation of the confluent Heun polynomials may proceed by first computing C_2 , C_3 , and C_4 . Then, for a chosen polynomial order, one would compute $\hat{\mathbf{A}}_p$ [Eq. (81)], so that its eigenvalues and vectors may be determined. Since scalar products between polynomials is a weighted sum over monomial moments [Eq. (64)], it is useful to precompute and store all of the monomial moments relevant to a given polynomial order. This allows all polynomial scalar products to be evaluated without explicit integration [Eq. (47)]. From this perspective, numerical evaluation of confluent Heun polynomials may be accomplished on-the-fly via the appropriate sum over monomials [Eq. (74)]. Here, this procedure is used to provide select numerical examples.

Figure 4 shows example confluent Heun polynomials of orders $p \in \{0, 1, 2, 3\}$. Only the polynomials for which $p = k$ are shown; others have qualitatively similar behavior. As expected, the real and imaginary parts of each case behave as polynomials in ξ . However, since each polynomial has complex value coefficients, the polynomials' amplitudes and phases are nonpolynomial functions of ξ . Similarly, since the polynomials are complex valued, we should not expect the roots of their respective real and imaginary parts to be simultaneous. For example, the top left panel of Fig. 4 shows $\text{Re} y_{11}$ having a root near $\xi = 0.5$, but $\text{Im} y_{11}$ having no root at all on the real line.

Figure 5 shows the distribution of polynomial roots for three cases. In focus is the distribution of roots in the complex plane as polynomial order and overtone label

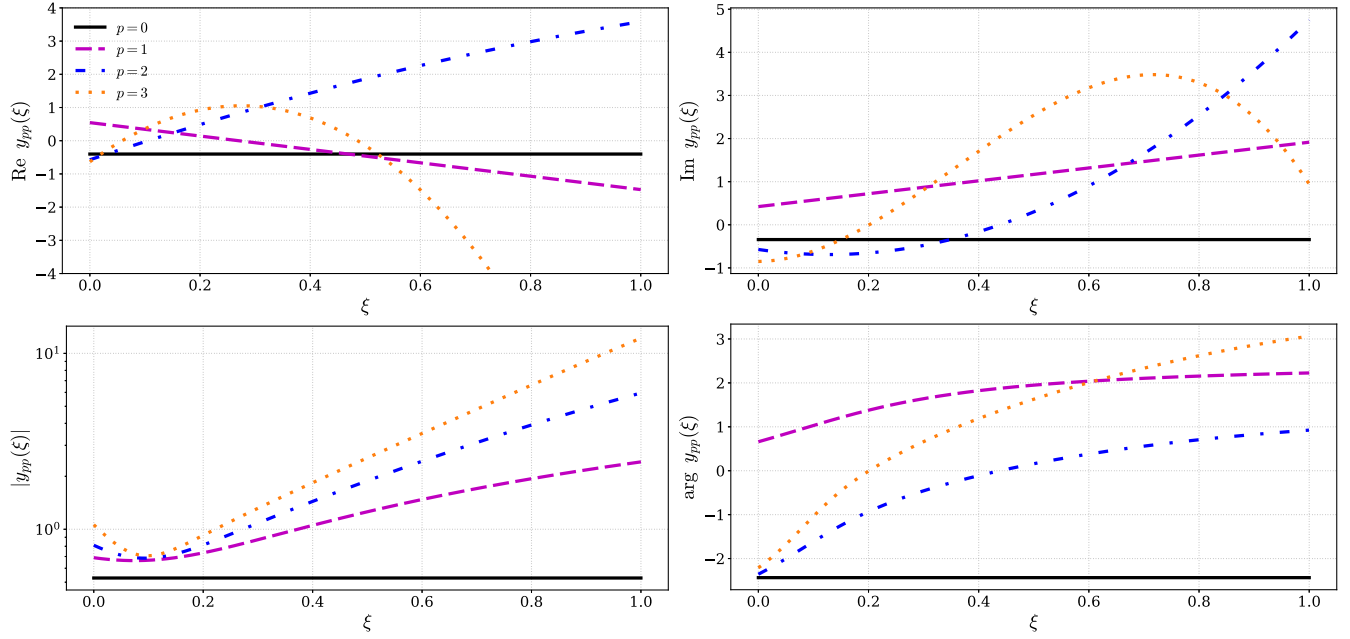


FIG. 4. The first four confluent Heun polynomials for $p = k$, spin weight $s = -2$, black hole dimensionless spin $a/M = 0.86$, and $\tilde{\omega}$ defined by the $(\ell, m, n) = (3, 3, 1)$ QNM. The corresponding QNM frequency is $M\tilde{\omega} = 0.9840 - 0.2140i$. Polynomial orders $p \in \{0, 1, 2, 3\}$ are shown as a function of the physically valued fractional radius ξ . The top left and right panels shows the real and imaginary parts of each polynomial, respectively. Bottom left and right panels show the corresponding polynomial amplitudes and phases, respectively.

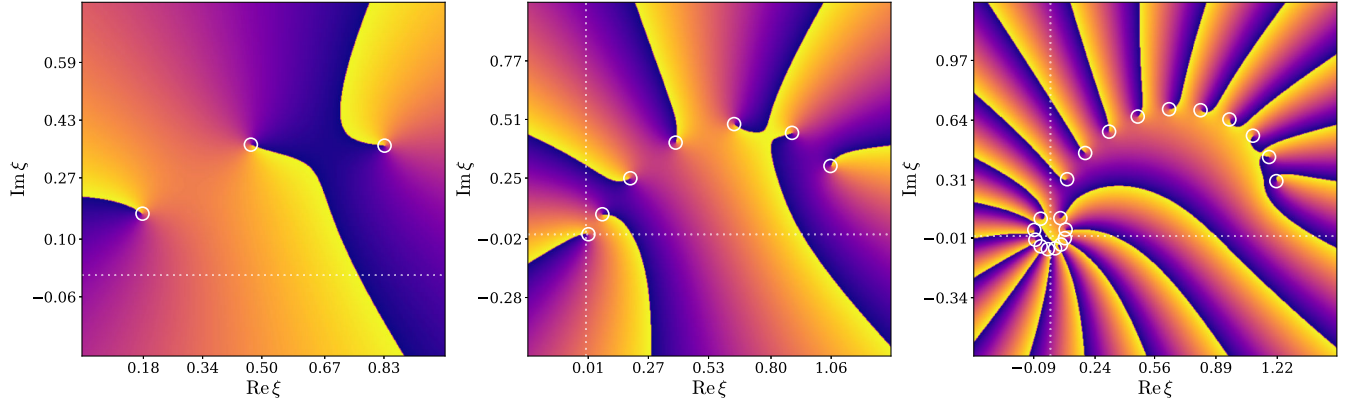


FIG. 5. Distribution of roots for confluent Heun polynomials, $y_{pk}(\xi)$, for three cases, all with $s = -2$, $a/M = 0.7$, and $\ell = m = 2$. The three cases differ in polynomial order, p , and the frequency, $\tilde{\omega}$, used: (left) y_{33} , the $n = 0$ QNM frequency, $M\tilde{\omega} = 0.5326 - 0.0808i$, was used, (center) y_{77} , the $n = 0$ QNM frequency, $M\tilde{\omega} = 0.4713 - 0.5843i$, was used, and (right) $y_{20,20}$, the $n = 12$ QNM frequency, $M\tilde{\omega} = 0.4155 - 2.5050i$, was used. Polynomial roots are marked by white open circles. Roots were calculated using `numpy.roots` [159,160]. Values of polynomial phase, $\arg(y_{pk})$, are shaded from blue to yellow, indicating values between $-\pi$ and π , respectively. Dotted horizontal and vertical lines mark the respective locations of the real and imaginary axes.

increase. As expected, roots are typically not real valued. For large n and large polynomial orders, roots are found to form two families, one situated around the origin, and another dispersed across the upper-right quadrant. This is seen in the right panel of Fig. 5. In all panels of Fig. 5, each root is seen along with its associated phase boundary. Any smooth curve, e.g., $\xi(z)$, may be defined to have a

continuous phase by adding the appropriate integer factor of 2π at each phase boundary.

Figure 6 shows examples of orthogonality at fixed polynomial order,

$$p = 20. \quad (100)$$

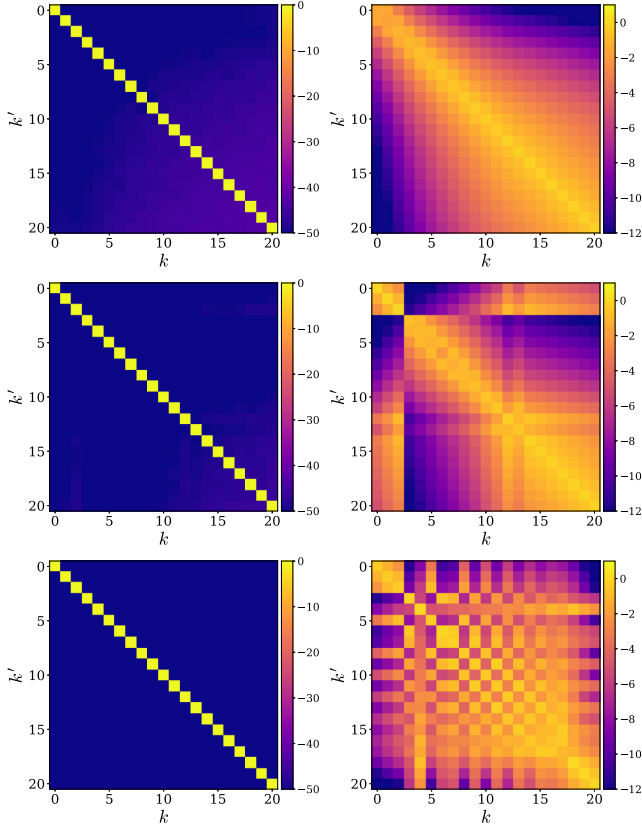


FIG. 6. Gramian matrices for three QNM overtone labels: $n = 0$ (top row), $n = 6$ (middle row), and $n = 12$ (bottom row). Considered here are the order $p = 20$ normalized confluent Heun polynomials with $s = 2$, $a/M = 0.70$ and QNM frequency labels $(\ell, m) = (2, 2)$. The QNM frequencies for top middle and bottom rows are $M\tilde{\omega} = 0.5326 - 0.0808i$, $M\tilde{\omega} = 0.4239 - 1.0954i$ and $M\tilde{\omega} = 0.4098 - 3.1270i$, respectively. The left column shows the absolute values of $\langle y_{pk'} | y_{pk} \rangle$; there, note that the color bar is log scaled between 10^{-50} and 10^0 . The right column shows absolute values of $\langle y_{pk'} | \xi | y_{pk} \rangle$; there, the color bar is logs scaled between 10^{-12} and 10^0 to accentuate graph structure.

for three cases. At focus, is the effect of increasing QNM overtone label or, equivalent, decreasing the QNM decay time τ , such that $1/\tau$ increases within $\tilde{\omega} = \omega - i/\tau$. The $p = 20$ polynomials are considered to be somewhat extreme cases: for this and all lower polynomial orders, no appreciable numerical artifacts are encountered for the numerical and physical parameters used here.

