Modular self-duality, symmetrized relative entropy, and Bogoliubov-Kubo-Mori susceptibility in quantum field theory

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We develop an operator-algebraic framework for modular self-duality, symmetrized relative entropy, and Bogoliubov-Kubo-Mori (BKM) susceptibility of local states in quantum field theory. In finite dimensions, modular self-duality singles out fixed points at which a state coincides with its modularly reflected partner. At such points, the natural comparison functional is the symmetrized Umegaki relative entropy. It vanishes at coincidence, and its Hessian is governed by the BKM quantum Fisher information along the reflected tangent direction. We then extend this fixed-point construction to the local type III von Neumann algebras that arise in quantum field theory. Here, a local state is compared with the modular pullback of its commutant restriction, and the intrinsic comparison functional is the symmetrized Araki relative entropy. For sufficiently regular state deformations, the fixed-localization Hessian at the self-dual point defines a type III BKM susceptibility. This coefficient is obtained by evaluating the BKM bilinear form on the tangent selected by the modular pairing. Exact coherent-state realizations are obtained for the free scalar field on wedge algebras and for the chiral $U(1)$ current on half-line algebras. In both examples, the comparison functional is exactly quadratic in the deformation parameter, and the susceptibility coefficients admit explicit boost-energy, stress-tensor, or half-line integral representations. We also apply the construction to a ball-shaped boundary region in the vacuum state of a conformal field theory with a semiclassical anti–de Sitter (AdS) gravitational dual. For linear bulk coherent excitations in an exact complementary-recovery model of this sector, bulk-boundary modular intertwining identifies the modular partner with the AdS-Rindler-reflected coherent state, and the symmetrized relative entropy is exactly quadratic. Its Hessian is twice the bulk canonical energy of the reflected-difference perturbation.

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I. INTRODUCTION

A basic question in quantum theory is how to quantify state distinguishability and second-order response under smooth deformations of a state. In finite-dimensional quantum systems, this question is usually formulated in terms of density matrices, relative entropy, and quantum Fisher metrics. In local quantum field theory, however, the analogous problem is subtler because observables localized in a spacetime region are typically described by type III von Neumann algebras, for which intrinsic density matrices do not exist [1,2]. One therefore needs a formulation of comparison and response that does not rely on finite-dimensional matrix language. The aim of this paper is to develop such a formulation using a common fixed-point structure that treats type I and type III von Neumann algebraic systems within a single framework. This operator-algebraic perspective builds on earlier work by the author [3,4], where modular self-duality and related entanglement questions were studied in finite-dimensional and quantum-field-theoretic settings using Tomita-Takesaki modular theory.

In the finite-dimensional part of the paper, we consider smooth one-parameter families of states together with their modularly reflected partners. Modular self-duality singles out special parameter values at which a state coincides with its reflected partner. The natural comparison functional near this locus is the symmetrized Umegaki relative entropy. It vanishes at coincidence, differentiability forces its first variation to vanish, and its Hessian is governed by the Bogoliubov-Kubo-Mori (BKM) quantum Fisher information along the reflected tangent direction [5–7]. In this sense, the finite-dimensional type I theory identifies a canonical BKM susceptibility selected by modular self-duality and symmetrized relative entropy. A related but distinct finite-dimensional connection between quantum Fisher information and entanglement curvature has recently

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been studied in two-qubit systems [8], although without the fixed-point comparison principle used here.

The local quantum-field-theoretic problem starts from a family of ambient states on the full theory and, for each localization region O_t, compares its restriction to the local algebra $\mathcal{A}(O_t)$ with the modular pullback of its commutant restriction. The resulting intrinsic comparison functional is the symmetrized Araki relative entropy [9,10]. At the self-dual locus, the paired local states coincide, so the first variation again vanishes. At fixed localization, the corresponding Hessian is twice the BKM Fisher-information scalar obtained by evaluating the type III BKM bilinear form on the tangent direction selected by the modularly paired family. This scalar is the local operator-algebraic BKM susceptibility. The identification is made under the standard second-order regularity assumption that Araki relative entropy admits a diagonal BKM expansion for the local state curves under consideration.

A central part of the paper is the construction of exact quantum-field-theoretic examples. For coherent states of the free scalar field on wedge algebras, known relative-entropy formulas [11–16], together with the Bisognano-Wichmann identification of wedge modular flow [17,18], yield an exact two-parameter comparison surface and explicit local susceptibility coefficients. These coefficients are determined by the reflected-difference profile selected by the modular pairing and admit boost-energy and stress-tensor representations. For the chiral $U(1)$ current, the same construction leads to explicit half-line formulas in the vacuum representation and to thermal half-line formulas in which the entropy kernel is explicit while the reflected profile is determined by the Borchers-Yngvason standard-subspace modular conjugation.

Beyond these exact quantum-field-theoretic realizations, we examine the physical interpretation of the construction in the AdS/CFT correspondence. The AdS/CFT correspondence relates a conformal field theory to a quantum-gravitational theory in an asymptotically anti–de Sitter spacetime [19–21]. In this setting, a boundary region and a semiclassical reference state determine a corresponding entanglement wedge in the bulk. This wedge is the domain of dependence of a bulk hypersurface bounded by the boundary region and its associated extremal surface [22,23]. For a vacuum boundary ball, subregion relative entropy and quantum Fisher information are related to canonical energy in the corresponding AdS-Rindler wedge [24–28]. The purpose of the holographic part of the paper is to determine how, for a chosen ambient perturbation, modular conjugation fixes its reflected comparison and how the resulting reflected-difference tangent is represented in the bulk.

This question is close to, but distinct from, holographic entanglement asymmetry. In that construction, a reduced state is compared with its average under an internal symmetry, and the leading relative-entropy coefficient is represented by Fisher information and bulk canonical energy [29]. The comparison considered here is instead fixed by the vacuum modular conjugation. It exchanges the ball algebra with its commutant, and the commutant restriction is pulled back before the two local states are compared. Although both quadratic coefficients admit a bulk canonical-energy representation in their respective regimes, they resolve different notions of asymmetry. The former probes internal-symmetry breaking, whereas the latter probes modular-reflection asymmetry.

There are independent physical reasons for regarding the modular pairing associated with the vacuum standard pair as distinguished. For a Minkowski wedge, the Bisognano-Wichmann theorem identifies modular flow with Lorentz boosts and modular conjugation with the corresponding charge conjugation, parity, and time reversal (CPT)-type wedge reflection [17,18]. For a ball in the vacuum of a conformal field theory, the Casini-Huerta-Myers conformal map sends the restricted vacuum to a thermal Kubo-Martin-Schwinger (KMS) state on the hyperbolic cylinder, whose holographic description is the AdS-Rindler wedge, equivalently the exterior of the massless hyperbolic black hole [24]. In this thermal frame, for regular exponential source deformations that admit a Kubo-Mori score representation, the BKM quadratic form is the Kubo-Mori equilibrium susceptibility [30,31]. The reflected-difference direction therefore has the physical meaning of the response direction that breaks the reflection symmetry of a geometrically selected equilibrium reference state.

The operator-algebraic input needed to identify the reflected tangent is supplied by the exact complementary-recovery setting of Ref. [32]. Standard complementary reconstructability (c-reconstructability) preserves relative entropy and intertwines the boundary and bulk modular conjugations. For the linear bulk coherent family considered here, the modular partner is therefore the AdS-Rindler-reflected coherent state, and the comparison depends on the difference between the original bulk perturbation and its modular reflection. The symmetrized relative entropy is exactly quadratic, and its Hessian equals twice the canonical energy of this reflected-difference perturbation.

The significance of these examples is therefore not that the underlying relative-entropy formulas are new in isolation. Rather, the point is that these known formulas become exact realizations of a single modular comparison principle. For each localization region, the vacuum modular conjugation selects a canonical comparison partner by pulling back the commutant restriction to the original local algebra. The self-dual point is then a coincidence point in the local state space, and the first nonvanishing departure from this coincidence is measured by a Bogoliubov-Kubo-Mori susceptibility. This susceptibility is not associated with an arbitrary perturbation of the state, but with the reflected-difference tangent selected by the modularly paired family. It therefore depends simultaneously on the

physical state deformation and on the localization geometry through the modular data of the local algebra. In this way, the wedge, chiral, and encoded AdS-Rindler coherent families realize the type III local version of the same fixed-point mechanism that appears in the finite-dimensional type I prototype.

II. MODULAR SELF-DUALITY AND SYMMETRIZED RELATIVE ENTROPY IN FINITE-DIMENSIONAL (TYPE I) SYSTEMS

A. The need for operator algebras

Consider a simple spin Hamiltonian on N sites,

$$H_N = -\sum_{i=1}^{N} \sigma_i^z, \tag{1}$$

with noninteracting sites and the standard Pauli matrix algebra at each site. If N is finite, the algebra of observables generated by the Pauli spin operators is a finite-dimensional matrix algebra $\mathfrak{A}_N$. More explicitly,

$$\mathfrak{A}_N = \bigotimes_{i=1}^{N} M_2(\mathbb{C}) \cong M_{2^N}(\mathbb{C}). \tag{2}$$

Up to unitary equivalence, this algebra has a unique irreducible representation. Every normal state is represented by a density matrix on a fixed finite-dimensional Hilbert space, and state comparison can be formulated directly in terms of density matrices and Umegaki relative entropy.

This uniqueness no longer persists when the number of degrees of freedom is infinite. For an infinite spin chain, a formal Hilbert space containing all possible infinite spin configurations as mutually orthogonal basis vectors would be nonseparable. Instead, one usually works in a separable Hilbert space generated from a chosen reference state. This choice of reference state selects a physical sector, and the weak operator closure of the represented local spin algebra depends on that sector.

Let $\mathfrak{A}_0$ denote the unital *-algebra generated by finite linear combinations of finite products of local Pauli operators. Let $\pi_\uparrow$ be the representation of $\mathfrak{A}_0$ generated from the all-spin-up reference vector

$$\Omega_\uparrow = |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle \otimes \cdots. \tag{3}$$

The finite-excitation domain is

$$\mathcal{D}_\uparrow := \pi_\uparrow(\mathfrak{A}_0)\Omega_\uparrow. \tag{4}$$

The corresponding Hilbert space is the completion

$$\mathcal{H}_\uparrow := \bar{\mathcal{D}}_\uparrow. \tag{5}$$

Equivalently, $\mathcal{H}_\uparrow$ has an orthonormal basis labelled by finite subsets of spin sites that have been flipped relative to $\Omega_\uparrow$, while $\mathcal{D}_\uparrow$ is the finite linear span of these basis vectors. Thus $\mathcal{D}_\uparrow$, not the full completion $\mathcal{H}_\uparrow$, consists of vectors differing from $\Omega_\uparrow$ at only finitely many sites.

The von Neumann algebra associated with the all-spin-up sector is the weak operator topology closure

$$\mathfrak{M}_\uparrow := \pi_\uparrow(\mathfrak{A}_0)'' = \overline{\pi_\uparrow(\mathfrak{A}_0)}^{\mathrm{WOT}} \subset \mathcal{B}(\mathcal{H}_\uparrow). \tag{6}$$

For the sequence used below, weak operator convergence means convergence of all matrix elements: a sequence A_i converges weakly to A if

$$\langle \psi, A_i \phi\rangle_{\mathcal{H}_\uparrow} \to \langle \psi, A\phi\rangle_{\mathcal{H}_\uparrow}, \qquad \psi, \phi \in \mathcal{H}_\uparrow. \tag{7}$$

The sector dependence can be seen explicitly. Define the spin raising and lowering operators by

$$\sigma_i^+ = \frac{1}{2}(\sigma_i^x + i\sigma_i^y), \qquad \sigma_i^- = \frac{1}{2}(\sigma_i^x - i\sigma_i^y), \tag{8}$$

so that

$$\begin{aligned} \sigma_i^+|\downarrow\rangle_i &= |\uparrow\rangle_i, & \sigma_i^+|\uparrow\rangle_i &= 0, \\ \sigma_i^-|\uparrow\rangle_i &= |\downarrow\rangle_i, & \sigma_i^-|\downarrow\rangle_i &= 0. \end{aligned} \tag{9}$$

The projection onto a down spin at site i is therefore

$$P_i^\downarrow := \sigma_i^- \sigma_i^+ = |\downarrow\rangle_i\langle\downarrow|_i. \tag{10}$$

In the all-spin-up sector, every vector in the dense domain $\mathcal{D}_\uparrow$ has $|\uparrow\rangle$ at all sufficiently far sites. Hence, for $\psi, \phi \in \mathcal{D}_\uparrow$, we obtain

$$\langle \psi, P_i^\downarrow \phi\rangle_{\mathcal{H}_\uparrow} = 0 \tag{11}$$

for all sufficiently large i. Since $\|P_i^\downarrow\| \le 1$, density extends this weak convergence to all $\psi, \phi \in \mathcal{H}_\uparrow$. Moreover, because $P_i^\downarrow$ is a projection, the same density argument applied to $\|P_i^\downarrow \psi\|$ gives

$$P_i^\downarrow \to 0 \quad \text{strongly, and hence weakly, on } \mathcal{H}_\uparrow. \tag{12}$$

If instead one starts from the all-spin-down reference vector

$$\Omega_\downarrow = |\downarrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle \otimes \cdots, \tag{13}$$

and lets $\pi_\downarrow$ denote the corresponding representation of $\mathfrak{A}_0$, then the finite-excitation domain and Hilbert space are

$$\mathcal{D}_\downarrow := \pi_\downarrow(\mathfrak{A}_0)\Omega_\downarrow, \qquad \mathcal{H}_\downarrow := \bar{\mathcal{D}}_\downarrow. \tag{14}$$

The associated von Neumann algebra is

$$\mathfrak{M}_{\downarrow} \coloneqq \pi_{\downarrow}(\mathfrak{A}_0)'' = \overline{\pi_{\downarrow}(\mathfrak{A}_0)}^{\mathrm{WOT}} \subset \mathcal{B}(\mathcal{H}_{\downarrow}). \tag{15}$$

In this sector, every vector in $\mathcal{D}_{\downarrow}$ has $|{\downarrow}\rangle$ at all sufficiently far sites. Therefore, for $\psi, \phi \in \mathcal{D}_{\downarrow}$, we obtain

$$\langle \psi, P_i^{\downarrow} \phi \rangle_{\mathcal{H}_{\downarrow}} = \langle \psi, \phi \rangle_{\mathcal{H}{\downarrow}} \tag{16}$$

for all sufficiently large i. Again, by density and uniform boundedness,

$$P_i^{\downarrow} \to \mathbf{1} \quad \text{weakly on } \mathcal{H}_{\downarrow}. \tag{17}$$

Since $\mathbf{1} - P_i^{\downarrow}$ is the projection onto an up spin at site i, the same density argument gives strong convergence,

$$P_i^{\downarrow} \to \mathbf{1} \quad \text{strongly on } \mathcal{H}_{\downarrow}. \tag{18}$$

Thus the same sequence of finite-site observables has different weak limits in the two representations:

$$\begin{aligned} P_i^{\downarrow} &\to 0 \quad \text{in the } \Omega_{\uparrow} \text{ sector,} \\ P_i^{\downarrow} &\to \mathbf{1} \quad \text{in the } \Omega_{\downarrow} \text{ sector.} \end{aligned} \tag{19}$$

This illustrates the basic need for the operator-algebraic point of view. In an infinite system, the represented algebra and its weak closure retain information about the physical sector or state used to realize the observables. The example is not meant to assert that these spin-chain sector closures are type III; it is meant to show why representations and weak closures become essential once infinitely many degrees of freedom are present.

This operator-algebraic viewpoint treats observables as elements of a C^*-algebra and states as positive normalized linear functionals on that algebra. If $\mathfrak{A}$ is the algebra of observables, then a state ω assigns expectation values by the pairing

$$A \longmapsto \omega(A), \qquad A \in \mathfrak{A}. \tag{20}$$

Dynamics are described algebraically by a one-parameter group of C^*-automorphisms, as follows:

$$\alpha_t \colon \mathfrak{A} \to \mathfrak{A}, \qquad \alpha_t(AB) = \alpha_t(A)\alpha_t(B), \qquad \alpha_t(A^*) = \alpha_t(A)^*. \tag{21}$$

This point of view separates the abstract observables from the particular Hilbert-space representation selected by a state.