The left columns of Fig. 6 show the absolute values of Gramian matrix elements, $\langle y_{pk'} | y_{pk} \rangle$, and the right columns show absolute values of $\langle y_{pk'} | \xi | y_{pk} \rangle$. Scalar products have been evaluated using analytic continuation [Eq. (64)] with 64 significant figures of precision via the `dps` setting of `MPMATH` [126].

Orthogonality of the polynomials at fixed order is seen through the diagonal structure of each Gramian. The top left panel, having $n = 0$, displays a slight deviation away

from orthogonality (i.e., values near 10^{-40} rather than 10^{-50} and below) as k nears its maximum value of 20.

This effect has two causes, one technical and the other mathematical. The technical cause of this feature is simply the value of the precision parameter, `dps` = 64, which means that numerical results near 10^{-64} are more prone to round-off error. The mathematical cause of this feature is that, like the Legendre polynomials, the confluent Heun polynomials have normalization constants that generally decrease as polynomial order increases.¹⁸ Consequently the action of normalization effectively scales each polynomial by a very large number, exacerbating finite precision errors. At worst, square normalization constants might be 10^{-64} or smaller and would thereby be corrupted by finite precision error. In turn, this would correspond to normalization constants 10^{-32} or smaller, and the act of normalization would be to scale the non-normalized polynomials by a corrupted number of order 10^{32} . This situation can be avoided by increasing numerical precision.

The confluent Heun polynomials, while orthogonal at fixed order, do not display a feature common among classical polynomials. In particular, they do not possess three-term-recursion relations between polynomials of different values of k [106,156]. This may be understood thusly: Since $\xi \times y_{pk}$ is a polynomial of order $p + 1$, it is to be expected that it cannot be written in terms of *only* polynomials of order p . Consequently, the matrix whose elements are $\langle y_{pk'} | \xi | y_{pk} \rangle$ will in general have no nonzero values, and therefore will not be band-diagonal.

This feature of the confluent Heun polynomials is shown in the right column of Fig. 6. There, the structure of $|\langle y_{pk'} | \xi | y_{pk} \rangle|$ is seen to increase nontrivially with overtone label, n . This may be partially understood in the context of the polynomial eigenvalues.

It was previously noted that the small deviations from orthogonality are linked to small normalization constants, and that increasing structure in the distribution of $|\langle y_{pk'} | \xi | y_{pk} \rangle|$ are linked to the polynomial eigenvalues. Polynomial normalization constants and eigenvalues are shown, respectively, in the left and right panels of Fig. 7. There, the same three cases seen in Fig. 6 are considered.

At focus in the left panel of Fig. 7 is the distribution of normalization constants. While normalization constants are found to generally decrease with polynomial order (not shown in figure), they are also found to generally decrease as k increases, and they are found to generally increase as the imaginary part $\tilde{\omega}$ increases.

At focus in the right panel of Fig. 7 is how the distribution of eigenvalues changes as the imaginary part of $\tilde{\omega}$ becomes more negative (i.e., as n increases). For $n = 0$, either $\text{Re}\tilde{\omega}$ or $\text{Im}\tilde{\omega}$ may be used to define equivalent

¹⁸For the Legendre polynomials, the analog of η_{pk}^2 [see Eq. (92)] is $(2p + 1)^{-1}$.

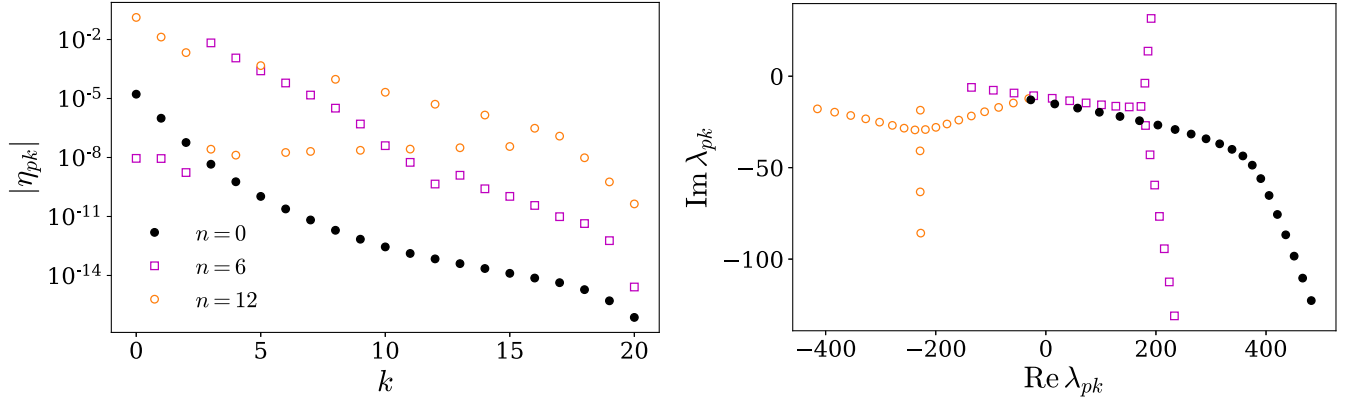


FIG. 7. Distribution of normalization constants, η_{pk} , and eigenvalues, $\lambda_{pk} = \text{Re}\lambda_{pk} + i\text{Im}\lambda_{pk}$, for three different QNM overtone labels: $n \in \{0, 6, 12\}$. For all values of n , the spin weight is $s = 2$, the black hole dimensionless spin is $a/M = 0.70$, and the QNM (ℓ, m) are $(2, 2)$. The left panel shows normalization constants, and the right panel shows eigenvalues. Black dots mark $n = 0$, orange circles mark $n = 6$, and magenta squares mark $n = 12$.

orders (i.e., maps between polynomials and the index k). This is not the case for $n \in \{6, 12\}$, where branches appear in the eigenvalue distribution. In these cases, the eigenvalues are still unique, so a well-defined order can still be constructed.

VIII. IMPLICATIONS FOR ANALYTIC SOLUTIONS TO TEUKOLSKY'S RADIAL EQUATION

We previously set out to apply two concepts to Teukolsky's radial problem. When stated in the context of the radial equation, these two concepts are

- (i) That a scalar product may be developed from the structure of the radial equation [i.e., (Q1)].
- (ii) That special functions may be developed from the structure of the radial equation, and that these functions *may* be used to exactly represent analytic solutions [i.e., (Q2)].

Previous sections show that concept (i) is indeed applicable to the radial equation, thus affirmatively answering (Q1). A radial scalar product was developed in Sec. IV. Methods for evaluation of the scalar product were developed in Sec. V. Connections between the scalar product and concept (ii) were discussed in Sec. VII. In particular, the scalar products connection with problem specific special functions (i.e., the confluent Heun polynomials) was detailed in Sec. VII B. Here, we take a first step toward applying confluent Heun polynomials to the representation of analytic solutions to Teukolsky's radial equation for QNMs. A nonclassical feature of the confluent Heun polynomials means that, for now, *only a preliminary and conceptual step* is taken.

Recall that Teukolsky's radial equation is a confluent Heun equation of the form,

$$\mathcal{L}_\xi = [C_0 + C_1(1 - \xi)] + \mathcal{D}_\xi. \quad (101)$$

In Eq. (101), the transformed radial operator, $\mathcal{L}_\xi$, is simply rewritten such the differential part is explicitly identified as $\mathcal{D}_\xi$ [see Eq. (72)]. In Sec. VII B, it was found that $\mathcal{D}_\xi$ generates confluent Heun polynomials, $y_{pk}(\xi)$, such that each $y_{pk}(\xi)$ satisfies a two parameter eigenvalue problem,

$$\mathcal{D}_\xi |y_{pk}\rangle = (\lambda_{pk} + \mu_p \xi) |y_{pk}\rangle, \quad (102)$$

where, for a given polynomial order p , there are $p + 1$ solutions, y_{pk} . This situation differs starkly from that of classical (e.g., Jacobi) polynomials, for which every *single* polynomial order maps to a *single* polynomial solution. In other words, for each polynomial order, classical polynomials have a *simplex* of solutions, while the confluent Heun polynomials have a *multiplex*.

The existence of a multiplex for each polynomial order prompts a nontrivial conceptual tension: The Jacobi polynomials' recursion relationships play a key role in the angular problem (see Appendix B); however, there cannot exist similar recursion relationships between the confluent Heun polynomials, y_{pk} . This is because such recursion relationships require an orthogonal polynomial sequence, wherein each polynomial has a unique order (i.e., is a simplex) [156]. Further, since each confluent Heun polynomial multiplex is composed of polynomials of like order, the collection of all such multiplexes would appear to contain *too many polynomials*, meaning that they allow for multiple equivalent representations (i.e., decompositions) of radial functions.