The same structural issue appears in quantum field theory. One first specifies an abstract algebra of local observables before choosing a particular Hilbert-space representation. Let $\mathfrak{A}$ denote the abstract C^*-algebra of observables, and let $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}$ denote the abstract local algebra associated with a spacetime region $\mathcal{O}$. A state on $\mathfrak{A}$ is again a positive normalized linear functional. Choosing a state determines a Hilbert-space representation through the Gelfand-Naimark-Segal (GNS) construction. For the vacuum state ω_0, this gives a triple

$$(\pi_0, \mathcal{H}, \Omega). \tag{22}$$

Here $\mathcal{H}$ is the vacuum Hilbert space, $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$, and

$$\pi_0 \colon \mathfrak{A} \to \mathcal{B}(\mathcal{H}) \tag{23}$$

is the representation that turns each abstract observable into a bounded operator on $\mathcal{H}$. The vector $\Omega \in \mathcal{H}$ is the vacuum vector. It represents the vacuum state by

$$\omega_0(A) = \langle \Omega, \pi_0(A)\Omega \rangle, \qquad A \in \mathfrak{A}, \tag{24}$$

and is cyclic for the represented algebra, as follows:

$$\overline{\pi_0(\mathfrak{A})\Omega} = \mathcal{H}. \tag{25}$$

For each region $\mathcal{O}$, the represented local von Neumann algebra is

$$\mathcal{A}(\mathcal{O}) \coloneqq \pi_0(\mathfrak{A}(\mathcal{O}))'' \subset \mathcal{B}(\mathcal{H}). \tag{26}$$

Equivalently, $\mathcal{A}(\mathcal{O})$ is the weak operator closure of $\pi_0(\mathfrak{A}(\mathcal{O}))$. The double-prime notation denotes the bicommutant, and the equivalence with weak closure is the content of von Neumann's bicommutant theorem (see Sec. III). The result is the concrete von Neumann algebra generated by the bounded local observables associated with $\mathcal{O}$ in the vacuum sector.

The local algebras obtained in relativistic quantum field theory are typically type III von Neumann algebras. They do not possess an intrinsic trace, do not admit canonical density matrices for local state restrictions, and do not come with a canonical tensor-product factorization into a region and its complement. Consequently, finite-dimensional notions such as reflected density matrices and Umegaki relative entropy must be replaced by modular-theoretic notions: standard pairs, modular conjugations, relative modular operators, and Araki relative entropy.

In the present paper, the finite-dimensional type I construction serves as a prototype. It explains the fixed-point mechanism in the familiar language of density matrices. The local type III construction then replaces density matrices by states on local von Neumann algebras and replaces ordinary relative entropy by Araki relative entropy.

B. Modular reflection and self-dual points

We start with a modular-theoretic idea that will be used in the finite-dimensional type I construction. The Tomita-Takesaki construction needed below is briefly reviewed in Sec. III. Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. A von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ is a unital *-subalgebra closed in the weak operator topology. Its commutant is

$$\mathfrak{M}' := \{B \in \mathcal{B}(\mathcal{H}) : BA = AB,\ \forall\ A \in \mathfrak{M}\}. \tag{27}$$

A vector $\Omega \in \mathcal{H}$ is cyclic for $\mathfrak{M}$ if $\mathfrak{M}\Omega$ is dense in $\mathcal{H}$, and separating for $\mathfrak{M}$ if $A\Omega = 0, A \in \mathfrak{M} \to A = 0$. A pair $(\mathfrak{M}, \Omega)$ with Ω cyclic and separating is called a standard pair. Tomita-Takesaki theory assigns to such a pair an antiunitary modular conjugation J. One of its basic structural consequences is

$$J\mathfrak{M}J = \mathfrak{M}'. \tag{28}$$

Thus, J implements an algebraic reflection in the sense that it exchanges the algebra with its commutant. For the finite-dimensional discussion below, only the idea of an antiunitary reflection is needed; the antiunitary J is fixed as part of the comparison data.

Let J be a fixed antiunitary involution on the finite-dimensional Hilbert space $\mathcal{H}$. Given a density matrix ρ, we define its reflected partner by

$$\rho_J := J\rho J. \tag{29}$$

A state is called modularly self-dual, relative to this reflection, if

$$\rho = \rho_J. \tag{30}$$

Equivalently, ρ is a fixed point of the reflected state map $\rho \mapsto J\rho J$.

For a smooth one-parameter family of density matrices $\rho(g)$, we assume that the reflection induces *an involution r of parameter space*, as follows:

$$\rho_J(g) := J\rho(g)J = \rho(r(g)), \qquad r(r(g)) = g. \tag{31}$$

A parameter value $g_\star$ is then a modular self-dual point if

$$r(g_\star) = g_\star, \qquad \rho(g_\star) = \rho_J(g_\star). \tag{32}$$

The next subsection studies the symmetrized Umegaki relative entropy between $\rho(g)$ and its reflected partner $\rho_J(g)$ near such a fixed point. The finite-dimensional construction below should be read as the type I prototype of the local modular-pullback construction developed later in the paper.

C. Symmetrized Umegaki relative entropy and the BKM metric at modular fixed points

The natural comparison functional for a state and its reflected pair is the symmetrized Umegaki relative entropy [33]

$$\mathfrak{S}_J(g) = S(\rho(g)\|\rho_J(g)) + S(\rho_J(g)\|\rho(g)), \tag{33}$$

with

$$S(\rho\|\sigma) = \mathrm{Tr}\rho(\log\rho - \log\sigma). \tag{34}$$

At the modular self-dual point,

$$\mathfrak{S}_J(g_\star) = 0, \tag{35}$$

because the two states coincide. Since $\mathfrak{S}_J(g) \geq 0$, differentiability gives

$$\partial_g \mathfrak{S}_J(g)|_{g=g_\star} = 0. \tag{36}$$

The first nontrivial local datum is therefore the Hessian

$$I_J(g_\star) = \partial_g^2 \mathfrak{S}_J(g)|_{g=g_\star}, \tag{37}$$

so that

$$\mathfrak{S}_J(g) = \frac{1}{2} I_J(g_\star)(g - g_\star)^2 + o((g - g_\star)^2). \tag{38}$$

To further identify the quadratic coefficient, we write

$$\begin{aligned} &\delta = g - g_\star, \qquad \rho_\star = \rho(g_\star), \\ &X_\star = \partial_g \rho(g)|_{g=g_\star}, \qquad Y_\star = \partial_g^2 \rho(g)|_{g=g_\star}. \end{aligned} \tag{39}$$

Near a fixed point $g_\star$, we choose the local coordinate $\delta := g - g_\star$ so that this involution acts as reflection about $g_\star$, as follows:

$$r(g_\star + \delta) = g_\star - \delta. \tag{40}$$

Then a C^2-differentiability assumption gives

$$\begin{aligned} \rho(g_\star + \delta) &= \rho_\star + \delta X_\star + \frac{1}{2}\delta^2 Y_\star + o(\delta^2), \\ \rho_J(g_\star + \delta) &= \rho(r(g_\star + \delta)) = \rho(g_\star - \delta) \\ &= \rho_\star - \delta X_\star + \frac{1}{2}\delta^2 Y_\star + o(\delta^2). \end{aligned} \tag{41}$$

Before applying the logarithm, it is useful to spell out the support convention used in this subsection. The operator logarithm and the BKM kernel are singular on zero eigenvalues. Thus the formulas below are meant either for strictly positive density matrices on the full Hilbert space, or more generally on a fixed common support. Concretely, let

$$P \coloneqq \operatorname{supp}\rho_\star. \tag{42}$$

We assume that, for $|\delta|$ sufficiently small,

$$\operatorname{supp}\rho(g_\star+\delta) = \operatorname{supp}\rho_J(g_\star+\delta) = P. \tag{43}$$

All logarithms, Kubo-Mori maps, traces, and spectral sums are then understood after restricting the density matrices and tangent operators to the smaller Hilbert space $\mathcal{PH}$. On this reduced Hilbert space, the reference state $\rho_\star$ is faithful, so the ordinary finite-dimensional formulas apply. The discarded sector is a parameter-independent null sector: no probability weight moves into or out of it along the family. If the support were allowed to change with g, additional singular or nonsmooth contributions to the relative entropy could appear, and the Hessian formula would require a separate analysis.

Let $D(\log)_{\rho_\star}[X]$ denote the Fréchet derivative of the operator logarithm map $\rho \mapsto \log\rho$, evaluated at the reference state $\rho_\star$ and applied to the perturbation direction X. After restricting to the fixed support $\mathcal{PH}$, the reference state $\rho_\star$ has no zero eigenvalues. Therefore, the following Fréchet derivative of the logarithm is well defined on $\mathcal{PH}$, as follows:

$$D(\log)_{\rho_\star}[X] = \int_0^\infty du(\rho_\star+uI)^{-1}X(\rho_\star+uI)^{-1}. \tag{44}$$

Equivalently, for small ε, we obtain

$$\log(\rho_\star+\varepsilon X) = \log\rho_\star + \varepsilon D(\log)_{\rho_\star}[X] + O(\varepsilon^2). \tag{45}$$

It is convenient to package this in the Kubo-Mori map, as follows:

$$\mathcal{K}_{\rho_\star}(X) = \int_0^1 ds\rho_\star^s X\rho_\star^{1-s}, \tag{46}$$

whose inverse is exactly the Fréchet derivative of the logarithm $\mathcal{K}^{-1}_{\rho_\star}(X) = D(\log)_{\rho_\star}[X]$.

The BKM bilinear form at $\rho_\star$ is then defined by

$$\gamma^{\mathrm{BKM}}_{\rho_\star}(X,Y) = \mathrm{Tr}[X\mathcal{K}^{-1}_{\rho_\star}(Y)]. \tag{47}$$

The associated BKM quantum Fisher information along the family $g \mapsto \rho(g)$ is [6,30]

$$\begin{aligned} F_{\mathrm{BKM}}(g_\star) &= \gamma^{\mathrm{BKM}}_{\rho_\star}(X_\star,X_\star) = \mathrm{Tr}[X_\star\mathcal{K}^{-1}_{\rho_\star}(X_\star)] \\ &= \int_0^\infty du\mathrm{Tr}[X_\star(\rho_\star+uI)^{-1}X_\star(\rho_\star+uI)^{-1}]. \end{aligned} \tag{48}$$

If $\rho_\star = \sum_i p_i|i\rangle\langle i|$ is the spectral decomposition, then the BKM form is given by [34,35]

$$\gamma^{\mathrm{BKM}}_{\rho_\star}(X,Y) = \sum_{i,j} c_{\mathrm{BKM}}(p_i,p_j)X_{ij}Y_{ji}, \tag{49}$$

with kernel

$$c_{\mathrm{BKM}}(x,y) = \frac{\log x - \log y}{x-y} \quad (x\neq y), \qquad c_{\mathrm{BKM}}(x,x) = \frac{1}{x}. \tag{50}$$

This makes positivity manifest, since $\gamma^{\mathrm{BKM}}_{\rho_\star}(X,X) \geq 0$ for Hermitian X.

To extract the quadratic term, we expand the difference of the two logarithms directly. Because log is analytic on positive definite operators, and because the quadratic terms in Eq. (41) coincide, one obtains

$$\log\rho(g_\star+\delta) - \log\rho_J(g_\star+\delta) = 2\delta\mathcal{K}^{-1}_{\rho_\star}(X_\star) + o(\delta^2). \tag{51}$$

Substituting Eqs. (41) and (51) into the first relative entropy yields

$$\begin{aligned} &S(\rho(g_\star+\delta)\|\rho_J(g_\star+\delta)) \\ &= \mathrm{Tr}\left[\left(\rho_\star+\delta X_\star+\frac{1}{2}\delta^2 Y_\star+o(\delta^2)\right)(2\delta K^{-1}_{\rho_\star}(X_\star)+o(\delta^2))\right] \\ &= 2\delta\mathrm{Tr}[\rho_\star K^{-1}_{\rho_\star}(X_\star)] + 2\delta^2\mathrm{Tr}[X_\star K^{-1}_{\rho_\star}(X_\star)] + o(\delta^2). \end{aligned} \tag{52}$$

Since $\mathrm{Tr}X_\star = 0$ and, in the spectral basis of $\rho_\star$, we have

$$\begin{aligned} \mathrm{Tr}[\rho_\star\mathcal{K}^{-1}_{\rho_\star}(X_\star)] &= \sum_i p_i c_{\mathrm{BKM}}(p_i,p_i)(X_\star)_{ii} \\ &= \sum_i (X_\star)_{ii} = \mathrm{Tr}X_\star = 0, \end{aligned} \tag{53}$$

the linear term vanishes and we obtain

$$S(\rho(g_\star+\delta)\|\rho_J(g_\star+\delta)) = 2F_{\mathrm{BKM}}(g_\star)\delta^2 + o(\delta^2). \tag{54}$$

The same expansion holds after exchanging the two arguments, so

$$\mathfrak{S}_J(g_\star+\delta) = 4F_{\mathrm{BKM}}(g_\star)\delta^2 + o(\delta^2). \tag{55}$$

Comparing with Eq. (38) gives

$$I_J(g_\star) = 8F_{\mathrm{BKM}}(g_\star). \tag{56}$$

Thus, with the normalization used in Eq. (38), the Hessian of the symmetrized Umegaki relative entropy is the BKM quantum Fisher information of the single tangent $X_\star$, multiplied by the universal factor 8. The factor 8 comes from two sources. First, the reflected comparison uses the tangent difference $X_\star - (-X_\star) = 2X_\star$. Second, the functional is symmetrized, so the two relative-entropy terms contribute equally. This is the direct type I precursor of the

local type III Hessian built later from symmetrized Araki relative entropy.

D. Phase-damped single-excitation X-state family

As a concrete mixed-state illustration, consider the phase-damped single-excitation family

$$|\psi(g)\rangle = \cos\left(\frac{\pi}{4}+g\right)|01\rangle + \sin\left(\frac{\pi}{4}+g\right)|10\rangle, \qquad |g| < \frac{\pi}{4}, \tag{57}$$

with even dephasing profile

$$\Lambda(g) = e^{-(\gamma_0+\gamma_2 g^2)\tau}, \quad \gamma_0 > 0, \quad \gamma_2 \geq 0, \quad \tau > 0. \tag{58}$$

The resulting density matrix in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is

$$\rho(g) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta & \Lambda(g)\sin\theta\cos\theta & 0 \\ 0 & \Lambda(g)\sin\theta\cos\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \theta = \tfrac{\pi}{4} + g. \tag{59}$$

Its support is the fixed two-dimensional subspace

$$P\mathcal{H} = \mathrm{span}\{|01\rangle, |10\rangle\}. \tag{60}$$

For $\tau > 0$, one has $0 < \Lambda(g) < 1$, and since $|g| < \pi/4$ also $\sin\theta\cos\theta > 0$. Hence, the restricted state $P\rho(g)P$, viewed as a 2×2 density matrix on $P\mathcal{H}$, is faithful for all such g. The two zero rows and columns in Eq. (59) therefore do not enter the logarithms. They represent a fixed null sector outside $P\mathcal{H}$, not eigenvalues that are varied through zero. All Umegaki entropies and BKM quantities in this example are computed on $P\mathcal{H}$.

On $P\mathcal{H}$, let K_P denote complex conjugation in the ordered basis $\{|01\rangle, |10\rangle\}$, and let S_P be the unitary swap satisfying $S_P|01\rangle = |10\rangle$ and $S_P|10\rangle = |01\rangle$. Define the antiunitary involution $J := S_P K_P$. Since $\Lambda(g)$ is even, this reflection satisfies

$$\rho_J(g) := J\rho(g)J = \rho(-g), \qquad g_\star = 0. \tag{61}$$

Thus, $g = 0$ is the modular self-dual point. The family of states takes the effective qubit form

$$\rho(g) = \frac{1}{2}(\mathbf{1} + \Lambda(g)\cos(2g)\sigma_x - \sin(2g)\sigma_z), \tag{62}$$

where σ_x and σ_z are the Pauli matrices. At the fixed point,

$$\lambda_0 = e^{-\gamma_0\tau} \in (0,1),$$
$$\rho_0 = \rho(0) = \frac{1}{2}(\mathbf{1} + \lambda_0\sigma_x),$$
$$X_0 = \partial_g\rho(g)|_{g=0} = -\sigma_z, \tag{63}$$

where the last identity uses $\Lambda'(0) = 0$ by evenness.