This does not, however, preclude the possibility that there are special combinations of confluent Heun polynomials allow for unique representations, and classical recursion relationships. While this possibility is perhaps too open ended to be fully pursued here, we may *assume* (somewhat adventurously) that such polynomials exist, and

then contemplate implications for Teukolsky's radial equation:

Suppose that solutions to the radial equation, $|f\rangle$, can be exactly represented in terms of orthonormal polynomials, u_n ,

$$|f\rangle = \sum_{p=0}^{\infty} a_p |u_p\rangle, \quad (103a)$$

where it is additionally supposed that

$$|u_p\rangle = \sum_{k=0}^p c_{pk} |y_{pk}\rangle, \quad (103b)$$

$$\mathbb{I} = \sum_{p=0}^{\infty} |u_p\rangle \langle u_p|, \quad (103c)$$

and

$$\xi |u_p\rangle = \sigma_{p0} |u_{p-1}\rangle + \sigma_{p1} |u_p\rangle + \sigma_{p2} |u_{p+1}\rangle. \quad (103d)$$

In Eqs. (103a)–(103d), despite not being classical, u_n are constructed to have the essential properties of classical polynomials, namely completeness, orthonormality and three-term recursion [148,156]. In this sense, we may think of u_n as having *canonical* properties. This motivates their being referred to as *canonical confluent Heun polynomials*.

Further, in Eqs. (103a)–(103d), a_p , c_{pk} , σ_{p0} , σ_{p1} and σ_{p2} are assumed to be generally nonzero complex valued constants. Henceforth, σ_{pj} will refer to complex valued constants that are assumed to be known if u_p are known. In Eq. (103c), $\mathbb{I}$ is the identity operator for the space of all confluent Heun functions of the fractional radial coordinate, ξ . Equation (103c) requires that the expansion coefficients present in Eq. (103a) are $a_p = \langle u_p | f \rangle$. The three-term-recursion relation given by Eq. (103d) is a universal feature of all (simplicial) orthogonal polynomial sequences, and it is a direct consequence of Eq. (103c) (i.e., completeness) along with the fact that $\langle u_j | \xi | u_k \rangle = \langle \xi u_j | u_k \rangle = \langle u_j | \xi u_k \rangle$ [156].

Naively, if each order p polynomial multiplex is a $p+1$ dimensional orthonormal basis, then it should be capable of exactly representing any order p polynomial on the same domain [139]. Thus it may be possible to construct u_p such that Eqs. (103b)–(103d) hold.

In effect, Eq. (103) fully defines a solution ansatz. This solution ansatz may be used to investigate whether the scalar product and confluent Heun polynomials imply that Teukolsky's radial equation is, like the angular equation, exactly tridiagonalizable. To this end, we may attempt to represent the radial equation in the basis defined by $\{u_p\}_{p=0}^{\infty}$.

In bra-ket notation, the transformed radial equation is

$$\mathcal{L}_{\xi} |f\rangle = A |f\rangle. \quad (104)$$

Applying Eq. (103c) such that $\mathcal{L}_{\xi}$ is represented as a matrix, $\hat{L}$, and $|f\rangle$ a vector, $\vec{f}$, gives

$$\begin{aligned} \sum_{p,p'} |u_{p'}\rangle \langle u_{p'} | \mathcal{L}_{\xi} | u_p \rangle \langle u_p | f \rangle \\ = A \sum_p |u_p\rangle \langle u_p | f \rangle, \end{aligned} \quad (105a)$$

$$\hat{L} \vec{f} = A \vec{f}. \quad (105b)$$

In Eq. (105b), the elements of $\hat{L}$ are $\langle u_{p'} | \mathcal{L}_{\xi} | u_p \rangle$, and the elements of $\vec{f}$ are $\langle u_p | f \rangle$. Henceforth, p' will label rows, and p will label columns, meaning that $p \geq p'$ denotes the upper triangle of $\hat{L}$, and $p \leq p'$ its lower. To compute the matrix elements, $\langle u_{p'} | \mathcal{L}_{\xi} | u_p \rangle$, the action of $\mathcal{L}_{\xi}$ on $|u_p\rangle$ must be considered in more detail.

Rewriting $\mathcal{L}_{\xi}$ in terms of $\mathcal{D}_{\xi}$ (see Eq. (101)) allows $\mathcal{L}_{\xi} |u_p\rangle$ to be expanded as

$$\mathcal{L}_{\xi} |u_p\rangle = (C_0 + C_1 [1 - \xi]) |u_p\rangle + \mathcal{D}_{\xi} |u_p\rangle. \quad (106)$$

In Eq. (106), $\mathcal{D}_{\xi} |u_p\rangle$ can be rewritten using the definition of u_p in terms of confluent Heun polynomials [Eq. (103b)], and the two parameter eigenvalue relationship [Eq. (102)]. Doing so yields

$$\mathcal{D}_{\xi} |u_p\rangle = \sum_{k=0}^p c_{pk} \mathcal{D}_{\xi} |y_{pk}\rangle \quad (107a)$$

$$= \sum_{k=0}^p c_{pk} (\lambda_{pk} + \mu_p \xi) |y_{pk}\rangle, \quad (107b)$$

$$= |v_p\rangle + \mu_p \xi |u_p\rangle. \quad (107c)$$

In going from Eqs. (107b) to (107c), the term dependent on the confluent Heun eigenvalues, λ_{pk} , has been used to define a new polynomial,

$$|v_p\rangle = \sum_{k=0}^p c_{pk} \lambda_{pk} |y_{pk}\rangle, \quad (108)$$

and it has been noticed that the remaining term in Eq. (107b) is simply proportional to $|u_p\rangle$.

It follows from Eq. (107c) that $\mathcal{L}_{\xi} |u_p\rangle$ may be rewritten as

$$\mathcal{L}_{\xi} |u_p\rangle = (C_0 + C_1) |u_p\rangle + |v_p\rangle + (\mu_p - C_1) \xi |u_p\rangle, \quad (109)$$

and that the matrix element, $\langle u_{p'} | \mathcal{L}_\xi | u_p \rangle$, is

$$\langle u_{p'} | \mathcal{L}_\xi | u_p \rangle = (C_0 + C_1) \langle u_{p'} \rangle u_p \quad (110a)$$

$$+ (\mu_p - C_1) \langle u_{p'} | \xi | u_p \rangle \quad (110b)$$

$$+ \langle u_{p'} \rangle v_p. \quad (110c)$$

Since $|u_p\rangle$ are orthonormal (i.e., $\langle u_{p'} \rangle u_p = \delta_{p'p}$), Eq. (110a) only contributes to diagonal matrix elements. Since $|u_p\rangle$ possess a three term recursion relation given by Eqs. (103d), (110b) will only contribute when $|p' - p| \leq 1$.

It happens that $\langle u_{p'} \rangle v_p$ is not only nonzero when $|p' - p| \leq 1$, but also restricted to the upper triangle (i.e., $p \geq p'$). This follows from the fact that $\mathcal{D}_\xi$ is self-adjoint with respect to the scalar product, meaning that

$$\langle u_{p'} \rangle \mathcal{D}_\xi u_p = \langle \mathcal{D}_\xi u_{p'} \rangle u_p. \quad (111)$$

Using Eq. (107c) to expand both sides of Eq. (111) yields,

$$\langle u_{p'} \rangle v_p - \langle v_{p'} \rangle u_p = (\mu_p - \mu_{p'}) \langle u_{p'} \rangle \xi | u_p. \quad (112)$$

Equation (112) may be understood thusly: Since $u_{p'}$ is a member of a complete orthonormal sequence, its scalar product with any order p polynomial will be nonzero if $p \geq p'$ and zero otherwise [156]. In other words, any order p polynomial may be expressed as a linear combination of $u_{p'}$, where $p' \leq p$ is required since the order of interest is p . Thus $\langle u_{p'} \rangle v_p$ and $\langle v_{p'} \rangle u_p$ only have nonzero values in the respective upper ($p \geq p'$) and lower ($p \leq p'$) triangle of $\hat{L}$. Further, ξu_p is a linear combination of $\{u_{p-1}, u_p, u_{p+1}\}$ due to Eq. (103d), so the matrix whose elements are $\langle u_{p'} \rangle \xi | u_p$ will be tridiagonal. Together, these points mean that the only way that Eq. (112) can hold, i.e., the only way that the upper and lower triangular structure of $\langle u_{p'} \rangle v_p$ and $\langle v_{p'} \rangle u_p$ can be equal to the tridiagonal structure of $\langle u_{p'} \rangle \xi | u_p$, is if $\{\langle u_{p'} \rangle v_p\}_{p'p}$ is upper triangular and tridiagonal. This means that $|v_p\rangle$ must be a linear combination of u_{p-1} and u_p ,

$$|v_p\rangle = \sigma_{p3} |u_{p-1}\rangle + \sigma_{p4} |u_p\rangle, \quad (113)$$

where $\sigma_{p3} = \langle u_{p-1} \rangle v_p$ and $\sigma_{p4} = \langle u_p \rangle v_p$ are complex valued constants which are determined by c_{pk} (i.e., u_p) and the confluent Heun eigenvalues λ_{pk} .

Together, Eqs. (103d), (110), and (113) allow $\mathcal{L}_\xi | u_p \rangle$ to be written as

$$\mathcal{L}_\xi | u_p \rangle = \sigma_{p5} |u_{p-1}\rangle + \sigma_{p6} |u_p\rangle + \sigma_{p7} |u_{p+1}\rangle, \quad (114)$$

where

$$\sigma_{p5} = (\mu_p - C_1) \sigma_{p0} + \sigma_{p3}, \quad (115a)$$

$$\sigma_{p6} = (\mu_p - C_1) \sigma_{p1} + \sigma_{p4} + (C_0 + C_1), \quad (115b)$$

$$\sigma_{p7} = (\mu_p - C_1) \sigma_{p2}. \quad (115c)$$

The implication of Eqs. (114)–(115) is that, if the canonical polynomials u_n can be determined, then Teukolsky's radial operator is exactly tridiagonalizable.

IX. DISCUSSION

The primary results of this article are (i) the introduction of a radial scalar product for QNMs (Sec. IV), (ii) the development of QNM specific confluent Heun polynomials that may be defined for arbitrary frequency parameter (Sec. VII B), and (iii) the demonstration that, in principle, the scalar product and proposed canonical confluent Heun polynomials enable the exact tridiagonalization of Teukolsky's radial equation for QNMs (Sec. VIII).

It is also of note that the scalar product may be evaluated by direct integration with a complex valued radius, or by more standard analytic extension using confluent hypergeometric functions (See Sec. VII A). While analytic extension was found to be the more practical route, the ability to construct a complex radial coordinate, $\xi(z)$, for evaluation of the scalar product also implies [via Eq. (30)] that the QNMs themselves are spatially bounded on $\xi(z)$. Since $\xi(z)$ must be constructed to coincide with Sturm-Liouville boundary conditions, a QNM on $\xi(z)$ would have physically relevant values at both the event horizon and spatial infinity.

Regarding the canonical polynomials, a clear next step is to investigate their computation. Like the Gram-Schmidt polynomials, which are unique once certain processing choices are made, one might expect that there are multiple constructions of the canonical polynomials. This a topic of ongoing research that is near conclusion [161].

Similarly, the tridiagonalization of Teukolsky's radial equation directly implies that its solutions (for fixed frequency parameter) are orthogonal. This would be directly analogous to the orthogonality of the spheroidal harmonics at fixed frequency (i.e., fixed “oblateness”) [56]. It would also be compatible with the conclusions of complementary studies of linear perturbations of Kerr that have encountered a kind of QNM orthogonality [85].

An adjacent implication is that the radial functions are complete (i.e., isomorphic to canonical confluent Heun polynomials). If the radial functions are both orthogonal and complete for given value of the frequency parameter $\tilde{\omega}$, then this would strongly imply the existence of a bi-orthogonal dual for the radial solutions determined by the QNM frequencies. Again, this would be directly analogous to the QNM's spheroidal harmonics, where each spheroidal harmonic requires input from a different QNM frequency [56]. A consequence would be that the QNMs are

“quasicomplete,” i.e., complete in the radial and angular domains, but not complete (specifically under- or over-complete) in the time domain. Investigation of these implications is also a topic of ongoing work that is near conclusion [162].

Lastly, these ideas imply a potentially useful direction of research for, e.g., BBH postmerger signals: rather than attempting to fit various QNMs’ within numerical data, it may be possible to *directly project* out individual QNMs’ time series using orthogonality. In the context of ongoing efforts in numerical relativity and signal modeling for future gravitational wave detectors, this may be a fruitful direction for future research.

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APPENDIX A: NOTATION

This section lists explicit expressions for the main text’s schematic notation. The details here are provided for convenience and completeness.

In Boyer-Lindquist coordinates, Teukolsky’s master differential operator, $\mathcal{L}_{tr\theta\phi}$, is

$$\begin{aligned} \mathcal{L}_{tr\theta\phi} = & \left[\frac{(r^2 + a^2)^2}{\Delta(r)} - a^2 \sin^2 \theta \right] \frac{\partial^2}{\partial t^2} \\ & + \frac{4Mar}{\Delta(r)} \frac{\partial^2}{\partial t \partial \phi} + \left[\frac{a^2}{\Delta(r)} - \frac{1}{\sin^2 \theta} \right] \frac{\partial^2}{\partial \phi^2} \\ & - \Delta(r)^{-s} \frac{\partial}{\partial r} \left(\Delta(r)^{s+1} \frac{\partial}{\partial r} \right) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) \\ & - 2s \left[\frac{a(r-M)}{\Delta(r)} + \frac{i \cos \theta}{\sin^2 \theta} \right] \frac{\partial}{\partial \phi} \\ & - 2s \left[\frac{M(r^2 - a^2)}{\Delta(r)} - r - ia \cos \theta \right] \frac{\partial}{\partial t} + (s^2 \cot^2 \theta - s), \end{aligned} \quad (\text{A1})$$

where

$$\Delta(r) = (r - r_+)(r - r_-), \quad (\text{A2a})$$

and

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}. \quad (\text{A2b})$$

Upon separation via a separable ansatz or integral transform (see Sec. III), Teukolsky’s angular differential operator is

encapsulated by the following expression,

$$u = \cos(\theta), \quad (\text{A3a})$$

$$\mathcal{L}_u = V_S(u) + \partial_u(1 - u^2)\partial_u, \quad (\text{A3b})$$

$$V_S(u) = s + ua\tilde{\omega}(ua\tilde{\omega} - 2s) - \frac{(m + su)^2}{1 - u^2}. \quad (\text{A3c})$$

Similarly, Teukolsky’s radial differential operator is commonly written as

$$\mathcal{L}_r = V_R(r) + \Delta(r)^{-s} \partial_r [\Delta(r)^{s+1} \partial_r], \quad (\text{A4})$$

where

$$\begin{aligned} V_R(r) = & \frac{K(r)^2 - 2is(r - M)K(r)}{\Delta(r)} + 4is\tilde{\omega}r \\ & - a^2\tilde{\omega}^2 + 2am\tilde{\omega} \end{aligned} \quad (\text{A5a})$$

$$K(r) = (r^2 + a^2)\tilde{\omega} - am. \quad (\text{A5b})$$

The expressions above may also be found in Refs. [66,81,115].

APPENDIX B: SOLVING TEUKOLSKY’S ANGULAR EQUATION WITH JACOBI POLYNOMIALS

Teukolsky’s angular equation is

$$\mathcal{L}_u S(u) = -AS(u). \quad (\text{B1})$$

In Eq. (B1),

$$u = \cos(\theta), \quad (\text{B2a})$$

$$\mathcal{L}_u = V_S(u) + \partial_u(1 - u^2)\partial_u, \quad (\text{B2b})$$

where the equation’s potential, V_S , is

$$V_S(u) = s + u\gamma(u\gamma - 2s) - \frac{(m + su)^2}{1 - u^2} \quad (\text{B2c})$$

with

$$\gamma = a\tilde{\omega}. \quad (\text{B2d})$$

Like Teukolsky’s radial equation, Eq. (B1) is a confluent Heun equation; however, Eq. (B1) differs from the radial case in two ways of current relevance:

- (i) Equation (B1) has two regular singular points at $u \in \{-1, 1\}$, and one irregular singular point at $u \rightarrow \infty$. The irregular singular point at $u \rightarrow \infty$ is not within the physical domain (i.e., there is no

boundary condition defined there). The relevant asymptotic boundary conditions are essentially that $S(u)$ are regular when $u \in \{-1, 1\}$.

- (ii) The degeneration of Eq. (B1) to a hypergeometric equation occurs at a physical limit ($a \rightarrow 0$), whereas the degeneration of the radial equation to a hypergeometric one occurs at a nonphysical limit for QNMs ($\tilde{\omega} \rightarrow 0$).

Respectively, these differences mean that (i) global¹⁹ solutions to the angular equation can be solved with high accuracy using nonconfluent basis functions, and that (ii) the optimal basis functions may be constructed from physical solutions at the $a \rightarrow 0$ limit.

The construction of analytic solutions begins by finding a similarity transformation that results in the transformed potential being regular when $u \in [-1, 1]$. The similarity transformation is equivalent to taking the solution ansatz,

$$S(u) = \eta(u)g(u), \quad (\text{B3})$$

where,

$$\eta(u) = e^{uk_2}(1-u)^{k_0}(1+u)^{k_1}, \quad (\text{B4})$$

and then applying it to the angular equation thusly,

$$[\eta(u)^{-1}\mathcal{L}_u\eta(u)]g(u) = -[\eta(u)^{-1}A\eta(u)]g(u) \quad (\text{B5a})$$

$$\mathcal{L}'_u = [\eta(u)^{-1}\mathcal{L}_u\eta(u)] \quad (\text{B5b})$$

$$\mathcal{L}'_u g(u) = -Ag(u). \quad (\text{B5c})$$

In Eq. (B5a), the similarity transformation of $\mathcal{L}_u$ is explicitly performed. In Eq. (B5b), the transformed operator, $\mathcal{L}'_u$, is defined, and in Eq. (B5c) its eigenvalue problem is stated.

The values of k_0 , k_1 and k_2 that result in the potential of $\mathcal{L}'_u$ being regular and linear on $u \in \{-1, 1\}$ are,

$$k_0 = \frac{1}{2}|m+s|, \quad (\text{B6a})$$

$$k_1 = \frac{1}{2}|m-s|, \quad (\text{B6b})$$

$$k_2 = \gamma. \quad (\text{B6c})$$

Equivalently, k_0 , k_1 and k_2 are the singular exponents that may be found via the method of Frobenius [106,108,128]. In effect, k_0 and k_1 remove divergences from the potential at the regular singular points, and k_2 removes the potential's

quadratic dependence on u . The appearance of the absolute values, e.g., $|m+s|$, ensures that η is always finite when $u \in \{-1, 1\}$, regardless of the values of m and s . This is equivalent to enforcing asymptotic boundary conditions on $S(u)$. In Eq. (B6c), $-\gamma$ is also a viable solution for k_2 ; however, $k_2 = +\gamma$ is chosen by convention [66]. The result for $\mathcal{L}'_u$ has two schematic forms of relevance,

$$\mathcal{L}'_u = (a_0 + a_1 u) \quad (\text{B7a})$$

$$+ (a_2 + a_3 u + a_4[1 - u^2])\partial_u \quad (\text{B7b})$$

$$+ (1 - u^2)\partial_u^2, \quad (\text{B7c})$$

and

$$\mathcal{L}'_u = (a_0 + a_1 u) + \mathcal{D}'_u. \quad (\text{B8})$$

In Eq. (B7), a_0 through a_4 are constant coefficients defined in Eqs. (B9a)–(B9e). They should not be confused with the similarly labeled coefficients defined in the main text.

$$a_0 = -k_0(2k_1 + 1) - k_1 - \frac{1}{2}[m^2 + s(s+2)] + \gamma^2 + 2\gamma(k_1 - k_0) \quad (\text{B9a})$$

$$a_1 = -2\gamma(k_0 + k_1 + s + 1) \quad (\text{B9b})$$

$$a_2 = -2(k_0 - k_1) \quad (\text{B9c})$$

$$a_3 = -2(k_0 + k_1 + 1) \quad (\text{B9d})$$

$$a_4 = 2\gamma \quad (\text{B9e})$$

Equation (B8) defines $\mathcal{D}'_u$ to be the differential part of Eq. (B7) [i.e., Eqs. (B7b) and (B7c)]. The operator, $\mathcal{D}'_u$, is useful for understanding the eigenproblem [Eq. (B5c)] in the nonspinning limit (i.e., $\gamma = a\tilde{\omega} = 0$), and for general spins.