In the eigenbasis of σ_x, the fixed-point state is diagonal,

$$\rho_0 = \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix}, \qquad p_\pm = \frac{1\pm\lambda_0}{2}, \tag{64}$$

while the tangent takes the form

$$X_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{65}$$

The BKM spectral formulas (49) and (50) then give [34,35]

$$F_{\mathrm{BKM}}(0) = 2c_{\mathrm{BKM}}(p_+, p_-) = 2\frac{\log p_+ - \log p_-}{p_+ - p_-} = \frac{2}{\lambda_0}\log\frac{1+\lambda_0}{1-\lambda_0}. \tag{66}$$

Hence, the reflected Hessian is

$$I_J(0) = 8F_{\mathrm{BKM}}(0) = \frac{16}{\lambda_0}\log\frac{1+\lambda_0}{1-\lambda_0}. \tag{67}$$

The same coefficient can also be extracted directly from the symmetrized Umegaki functional (33) [33]. Writing

$$\rho(g) = \frac{1}{2}(\mathbf{1} + \vec{R}(g)\cdot\vec{\sigma}), \qquad \vec{R}(g) = (\Lambda(g)\cos 2g, 0, -\sin 2g), \tag{68}$$

one has $|\vec{R}(g)| = |\vec{R}(-g)|$. Let

$$\mathcal{R}(g) = |\vec{R}(g)| = \sqrt{\Lambda(g)^2\cos^2(2g) + \sin^2(2g)}. \tag{69}$$

For a faithful qubit state $\rho = \frac{1}{2}(\mathbf{1} + \vec{R}\cdot\vec{\sigma})$ with $0 < \mathcal{R} = |\vec{R}| < 1$, one has

$$\log\rho = a(\mathcal{R})\mathbf{1} + b(\mathcal{R})\hat{R}\cdot\vec{\sigma}, \tag{70}$$

with

$$a(\mathcal{R}) = \frac{1}{2}\log\frac{1-\mathcal{R}^2}{4}, \qquad b(\mathcal{R}) = \mathrm{arctanh}(\mathcal{R}), \qquad \hat{R} = \frac{\vec{R}}{\mathcal{R}}. \tag{71}$$

Since $\rho(g)$ and $\rho(-g)$ have the same Bloch radius, the scalar part cancels in the relative entropy. Using $\mathrm{Tr}[(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})] = 2\vec{a}\cdot\vec{b}$, a direct calculation gives

$$S(\rho(g)\|\rho(-g)) = \frac{\operatorname{arctanh}\mathcal{R}(g)}{\mathcal{R}(g)}(\mathcal{R}(g)^2 - \vec{R}(g)\cdot\vec{R}(-g)). \tag{72}$$

Now,

$$\mathcal{R}(g)^2 - \vec{R}(g)\cdot\vec{R}(-g) = 2\sin^2(2g), \tag{73}$$

so the symmetrized Umegaki entropy is exactly

$$\mathfrak{S}_J(g) = \frac{4\operatorname{arctanh}\mathcal{R}(g)}{\mathcal{R}(g)}\sin^2(2g). \tag{74}$$

Since $\mathcal{R}(g)$ is even and $\mathcal{R}(0) = \lambda_0$, one has

$$\frac{\operatorname{arctanh}\mathcal{R}(g)}{\mathcal{R}(g)} = \frac{\operatorname{arctanh}\lambda_0}{\lambda_0} + O(g^2),$$
$$\sin^2(2g) = 4g^2 + O(g^4). \tag{75}$$

Therefore,

$$\begin{aligned}\mathfrak{S}_J(g) &= \frac{16\operatorname{arctanh}\lambda_0}{\lambda_0}g^2 + O(g^4)\\ &= \frac{8}{\lambda_0}\log\frac{1+\lambda_0}{1-\lambda_0}g^2 + O(g^4)\\ &= \frac{1}{2}I_J(0)g^2 + O(g^4).\end{aligned} \tag{76}$$

This example therefore shows explicitly that the coefficient of g^2 in the reflected Umegaki functional is $4F_{\mathrm{BKM}}(0)$ [6,30], or equivalently that the Hessian is $I_J(0) = 8F_{\mathrm{BKM}}(0)$. It also shows that the parameter γ_2 in the even dephasing profile affects only quartic and higher even orders. The Hessian itself is fixed entirely by the self-dual state ρ_0 and the tangent X_0.

III. VON NEUMANN ALGEBRAS AND MODULAR THEORY

We now recast the finite-dimensional discussion of Sec. II in the operator-algebraic language appropriate to local quantum field theory. The essential point is that local observables in relativistic quantum field theory are not described by finite-dimensional matrix algebras, or even by type I factors with intrinsic density matrices. They are described by local von Neumann algebras, which are generically type III. Consequently, the finite-dimensional notions of density matrices, reflected density matrices, and Umegaki relative entropy must be replaced by modular-theoretic objects: standard pairs, modular conjugations, relative modular operators, and Araki relative entropy. The relative modular and Araki-entropy conventions used below are fixed in Appendix A.

A. Smeared fields, Weyl operators, and local nets

For a free real scalar field on Minkowski spacetime $\mathcal{M}$, it is convenient to work with smeared fields $\Phi(f)$, where $f \in C_0^\infty(\mathcal{M})$ is a smooth compactly supported test function. Formally,

$$\Phi(f) = \int_{\mathcal{M}} \phi(x)f(x)d\mu(x), \tag{77}$$

where ϕ is the operator-valued distribution associated with the scalar field. The smeared fields satisfy linearity,

$$\Phi(\alpha f_1 + \beta f_2) = \alpha\Phi(f_1) + \beta\Phi(f_2), \qquad \alpha,\beta\in\mathbb{C}, \tag{78}$$

and the adjoint relation

$$\Phi(f)^* = \Phi(\bar{f}). \tag{79}$$

In particular, if f is real valued, then $\Phi(f)$ is symmetric; in the vacuum representation it is essentially self-adjoint on the standard finite-particle domain.

Let

$$P := \Box + m^2 \tag{80}$$

denote the Klein-Gordon operator, and let

$$E := E_{\mathrm{ret}} - E_{\mathrm{adv}} \tag{81}$$

be the causal propagator. The field equation and the covariant commutation relations are

$$\Phi(Pf) = 0, \tag{82}$$

$$[\Phi(f), \Phi(g)] = iE(f,g)\mathbf{1}, \tag{83}$$

where

$$E(f,g) := \int_{\mathcal{M}\times\mathcal{M}} f(x)E(x,y)g(y)d\mu(x)d\mu(y) \tag{84}$$

is the induced antisymmetric bilinear form. The relation $\Phi(Pf) = 0$ means that the field depends only on the equivalence class of f modulo $PC_0^\infty(\mathcal{M})$. Equivalently, one may regard the basic real test-function space as the quotient

$$\mathcal{K}_{\mathbb{R}} := C_0^\infty(\mathcal{M},\mathbb{R})/PC_0^\infty(\mathcal{M},\mathbb{R}), \tag{85}$$

equipped with the symplectic form induced by E.

Because the smeared fields are generally unbounded, it is standard to pass to the bounded Weyl operators. For real-valued f, we define

$$W(f) := e^{i\Phi(f)}. \tag{86}$$

These unitaries satisfy the Weyl relations

$$W(f)W(g) = e^{-\frac{i}{2}E(f,g)}W(f+g), \tag{87}$$

or equivalently,

$$W(f)W(g) = e^{-iE(f,g)}W(g)W(f), \tag{88}$$

together with

$$W(f)^* = W(-f). \tag{89}$$

Equation (88) is the Weyl form of the canonical commutation relations (CCR). Since $\Phi(Pf) = 0$, the Weyl operator $W(f)$ also depends only on the equivalence class of f in $\mathcal{K}_{\mathbb{R}}$, although we continue to write $W(f)$ for simplicity.

Let $\mathfrak{A}$ denote the global Weyl C^*-algebra, that is, the universal C^*-algebra generated by the symbols $W(f)$ subject to the Weyl relations above. For an open region $\mathcal{O} \subset \mathcal{M}$, we define the local C^*-subalgebra

$$\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A} \tag{90}$$

to be the C^*-algebra generated by those Weyl operators $W(f)$ with $\operatorname{supp} f \subset \mathcal{O}$. The assignment

$$\mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O}) \tag{91}$$

is the abstract local Weyl net. Here, "net" simply means a region-by-region assignment of algebras, ordered by inclusion of regions. In particular, if

$$\mathcal{O}_1 \subset \mathcal{O}_2, \tag{92}$$

then isotony gives

$$\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2). \tag{93}$$

Moreover, if $\mathcal{O}'$ denotes the causal complement of $\mathcal{O}$, then locality means

$$\mathcal{O}_1 \subset \mathcal{O}_2' \Rightarrow [A_1, A_2] = 0, \qquad A_i \in \mathfrak{A}(\mathcal{O}_i). \tag{94}$$

This is the algebraic expression of Einstein causality: observables localized in spacelike separated regions commute. The regions relevant later in the paper include both bounded regions and unbounded ones such as wedges.

To obtain concrete operators on a Hilbert space, one must choose a state and pass to its GNS representation. Recall that a state on $\mathfrak{A}$ is a positive normalized linear functional. For the free scalar field, there is a distinguished vacuum state, denoted ω_0, and its GNS construction yields a triple

$$(\pi_0, \mathcal{H}, \Omega), \tag{95}$$

called the vacuum representation. Here, $\pi_0 \colon \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a $*$-representation of the abstract algebra by bounded operators on a Hilbert space $\mathcal{H}$, and $\Omega \in \mathcal{H}$ is a cyclic vector satisfying

$$\omega_0(A) = \langle \Omega, \pi_0(A)\Omega \rangle, \qquad A \in \mathfrak{A}. \tag{96}$$

Thus, the vacuum state is realized as a vector state in the Hilbert-space representation, and Ω is the vacuum vector.

For each region $\mathcal{O}$, we then define the represented local von Neumann algebra by $\mathcal{A}(\mathcal{O}) \coloneqq \pi_0(\mathfrak{A}(\mathcal{O}))'' \subset \mathcal{B}(\mathcal{H})$. The double commutant here is equivalent, by von Neumann's double commutant theorem, to the weak operator closure of $\pi_0(\mathfrak{A}(\mathcal{O}))$. In other words, $\mathcal{A}(\mathcal{O})$ is the concrete algebra of bounded local observables associated with $\mathcal{O}$ in the vacuum sector.

This distinction between the abstract net and its vacuum realization is important throughout the paper. The notation $\mathfrak{A}(\mathcal{O})$ refers to the abstract local Weyl C^*-algebra, while $\mathcal{A}(\mathcal{O})$ refers to the corresponding represented local von Neumann algebra acting on the vacuum Hilbert space. The former is part of the model-independent C^*-algebraic description, whereas the latter is the concrete operator algebra used in the vacuum representation. Throughout the main text, local algebras are understood in this represented sense. In the free scalar theory, this includes, in particular, the wedge algebras $\mathcal{A}(W_t)$ used later, and in the chiral setting, the analogous notation is used for local algebras associated with intervals or half-lines.

B. Tomita-Takesaki theory

For extensive reviews of the topics summarized below, see Refs. [1,2,31,36]. As above, we work in the vacuum representation $(\pi_0, \mathcal{H}, \Omega)$ of the abstract local net and write

$$\mathcal{A}(\mathcal{O}) \coloneqq \pi_0(\mathfrak{A}(\mathcal{O}))'' \subset \mathcal{B}(\mathcal{H}) \tag{97}$$

for the represented local von Neumann algebra associated with an open spacetime region $\mathcal{O} \subset \mathcal{M}$. Here, $\mathcal{M}$ denotes Minkowski spacetime, while $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ denotes a generic von Neumann algebra. All modular and relative modular objects used below are attached to these represented local algebras.

A von Neumann algebra $\mathfrak{M}$ is a unital *-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the weak operator topology, or equivalently satisfies the bicommutant relation

$$\mathfrak{M} = \mathfrak{M}''. \tag{98}$$

In finite-dimensional quantum mechanics, the basic example is the type I factor

$$\mathfrak{M} = \mathcal{B}(\mathcal{H}), \tag{99}$$

where states are represented by density matrices and relative entropy is given by the usual trace formula. In

local quantum field theory, by contrast, the local algebras $\mathcal{A}(\mathcal{O})$ are generically type III factors. They do not possess an intrinsic trace, do not admit a canonical density-matrix description of local state restrictions, and do not come with a canonical Hilbert-space tensor factorization into a region and its complement. This is the structural reason why the finite-dimensional construction must be reformulated in modular terms.

The relevant modular input begins with a standard pair $(\mathfrak{M}, \Omega)$, where $\Omega \in \mathcal{H}$ is cyclic and separating for $\mathfrak{M}$. Cyclicity means that $\mathfrak{M}\Omega$ is dense in $\mathcal{H}$, while separating means that

$$A\Omega = 0, \qquad A \in \mathfrak{M} \Rightarrow A = 0. \tag{100}$$

For such a pair, one defines the Tomita operator on the dense domain $\mathfrak{M}\Omega$ by

$$S_0(A\Omega) = A^*\Omega, \qquad A \in \mathfrak{M}. \tag{101}$$

Its closure S has polar decomposition

$$S = J\Delta^{1/2}, \tag{102}$$

where J is the modular conjugation and Δ is the modular operator. Tomita-Takesaki theory gives

$$J\mathfrak{M}J = \mathfrak{M}', \tag{103}$$

and the modular automorphism group

$$\sigma_s^{\Omega}(A) = \Delta^{is} A \Delta^{-is}, \qquad A \in \mathfrak{M}, \qquad s \in \mathbb{R}, \tag{104}$$

satisfies $\sigma_s^{\Omega}(\mathfrak{M}) = \mathfrak{M}$. Thus the modular operator generates intrinsic dynamics of the algebra, while the modular conjugation exchanges the algebra with its commutant.

In the local quantum-field-theoretic setting, the Reeh-Schlieder property implies that the vacuum vector Ω is cyclic and separating for the local algebras associated with the regions considered here. Hence, each pair

$$(\mathcal{A}(\mathcal{O}), \Omega) \tag{105}$$

is standard, and the modular data are available intrinsically.

For the local comparison problem below, it is useful to write the relevant local algebra in the uniform form

$$M_t \coloneqq \mathcal{A}(O_t), \qquad J_t \coloneqq J_{M_t,\Omega}, \qquad \Delta_t \coloneqq \Delta_{M_t,\Omega}. \tag{106}$$

Here, O_t denotes the localization region under consideration. In the scalar-field example of Sec. V, $O_t = W_t$ is a wedge. In the chiral-current example of Sec. VI, $O_t = (-\infty, t)$ is a half-line. For every such standard pair, Tomita-Takesaki theory gives

$$J_t M_t J_t = M_t'. \tag{107}$$

When $O_t = W_t$ is a wedge and one identifies M_t' with the algebra of the opposite wedge, one invokes wedge duality. When the modular objects are given a geometric interpretation in the scalar wedge model, one additionally invokes the Bisognano-Wichmann theorem.

A state on a von Neumann algebra $\mathfrak{M}$ is a positive normalized linear functional

$$\omega\colon \mathfrak{M} \to \mathbb{C}, \qquad \omega(A^*A) \geq 0, \qquad \omega(\mathbf{1}) = 1. \tag{108}$$

It is faithful if $\omega(A^*A) = 0$ implies $A = 0$, and normal if it is ultraweakly continuous. In finite-dimensional type I settings, normal states are precisely those represented by density matrices. In type III local algebras, faithful normal states are still well defined, but there is no intrinsic local density matrix with respect to a canonical trace.

To compare two faithful normal states ω and ϕ on $\mathfrak{M}$, one uses relative modular theory. In standard form, each faithful normal state is represented by a vector ξ_ω in the natural cone. With the convention fixed in Appendix A, the relative modular operator $\Delta_{\phi|\omega}$ determines the Araki relative entropy [9,10] as follows:

$$S_{\mathfrak{M}}(\omega\|\phi) \coloneqq -\langle \xi_\omega, \log \Delta_{\phi|\omega} \xi_\omega \rangle. \tag{109}$$

This is the intrinsic notion of relative entropy appropriate to local operator algebras. In the type I case, if

$$\omega(A) = \mathrm{Tr}(\rho A), \qquad \phi(A) = \mathrm{Tr}(\sigma A), \tag{110}$$

with faithful density matrices ρ and σ, then

$$S_{\mathfrak{M}}(\omega\|\phi) = \mathrm{Tr}\rho(\log\rho - \log\sigma), \tag{111}$$

so Araki relative entropy reduces to Umegaki relative entropy [33].

The transition from the type I construction to the local type III setting can now be stated conceptually. In finite dimensions, the reflected partner of a state is another density matrix obtained by applying the chosen antiunitary within the same matrix algebra. For a local type III algebra, the vacuum modular conjugation instead carries the algebra to its commutant. Thus the reflected comparison cannot be formulated as an internal density-matrix operation on the local algebra. One must restrict the ambient state to the commutant and then pull that restriction back to the original local algebra by the modular anti-isomorphism associated with the vacuum standard pair. This construction is formulated in the next section for the local algebras relevant to both examples: wedge algebras in the scalar-field model and half-line algebras in the chiral-current model.

IV. MODULAR SELF-DUALITY AND SYMMETRIZED RELATIVE ENTROPY IN LOCAL ALGEBRAS

Having fixed the operator-algebraic conventions, we now formulate the local type III analog of the reflected construction of Sec. II. The central replacement is that the finite-dimensional reflected density matrix $J\rho(g)J$ is replaced by a modularly pulled-back state on the same local algebra. The comparison functional is then the symmetrized Araki relative entropy, and its quadratic coefficient at a self-dual point is governed by the local Bogoliubov-Kubo-Mori geometry. Table I summarizes the correspondence between the finite-dimensional reflected-state construction and the local type III modular-pullback construction, including the different normalizations of the associated BKM susceptibility coefficients.