In the nonspinning limit, $\mathcal{D}'_u$ is simply $\mathcal{D}_u^{\text{Jac}}$, the differential operator for which the Jacobi polynomials are eigenfunctions

$$\mathcal{D}'_u|_{\gamma=0} = \mathcal{D}_u^{\text{Jac}}. \quad (\text{B10})$$

The Jacobi operator is

$$\mathcal{D}_u^{\text{Jac}} = -(\alpha - \beta + (\alpha + \beta + 2)u)\partial_u + (1 - u^2)\partial_u^2, \quad (\text{B11})$$

where

$$\alpha = 2k_0, \quad (\text{B12})$$

$$\beta = 2k_1. \quad (\text{B13})$$

¹⁹By *global*, one means a solution that applies to the entire domain. This is in contrast to local solutions, such as those use in the MST method, that are only valid in the vicinity of a singular point [100,142].

The simplest functions generated by $\mathcal{D}_u^{\text{Jac}}$ are the Jacobi polynomials, $p_n(u)$. Their eigenvalue relationship is

$$\mathcal{D}_u^{\text{Jac}} p_n(u) = -n(\alpha + \beta + n + 1)p_n(u). \quad (\text{B14})$$

In Eq. (B14), the eigenvalue is determined by the requirement that the series solution for $p_n(u)$ terminates after n terms.

The zero-spin eigenvalue, $A|_{a=0} = A|_{\gamma=0}$, can be determined by requiring equivalence between the physical eigenvalue problem [Eq. (B5c)], and the Jacobi one [Eq. (B14)]. This amounts to holding the nonderivative terms in each eigenvalue relation to be equivalent in the zero spin limit,

$$(a_0 + A)|_{\gamma=0} = n(\alpha + \beta + n + 1) \quad (\text{B15})$$

Equation (B15) can be solved for the eigenvalue,

$$A_0 = A|_{\gamma=0} \quad (\text{B16a})$$

$$= -(\ell - s)(\ell + s + 1). \quad (\text{B16b})$$

In Eq. (B16b), A_0 is the usual spin weighted spherical harmonic eigenvalue [25,66,77], and index ℓ is related to the polynomial order, n , via

$$\ell = \ell_{\min} + n, \quad (\text{B17a})$$

$$\ell_{\min} = k_0 + k_1. \quad (\text{B17b})$$

For nonzero spin, term by term comparison of $\mathcal{D}'_u$ and $\mathcal{D}_u^{\text{Jac}}$ yields

$$\mathcal{D}'_u = \mathcal{D}_u^{\text{Jac}} + a_4(1 - u^2)\partial_u. \quad (\text{B18})$$

Equation (B18), in effect, casts $\mathcal{D}'_u$ as a deformation of the Jacobi operator. This implies that it is natural to consider the spheroidal functions $g(u)$ to be deformations of the Jacobi polynomials. In particular, we will take $g(u)$ to be *exactly represented* by Jacobi polynomials,

$$g(u) = \sum_{n=0}^{\infty} a_n p_n(u). \quad (\text{B19})$$

Equation (B19) is justified not only because the Jacobi polynomials are known to be complete, but also because $g(u)$ and $p_n(u)$ satisfy the same asymptotic boundary conditions on the same domain.

Equation (B19) is an analytic solution for $g(u)$ that is motivated by the structure of Teukolsky's angular equation. In effect, Eq. (B19) is a spectral decomposition of $g(u)$ into Jacobi moments, a_n . Thus, finding the values of each a_n is equivalent to finding $g(u)$.

To do so, the Jacobi polynomials' recursive and orthogonality properties must be used. The relevant recursion rules for Jacobi polynomials are

$$(1 - u^2)\partial_u p_n = b_0 p_{n-1} + b_1 p_n + b_2 p_{n+1}, \quad (\text{B20a})$$

$$u p_n = b_3 p_{n-1} + b_4 p_n + b_5 p_{n+1}. \quad (\text{B20b})$$

Normalization of the Jacobi polynomials is defined with respect to the weighted inner product,

$$\langle p_n | p_{n'} \rangle = \int_{-1}^1 p_n(u) p_{n'}(u) W^{\text{Jac}}(u) du \quad (\text{B21a})$$

$$= b_6 \delta_{n'n}, \quad (\text{B21b})$$

where the Jacobi weight function, $W^{\text{Jac}}(u)$, is

$$W^{\text{Jac}} = (1 - u)^\alpha (1 + u)^\beta. \quad (\text{B22})$$

In Eqs. (B20) and (B21), b_0 through b_6 are n dependent constants defined in Eqs. (B23a)–(B23g).

$$b_0(n) = -\frac{2(\alpha + n)(\beta + n)(\alpha + \beta + n + 1)}{(\alpha + \beta + 2n)(\alpha + \beta + 2n + 1)} \quad (\text{B23a})$$

$$b_1(n) = -\frac{2n(\beta - \alpha)(\alpha + \beta + n + 1)}{(\alpha + \beta + 2n)(\alpha + \beta + 2n + 2)} \quad (\text{B23b})$$

$$b_2(n) = \frac{2n(n + 1)(\alpha + \beta + n + 1)}{(\alpha + \beta + 2n + 1)(\alpha + \beta + 2n + 2)} \quad (\text{B23c})$$

$$b_3(n) = -\frac{2(\alpha + n)(\beta + n)}{(\alpha + \beta + 2n)(\alpha + \beta + 2n + 1)} \quad (\text{B23d})$$

$$b_4(n) = -\frac{\alpha^2 - \beta^2}{(\alpha + \beta + 2n)(\alpha + \beta + 2n + 2)} \quad (\text{B23e})$$

$$b_5(n) = -\frac{2(n + 1)(\alpha + \beta + n + 1)}{(\alpha + \beta + 2n + 1)(\alpha + \beta + 2n + 2)} \quad (\text{B23f})$$

$$b_6(n) = \frac{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{(\alpha + \beta + 2n + 1)\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)} \quad (\text{B23g})$$

Observe that $W^{\text{Jac}}(u)$ is very similar to the scaling function $\eta(u)$ introduced in Eq. (B4). This can be understood as resulting from the fact that the transformation present in Eq. (B5a) is conformal, meaning that

$$\int_{-1}^1 p_n(u) p_{n'}(u) W^{\text{Jac}}(u) du = \quad (\text{B24a})$$

$$\int_{-1}^1 ([p_n(u)\eta(u)][p_{n'}(u)\eta(u)])|_{\gamma=0} du \quad (\text{B24b})$$

There is no weight function in Eq. (B24b) because Eq. (B2b) is explicitly in Sturm-Liouville form (with

weight function 1) [106]. The integrands of Eqs. (B24a) and (B24b) must be equivalent, meaning that $W^{\text{Jac}}(u)$ and $\eta(u)|_{\gamma=0}$ are related,

$$W^{\text{Jac}}(u) = \eta(u)^2|_{\gamma=0}. \quad (\text{B25})$$

In general, it can be shown that if f_1 satisfies the second order linear differential equation $\mathcal{L}_1 f_1 = 0$, where $\mathcal{L}$ is self-adjoint with weight function λ_1 , then $f_2 = \eta^{-1} f_1$ will satisfy $\mathcal{L}_2 g = 0$, where $\mathcal{L}_2 = \eta^{-1} \mathcal{L}_1 \eta$ is self-adjoint with new weight function $\lambda_2 = \eta^2 \lambda_1$.

Equations (B20)–(B23) can be used to determine the values of a_n as follows. Given the scalar product, Eq. (B21a), we will use bra-ket notation to simplify to representation of various steps. In that notation, the completeness of the Jacobi polynomials is expressed as

$$\sum_{n=0}^{\infty} |p_n\rangle\langle p_n| = \mathbb{I}. \quad (\text{B26})$$

In Eq. (B26), $\mathbb{I}$ is the identity operator for all functions that have the same asymptotic behavior as the Jacobi polynomials on $u \in [-1, 1]$, and the polynomials $|p_n\rangle$ are implicitly normalized so that $\langle p_n | p_n \rangle = 1$. The bra-ket notation form of the physical eigenvalue relation is,

$$\mathcal{L}'_u |g\rangle = -A |g\rangle. \quad (\text{B27})$$

From this perspective, the Jacobi moments, a_n , are simply the components of $g(u)$ in the basis of Jacobi polynomials. This is made explicit by using Eq. (B26) to express both sides of Eq. (B27) in the Jacobi polynomial basis,

$$\sum_{n,n'} |p_{n'}\rangle\langle p_{n'}| \mathcal{L}'_u |p_n\rangle\langle p_n| g = -A \sum_n |p_n\rangle\langle p_n| g. \quad (\text{B28})$$

Note that the right-hand side of Eq. (B28) is equivalent to that of Eq. (B18) (i.e., $a_n = \langle p_n | g \rangle$). Equation (B28) may be recast as a simple matrix equation

$$\hat{L} \vec{g} = -A \vec{g}, \quad (\text{B29})$$

where $\hat{L}$ and $\vec{g}$ are defined from Eq. (B28) as

$$\hat{L} = \{\langle p_{n'} | \mathcal{L}'_u | p_n \rangle\}_{n,n'}, \quad (\text{B30a})$$

$$\vec{g} = \{\langle p_n | g \rangle\}_n. \quad (\text{B30b})$$

Thus, taking a finite dimensional truncation of $\hat{L}$ allows $\vec{g}$ (i.e., a_n) to be determined using standard linear algebraic methods [139,149].

The essential feature of this approach is that, starting with Eq. (B29), the recursion and orthogonality relationships, Eqs. (B20)–(B23), can be used to show that $\hat{L}$ is at

most *tridiagonal*. In particular, when $\mathcal{L}'_u$ acts on $|p_n\rangle$, only lower to upper diagonal terms contribute,

$$\mathcal{L}'_u |p_n\rangle = \sigma_n |p_{n-1}\rangle + \mu_n |p_n\rangle + \nu_n |p_{n+1}\rangle. \quad (\text{B31})$$

In Eq. (B31), the n dependent constants, σ_n , μ_n , and ν_n are

$$\sigma_n = \frac{b_6(n-1)}{b_6(n)} [a_1 b_3(n) + a_4 b_0(n)] \quad (\text{B32a})$$

$$\begin{aligned} \mu_n &= a_0 + a_1 b_4(n) + a_4 b_1(n) \\ &\quad - n(\alpha + \beta + n + 1) \end{aligned} \quad (\text{B32b})$$

$$\nu_n = \frac{b_6(n+1)}{b_6(n)} [a_1 b_5(n) + a_4 b_2(n)] \quad (\text{B32c})$$

For any N dimensional truncation of $\hat{L}$, say $\hat{L}_N$, Eq. (B32) provides the matrix elements,

$$\hat{L}_N = \begin{pmatrix} \mu_0 & \nu_0 & 0 & \cdots & 0 \\ \sigma_1 & \mu_1 & \nu_1 & \ddots & 0 \\ 0 & \sigma_2 & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \nu_{N-1} \\ 0 & 0 & \cdots & \sigma_N & \mu_N \end{pmatrix}. \quad (\text{B33})$$

The use of Eq. (B33) to determine angular functions $S(u) = \eta(u)g(u)$, and their eigenvalues $A(u)$, amounts to determining the eigenvectors and eigenvalues of $\hat{L}_N$. Since this method explicitly estimates the point spectra of $\hat{L}$, it is a purely spectral method (e.g., not a generalized spectral method) for the construction of analytic solutions to Teukolsky's angular equation.

This construction facilitates the following conclusions: Since the transformed angular equation degenerates to the Jacobi equation when $\gamma = 0$, the spheroidal functions $g(u)$ may be exactly identified with Jacobi polynomials in that limit. For general values of γ , the fact that $\hat{L}$ is tridiagonal means that for every $g(u)$ associated with index ℓ , there always exists a unique Jacobi polynomial with that same index [56]. In this way, the QNM index ℓ is inherently *linked* to a Jacobi polynomial of order $n = \ell - \ell_{\min}$.

APPENDIX C: THE CONFLUENT HEUN EQUATION IN NATURAL GENERAL FORM

The confluent Heun equation has many forms that differ with respect to transformations of the independent and dependent variables [100]. Here, we briefly outline the connection between Teukolsky's radial equation and the confluent Heun equation in *natural general form*.

Perhaps the most referenced (e.g., Refs. [95,103,127]) form of confluent Heun equation is,

$$\mathcal{L}_{\text{CH}}^{(1)} y(z) = 0, \quad (\text{C1a})$$

where

$$\begin{aligned} \mathcal{L}_{\text{CH}}^{(1)} &= (\alpha \nu z - \sigma) \\ &+ [\gamma(z-1) + z(\delta + \nu(z-1))] \partial_z \\ &+ (z-1)z \partial_z^2. \end{aligned} \quad (\text{C1b})$$

Equation (C1) may be derived from the Heun equation (which has 4 regular singular points) by confluence of two singular points (see, e.g., [106]) into a single irregular one. Equation (C1a) is simply a definition of the confluent Heun equation in terms of its associated linear differential operator, $\mathcal{L}_z^{(1)}$. In Eq. (C1b), $\alpha, \nu, \sigma, \gamma$ and δ are complex valued parameters, and z is nominally real valued.

Equation (C1), is the confluent Heun equation in what is called *nonsymmetrical canonical form* [100,121,157]. It happens that this is actually not the most general form of the confluent Heun equation. As a result, a naive comparison between Eq. (C1) and Teukolsky's radial equation may generate unnecessary confusion. To facilitate the upcoming discussion, we will henceforth use r rather than z , and we will discuss forms of the confluent Heun equation by referring to the related differential operators, $\mathcal{L}_{\text{CH}}^{(i)}$, such that

$$\mathcal{L}_{\text{CH}}^{(i)} y = 0. \quad (\text{C2})$$

The most general form of the confluent Heun differential operator with an irregular singular point at infinity is (see, e.g., Ref. [100])

$$\begin{aligned} \mathcal{L}_{\text{CH}}^{(2)} &= \left(\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \frac{\alpha_4}{r-r_-} + \frac{\alpha_3}{r-r_+} \right) \\ &+ (\alpha_5 + \alpha_6 r + \alpha_7 r^2) \partial_r \\ &+ (r-r_-)(r-r_+) \partial_r^2. \end{aligned} \quad (\text{C3})$$

In Eq. (C3), $\mathcal{L}_{\text{CH}}^{(2)}$ corresponds to the confluent Heun equation in what is called *natural general form* [100].

In Eq. (C3), $\mathcal{L}_{\text{CH}}^{(2)}$ has been formatted to ease comparison with Teukolsky's radial differential operator. In particular, the two regular singular points have been placed at r_+ and r_- , just as they are in the case of Teukolsky's radial differential operator [Eq. (A4)], which can be formatted as

$$\begin{aligned} \mathcal{L}_r &= \left(A_0 + A_1 r + A_2 r^2 + \frac{A_3}{r-r_-} + \frac{A_4}{r-r_+} \right) \\ &+ (A_5 + A_6 r) \partial_r \\ &+ (r-r_-)(r-r_+) \partial_r^2. \end{aligned} \quad (\text{C4})$$

For completeness, parameters A_1 through A_6 are defined in terms of r_+ , r_- and s as

$$A_0 = \tilde{\omega}(r_- + r_+)(\tilde{\omega}(r_- + r_+) - is) \quad (\text{C5a})$$

$$A_1 = \tilde{\omega}(\tilde{\omega}(r_- + r_+) + 2is) \quad (\text{C5b})$$

$$A_2 = \tilde{\omega}^2 \quad (\text{C5c})$$

$$\begin{aligned} A_3 &= \frac{r_-^2(m^2 + 4r_-^2 \tilde{\omega}^2)}{r_- - r_+} \\ &- r_-(m^2 + r_- \tilde{\omega}(3r_- \tilde{\omega} + is)) \\ &+ \frac{4mr_-^2 \tilde{\omega} \sqrt{r_- r_+}}{r_+ - r_-} + ims \sqrt{r_- r_+} \\ &+ 2mr_- \tilde{\omega} \sqrt{r_- r_+} - r_- r_+ \tilde{\omega}(r_- \tilde{\omega} + is) \end{aligned} \quad (\text{C5d})$$

$$\begin{aligned} A_4 &= -\frac{r_+^2(m^2 + 4r_+^2 \tilde{\omega}^2)}{r_- - r_+} \\ &- r_+(m^2 + r_+ \tilde{\omega}(3r_+ \tilde{\omega} + is)) \\ &+ \frac{4mr_+^2 \tilde{\omega} \sqrt{r_- r_+}}{r_- - r_+} + ims \sqrt{r_- r_+} \\ &+ 2mr_+ \tilde{\omega} \sqrt{r_- r_+} - r_- r_+ \tilde{\omega}(r_+ \tilde{\omega} + is) \end{aligned} \quad (\text{C5e})$$

$$A_5 = (s+1)(-(r_- + r_+)) \quad (\text{C5f})$$

$$A_6 = 2(s+1). \quad (\text{C5g})$$

The fact that Teukolsky's radial operator is a special case of Heun's confluent operator is clear by comparing Eq. (C3) to (C4). The confluent Heun nature of the operator is not changed under transformations of the dependent or independent variables, but such transformations can impact the physical nature of solutions. For example, application of the QNM boundary conditions [see, e.g., Eqs. (11)–(13)] has the effect of transforming Eq. (C4) into a *different version of* Eq. (C3). However, if the solution of the transformed equation [e.g., Eq. (15)] is to be consistent with QNM boundary conditions, then it cannot have asymptotic behavior that affects a transformation into yet *another* version of the confluent Heun equation. Application of a compactified coordinate [see Eq. (10)] has the effect of changing the coordinate position of singular points, but not their regular or irregular nature.