For the remainder of this section and in the explicit models below, we work in the vacuum representation. Let O_t denote the localization region under consideration and set

$$M_t := \mathcal{A}(O_t) \subset \mathcal{B}(\mathcal{H}). \tag{112}$$

In the scalar-field example of Sec. V, $O_t = W_t$ is a wedge. In the chiral-current example of Sec. VI, $O_t = (-\infty, t)$ is a half-line. We assume that (M_t, Ω) is a standard pair, so that the modular objects J_t and Δ_t are well defined. Thus

$$J_t M_t J_t = M_t'. \tag{113}$$

The induced local states considered below are assumed to be faithful and normal on M_t. When $O_t = W_t$ is a wedge and we identify M_t' with the algebra of the opposite wedge, we invoke wedge duality. When the modular objects are given a geometric interpretation in the wedge model, we additionally invoke the Bisognano-Wichmann theorem. At the abstract level, the definition of the self-dual locus and of the symmetrized comparison functional requires only the standard-pair structure. The identification of the quadratic coefficient with a BKM form requires the second-order differentiability assumption stated below.

In finite-dimensional type I systems, the antiunitary J acts within a single algebra and directly produces the reflected density matrix $J\rho(g)J$. In the local type III setting, by contrast, there is no intrinsic density matrix on M_t, and the vacuum modular conjugation maps M_t to its commutant. The local analog therefore starts from one ambient

TABLE I. Structural correspondence between the finite-dimensional type I prototype and its intrinsic local type III extension. In the type I theory, modular self-duality compares a state with its reflected partner through the symmetrized Umegaki relative entropy. In the local type III theory, the comparison is between the local state $\omega_{g,t}$ on $M_t = A(\mathcal{O}_t)$ and the modular pullback $\tilde{\omega}_{g,t}$ of the commutant restriction. The same fixed-point structure holds in both settings, but the type III partner requires the modular anti-isomorphism $j_t(A) = J_t A^* J_t$. The table also distinguishes the full BKM bilinear form from the Fisher-information scalar obtained by evaluating that form on the tangent selected by the one-parameter family. The type III BKM identification is understood under the diagonal second-order regularity assumption stated in the text.

Concept	Type I/finite-dimensional system	Intrinsic type III/local operator-algebraic extension
State family	Density-matrix family $\rho(g)$	$M_t := \mathcal{A}(\mathcal{O}_t)$, (M_t, Ω) standard, $\omega_{g,t} := \varpi_g \restriction_{M_t}$.
Partner construction	Reflected state $\rho_J(g) := J\rho(g)J$	$\tilde{\omega}_{g,t} := (\varpi_g \restriction_{M_t'}) \circ j_t$, $j_t : M_t \to M_t'$, $j_t(A) := J_t A^* J_t$.
Self-dual locus	$\rho(g_\star) = \rho_J(g_\star)$	$\omega_{0,t} = \tilde{\omega}_{0,t} =: \omega_t$
Parameters	Deformation parameter g	Ambient deformation parameter g, localization parameter t.
Localization	No local-algebra parameter	$O_t = W_t$ for wedge algebras, $O_t = (-\infty, t)$ for chiral half-lines.
Comparison functional	$S_J(g) := S(\rho(g)\|\|\rho_J(g)) + S(\rho_J(g)\|\|\rho(g))$.	$S(g,t) := S_{M_t}(\omega_{g,t}\|\|\tilde{\omega}_{g,t}) + S_{M_t}(\tilde{\omega}_{g,t}\|\|\omega_{g,t})$.
Vanishing linear term	$S_J(g_\star) = 0$, $S_J(g) \geq 0$, $\partial_g S_J(g)\|_{g=g_\star} = 0$ if differentiable.	$S(0,t) = 0$, $S(g,t) \geq 0$, $\partial_g S(g,t)\|_{g=0} = 0$ if differentiable.
Second-order coefficient	$I_J(g_\star) := \partial_g^2 S_J(g)\|_{g=g_\star}$, $S_J(g) = \frac{1}{2} I_J(g_\star)(g-g_\star)^2 + o((g-g_\star)^2)$.	$I_A(t) := \partial_g^2 S(g,t)\|_{g=0}$, $S(g,t) = \frac{1}{2} I_A(t) g^2 + o(g^2)$.
Underlying BKM geometry	$\gamma^{\mathrm{BKM}}_{\rho_\star}(X,Y)$, $\rho_\star := \rho(g_\star)$	$\gamma^{\mathrm{BKM}}_{\omega_t}(u,v)$, $\omega_t := \omega_{0,t} = \tilde{\omega}_{0,t}$
Family-selected tangent	$\dot{\rho}_\star := \partial_g \rho(g)\|_{g=g_\star}$	$u_t := \partial_g \omega_{g,t}\|_{g=0} - \partial_g \tilde{\omega}_{g,t}\|_{g=0}$, $u_t \in (M_t)^{\mathrm{sa}}_{*,0}$.
Family-selected sensitivity	$F_{\mathrm{BKM}}(g_\star) := \gamma^{\mathrm{BKM}}_{\rho_\star}(\dot{\rho}_\star, \dot{\rho}_\star)$, $I_J(g_\star) = 8F_{\mathrm{BKM}}(g_\star)$.	$F^{\mathcal{A}}_{\mathrm{BKM}}(t) := \gamma^{\mathrm{BKM}}_{\omega_t}(u_t, u_t)$, $I_A(t) = 2F^{\mathcal{A}}_{\mathrm{BKM}}(t)$.

state family and uses J_t to pull the commutant restriction back to the original local algebra M_t.

Let $\{\varpi_g\}_{g\in I}$ be a C^2 family of normal states on $\mathcal{B}(\mathcal{H})$, where g is the state-deformation parameter. For each localization region O_t, define the local restriction

$$\omega_{g,t} := \varpi_g\restriction_{M_t}. \tag{114}$$

Next introduce the canonical linear unital *-anti-isomorphism

$$j_t\colon M_t \to M_t', \qquad j_t(A) := J_t A^* J_t. \tag{115}$$

The modularly pulled-back partner of $\omega_{g,t}$ is then

$$\tilde{\omega}_{g,t} := (\varpi_g\restriction_{M_t'}) \circ j_t; \tag{116}$$

that is,

$$\tilde{\omega}_{g,t}(A) = \varpi_g(J_t A^* J_t), \qquad A \in M_t. \tag{117}$$

Thus, both $\omega_{g,t}$ and $\tilde{\omega}_{g,t}$ are states on the same local algebra M_t. This is the local type III replacement for comparing two density matrices on a single type I algebra. In the wedge case $O_t = W_t$, wedge duality identifies

$$M_t' = \mathcal{A}(W_t)' = \mathcal{A}(W_t'), \tag{118}$$

where W_t' denotes the causal complement, equivalently the opposite wedge. In that case, $\tilde{\omega}_{g,t}$ may be viewed as the restriction of the same ambient state to the opposite-wedge algebra, pulled back to $\mathcal{A}(W_t)$ by the modular anti-isomorphism $j_t(A) = J_t A^* J_t$. For other localization regions, such as the chiral half-lines used below, the definition remains $\tilde{\omega}_{g,t} := (\varpi_g\restriction_{M_t'}) \circ j_t$, without requiring a separate wedge-duality interpretation.

We use the Hilbert-space convention that $\langle\cdot,\cdot\rangle$ is anti-linear in the first argument and linear in the second. Thus, for an antiunitary involution J, we obtain

$$\langle J\xi, J\eta\rangle = \langle \eta, \xi\rangle. \tag{119}$$

Consequently, for any bounded operator A and any vector ξ, we obtain

$$\langle \xi, JA^*J\xi\rangle = \langle A^*J\xi, J\xi\rangle = \langle J\xi, AJ\xi\rangle. \tag{120}$$

In particular, if the ambient state is represented by a vector ξ_g, so that

$$\varpi_g(B) = \langle \xi_g, B\xi_g\rangle, \qquad B \in \mathcal{B}(\mathcal{H}), \tag{121}$$

then the modularly pulled-back state is the vector state on M_t implemented by $J_t\xi_g$, as follows:

$$\tilde{\omega}_{g,t}(A) = \langle \xi_g, J_t A^* J_t \xi_g\rangle = \langle J_t\xi_g, AJ_t\xi_g\rangle, \quad A \in M_t. \tag{122}$$

The parameter g deforms the ambient state family, while the geometric parameter t deforms the localization region and hence the algebra M_t together with its modular conjugation J_t. The local modular self-duality condition is the fixed-point relation

$$\omega_{0,t} = \tilde{\omega}_{0,t}. \tag{123}$$

Equivalently,

$$\varpi_0(A) = \varpi_0(J_t A^* J_t), \qquad A \in M_t. \tag{124}$$

At such a point, the local restriction of the ambient state coincides with the modular pullback of its commutant restriction. This is the local type III analog of the finite-dimensional fixed-point condition $\rho = J\rho J$. The self-dual locus is indexed by the geometric parameter t.

In standard form, each faithful normal state has a unique natural-cone representative, and the natural cone is fixed pointwise by the standard-form modular conjugation; see Eq. (A2). The comparison used here keeps the vacuum standard pair (M_t, Ω), its modular conjugation J_t, and the actual ambient preparation ϖ_g fixed. Self-duality is therefore a relational condition on the region and commutant data of that ambient state. The nontrivial response is the reflected-difference tangent obtained from this fixed pairing, rather than the availability of a canonical representative for the local restriction alone.

To compare the two local states at fixed t, we use the Araki relative entropy on the local algebra M_t, with the convention fixed in Appendix A. Thus,

$$S_{M_t}(\omega_{g,t}\|\tilde{\omega}_{g,t}) = -\langle \xi_{\omega_{g,t}}, \log \Delta_{\tilde{\omega}_{g,t}|\omega_{g,t}} \xi_{\omega_{g,t}}\rangle, \tag{125}$$

where $\xi_{\omega_{g,t}}$ is the natural-cone representative of $\omega_{g,t}$. This is the intrinsic local substitute for the finite-dimensional Umegaki expression.

We define the symmetrized local comparison functional by

$$S(g,t) := S_{M_t}(\omega_{g,t}\|\tilde{\omega}_{g,t}) + S_{M_t}(\tilde{\omega}_{g,t}\|\omega_{g,t}). \tag{126}$$

By construction, $S(g,t) \geq 0$. At a modular self-dual point, we write

$$\omega_{0,t} = \tilde{\omega}_{0,t} =: \omega_t. \tag{127}$$

Therefore,

$$S(0,t) = 0. \tag{128}$$

Thus, $g = 0$ is a local minimum of the symmetrized Araki comparison functional. Whenever $g \mapsto S(g,t)$ is differentiable at $g = 0$, one has

$$\partial_g S(g,t)|_{g=0} = 0. \tag{129}$$

We now make explicit the regularity assumption needed to identify the quadratic coefficient with the local BKM form. Fix t. Let $(M_t)_*$ denote the predual of M_t, and define

$$(M_t)^{\mathrm{sa}}_{*,0} := \{\nu \in (M_t)^{\mathrm{sa}}_* :\ \nu(\mathbf{1}) = 0\}. \tag{130}$$

Every C^1 curve of normal states through ω_t has tangent in this real Banach space. More explicitly, if η_g is such a curve with $\eta_0 = \omega_t$, then

$$\partial_g \eta_g|_{g=0} \in (M_t)^{\mathrm{sa}}_{*,0}. \tag{131}$$

We do not assume, however, that every element of $(M_t)^{\mathrm{sa}}_{*,0}$ is an entropy-admissible tangent direction. Instead, we restrict attention to directions realized by curves for which Araki relative entropy has a second-order expansion at the diagonal.

We assume that the paired local curves

$$g \mapsto \omega_{g,t}, \qquad g \mapsto \tilde{\omega}_{g,t}, \tag{132}$$

are C^2 as maps into $(M_t)_*$, take values in faithful normal states for $|g|$ sufficiently small, and satisfy

$$\omega_{0,t} = \tilde{\omega}_{0,t} = \omega_t. \tag{133}$$

We further assume that, on the entropy-admissible tangent directions generated by the curves considered here, Araki relative entropy is twice differentiable at the diagonal and defines a positive symmetric bilinear form

$$\gamma^{\mathrm{BKM}}_{\omega_t}(\cdot,\cdot). \tag{134}$$

Equivalently, for any admissible C^2 curves

$$a \mapsto \varphi_a, \qquad b \mapsto \psi_b, \tag{135}$$

of faithful normal states on M_t, with

$$\varphi_0 = \omega_t, \tag{136}$$

$$\psi_0 = \omega_t, \tag{137}$$

$$\partial_a \varphi_a|_{a=0} = u, \tag{138}$$

$$\partial_b \psi_b|_{b=0} = v, \tag{139}$$

one has the diagonal expansion

$$S_{M_t}(\varphi_a \| \psi_b) = \frac{1}{2}\gamma^{\mathrm{BKM}}_{\omega_t}(au - bv, au - bv) + o(a^2 + b^2). \tag{140}$$

Equivalently, the mixed Hessian is

$$\gamma^{\mathrm{BKM}}_{\omega_t}(u,v) := -\frac{\partial^2}{\partial a \partial b} S_{M_t}(\varphi_a \| \psi_b)|_{a=b=0}. \tag{141}$$

The bilinear form is assumed to be independent of the admissible curves used to realize the tangents. This is the local type III Bogoliubov-Kubo-Mori bilinear form at ω_t, understood in the sense of operator-algebraic information geometry and quasientropy theory [5,7,37].

Under this hypothesis, we define the two tangents:

$$u_t^{(1)} := \partial_g \omega_{g,t}|_{g=0}, \tag{142}$$

$$u_t^{(2)} := \partial_g \tilde{\omega}_{g,t}|_{g=0}. \tag{143}$$

The reflected-difference tangent is

$$u_t := u_t^{(1)} - u_t^{(2)} \in (M_t)^{\mathrm{sa}}_{*,0}. \tag{144}$$

Applying the second-order expansion to

$$\varphi_g = \omega_{g,t}, \tag{145}$$

$$\psi_g = \tilde{\omega}_{g,t}, \tag{146}$$

gives

$$S_{M_t}(\omega_{g,t} \| \tilde{\omega}_{g,t}) = \frac{1}{2}\gamma^{\mathrm{BKM}}_{\omega_t}(u_t, u_t)g^2 + o(g^2). \tag{147}$$

Exchanging the two arguments gives

$$S_{M_t}(\tilde{\omega}_{g,t} \| \omega_{g,t}) = \frac{1}{2}\gamma^{\mathrm{BKM}}_{\omega_t}(u_t, u_t)g^2 + o(g^2). \tag{148}$$

Therefore, the symmetrized local comparison functional satisfies

$$S(g,t) = \gamma^{\mathrm{BKM}}_{\omega_t}(u_t, u_t)g^2 + o(g^2). \tag{149}$$

Consequently, the local modular Hessian

$$I_A(t) := \partial_g^2 S(g,t)|_{g=0} \tag{150}$$

is

$$I_A(t) = 2\gamma^{\mathrm{BKM}}_{\omega_t}(u_t, u_t). \tag{151}$$

It is useful to define the local BKM Fisher-information scalar of the modularly paired family by

$$F^{\mathcal{A}}_{\mathrm{BKM}}(t) := \gamma^{\mathrm{BKM}}_{\omega_t}(u_t, u_t). \tag{152}$$

With this convention,

$$I_A(t) = 2F^{\mathcal{A}}_{\mathrm{BKM}}(t). \tag{153}$$

Here $\gamma^{\mathrm{BKM}}_{\omega_t}(u, v)$ is the local BKM bilinear form, whereas $F^{\mathcal{A}}_{\mathrm{BKM}}(t)$ is the scalar obtained by evaluating that bilinear form on the particular tangent selected by the modularly paired one-parameter family. This convention differs from the finite-dimensional formula $I_J = 8F_{\mathrm{BKM}}$ only because the finite-dimensional F_{BKM} there is defined using the single tangent $\dot{\rho}_\star$, while the symmetrized reflected comparison uses the tangent difference $2\dot{\rho}_\star$.

Operationally, $I_A(t)$ is the local second-order distinguishability susceptibility of the ambient deformation $g \mapsto \varpi_g$ against its modularly paired deformation at localization O_t. It is not a state function of ϖ_g alone: it depends on the chosen reference standard pair (M_t, Ω), hence on the modular conjugation associated with the region. This dependence is part of the construction. The quantity $I_A(t)$ measures the BKM norm of the reflected-difference tangent

$$u_t = \partial_g \omega_{g,t}|_{g=0} - \partial_g \tilde{\omega}_{g,t}|_{g=0}. \tag{154}$$

Only in the special case where the partner tangent equals the negative of the original tangent does this become literally the odd component of the state deformation. In the coherent-state examples below, the tangent difference is represented by the reflected difference profile $\delta_t h = h - \vartheta_t h$.