- [1] C. V. Vishveshwara, *Phys. Rev. D* **1**, 2870 (1970).
- [2] B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), *Phys. Rev. Lett.* **116**, 061102 (2016).
- [3] B. P. Abbott *et al.* (Virgo Collaboration and LIGO Scientific Collaboration), *Phys. Rev. Lett.* **116**, 221101 (2016).
- [4] R. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), *Phys. Rev. X* **11**, 021053 (2021).
- [5] R. Abbott *et al.* (LIGO Scientific Collaboration, VIRGO Collaboration, and KAGRA Collaboration), *Phys. Rev. X* **13**, 041039 (2023).
- [6] B. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), *Phys. Rev. X* **9**, 031040 (2019).
- [7] I. Kamaretsos, M. Hannam, and B. Sathyaprakash, *Phys. Rev. Lett.* **109**, 141102 (2012).
- [8] L. London, D. Shoemaker, and J. Healy, *Phys. Rev. D* **90**, 124032 (2014).
- [9] S. A. Hughes, A. Apte, G. Khanna, and H. Lim, *Phys. Rev. Lett.* **123**, 161101 (2019).
- [10] E. Berti, V. Cardoso, and C. M. Will, *Phys. Rev. D* **73**, 064030 (2006).
- [11] E. Berti, V. Cardoso, and A. O. Starinets, *Classical Quantum Gravity* **26**, 163001 (2009).
- [12] A. Maselli, K. Kokkotas, and P. Laguna, *Phys. Rev. D* **95**, 104026 (2017).
- [13] K. D. Kokkotas and B. G. Schmidt, *Living Rev. Relativity* **2**, 2 (1999).
- [14] T. Evstafyeva, M. Agathos, and J. L. Ripley, *Phys. Rev. D* **107**, 124010 (2023).
- [15] K. G. Arun *et al.* (LISA Collaboration), *Living Rev. Relativity* **25**, 4 (2022).
- [16] C. Pacilio and S. Bhagwat, *Phys. Rev. D* **107**, 083021 (2023).
- [17] M. Bailes *et al.*, *Nat. Rev. Phys.* **3**, 344 (2021).
- [18] B. Sathyaprakash *et al.*, *Classical Quantum Gravity* **29**, 124013 (2012); **30**, 079501(E) (2013).
- [19] M. Vallisneri, J. Kanner, R. Williams, A. Weinstein, and B. Stephens, *J. Phys. Conf. Ser.* **610**, 012021 (2015).
- [20] N. Karnesis *et al.*, *Int. J. Mod. Phys. D* **33**, 2450027 (2024).
- [21] B. P. Abbott *et al.* (Virgo Collaboration and LIGO Scientific Collaboration), *Phys. Rev. Lett.* **119**, 141101 (2017).
- [22] B. P. Abbott *et al.* (Virgo Collaboration and LIGO Scientific Collaboration), *Phys. Rev. Lett.* **116**, 241102 (2016).
- [23] C. W. Misner, K. S. Thorne, and J. A. Wheeler, edited by C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W.H. Freeman and Co., San Francisco, 1973).
- [24] M. Maggiore, *Gravitational Waves* (Oxford University Press, Oxford, 2007).
- [25] E. T. Newman and R. Penrose, *J. Math. Phys. (N.Y.)* **7**, 863 (1966).
- [26] E. Newman and R. Penrose, *J. Math. Phys. (N.Y.)* **3**, 566 (1962).
- [27] S. A. Teukolsky, *Phys. Rev. Lett.* **29**, 1114 (1972).
- [28] B. F. Whiting, *J. Math. Phys. (N.Y.)* **30**, 1301 (1989).
- [29] M. Giesler, M. Isi, M. A. Scheel, and S. Teukolsky, *Phys. Rev. X* **9**, 041060 (2019).
- [30] S. T. McWilliams, *Phys. Rev. Lett.* **122**, 191102 (2019).
- [31] G. Carullo, W. Del Pozzo, and J. Veitch, *Phys. Rev. D* **99**, 123029 (2019); **100**, 089903(E) (2019).
- [32] I. Ota and C. Chirenti, *Phys. Rev. D* **101**, 104005 (2020).
- [33] S. Bhagwat, X. J. Forteza, P. Pani, and V. Ferrari, *Phys. Rev. D* **101**, 044033 (2020).
- [34] X. J. Forteza, S. Bhagwat, P. Pani, and V. Ferrari, *Phys. Rev. D* **102**, 044053 (2020).
- [35] S. H. W. Leong, J. Calderón Bustillo, M. Gracia-Linares, and P. Laguna, *Phys. Rev. D* **108**, 124079 (2023).
- [36] N. Khera, A. Ribes Metidieri, B. Bonga, X. J. Forteza, B. Krishnan, E. Poisson, D. Pook-Kolb, E. Schnetter, and H. Yang, *Phys. Rev. Lett.* **131**, 231401 (2023).
- [37] N. Franchini and S. H. Völkel, [arXiv:2305.01696](https://arxiv.org/abs/2305.01696).
- [38] P. J. Nee, S. H. Völkel, and H. P. Pfeiffer, *Phys. Rev. D* **108**, 044032 (2023).
- [39] R. Cotesta, G. Carullo, E. Berti, and V. Cardoso, *Phys. Rev. Lett.* **129**, 111102 (2022).
- [40] V. Baibhav, M. H.-Y. Cheung, E. Berti, V. Cardoso, G. Carullo, R. Cotesta, W. Del Pozzo, and F. Duque, *Phys. Rev. D* **108**, 104020 (2023).
- [41] R. Abbott *et al.* (LIGO Scientific Collaboration and VIRGO Collaboration), *Phys. Rev. D* **109**, 022001 (2024).
- [42] B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), *Phys. Rev. D* **100**, 104036 (2019).
- [43] R. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), *Phys. Rev. D* **103**, 122002 (2021).
- [44] R. Abbott *et al.* (LIGO Scientific Collaboration, VIRGO Collaboration, and KAGRA Collaboration), *Phys. Rev. D* **112**, 084080 (2025).
- [45] E. Berti, A. Sesana, E. Barausse, V. Cardoso, and K. Belczynski, *Phys. Rev. Lett.* **117**, 101102 (2016).
- [46] C. K. Mishra, K. G. Arun, B. R. Iyer, and B. S. Sathyaprakash, *Phys. Rev. D* **82**, 064010 (2010).
- [47] T. Robson, N. J. Cornish, and C. Liu, *Classical Quantum Gravity* **36**, 105011 (2019).
- [48] A. Klein *et al.*, *Phys. Rev. D* **93**, 024003 (2016).
- [49] C. F. B. Macedo, P. Pani, V. Cardoso, and L. C. B. Crispino, *Astrophys. J.* **774**, 48 (2013).
- [50] E. Barausse, V. Cardoso, and P. Pani, *J. Phys. Conf. Ser.* **610**, 012044 (2015).
- [51] Y. Zhao, B. Sun, Z. Cao, K. Lin, and W.-L. Qian, *Phys. Rev. D* **109**, 044031 (2024).
- [52] E. Berti, V. Cardoso, K. D. Kokkotas, and H. Onozawa, *Phys. Rev. D* **68**, 124018 (2003).
- [53] E. Berti and A. Klein, *Phys. Rev. D* **90**, 064012 (2014).
- [54] E. Berti, V. Cardoso, and M. Casals, *Phys. Rev. D* **73**, 024013 (2006); **73**, 109902(E) (2006).
- [55] L. London, *Phys. Rev. D* **102**, 084052 (2020).
- [56] L. T. London, *Phys. Rev. D* **107**, 044056 (2023).
- [57] L. London and E. Fauchon-Jones, *Classical Quantum Gravity* **36**, 235015 (2019).
- [58] S. Husa, S. Khan, M. Hannam, M. Pürrer, F. Ohme, X. Jiménez Forteza, and A. Bohé, *Phys. Rev. D* **93**, 044006 (2016).
- [59] E. Hamilton, L. London, J. E. Thompson, E. Fauchon-Jones, M. Hannam, C. Kalaghatgi, S. Khan, F. Pannarale, and A. Vano-Vinuales, *Phys. Rev. D* **104**, 124027 (2021).
- [60] E. Hamilton, L. London, and M. Hannam, *Phys. Rev. D* **107**, 104035 (2023).
- [61] R. Cotesta, S. Marsat, and M. Pürrer, *Phys. Rev. D* **101**, 124040 (2020).
- [62] L. Pompili *et al.*, *Phys. Rev. D* **108**, 124035 (2023).
- [63] S. Ossokine *et al.*, *Phys. Rev. D* **102**, 044055 (2020).

- [64] M. Lagos and L. Hui, *Phys. Rev. D* **107**, 044040 (2023).
- [65] W. H. Press, *Astrophys. J. Lett.* **170**, L105 (1971).
- [66] E. Leaver, *Proc. R. Soc. A* **402**, 285 (1985).
- [67] W. H. Press and S. A. Teukolsky, *Astrophys. J.* **185**, 649 (1973).
- [68] A. Ghosh, N. K. Johnson-McDaniel, A. Ghosh, C. K. Mishra, P. Ajith, W. Del Pozzo, C. P. L. Berry, A. B. Nielsen, and L. London, *Classical Quantum Gravity* **35**, 014002 (2018).
- [69] H. Yang, D. A. Nichols, F. Zhang, A. Zimmerman, Z. Zhang, and Y. Chen, *Phys. Rev. D* **86**, 104006 (2012).
- [70] H. Yang, F. Zhang, A. Zimmerman, D. A. Nichols, E. Berti, and Y. Chen, *Phys. Rev. D* **87**, 041502 (2013).
- [71] A. Zimmerman and Z. Mark, *Phys. Rev. D* **93**, 044033 (2016); **93**, 089905(E) (2016).
- [72] C. Dariescu, M.-A. Dariescu, and C. Stelea, *Adv. High Energy Phys.* **2021**, 5512735 (2021).
- [73] P. P. Fiziev, *Classical Quantum Gravity* **27**, 135001 (2010).
- [74] Y. Hatsuda, *Gen. Relativ. Gravit.* **53**, 93 (2021).
- [75] F. Brauer, *Michigan Math. J.* **11**, 379 (1964).
- [76] O. Christensen, *An Introduction to Frames and Riesz Bases* (Birkhäuser, Boston, 2003).
- [77] E. D. Fackerell and R. G. Crossman, *J. Math. Phys. (N.Y.)* **18**, 1849 (2008).
- [78] J. N. Goldberg, A. J. MacFarlane, E. T. Newman, F. Rohrlich, and E. C. G. Sudarshan, *J. Math. Phys. (N.Y.)* **8**, 2155 (1967).
- [79] L. Blanchet, *Living Rev. Relativity* **17**, 2 (2014).
- [80] B. P. Abbott *et al.* (Virgo Collaboration and LIGO Scientific Collaboration), *Phys. Rev. X* **6**, 041015 (2016).
- [81] S. O'Sullivan and S. A. Hughes, *Phys. Rev. D* **90**, 124039 (2014); **91**, 109901(E) (2015).
- [82] M. Ruiz, R. Takahashi, M. Alcubierre, and D. Nunez, *Gen. Relativ. Gravit.* **40**, 2467 (2008).
- [83] R. A. Breuer, M. P. Ryan, and S. Waller, *Proc. R. Soc. A* **358**, 71 (1977).
- [84] L. London and S. A. Hughes, [arXiv:2206.15246](https://arxiv.org/abs/2206.15246).
- [85] S. R. Green, S. Hollands, L. Sberna, V. Toomani, and P. Zimmerman, *Phys. Rev. D* **107**, 064030 (2023).
- [86] E. W. Leaver, *Phys. Rev. D* **34**, 384 (1986).
- [87] S. A. Teukolsky, *Astrophys. J.* **185**, 635 (1973).
- [88] S. A. Hughes, *Phys. Rev. D* **61**, 084004 (2000); **63**, 049902 (E) (2001); **65**, 069902(E) (2002); **67**, 089901(E) (2003); **78**, 109902(E) (2008); **90**, 109904(E) (2014).
- [89] A. Taracchini, A. Buonanno, G. Khanna, and S. A. Hughes, *Phys. Rev. D* **90**, 084025 (2014).
- [90] T. Islam, S. E. Field, S. A. Hughes, G. Khanna, V. Varma, M. Giesler, M. A. Scheel, L. E. Kidder, and H. P. Pfeiffer, *Phys. Rev. D* **106**, 104025 (2022).
- [91] N. E. M. Rifat, G. Khanna, and L. M. Burko, *Phys. Rev. Res.* **1**, 033150 (2019).
- [92] M. Campanelli, G. Khanna, P. Laguna, J. Pullin, and M. P. Ryan, *Classical Quantum Gravity* **18**, 1543 (2001).
- [93] W. Krivan, P. Laguna, and P. Papadopoulos, *Phys. Rev. D* **54**, 4728 (1996).
- [94] M. Vishal, S. E. Field, K. Rink, S. Gottlieb, and G. Khanna, *Phys. Rev. D* **110**, 104009 (2024).
- [95] E. W. Leaver, *J. Math. Phys. (N.Y.)* **27**, 1238 (1986).
- [96] S. Isoyama, R. Fujita, A. J. K. Chua, H. Nakano, A. Pound, and N. Sago, *Phys. Rev. Lett.* **128**, 231101 (2022).
- [97] M. Casals and P. Zimmerman, *Phys. Rev. D* **100**, 124027 (2019).
- [98] M. Sasaki and H. Tagoshi, *Living Rev. Relativity* **6**, 6 (2003).
- [99] R. Fujita and H. Tagoshi, *Prog. Theor. Phys.* **112**, 415 (2004).
- [100] A. Ronveaux and F. Arscott, *Heun's Differential Equations*, Oxford science publications (Oxford University Press, New York, 1995).
- [101] C. Chen and J. Jing, *J. Cosmol. Astropart. Phys.* **11** (2023) 070.
- [102] P. P. Fiziev, *Phys. Rev. D* **80**, 124001 (2009).
- [103] S. Zarrinkamar and M. Hortaçsu, *Adv. High Energy Phys.* **2018**, 8621573 (2018).
- [104] A. P. Magnus, F. Ndayiragije, and A. Ronveaux, *J. Approx. Theory* **263**, 105522 (2021).
- [105] G. Bonelli, C. Iossa, D. P. Lichtig, and A. Tanzini, *Phys. Rev. D* **105**, 044047 (2022).
- [106] G. B. Arfken, H. J. Weber, and F. E. Harris, in *Mathematical Methods for Physicists (Seventh Edition)*, edited by G. B. Arfken, H. J. Weber, and F. E. Harris (Academic Press, Boston, 2013), ed., pp. 401–445.
- [107] G. Kristensson, *Second Order Differential Equations—Special Functions and Their Classification* (Springer, Germany, 2010).
- [108] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Wiley, New York, 1989).
- [109] W. Adkins and S. Weintraub, *Algebra: An Approach via Module Theory*, Graduate Texts in Mathematics (Springer, New York, 2012).
- [110] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1964), 9th dover printing, 10th gpo printing ed.
- [111] L. Motl and A. Neitzke, *Adv. Theor. Math. Phys.* **7**, 307 (2003).
- [112] N. Andersson, *Phys. Rev. D* **55**, 468 (1997).
- [113] J. L. Jaramillo, R. Panosso Macedo, and L. Al Sheikh, *Phys. Rev. X* **11**, 031003 (2021).
- [114] H.-P. Nollert and R. H. Price, *J. Math. Phys. (N.Y.)* **40**, 980 (1999).
- [115] S. A. Teukolsky and W. H. Press, *Astrophys. J.* **193**, 443 (1974).
- [116] R. M. Wald, *Gen. Relativ. Gravit.* **11**, 321 (1979).
- [117] O. J. C. Dias, M. Godazgar, and J. E. Santos, *J. High Energy Phys.* **07** (2022) 076.
- [118] A. K.-W. Chung, P. Wagle, and N. Yunes, *Phys. Rev. D* **107**, 124032 (2023).
- [119] A. K.-W. Chung, P. Wagle, and N. Yunes, *Phys. Rev. D* **109**, 044072 (2024).
- [120] B. Ghogh, F. Karray, and M. Crowley, [arXiv:1903.11240](https://arxiv.org/abs/1903.11240).
- [121] G. B. Cook and M. Zalutskiy, *Phys. Rev. D* **90**, 124021 (2014).
- [122] H.-P. Nollert, *Classical Quantum Gravity* **16**, R159 (1999).
- [123] H.-P. Nollert and R. H. Price, *J. Math. Phys. (N.Y.)* **40**, 980 (1999).
- [124] L. London, E. Fauchon, and E. Hamilton, *london6/POSITIVE: map* (2020), [10.5281/zenodo.3901856](https://zenodo.org/record/3901856).
- [125] L. C. Stein, *J. Open Source Software* **4**, 1683 (2019).

- [126] T. mpmath development team, MPMATH: A Python library for arbitrary-precision floating-point arithmetic (version 1.3.0) (2023), *transverse-traceless* (TTh)<http://mpmath.org/>.
- [127] G. B. Cook, *Phys. Rev. D* **102**, 024027 (2020).
- [128] Y. Pinchover and J. Rubinstein, Equations in high dimensions, in *An Introduction to Partial Differential Equations* (Cambridge University Press, Cambridge, England, 2005), pp. 226–281.
- [129] K. S. Thorne, *Rev. Mod. Phys.* **52**, 299 (1980).
- [130] R. H. Boyer and R. W. Lindquist, *J. Math. Phys. (N.Y.)* **8**, 265 (2004).
- [131] S. A. Hughes, *Phys. Rev. D* **62**, 044029 (2000); **67**, 089902 (E) (2003).
- [132] Y. Mino, M. Sasaki, M. Shibata, H. Tagoshi, and T. Tanaka, *Prog. Theor. Phys. Suppl.* **128**, 1 (1997).
- [133] T. Helfer, D. J. E. Marsh, K. Clough, M. Fairbairn, E. A. Lim, and R. Becerril, *J. Cosmol. Astropart. Phys.* **03** (2017) 055.
- [134] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, and J. van Meter, *Phys. Rev. Lett.* **96**, 111102 (2006).
- [135] E. Berti, J. Cardoso, V. Cardoso, and M. Cavaglia, *Phys. Rev. D* **76**, 104044 (2007).
- [136] E. Berti and V. Cardoso, *Phys. Rev. D* **74**, 104020 (2006).
- [137] H. T. Cho, A. S. Cornell, J. Doukas, and W. Naylor, *Phys. Rev. D* **80**, 064022 (2009).
- [138] T. Regge and J. A. Wheeler, *Phys. Rev.* **108**, 1063 (1957).
- [139] S. Axler, *Linear Algebra Done Right (eBook)* 3rd ed. (Springer, Cham, 2015).
- [140] P. Lax, *Functional Analysis* (Wiley, New York, 2002).
- [141] G. Cho, [arXiv:2008.12526](https://arxiv.org/abs/2008.12526).
- [142] S. Mano, H. Suzuki, and E. Takasugi, *Prog. Theor. Phys.* **95**, 1079 (1996).
- [143] J. L. Ripley, *Classical Quantum Gravity* **39**, 145009 (2022).
- [144] K. Lin and W.-L. Qian, *Classical Quantum Gravity* **40**, 085019 (2023).
- [145] A. Zimmerman, H. Yang, Z. Mark, Y. Chen, and L. Lehner, *Astrophys. Space Sci. Proc.* **40**, 217 (2015).
- [146] E. Seidel, *Classical Quantum Gravity* **6**, 1057 (1989).
- [147] I. Gohberg, P. Lancaster, and L. Rodman, *Indefinite Linear Algebra and Applications* (Birkhäuser, Basel, 2006).
- [148] Y. Chen and D. Dai, *J. Approx. Theory* **162**, 2149 (2010).
- [149] P. Virtanen *et al.* (SciPy 1.0 Contributors), *Nat. Methods* **17**, 261 (2020).
- [150] A. B. J. Kuijlaars, A. Martinez-Finkelshtein, and R. Orive, [arXiv:math/0301037](https://arxiv.org/abs/math/0301037).
- [151] L. V. Ahlfors, *Complex Analysis* 2nd ed. (McGraw-Hill Book Company, New York, 1966).
- [152] DLMF, NIST Digital Library of Mathematical Functions, <https://dlmf.nist.gov/>, Release 1.1.10 of 2023 – 06 – 15, edited by f. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain.
- [153] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge Mathematical Library (Cambridge University Press, Cambridge, England, 1996).
- [154] N. Andersson, *Classical Quantum Gravity* **11**, L39 (1994).
- [155] O. J. C. Dias and H. S. Reall, *Classical Quantum Gravity* **30**, 095003 (2013).
- [156] T. Chihara, *An Introduction to Orthogonal Polynomials*, Dover Books on Mathematics (Dover Publications, New York, 2011).
- [157] P. P. Fiziev, *J. Phys. A* **43**, 035203 (2009).
- [158] F. Atkinson and A. Mingarelli, *Multiparameter Eigenvalue Problems: Sturm-Liouville Theory* (CRC Press, Boca Raton, FL, 2011).
- [159] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, England, 1990).
- [160] C. R. Harris *et al.*, *Nature (London)* **585**, 357 (2020).
- [161] L. London and M. Gurevich, following paper, *Phys. Rev. D* **113**, 044009 (2026).
- [162] L. London and M. Gurevich (to be published).