This is the direct type III counterpart of the type I statement that the Hessian of the symmetrized relative entropy is governed by the BKM metric; the numerical factor depends on whether the Fisher information is defined using a single tangent or the reflected tangent difference. For the operator-algebraic information geometry of faithful normal states on a von Neumann algebra, including the noncommutative divergence structure whose second-order term yields the Kubo-Mori or Bogoliubov-Kubo-Mori geometry, see Ref. [7]. For the quasientropy framework on von Neumann algebras and its relation to Araki relative entropy, see Ref. [5]. In the present setting, the bilinear form $\gamma^{\mathrm{BKM}}_{\omega_t}(u, v)$ is the local type III BKM metric at the self-dual state ω_t, while $F^{\mathcal{A}}_{\mathrm{BKM}}(t)$ is the scalar obtained by evaluating that metric on the tangent difference u_t selected by the modularly paired family. Thus, $F^{\mathcal{A}}_{\mathrm{BKM}}(t)$ is the appropriate family-dependent BKM susceptibility associated with the modularly paired local deformation, and $I_A(t) = 2F^{\mathcal{A}}_{\mathrm{BKM}}(t)$ is the corresponding Hessian relation generated by the symmetrized Araki relative entropy.

A useful special case occurs when the modularly paired family is locally odd at the fixed point, in the sense that

$$\partial_g \tilde{\omega}_{g,t}|_{g=0} = -\partial_g \omega_{g,t}|_{g=0}. \tag{155}$$

Then,

$$u_t = 2\partial_g \omega_{g,t}|_{g=0}, \tag{156}$$

and Eq. (151) reduces to

$$I_A(t) = 8\gamma^{\mathrm{BKM}}_{\omega_t}(\partial_g \omega_{g,t}|_{g=0}, \partial_g \omega_{g,t}|_{g=0}). \tag{157}$$

In this odd-pairing case, the local type III formula is exactly parallel to the type I reflected formula $I_J = 8F_{\mathrm{BKM}}$. In the explicit coherent-state models below, the more general difference-tangent formula (151) is the relevant one, with the reflected difference profile playing the role of u_t.

The local susceptibility $I_A(t)$ depends both on the ambient state family and on the localization geometry through the local algebra M_t and its modular conjugation J_t. The comparison problem is therefore naturally organized as a two-parameter surface,

$$(g, t) \longmapsto S(g, t), \tag{158}$$

where g measures departure from modular self-duality in state space and t labels the localization geometry. Equivalently, t indexes a family of local faithful state manifolds, self-dual base points ω_t, and BKM bilinear forms $\gamma^{\mathrm{BKM}}_{\omega_t}$. In the coherent-state models studied below, $S(g, t)$ is exactly quadratic in g, so the coefficients computed there are exact evaluations of the local BKM Fisher information $F^{\mathcal{A}}_{\mathrm{BKM}}(t)$.

V. WEDGE COHERENT STATES AS AN EXACT TYPE III MODEL

We now present a concrete free-field realization of the local type III BKM-Araki fixed-point formalism introduced in Sec. IV. The conventions are those fixed in the Appendixes. The underlying coherent-state relative-entropy formulas are known from Refs. [11,14]. For the coherent family ω_{gh}, the modular pullback maps the profile h to $\vartheta_t h$. The comparison therefore depends only on the reflected difference profile $\delta_t h = h - \vartheta_t h$, and the known wedge entropy formula evaluates the corresponding local BKM coefficient exactly. In this sense, the wedge model furnishes an exact local-algebraic realization of the type III modular self-duality formalism.

We begin with the standard right wedge

$$W := \{x \in \mathbb{R}^{1,d-1} : x^1 > |x^0|\}, \tag{159}$$

and denote by

$$\mathcal{A}(W) \subset \mathcal{B}(\mathcal{H}) \tag{160}$$

the corresponding represented local von Neumann algebra in the vacuum representation. More generally, we write $\mathcal{A}(W_t)$ for the represented algebra of a wedge W_t in the family under consideration. For the class of wedges used

here, the vacuum vector Ω is cyclic and separating, so each pair

$$(\mathcal{A}(W_t), \Omega) \tag{161}$$

is standard.

In the undeformed case $W = W_0$, let U denote the vacuum unitary representation of the proper orthochronous Poincaré group, and let $\Lambda_W(\chi)$ denote the one-parameter Lorentz boost preserving the right wedge W. We use the rapidity convention

$$\begin{aligned}(\Lambda_W(\chi)x)^0 &= x^0\cosh\chi + x^1\sinh\chi,\\ (\Lambda_W(\chi)x)^1 &= x^1\cosh\chi + x^0\sinh\chi,\\ (\Lambda_W(\chi)x)^\perp &= x^\perp.\end{aligned} \tag{162}$$

With this convention, the Bisognano-Wichmann theorem identifies the modular group of the vacuum standard pair $(\mathcal{A}(W), \Omega)$ with the wedge-preserving boosts [17,18], as follows:

$$\Delta_W^{is} = U(\Lambda_W(-2\pi s)), \qquad s \in \mathbb{R}. \tag{163}$$

Equivalently, the modular automorphism group acts geometrically on wedge-localized observables $A \in \mathcal{A}(W)$ by

$$\sigma_s^\Omega(A) = \Delta_W^{is} A \Delta_W^{-is} = U(\Lambda_W(-2\pi s)) A U(\Lambda_W(-2\pi s))^{-1}. \tag{164}$$

The same theorem identifies the modular conjugation J_W with the antiunitary operator implementing the corresponding wedge reflection.

More generally, for each wedge W_t in the translated family, we write J_t for the vacuum modular conjugation of the standard pair $(\mathcal{A}(W_t), \Omega)$. On Weyl operators, J_t induces a real *Weyl-label involution* ϑ_t, characterized by

$$J_t W(f) J_t = W(\vartheta_t f), \qquad \vartheta_t^2 = \mathbf{1}. \tag{165}$$

The explicit form of ϑ_t depends on the Weyl-label convention and will be fixed below. These modular-theoretic ingredients underlie the coherent-state formulas used in this section.

For the free scalar field, Sec. III A fixes the Weyl operators in terms of the smeared field $\Phi(f)$. For $f \in C_0^\infty(\mathcal{M}, \mathbb{R})$, we write

$$W(f) := e^{i\Phi(f)}, \qquad \Omega_f := W(f)\Omega, \tag{166}$$

and denote by

$$\omega_f(B) := \langle \Omega_f, B\Omega_f \rangle, \qquad B \in \mathcal{B}(\mathcal{H}), \tag{167}$$

the corresponding coherent state. Its restriction to a local algebra $\mathcal{A}(W_t)$ is again denoted by ω_f. Since conjugation by $W(f)$ acts on local Weyl operators by a phase and hence preserves each local von Neumann algebra, $\omega_f|_{\mathcal{A}(W_t)}$ is obtained from the vacuum state by a local automorphism. Because the vacuum state is faithful on $\mathcal{A}(W_t)$, the coherent restriction is also faithful. It is normal in the vacuum representation, so the relative modular objects of Appendix A are well defined.

For coherent states, the Araki relative entropy satisfies the difference identity [14]

$$S_{\mathcal{A}(W_t)}(\omega_f \| \omega_g) = S_{\mathcal{A}(W_t)}(\omega_{f-g} \| \omega_0), \tag{168}$$

and, in particular,

$$S_{\mathcal{A}(W_t)}(\omega_f \| \omega_0) = S_{\mathcal{A}(W_t)}(\omega_0 \| \omega_f). \tag{169}$$

Thus, the comparison of two coherent states reduces to the comparison of a single coherent excitation with the vacuum.

For the undeformed wedge W, the relative entropy admits an exact boost-energy formula. Let

$$P := \Box + m^2, \qquad E := E_{\mathrm{ret}} - E_{\mathrm{adv}}, \tag{170}$$

and let

$$\Psi_f := Ef \tag{171}$$

be the classical Klein-Gordon solution generated by f, so that

$$(\Box + m^2)\Psi_f = 0. \tag{172}$$

For the wedge entropy formula, one introduces the left- and right-wedge components of the excitation determined by f. We write this schematically as

$$f = f_L + f_R, \qquad \mathrm{supp}\, f_L \subset W', \qquad \mathrm{supp}\, f_R \subset W, \tag{173}$$

where W' is the causal complement of W. f_R denotes the right-wedge component of the associated one-particle vector, equivalently of the Cauchy data of the classical solution Ef. Also, this notation is not meant as a literal decomposition of an arbitrary test function $f \in C_0^\infty(\mathcal{M})$ inside $C_0^\infty(\mathcal{M})$. Rather, it is the standard coherent-state wedge decomposition at the level of the associated one-particle vector, equivalently of the Cauchy data of the classical solution Ψ_f on the surface $x^0 = 0$. In this sense, the wedge algebra is sensitive only to the right-wedge component, and the relative entropy depends only on f_R.

Let $\mathcal{H}_1$ denote the one-particle Hilbert space of the free scalar field, and let U_1 be the corresponding one-particle representation of the Poincaré group. The boost subgroup preserving the right wedge acts on $\mathcal{H}_1$ by a strongly

continuous one-parameter unitary group. With the boost convention fixed above, we define its self-adjoint generator k_1 by

$$U_1(\Lambda_W(\chi)) = \exp(i\chi k_1). \tag{174}$$

The induced Fock-space representation is the second quantization of the one-particle representation,

$$U(\Lambda_W(\chi)) = \Gamma(U_1(\Lambda_W(\chi))), \tag{175}$$

and therefore

$$U(\Lambda_W(\chi)) = \exp(i\chi K_1), \qquad K_1 = d\Gamma(k_1). \tag{176}$$

Here, $d\Gamma(k_1)$ denotes the usual second quantization of the one-particle generator [38]. Thus, k_1 is the one-particle boost generator, whereas K_1 is the corresponding boost generator on the symmetric Fock space. With Eqs. (163) and (176), one has

$$\Delta_W^{is} = \exp(-2\pi i s K_1), \tag{177}$$

or equivalently,

$$-\log \Delta_W = 2\pi K_1. \tag{178}$$

With this convention, the wedge relative entropy of a coherent excitation is [11,14]

$$S_{\mathcal{A}(W)}(\omega_f \| \omega_0) = 2\pi \langle f_R, k_1 f_R \rangle. \tag{179}$$

This identity is understood as a quadratic-form identity on the form domain of the one-particle boost generator. Equivalently, since $-\log \Delta_W = 2\pi K_1$, it is the coherent-state expectation of the wedge modular Hamiltonian, with the vacuum contribution subtracted. The quantity $\langle f_R, k_1 f_R \rangle$ is the corresponding one-particle boost-energy quadratic form.

The relation between Eq. (179) and the stress-tensor expression is the standard identification of the one-particle boost quadratic form with the classical Noether charge for the Lorentz boost preserving the right wedge. Let

$$\xi_W := x^1 \partial_0 + x^0 \partial_1 \tag{180}$$

be the boost Killing vector field associated with W. For a classical Klein-Gordon solution Ψ, define the corresponding boost charge on the wedge Cauchy surface by

$$Q_{\xi_W}[\Psi] := \int_{x^0=0, x^1>0} d\Sigma_\mu T^\mu{}_\nu[\Psi] \xi_W^\nu. \tag{181}$$

For the classical solution Ψ_{f_R} associated with the right-wedge one-particle vector f_R, the normalization used in Eq. (179) is

$$\langle f_R, k_1 f_R \rangle = Q_{\xi_W}[\Psi_{f_R}]. \tag{182}$$

Thus the term "boost energy" refers to the classical Noether charge whose one-particle quantization gives the quadratic form of k_1.

On the hypersurface $x^0 = 0$, the future-directed surface element is

$$d\Sigma_\mu = (1, 0, \ldots, 0) d^{d-1}x, \tag{183}$$

and the boost Killing vector restricts to

$$\xi_W|_{x^0=0} = x^1 \partial_0. \tag{184}$$

Therefore,

$$T^\mu{}_\nu[\Psi] \xi_W^\nu d\Sigma_\mu = T^0{}_0[\Psi] x^1 d^{d-1}x = x^1 T_{00}[\Psi] d^{d-1}x, \tag{185}$$

where in the last equality, we used the mostly-minus metric convention. Consequently,

$$\langle f_R, k_1 f_R \rangle = \int_{x^1>0} d^{d-1}x x^1 T_{00}[\Psi_{f_R}](0, \mathbf{x}). \tag{186}$$

Since the wedge relative entropy depends only on the right-wedge component of the excitation, one may write the resulting classical solution simply as Ψ_f, with the understanding that only its right-wedge component contributes. Substituting Eq. (186) into Eq. (179) gives

$$S_{\mathcal{A}(W)}(\omega_f \| \omega_0) = 2\pi \int_{x^1>0} d^{d-1}x x^1 T_{00}[\Psi_f](0, \mathbf{x}). \tag{187}$$

Here,

$$T_{\mu\nu}[\Psi_f] = \partial_\mu \Psi_f \partial_\nu \Psi_f - \frac{1}{2} \eta_{\mu\nu} (\partial^\alpha \Psi_f \partial_\alpha \Psi_f - m^2 \Psi_f^2) \tag{188}$$

is the classical Klein-Gordon stress tensor. In particular, for the mostly-minus metric convention,

$$T_{00}[\Psi_f] = \frac{1}{2}((\partial_0 \Psi_f)^2 + |\nabla \Psi_f|^2 + m^2 \Psi_f^2), \tag{189}$$

which yields the expanded form used below. In particular,

$$S_{\mathcal{A}(W)}(\omega_f \| \omega_0) = 2\pi \int_{x^1>0} d^{d-1}x x^1 \times \frac{1}{2}((\partial_0 \Psi_f)^2 + |\nabla \Psi_f|^2 + m^2 \Psi_f^2)|_{x^0=0}. \tag{190}$$

By covariance, the analogous formula holds for any wedge W_t in the family, with the corresponding modular boost generator and wedge-adapted Cauchy slice.

We now specialize the modular self-duality construction of Sec. IV to coherent states. Fix $h \in C_0^\infty(\mathcal{M}, \mathbb{R})$ and define the ambient coherent-state family on $\mathcal{B}(\mathcal{H})$ by

$$\varpi_g := \omega_{gh}. \tag{191}$$

For each wedge W_t, its restriction to the local algebra is

$$\omega_{g,t} := \varpi_g \restriction_{\mathcal{A}(W_t)} = \omega_{gh} \restriction_{\mathcal{A}(W_t)}. \tag{192}$$

In accordance with Sec. IV, the local partner is defined by modular pullback of the commutant restriction,

$$\tilde{\omega}_{g,t} := (\varpi_g \restriction_{\mathcal{A}(W_t)'}) \circ j_t, \qquad j_t(A) := J_t A^* J_t. \tag{193}$$

Thus, for $A \in \mathcal{A}(W_t)$, we obtain

$$\tilde{\omega}_{g,t}(A) = \varpi_g(J_t A^* J_t) = \langle \Omega_{gh}, J_t A^* J_t \Omega_{gh} \rangle. \tag{194}$$

Using the antiunitary identity (120), this becomes

$$\tilde{\omega}_{g,t}(A) = \langle J_t \Omega_{gh}, A J_t \Omega_{gh} \rangle. \tag{195}$$

Using $J_t \Omega = \Omega$, $J_t W(f) J_t = W(\vartheta_t f)$, and $g \in \mathbb{R}$, one obtains

$$\begin{aligned} J_t \Omega_{gh} &= J_t W(gh)\Omega = (J_t W(gh) J_t) J_t \Omega \\ &= W(g\vartheta_t h)\Omega = \Omega_{g\vartheta_t h}. \end{aligned} \tag{196}$$

Therefore,

$$\tilde{\omega}_{g,t}(A) = \langle \Omega_{g\vartheta_t h}, A\Omega_{g\vartheta_t h} \rangle = \omega_{g\vartheta_t h}(A), \quad \mathcal{A} \in A(W_t). \tag{197}$$

So the modularly pulled-back partner is again a coherent state on the same local algebra, now with profile $\vartheta_t h$. At $g = 0$, the two local states coincide, as follows:

$$\omega_{0,t} = \tilde{\omega}_{0,t} = \omega_0 \restriction_{\mathcal{A}(W_t)}. \tag{198}$$

Thus, $g = 0$ is the modular self-dual locus.

We introduce the t-dependent reflected difference profile

$$\delta_t h := h - \vartheta_t h. \tag{199}$$

To connect with the general type III BKM formalism, we define for each $f \in C_0^\infty(\mathcal{M}, \mathbb{R})$ the tangent functional

$$\nu_{f,t}(A) := \partial_g \omega_{gf}(A)|_{g=0}, \qquad A \in \mathcal{A}(W_t). \tag{200}$$

Equivalently,

$$\omega_{gf}(A) = \langle W(gf)\Omega, A W(gf)\Omega \rangle. \tag{201}$$

Since Ω lies in the domain of the smeared field $\Phi(f)$, the vector curve $g \mapsto W(gf)\Omega$ is norm differentiable at $g = 0$, with

$$\frac{d}{dg} W(gf)\Omega|_{g=0} = i\Phi(f)\Omega. \tag{202}$$

Therefore, the derivative of the vector state is

$$\begin{aligned} \nu_{f,t}(A) &= \frac{d}{dg} \langle W(gf)\Omega, A W(gf)\Omega \rangle|_{g=0} \\ &= \langle i\Phi(f)\Omega, A\Omega \rangle + \langle \Omega, A i\Phi(f)\Omega \rangle. \end{aligned} \tag{203}$$

Using the convention that the Hilbert-space inner product is antilinear in the first argument and using the symmetry of $\Phi(f)$, this becomes

$$\nu_{f,t}(A) = i\langle \Omega, [A, \Phi(f)]\Omega \rangle. \tag{204}$$

The last expression is understood as the normal functional obtained from the differentiable coherent-state curve. Equivalently, the commutator formula may first be read on the usual finite-particle core, or on bounded Weyl-polynomial observables, and then extended by normality.

In particular, the tangent difference selected by the modularly paired family is

$$\begin{aligned} u_t(A) &:= \partial_g \omega_{g,t}(A)|_{g=0} - \partial_g \tilde{\omega}_{g,t}(A)|_{g=0} \\ &= \nu_{h,t}(A) - \nu_{\vartheta_t h,t}(A) = \nu_{\delta_t h,t}(A). \end{aligned} \tag{205}$$

Thus the local type III BKM tangent is generated exactly by the reflected difference profile $\delta_t h$. In general, $\vartheta_t h \neq -h$, so the relevant formula is the general difference-tangent relation of Sec. IV rather than the locally odd special case.

The coherent-state identities above now give

$$S_{\mathcal{A}(W_t)}(\omega_{g,t} \| \tilde{\omega}_{g,t}) = S_{\mathcal{A}(W_t)}(\omega_{g\delta_t h} \| \omega_0). \tag{206}$$

By homogeneity of the coherent-state entropy,

$$S_{\mathcal{A}(W_t)}(\omega_{g\delta_t h} \| \omega_0) = g^2 S_{\mathcal{A}(W_t)}(\omega_{\delta_t h} \| \omega_0), \tag{207}$$

and the same expression is obtained after exchanging the two arguments. Therefore, the symmetrized local comparison functional becomes

$$S(g, t) = 2g^2 S_{\mathcal{A}(W_t)}(\omega_{\delta_t h} \| \omega_0). \tag{208}$$

Comparing Eq. (208) with the general fixed-point expansion

$$S(g, t) = F^{\mathcal{A}}_{\mathrm{BKM}}(t) g^2 + o(g^2), \qquad F^{\mathcal{A}}_{\mathrm{BKM}}(t) := \gamma^{\mathrm{BKM}}_{\omega_t}(u_t, u_t), \tag{209}$$

where $\omega_t = \omega_{0,t}$, we obtain the exact identification

$$F^{\mathcal{A}}_{\mathrm{BKM}}(t) = 2S_{\mathcal{A}(W_t)}(\omega_{\delta_t h}\|\omega_0). \tag{210}$$

Accordingly, the local Araki Hessian is

$$I_A(t) \coloneqq \partial_g^2 S(g,t)|_{g=0} = 2F^{\mathcal{A}}_{\mathrm{BKM}}(t) = 4S_{\mathcal{A}(W_t)}(\omega_{\delta_t h}\|\omega_0). \tag{211}$$

Thus, in the modularly defined coherent-state model, the coefficient computed from the symmetrized Araki entropy is exactly the local BKM Fisher information of the reflected-difference tangent, and the Hessian is twice that Fisher information.

For the undeformed wedge $W = W_0$, substitution of the boost-energy formula gives

$$F^{\mathcal{A}}_{\mathrm{BKM}}(0) = 4\pi\langle(\delta_0 h)_R, k_1(\delta_0 h)_R\rangle, \tag{212}$$

where $(\delta_0 h)_R$ denotes the right-wedge component of $\delta_0 h = h - \vartheta_0 h$. Equivalently,

$$F^{\mathcal{A}}_{\mathrm{BKM}}(0) = 4\pi\int_{x^1>0} d^{d-1}x x^1 T_{00}[\Psi_{\delta_0 h}](0,\mathbf{x}), \tag{213}$$

or, after expanding T_{00}, we have

$$F^{\mathcal{A}}_{\mathrm{BKM}}(0) = 2\pi\int_{x^1>0} d^{d-1}x x^1 \times((\partial_0\Psi_{\delta_0 h})^2 + |\nabla\Psi_{\delta_0 h}|^2 + m^2\Psi^2_{\delta_0 h})|_{x^0=0}, \tag{214}$$

with

$$\Psi_{\delta_0 h} \coloneqq E(\delta_0 h). \tag{215}$$

Hence,

$$I_A(0) = 2F^{\mathcal{A}}_{\mathrm{BKM}}(0) = 8\pi\langle(\delta_0 h)_R, k_1(\delta_0 h)_R\rangle \tag{216}$$

and

$$I_A(0) = 8\pi\int_{x^1>0} d^{d-1}x x^1 T_{00}[\Psi_{\delta_0 h}](0,\mathbf{x}). \tag{217}$$

Thus, both the local BKM Fisher information and the local Hessian are governed by positive boost-energy forms of the reflected difference profile.

To analyze the geometric dependence of these quantities, we now restrict the localization parameter t to the one-parameter family of wedges obtained by null translations of the right wedge, as follows:

$$W_t \coloneqq W + tv, \qquad v \coloneqq (1,1,0,\ldots,0). \tag{218}$$

Thus, the edge of the wedge is translated along the null direction v. Let

$$\Sigma_t \coloneqq W_t \cap \{x^0 = t\} \tag{219}$$

denote the spatial section of the translated wedge on the Cauchy surface $x^0 = t$. In coordinates,

$$\Sigma_t = \{(t,\mathbf{x}) : x^1 > t\}. \tag{220}$$

For this family,

$$J_t = U(tv)J_W U(-tv), \tag{221}$$

and the corresponding involution on Weyl labels is

$$\vartheta_t = \tau_{tv}\circ\vartheta_0\circ\tau_{-tv}, \tag{222}$$

where τ_{tv} denotes spacetime translation by tv. Here, τ_a acts on test functions by

$$(\tau_a f)(x) \coloneqq f(x-a). \tag{223}$$

Equivalently, if

$$r_0(x^0,x^1,\mathbf{x}_\perp) = (-x^0,-x^1,\mathbf{x}_\perp) \tag{224}$$

denotes the geometric wedge reflection associated with W, then the translated geometric reflection is

$$r_t(x) = tv + r_0(x - tv) \tag{225}$$

or

$$r_t(x^0,x^1,\mathbf{x}_\perp) = (2t - x^0, 2t - x^1, \mathbf{x}_\perp). \tag{226}$$

It satisfies $r_t^2 = \mathrm{id}$.

It is important to distinguish the geometric action on the scalar field from the induced action on Weyl labels. For the real scalar field, there is no additional internal sign in the geometric field transformation:

$$J_t\Phi(f)J_t = \Phi(f\circ r_t^{-1}). \tag{227}$$

However, J_t is antiunitary, while the Weyl operators are defined by

$$W(f) = e^{i\Phi(f)}. \tag{228}$$

Therefore,

$$J_t W(f) J_t = J_t e^{i\Phi(f)} J_t = e^{-iJ_t\Phi(f)J_t} = e^{-i\Phi(f\circ r_t^{-1})} = W(-f\circ r_t^{-1}). \tag{229}$$

Thus, the involution ϑ_t appearing in $J_t W(f) J_t = W(\vartheta_t f)$ is the Weyl-label involution

$$(\vartheta_t h)(x) \coloneqq -h(r_t^{-1}x) = -h(r_t x), \tag{230}$$

where the last equality uses $r_t^2 = \mathrm{id}$. The minus sign is not an internal sign of the real scalar field; it is the sign forced on the Weyl label by the antiunitarity of J_t. Hence, the reflected difference profile entering the local susceptibility is

$$\delta_t h(x) := h(x) - (\vartheta_t h)(x) = h(x) + h(r_t x). \tag{231}$$

By covariance of the coherent-state entropy formula, one has for each fixed t, the following expression

$$S_{\mathcal{A}(W_t)}(\omega_f \| \omega_0) = 2\pi \int_{\Sigma_t} d^{d-1}x (x^1 - t) T_{00}[\Psi_f](t, \mathbf{x}). \tag{232}$$

Applying this with $f = \delta_t h$ yields

$$F^{\mathcal{A}}_{\mathrm{BKM}}(t) = 4\pi \int_{\Sigma_t} d^{d-1}x (x^1 - t) T_{00}[\Psi_{\delta_t h}](t, \mathbf{x}), \tag{233}$$

or equivalently

$$F^{\mathcal{A}}_{\mathrm{BKM}}(t) = 2\pi \int_{x^1 > t} d^{d-1}x (x^1 - t) \times ((\partial_0 \Psi_{\delta_t h})^2 + |\nabla \Psi_{\delta_t h}|^2 + m^2 \Psi^2_{\delta_t h})|_{x^0 = t}. \tag{234}$$

Therefore,

$$I_A(t) = 2F^{\mathcal{A}}_{\mathrm{BKM}}(t) = 8\pi \int_{\Sigma_t} d^{d-1}x (x^1 - t) T_{00}[\Psi_{\delta_t h}](t, \mathbf{x}), \tag{235}$$

or equivalently,

$$I_A(t) = 4\pi \int_{x^1 > t} d^{d-1}x (x^1 - t) \times ((\partial_0 \Psi_{\delta_t h})^2 + |\nabla \Psi_{\delta_t h}|^2 + m^2 \Psi^2_{\delta_t h})|_{x^0 = t}. \tag{236}$$

This is the precise local type III BKM–Araki coefficient selected by the modular construction. The dependence on t enters simultaneously through the translated wedge algebra $\mathcal{A}(W_t)$ and through the reflected profile $\vartheta_t h$ appearing in $\delta_t h$. The geometric dependence is therefore encoded in the boost energy of the t-dependent reflected difference profile. Taken together, these formulas furnish the wedge realization of the local type III fixed-point formalism.

VI. THE CHIRAL $U(1)$ CURRENT AS AN EXACT TYPE III MODEL

We next consider the chiral $U(1)$ current, a standard exactly solvable chiral conformal quantum field theory on the light ray. It may be viewed as the left- or right-moving current sector of the two-dimensional massless free boson. In light-cone coordinates, one retains a single null coordinate, denoted here by x, and works with a Hermitian current field $j(x)$. Formally, this current is the derivative of a chiral scalar field,

$$j(x) = \partial_x \phi(x). \tag{237}$$

Thus, only one chiral sector is retained; there are no independent left- and right-moving degrees of freedom within the reduced model.

The local observables are generated by smeared currents

$$J(f) := \int_{\mathbb{R}} j(x) f(x) dx, \qquad f \in C_c^\infty(\mathbb{R}, \mathbb{R}). \tag{238}$$

We use the symplectic convention

$$[J(f), J(g)] = 2i\sigma(f, g)\mathbf{1}, \qquad \sigma(f, g) = \int_{\mathbb{R}} f(x) g'(x) dx, \tag{239}$$

with the Weyl operators

$$\begin{aligned} &W(f) := e^{iJ(f)}, \\ &W(f)W(g) = e^{-i\sigma(f,g)} W(f+g), \\ &W(f)^* = W(-f); \end{aligned} \tag{240}$$

this is the CCR normalization used in the standard-subspace relative-entropy formulas below. For an interval or half-line $I \subset \mathbb{R}$, the local algebra $\mathcal{A}(I)$ is generated by those $W(f)$ with $\mathrm{supp}\, f \subset I$.

The coherent states used below are obtained by Weyl displacement of the vacuum. For $h \in C_c^\infty(\mathbb{R}, \mathbb{R})$, we define

$$\omega_h(B) := \langle \Omega, W(h)^* B W(h) \Omega \rangle, \qquad B \in \mathcal{B}(\mathcal{H}). \tag{241}$$

Formally, this displacement shifts the smeared current by

$$\begin{aligned} W(h)^* J(f) W(h) &= J(f) + 2\sigma(h, f)\mathbf{1} \\ &= J(f) - 2 \int_{\mathbb{R}} h'(x) f(x) dx \mathbf{1}, \end{aligned} \tag{242}$$

where the last equality follows by integration by parts. Thus, up to the normalization fixed by the Weyl convention above, the Weyl label h determines a classical current profile proportional to $h'(x)$. This is why the half-line relative-entropy formula below is naturally expressed as a quadratic functional of h'.

The restriction of ω_h to each local algebra is faithful and normal. Indeed, by the Weyl relations,

$$W(h)^* W(f) W(h) = e^{2i\sigma(h,f)} W(f), \qquad \mathrm{supp}\, f \subset I, \tag{243}$$

so conjugation by $W(h)$ preserves the local algebra $\mathcal{A}(I)$. Hence, $\omega_h|_{\mathcal{A}(I)}$ is obtained from the faithful vacuum state

by an automorphism of the local algebra. Since it is a vector state in the vacuum representation, it is normal.

The vacuum representation of the chiral current is a positive-energy representation of the translation-dilation symmetry on the line. For half-lines, the vacuum modular group acts geometrically, just as the Bisognano-Wichmann modular group acts geometrically on wedge algebras in the scalar-field example of Sec. V. This makes the chiral current a useful complementary exact model: the same local type III BKM-Araki fixed-point construction appears, but the final sensitivity coefficient is a closed one-dimensional half-line functional rather than a spacetime boost-energy integral.

We now recall the operator-algebraic realization of this model, following the standard-subspace formulation of Refs. [14,16]. In the vacuum representation, the one-particle standard subspaces associated with the half-lines $(-\infty, t)$ are

$$L_t^{U(1)} := \overline{C_c^\infty((-\infty, t), \mathbb{R})}, \tag{244}$$

where the closure is taken in the one-particle Hilbert space. Their second quantization yields the represented local von Neumann algebras

$$\mathcal{A}((-\infty, t)) \subset \mathcal{B}(\mathcal{H}), \tag{245}$$

acting on the vacuum Hilbert space $\mathcal{H}$ with vacuum vector Ω. The modular flow acts geometrically on the standard subspaces $L_t^{U(1)}$, and hence on the half-line algebras. We write J_t for the modular conjugation of the standard pair

$$(\mathcal{A}((-\infty, t)), \Omega). \tag{246}$$

For the vacuum half-line

$$I_t := (-\infty, t), \tag{247}$$

the modular conjugation is the geometric orientation reversal of the light ray about the endpoint t. We write this reflection as

$$r_t(x) := 2t - x, \qquad r_t(I_t) = (t, \infty). \tag{248}$$

The chiral current is treated as a one-form under this orientation reversal.

In the following formulas, J_t denotes the modular conjugation, whereas $J(f)$ denotes the smeared current. With the symplectic convention fixed above, this means that the current distribution transforms as

$$J_t j(x) J_t = -j(r_t x). \tag{249}$$

Consequently, for smeared currents,

$$J_t J(f) J_t = -J(f \circ r_t). \tag{250}$$

Since J_t is antiunitary, $J_t i J_t = -i$. Therefore, the induced action on Weyl operators is

$$J_t W(f) J_t = W(\vartheta_t f), \qquad (\vartheta_t f)(x) := (f \circ r_t)(x) = f(2t - x). \tag{251}$$

This convention is orientation-sensitive: the map ϑ_t is antisymplectic, as follows:

$$\sigma(\vartheta_t f, \vartheta_t g) = -\sigma(f, g), \tag{252}$$

which is precisely compensated by the antiunitarity of J_t in the Weyl relations. Thus, the vacuum reflected difference profile is

$$\delta_t h(x) := h(x) - (\vartheta_t h)(x) = h(x) - h(2t - x), \tag{253}$$

and hence,

$$(\delta_t h)'(x) = h'(x) + h'(2t - x). \tag{254}$$

In the standard-subspace literature, the quantity often denoted $S_{L_t^{U(1)}}(h, h)$ is, in the present notation, the Araki relative entropy

$$S_{\mathcal{A}((-\infty,t))}(\omega_h \| \omega_0), \tag{255}$$

where ω_0 is the vacuum state restricted to the half-line algebra. The key simplification in this model is that this quantity is explicitly computable [14,16], as follows:

$$S_{\mathcal{A}((-\infty,t))}(\omega_h \| \omega_0) = 2\pi \int_{-\infty}^{t} (t - x)[h'(x)]^2 dx. \tag{256}$$

Equivalently, this formula may be viewed as the coherent-state increase of the vacuum modular Hamiltonian for the half-line. In traditional CFT language, the vacuum modular Hamiltonian for $(-\infty, t)$ is the geometric generator

$$K_t = 2\pi \int_{-\infty}^{t} (t - x) T(x) dx, \tag{257}$$

where $T(x)$ is the chiral stress tensor. For the $U(1)$ current, $T(x)$ is the Sugawara energy density, quadratic in the current, with the overall normalization chosen consistently with the symplectic convention in Eq. (239) and the relative-entropy formula (256). The coherent displacement shifts the current by the classical profile proportional to $h'(x)$, and hence the relative entropy reduces to the positive boundary-weighted quadratic form in Eq. (256).

We now implement the local type III construction of Sec. IV. Fix $h \in C_c^\infty(\mathbb{R}, \mathbb{R})$ and define the ambient coherent-state family on $\mathcal{B}(\mathcal{H})$ by

$$\varpi_g := \omega_{gh}. \tag{258}$$

For each half-line $(-\infty, t)$, its restriction to the local algebra is

$$\omega_{g,t} := \varpi_g \restriction_{\mathcal{A}((-\infty,t))} = \omega_{gh} \restriction_{\mathcal{A}((-\infty,t))}. \tag{259}$$

As in Sec. IV, the local partner is defined by modular pullback of the commutant restriction,

$$\tilde{\omega}_{g,t} := (\varpi_g \restriction_{\mathcal{A}((-\infty,t))'}) \circ j_t, \qquad j_t(A) := J_t A^* J_t. \tag{260}$$

Writing

$$\Omega_{gh} := W(gh)\Omega, \tag{261}$$

and using Eq. (120), we obtain

$$\tilde{\omega}_{g,t}(A) = \langle J_t \Omega_{gh}, A J_t \Omega_{gh} \rangle, \qquad A \in \mathcal{A}((-\infty, t)). \tag{262}$$

Since $J_t\Omega = \Omega$ and $J_t W(f) J_t = W(\vartheta_t f)$, we obtain

$$J_t \Omega_{gh} = J_t W(gh)\Omega = W(g\vartheta_t h)\Omega = \Omega_{g\vartheta_t h}. \tag{263}$$

Hence, one has

$$\tilde{\omega}_{g,t} = \omega_{g\vartheta_t h} \restriction_{\mathcal{A}((-\infty,t))}. \tag{264}$$

Thus, the modularly pulled-back partner is again a coherent state on the same half-line algebra, now with reflected profile $\vartheta_t h$.

At $g = 0$, the two local states coincide, as follows:

$$\omega_{0,t} = \tilde{\omega}_{0,t} = \omega_0 \restriction_{\mathcal{A}((-\infty,t))}. \tag{265}$$

Thus, $g = 0$ is the modular self-dual locus. We introduce the reflected difference profile

$$\delta_t h := h - \vartheta_t h. \tag{266}$$

To connect with the general type III BKM formalism, we define for each $f \in C_c^\infty(\mathbb{R}, \mathbb{R})$ the tangent functional

$$\nu_{f,t}(A) := \partial_g \omega_{gf}(A)|_{g=0}, \qquad A \in \mathcal{A}((-\infty, t)). \tag{267}$$

Then the tangent difference selected by the modularly paired family is

$$\begin{aligned} u_t(A) &:= \partial_g \omega_{g,t}(A)|_{g=0} - \partial_g \tilde{\omega}_{g,t}(A)|_{g=0} \\ &= \nu_{h,t}(A) - \nu_{\vartheta_t h,t}(A) = \nu_{\delta_t h,t}(A). \end{aligned} \tag{268}$$

Thus, the local type III BKM tangent is generated exactly by the reflected difference profile $\delta_t h$.

The coherent-state identities now reduce the local comparison problem to a one-sided vacuum entropy. For coherent states of the CCR net, we obtain

$$\begin{aligned} S_{\mathcal{A}((-\infty,t))}(\omega_f \| \omega_g) &= S_{\mathcal{A}((-\infty,t))}(\omega_{f-g} \| \omega_0), \\ S_{\mathcal{A}((-\infty,t))}(\omega_f \| \omega_0) &= S_{\mathcal{A}((-\infty,t))}(\omega_0 \| \omega_f). \end{aligned} \tag{269}$$

Therefore,

$$S_{\mathcal{A}((-\infty,t))}(\omega_{g,t} \| \tilde{\omega}_{g,t}) = S_{\mathcal{A}((-\infty,t))}(\omega_{g\delta_t h} \| \omega_0). \tag{270}$$

By homogeneity of the coherent-state relative entropy, we have

$$S_{\mathcal{A}((-\infty,t))}(\omega_{g\delta_t h} \| \omega_0) = g^2 S_{\mathcal{A}((-\infty,t))}(\omega_{\delta_t h} \| \omega_0). \tag{271}$$

The same expression is obtained after exchanging the two arguments. Consequently, the symmetrized local comparison functional becomes

$$S(g, t) = 2g^2 S_{\mathcal{A}((-\infty,t))}(\omega_{\delta_t h} \| \omega_0). \tag{272}$$

Comparing Eq. (272) with the general fixed-point expansion

$$S(g, t) = F^{\mathcal{A}}_{\mathrm{BKM}}(t) g^2 + o(g^2), \qquad F^{\mathcal{A}}_{\mathrm{BKM}}(t) := \gamma^{\mathrm{BKM}}_{\omega_t}(u_t, u_t), \tag{273}$$

where $\omega_t = \omega_{0,t}$, yields the exact identification

$$F^{\mathcal{A}}_{\mathrm{BKM}}(t) = 2 S_{\mathcal{A}((-\infty,t))}(\omega_{\delta_t h} \| \omega_0). \tag{274}$$

Accordingly,

$$I_A(t) := \partial_g^2 S(g, t)|_{g=0} = 2 F^{\mathcal{A}}_{\mathrm{BKM}}(t). \tag{275}$$

Thus, in the modularly defined chiral model, the coefficient extracted from the symmetrized Araki entropy is exactly the local BKM Fisher information of the reflected-difference tangent, and the Hessian is twice that Fisher information.

Substituting the explicit reflection convention (254) into the half-line entropy formula gives

$$F^{\mathcal{A}}_{\mathrm{BKM}}(t) = 4\pi \int_{-\infty}^{t} (t - x)[h'(x) + h'(2t - x)]^2 dx. \tag{276}$$

Accordingly,

$$I_A(t) = 8\pi \int_{-\infty}^{t} (t - x)[h'(x) + h'(2t - x)]^2 dx. \tag{277}$$

The vacuum chiral $U(1)$ current therefore provides an exact one-dimensional realization of the local BKM-Araki fixed-point principle. The modular susceptibility is the positive half-line quadratic form in Eq. (277), obtained from the

reflected difference profile $\delta_t h(x) = h(x) - h(2t - x)$ and its derivative $(\delta_t h)'(x) = h'(x) + h'(2t - x)$.

The thermal half-line extension is described in Appendix B, where the reflected profile is defined by the Borchers-Yngvason [39] standard-subspace modular conjugation rather than by a pointwise light-ray reflection.

VII. MODULAR REFLECTION AND CANONICAL ENERGY IN AdS/CFT

AdS/CFT relates a conformal field theory in d spacetime dimensions to a gravitational theory in an asymptotically AdS_{d+1} spacetime [19–21]. The conformal field theory is defined on the conformal boundary of the higher-dimensional spacetime. A boundary state and a boundary subregion determine a corresponding bulk region known as the entanglement wedge. For a ball-shaped boundary region in the conformal field theory vacuum, the modular susceptibility is twice the canonical energy of the reflected-difference bulk perturbation in the associated AdS-Rindler wedge. For such a region, conformal symmetry makes the modular flow of the vacuum state geometric. A conformal transformation maps the causal development of the ball to a hyperbolic cylinder, where the restricted vacuum is a thermal state. The corresponding bulk region is an AdS-Rindler wedge, and the modular flow is generated by an AdS-Rindler Killing field that vanishes on the associated extremal surface [24]. The canonical energy is the quadratic energy of a linearized bulk perturbation associated with this Killing flow. The Jafferis-Lewkowycz-Maldacena-Suh (JLMS) relation and the holographic quantum Fisher information construction identify the second variation of boundary relative entropy with this bulk canonical energy [25–27].

The canonical energy does not by itself determine which bulk perturbation represents the modularly paired comparison studied here. The boundary modular conjugation does not act internally on the ball algebra. It maps the ball algebra to its commutant, after which the commutant restriction is pulled back to the original algebra. When bulk and boundary modular conjugations intertwine, the corresponding bulk operation is the AdS-Rindler modular reflection. The tangent selected by the boundary comparison is therefore represented by the difference between a bulk perturbation and its AdS-Rindler reflection. Equivalently, the susceptibility depends only on the modular-odd part of the bulk perturbation, which fixes the response direction on which the canonical energy is evaluated.

In an exact complementary-recovery model for a linear scalar field on a fixed AdS background, relative entropy is preserved and the bulk and boundary modular conjugations intertwine [32]. The modular partner is then exactly the reflected bulk coherent state, and the symmetrized relative entropy is exactly quadratic in the deformation parameter. For coupled matter and metric perturbations, the corresponding second-order semiclassical quadratic form is the Hollands-Wald canonical energy of the reflected-difference solution.

A. Holographic background for the ball construction

Anti–de Sitter space in $d+1$ dimensions, denoted by AdS_{d+1}, is the maximally symmetric Lorentzian spacetime with constant negative curvature. In Poincaré coordinates (z, x^μ), where $z > 0$ and $\mu = 0, \ldots, d-1$, its metric is

$$ds^2 = \frac{L_{\mathrm{AdS}}^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2),$$
$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1). \tag{278}$$

Here, L_{AdS} is the AdS curvature radius and z is a spatial coordinate. The limit $z \to 0$ defines the conformal boundary. Although the metric diverges there, the rescaled metric $z^2 ds^2 / L_{\mathrm{AdS}}^2$ has the finite boundary representative $\eta_{\mu\nu}dx^\mu dx^\nu$. Only its conformal class is intrinsic. The isometry group of AdS_{d+1} is locally $SO(2, d)$, which is also the global conformal group of d-dimensional Minkowski spacetime.

For a scalar bulk field ϕ, the near-boundary expansion in standard quantization has the form

$$\phi(z,x) = z^{d-\Delta}\phi_{(0)}(x) + z^{\Delta}\phi_{(\Delta)}(x) + \cdots, \quad z \to 0, \tag{279}$$

with the usual logarithmic modifications at exceptional values of Δ. The coefficient $\phi_{(0)}$ is the boundary source for a scalar CFT operator $\mathcal{O}$ of scaling dimension Δ, while $\phi_{(\Delta)}$ contains the normalizable response. The bulk mass and boundary scaling dimension obey

$$m^2 L_{\mathrm{AdS}}^2 = \Delta(\Delta - d). \tag{280}$$

The generating-functional relation is schematically

$$Z_{\mathrm{CFT}}[\phi_{(0)}] = Z_{\mathrm{bulk}}\left[\lim_{z\to 0} z^{\Delta-d}\phi(z,x) = \phi_{(0)}\right]. \tag{281}$$

The semiclassical regime used here evaluates the bulk functional perturbatively around a classical AdS solution. A Euclidean boundary source may be present on the contour that prepares the state. After that source is switched off, the Lorentzian perturbations obey the normalizable reflecting boundary condition $\phi_{(0)} = 0$. The preparation source and the subsequent Lorentzian boundary condition therefore play distinct roles.

Let B be a ball of radius R_B, centered at $\mathbf{x}_B$ on the boundary time slice $x^0 = 0$, and let $D(B)$ denote its causal development. The correspondence also associates bulk regions with boundary subregions. The bulk region associated with B is its entanglement wedge. In the vacuum, a constant-time bulk hypersurface is bounded by B and by the Ryu-Takayanagi extremal surface χ_B, which is

anchored on ∂B [22]. The Lorentzian entanglement wedge is the domain of dependence of this hypersurface. Within the semiclassical reconstruction sector, bulk observables in the wedge can be represented by boundary observables localized in B [23].

For the ball B in the CFT vacuum, the entanglement wedge is an AdS-Rindler wedge. It is a coordinate patch of pure AdS adapted to the modular flow of the ball. In suitable coordinates,

$$ds^2 = \left(\frac{\rho^2}{L_{\mathrm{AdS}}^2} - 1\right) d\tau^2 - \frac{d\rho^2}{\frac{\rho^2}{L_{\mathrm{AdS}}^2} - 1} - \rho^2 dH_{d-1}^2, \tag{282}$$

where dH_{d-1}^2 is the metric of unit $(d-1)$-dimensional hyperbolic space. The surface $\rho = L_{\mathrm{AdS}}$ is a Killing horizon whose bifurcation surface is χ_B. The coordinate τ is the bulk counterpart of the modular-flow parameter of the boundary ball.

The Casini-Huerta-Myers conformal transformation maps the causal development of the ball to a hyperbolic cylinder [24]. Under this map, the vacuum restricted to the ball becomes a thermal state, and the AdS-Rindler wedge becomes the exterior of the massless hyperbolic black hole described by Eq. (282).

We write $x = (x^0, \mathbf{x})$ for the boundary coordinates. Repeated spatial indices $i = 1, \ldots, d-1$ are summed, $\partial_\mu := \partial/\partial x^\mu$, and $|\cdot|$ denotes the Euclidean norm on a constant-time slice. The modular flow of the vacuum state restricted to B is generated by the conformal Killing field

$$\zeta_B = \frac{\pi}{R_B}[(R_B^2 - (x^0)^2 - |\mathbf{x} - \mathbf{x}_B|^2)\partial_0 - 2x^0(x^i - x_B^i)\partial_i]. \tag{283}$$

On the slice $x^0 = 0$, the nontrivial part of the modular Hamiltonian of the vacuum state restricted to B is

$$K_B^{\mathrm{CFT}} = 2\pi \int_B d^{d-1}x \frac{R_B^2 - |\mathbf{x} - \mathbf{x}_B|^2}{2R_B} T_{00}(0, \mathbf{x}) \tag{284}$$

up to an additive scalar [24]. Here, T_{00} is the boundary CFT energy density. In the hyperbolic-cylinder frame, the vacuum state restricted to B is a KMS state with

$$\beta_B = 2\pi R_B, \qquad T_B = \frac{1}{2\pi R_B}. \tag{285}$$

The AdS-Rindler Killing field ξ_B extends ζ_B into the bulk and vanishes on the bifurcation surface χ_B [24,25].

The bulk and boundary relative entropies are connected by the JLMS relation [27]. At the semiclassical order under consideration, it identifies the relative entropy of two boundary states restricted to B with the relative entropy of their bulk states restricted to the entanglement wedge. The gravitational area contribution appears separately in the modular Hamiltonian and in the holographic entropy but cancels from their relative-entropy difference. Consequently, the second variation of boundary relative entropy can be calculated from the corresponding bulk perturbations. For perturbations about the vacuum, this bulk quadratic form is the canonical energy associated with the AdS-Rindler Killing flow [25,26].

Consider a neutral linear scalar field on the fixed AdS background. It is described by the same Weyl-algebra construction used in Sec. III. Let

$$P := \Box_g + m^2 \tag{286}$$

be its Klein-Gordon operator. Since the conformal boundary of AdS is timelike, a boundary condition must be specified in addition to the field equation. A normalizable reflecting boundary condition compatible with the chosen AdS vacuum is imposed, with a well-posed initial-boundary-value problem.

The real symplectic space underlying the bulk Weyl algebra is

$$\mathcal{K}_{\mathbb{R}}^{\mathrm{bulk}} := C_0^\infty(\mathrm{AdS}, \mathbb{R})/PC_0^\infty(\mathrm{AdS}, \mathbb{R}), \tag{287}$$

with the chosen AdS boundary condition understood. As in Sec. III A, f denotes a real test function and $[f]$ its class in $\mathcal{K}_{\mathbb{R}}^{\mathrm{bulk}}$. Since $\Phi(Pk) = 0$ for every $k \in C_0^\infty(\mathrm{AdS}, \mathbb{R})$, the test functions f and $f + Pk$ define the same smeared quantum field and the same Weyl operator,

$$W(f) := e^{i\Phi(f)}. \tag{288}$$

The brackets on $[f]$ will be suppressed when no ambiguity can arise. The notation also includes the admissible vacuum-representation completion of $\mathcal{K}_{\mathbb{R}}^{\mathrm{bulk}}$. Throughout, labels in this completion are restricted to the finite domain of the AdS-Rindler coherent-state relative-entropy form, equivalently, after the identification below, of the canonical-energy form.

Let

$$E := E_{\mathrm{ret}} - E_{\mathrm{adv}} \tag{289}$$

be the causal propagator determined by the same boundary condition. The test function f determines the real classical solution,

$$\Psi_f := Ef. \tag{290}$$

Because $EPk = 0$, the solution depends only on $[f]$. The test-function class $[f]$ specifies the Weyl displacement, while Ψ_f is the corresponding classical bulk perturbation.

The Klein-Gordon symplectic current and symplectic form are

$$\omega^A_{\mathrm{KG}}(\Psi_1,\Psi_2) \coloneqq \Psi_1 \nabla^A \Psi_2 - \Psi_2 \nabla^A \Psi_1 \tag{291}$$

and

$$\sigma_{\mathrm{KG}}(\Psi_1,\Psi_2) \coloneqq \int_\Sigma d\Sigma_A \omega^A_{\mathrm{KG}}(\Psi_1,\Psi_2). \tag{292}$$

The field equation and the reflecting boundary condition make the integral independent of Σ.

For $g \in \mathbb{R}$, the vector

$$\Omega^{\mathrm{bulk}}_{g,f} \coloneqq W(gf)\Omega_{\mathrm{bulk}} \tag{293}$$

is obtained by applying the Weyl operator to the bulk vacuum and represents the field-theoretic counterpart of a harmonic-oscillator coherent state. In the oscillator construction $|\alpha\rangle = D(\alpha)|0\rangle$, the displacement operator $D(\alpha)$ translates the oscillator phase-space variables. Here, $W(gf)$ plays the corresponding role for the bulk field. The Weyl label gf corresponds to the classical solution $\Psi_{gf} = g\Psi_f$, while its connected correlation functions of order 2 and higher agree with those of the vacuum. The test-function class $[f]$ need not belong entirely to the real standard subspace of the AdS-Rindler wedge or to its symplectic complement.

B. The ball algebra and its exact modular partner

Let the boundary CFT be represented on its vacuum Hilbert space $\mathcal{H}_{\mathrm{CFT}}$, with vacuum vector Ω. The represented algebra associated with the causal development $D(B)$ is

$$M_B \coloneqq \mathcal{A}_{\mathrm{CFT}}(D(B)). \tag{294}$$

Under the standard assumptions of local quantum field theory, the vacuum is cyclic and separating for M_B [1,36]. Hence, (M_B,Ω) is standard, and its modular conjugation J_B satisfies

$$J_B M_B J_B = M'_B, \qquad J_B\Omega = \Omega. \tag{295}$$

The associated modular anti-isomorphism is

$$j_B(A) \coloneqq J_B A^* J_B, \qquad j_B\colon M_B \to M'_B. \tag{296}$$

When Haag duality holds, M'_B is the algebra of the causal complement of $D(B)$, understood on the conformal compactification.

Let $\psi \in \mathcal{H}_{\mathrm{CFT}}$ be a unit vector, and let ϖ_ψ denote the corresponding normal vector state on $\mathcal{B}(\mathcal{H}_{\mathrm{CFT}})$, defined by

$$\varpi_\psi(X) \coloneqq \langle \psi, X\psi\rangle, \qquad X \in \mathcal{B}(\mathcal{H}_{\mathrm{CFT}}).$$

Its restriction to the ball algebra is

$$\omega_{\psi,B}(A) \coloneqq \varpi_\psi(A) = \langle \psi, A\psi\rangle, \qquad A \in M_B. \tag{297}$$

Its modular partner is the pullback of the commutant restriction,

$$\tilde{\omega}_{\psi,B} \coloneqq (\varpi_\psi|_{M'_B}) \circ j_B. \tag{298}$$

Antiunitarity of J_B gives

$$\tilde{\omega}_{\psi,B}(A) = \langle \psi, J_B A^* J_B \psi\rangle = \langle J_B\psi, A J_B\psi\rangle. \tag{299}$$

Thus, the modular partner is the restriction to M_B of the vector state implemented by $J_B\psi$. For the vacuum,

$$\omega_{\Omega,B} = \tilde{\omega}_{\Omega,B} =: \omega_B. \tag{300}$$

The geometric meaning of J_B follows from conformal modular covariance. Let C_B map a Rindler wedge W to $D(B)$, and let r_W be the Bisognano-Wichmann wedge reflection. Up to the theory-dependent CPT action on spin and internal indices,

$$J_B = U(C_B) J_W U(C_B)^*, \qquad r_B \coloneqq C_B \circ r_W \circ C_B^{-1}. \tag{301}$$

The map r_B sends $D(B)$ to its causal complement. This double-cone modular action was established explicitly for the free massless scalar field in Ref. [40] and operator-algebraically for conformal nets in Ref. [41].

Equation (299) also fixes the reflected source without a separate pointwise prescription. Let η be a real Lorentzian boundary source, and let $X_\eta = X^*_\eta$ denote the corresponding smeared boundary operator. For $g \in \mathbb{R}$, the associated unitary preparation of the boundary CFT vacuum Ω is

$$\Omega_{g,\eta} = e^{igX_\eta}\Omega. \tag{302}$$

The reflected source operator is defined by

$$X_{\eta^{J_B}} \coloneqq -J_B X_\eta J_B. \tag{303}$$

The minus sign follows from $J_B i J_B = -i$, and

$$J_B e^{igX_\eta}\Omega = e^{igX_{\eta^{J_B}}}\Omega. \tag{304}$$

For a Euclidean preparation, the reflected source is obtained by the corresponding contour reflection, complex conjugation, and CPT action on the inserted operator. Conformal weights, Jacobians, and spin matrices are included by defining the reflected smeared operator through Eq. (303).

If $U_g \in M_B$ is unitary, then $J_B U_g J_B \in M'_B$ and commutes with every $A \in M_B$. Consequently,

$$\tilde{\omega}_{U_g\Omega,B}(A) = \langle \Omega, A\Omega\rangle, \qquad A \in M_B. \tag{305}$$

A unitary excitation localized entirely in the ball therefore has the vacuum as its modular partner on M_B. A nonvacuum modular partner requires a preparation whose restriction to M'_B differs from that of the vacuum.

C. AdS-Rindler representation of the reflected-difference tangent

Let

$$\mathcal{N}_B \coloneqq \mathcal{A}_{\rm bulk}(\mathcal{W}_B) \tag{306}$$

be the represented von Neumann algebra generated by the linear bulk field in the AdS-Rindler wedge $\mathcal{W}_B$, acting on $\mathcal{H}_{\rm bulk}$. The bulk vacuum $\Omega_{\rm bulk}$ is cyclic and separating for $\mathcal{N}_B$, so $(\mathcal{N}_B, \Omega_{\rm bulk})$ is standard. We denote its modular conjugation by $J_B^{\rm bulk}$.

For the AdS-Rindler vacuum, the bulk modular conjugation implements a geometric wedge reflection. Let $\mathcal{R}_B^{\rm bulk}$ denote the AdS isometry induced by the boundary reflection r_B. It reverses the two normal directions to the bifurcation surface and exchanges the wedge with its causal complement,

$$\mathcal{R}_B^{\rm bulk}(\mathcal{W}_B) = \mathcal{W}'_B, \qquad (\mathcal{R}_B^{\rm bulk})^2 = 1. \tag{307}$$

For a neutral scalar, the corresponding real-linear action on classical solutions is

$$(\mathcal{J}_B^{\rm cl}\Psi)(X) \coloneqq \Psi((\mathcal{R}_B^{\rm bulk})^{-1}X), \qquad (\mathcal{J}_B^{\rm cl})^2 = 1. \tag{308}$$

For fields with spin or internal quantum numbers, the appropriate tensor, spin, and CPT actions are included in $\mathcal{J}_B^{\rm cl}$.

Let

$$V\colon \mathcal{H}_{\rm bulk} \to \mathcal{H}_{\rm CFT}, \qquad V\Omega_{\rm bulk} = \Omega, \tag{309}$$

be an isometry from the bulk reconstruction sector into the boundary CFT Hilbert space. Assume that $\mathcal{N}_B$ is standardly c-reconstructable from M_B in the sense of Ref. [32]. Theorem 2 of that reference then implies preservation of relative entropy for normal states in the reconstruction sector and the bulk-boundary modular-intertwining relation

$$VJ_B^{\rm bulk} = J_B V. \tag{310}$$

These conclusions are exact in the complementary-recovery model. Their extension to coupled matter and metric perturbations, discussed in Sec. VII F, requires additional assumptions at the retained semiclassical order.

The action of $J_B^{\rm bulk}$ on the Weyl algebra induces an involution ϑ_B on $\mathcal{K}_{\mathbb{R}}^{\rm bulk}$, as follows:

$$J_B^{\rm bulk}W(f)J_B^{\rm bulk} = W(\vartheta_B f), \qquad \vartheta_B^2 = 1. \tag{311}$$

As in the Minkowski-wedge construction of Sec. V, ϑ_B includes the sign required by

$$J_B^{\rm bulk} i J_B^{\rm bulk} = -i.$$

Let $\mathcal{R}_{B*}^{\rm bulk}$ denote the pullback induced by $\mathcal{R}_B^{\rm bulk}$ on scalar functions. We use the same symbol for its action on test functions and on classical solutions. For the Weyl convention $W(f) = e^{i\Phi(f)}$, antiunitarity gives

$$\vartheta_B f = -\mathcal{R}_{B*}^{\rm bulk} f \tag{312}$$

for a CPT-even neutral scalar. Since $\mathcal{R}_B^{\rm bulk}$ reverses time orientation, it exchanges the retarded and advanced propagators. Consequently,

$$E\mathcal{R}_{B*}^{\rm bulk} = -\mathcal{R}_{B*}^{\rm bulk}E. \tag{313}$$

The two signs cancel when the reflected Weyl label is mapped to its classical solution:

$$\begin{aligned}\Psi_{\vartheta_B f} &= E\vartheta_B f \\ &= \mathcal{R}_{B*}^{\rm bulk}Ef \\ &= \mathcal{J}_B^{\rm cl}\Psi_f.\end{aligned} \tag{314}$$

For fields with nontrivial CPT parity, spin, or internal quantum numbers, the corresponding actions are included in both ϑ_B and $\mathcal{J}_B^{\rm cl}$.

The difference between a Weyl label and its modular reflection is

$$\delta_B f \coloneqq f - \vartheta_B f. \tag{315}$$

It vanishes for a modular-even label and is twice the ϑ_B-odd part of f.

If the one-particle vector determined by $[f]$ lies in the real standard subspace associated with $\mathcal{W}_B$, then

$$W(f) \in \mathcal{N}_B, \qquad W(\vartheta_B f) \in \mathcal{N}'_B.$$

Consequently, for every $A \in \mathcal{N}_B$, we have

$$\langle W(g\vartheta_B f)\Omega_{\rm bulk},\ AW(g\vartheta_B f)\Omega_{\rm bulk}\rangle = \langle \Omega_{\rm bulk},\ A\Omega_{\rm bulk}\rangle. \tag{316}$$

The reflected coherent state therefore restricts to the vacuum on $\mathcal{N}_B$. In this localized case, the modular comparison reduces to the usual comparison between an excitation in the ball and the vacuum. A nonvacuum modular partner requires a Weyl label that is not confined to the wedge standard subspace.

For $f \in \mathcal{K}_{\mathbb{R}}^{\rm bulk}$ in the reconstruction sector, we define the corresponding boundary vector by

$$\Omega_{g,f}^{\rm CFT} \coloneqq V\Omega_{g,f}^{\rm bulk} = VW(gf)\Omega_{\rm bulk} \tag{317}$$

and its restriction to the boundary ball by

$$\omega_{g,f;B}(A) \coloneqq \langle \Omega^{\mathrm{CFT}}_{g,f}, A\Omega^{\mathrm{CFT}}_{g,f}\rangle, \qquad A \in M_B. \tag{318}$$

Since $J_B^{\mathrm{bulk}}\Omega_{\mathrm{bulk}} = \Omega_{\mathrm{bulk}}$, the modular-intertwining relation gives

$$\begin{aligned} J_B\Omega^{\mathrm{CFT}}_{g,f} &= J_B VW(gf)\Omega_{\mathrm{bulk}} \\ &= VJ_B^{\mathrm{bulk}}W(gf)\Omega_{\mathrm{bulk}} \\ &= VW(g\vartheta_B f)\Omega_{\mathrm{bulk}}. \end{aligned} \tag{319}$$

Together with Eq. (299), this yields the exact boundary modular partner

$$\tilde{\omega}_{g,f;B} = \omega_{g,\vartheta_B f;B}. \tag{320}$$

Thus, the two boundary families entering the modular comparison are the images under V of the bulk coherent families with labels f and $\vartheta_B f$. At $g = 0$, both families reduce to the restricted boundary vacuum,

$$\omega_{0,f;B} = \tilde{\omega}_{0,f;B} = \omega_B. \tag{321}$$

Let

$$v_{B,f} \coloneqq \partial_g \omega_{g,f;B}\big|_{g=0} \tag{322}$$

denote the tangent generated by the coherent family with label f. The tangent selected by the modularly paired comparison is

$$u_{B,f}(A) \coloneqq \frac{d}{dg}\left[\omega_{g,f;B}(A) - \tilde{\omega}_{g,f;B}(A)\right]\Big|_{g=0}. \tag{323}$$

The exact partner relation $\tilde{\omega}_{g,f;B} = \omega_{g,\vartheta_B f;B}$, together with the real linearity of the first variation in the Weyl label, gives

$$u_{B,f} = v_{B,f} - v_{B,\vartheta_B f} = v_{B,\delta_B f}. \tag{324}$$

Under the real-linear correspondence $f \mapsto \Psi_f = Ef$, the difference label $\delta_B f$ corresponds to the reflected-difference solution

$$\begin{aligned} \delta_J\Psi_f &\coloneqq \Psi_f - \mathcal{J}_B^{\mathrm{cl}}\Psi_f \\ &= \Psi_f - \Psi_{\vartheta_B f} \\ &= \Psi_{\delta_B f}. \end{aligned} \tag{325}$$

Since $\vartheta_B^2 = 1$, we obtain

$$\vartheta_B\delta_B f = -\delta_B f, \qquad \mathcal{J}_B^{\mathrm{cl}}\Psi_{\delta_B f} = -\Psi_{\delta_B f}. \tag{326}$$

Thus the reflected-difference perturbation is modular odd.

Once the ambient coherent family is fixed, the label f is fixed. The modular pullback fixes its partner label $\vartheta_B f$, and the difference of the two families fixes the tangent $\delta_B f = f - \vartheta_B f$. The response direction is therefore a consequence of the modular pairing rather than an independently chosen bulk perturbation.

D. Exact relative entropy and the canonical energy of the reflected-difference tangent

Let $\omega^{\mathrm{bulk}}_{g,f}$ denote the restriction to $\mathcal{N}_B$ of the coherent vector state represented by $\Omega^{\mathrm{bulk}}_{g,f}$, and let ω_0^{bulk} denote the restricted bulk vacuum. The diagonal normalization of the scalar canonical-energy form is fixed by

$$E^{\mathcal{W}_B}_{\mathrm{can}}(\Psi_f, \Psi_f) \coloneqq \frac{d^2}{dg^2} S_{\mathcal{N}_B}(\omega^{\mathrm{bulk}}_{g,f} \| \omega_0^{\mathrm{bulk}})\Big|_{g=0}. \tag{327}$$

Its symmetric bilinear extension is obtained by polarization. In terms of the Klein-Gordon symplectic current defined in Eq. (291), it is

$$E^{\mathcal{W}_B}_{\mathrm{can}}(\Psi_{f_1}, \Psi_{f_2}) = -\int_{\Sigma_B} d\Sigma_A \omega^A_{\mathrm{KG}}(\Psi_{f_1}, \mathcal{L}_{\xi_B}\Psi_{f_2}), \tag{328}$$

where Σ_B is a Cauchy surface of $\mathcal{W}_B$ and $d\Sigma_A$ is future directed. Conservation of the symplectic current, the Killing property of ξ_B, and the reflecting boundary condition make this form symmetric on the stated domain. The minus sign follows from the metric and symplectic-current conventions and makes the diagonal form positive. With ξ_B normalized as the bulk extension of the boundary modular-flow generator ζ_B, Eq. (328) agrees with the relative-entropy Hessian in Eq. (327). The identification of the ball-region quantum Fisher form with AdS-Rindler canonical energy was established in Refs. [25,26].

We now apply relative-entropy preservation to the modularly paired boundary states. Equation (320) identifies them with the boundary images of the bulk coherent states labeled by f and $\vartheta_B f$. Consequently,

$$\begin{aligned} S_{M_B}(\omega_{g,f;B} \| \tilde{\omega}_{g,f;B}) &= S_{\mathcal{N}_B}(\omega^{\mathrm{bulk}}_{g,f} \| \omega^{\mathrm{bulk}}_{g,\vartheta_B f}), \\ S_{M_B}(\tilde{\omega}_{g,f;B} \| \omega_{g,f;B}) &= S_{\mathcal{N}_B}(\omega^{\mathrm{bulk}}_{g,\vartheta_B f} \| \omega^{\mathrm{bulk}}_{g,f}). \end{aligned} \tag{329}$$

Relative entropy between equal-covariance coherent states of a CCR algebra depends only on the difference of their test-function classes [14]. The two bulk relative entropies in Eq. (329) therefore reduce to

$$\begin{aligned} S_{\mathcal{N}_B}(\omega^{\mathrm{bulk}}_{g,f} \| \omega^{\mathrm{bulk}}_{g,\vartheta_B f}) &= S_{\mathcal{N}_B}(\omega^{\mathrm{bulk}}_{g,\delta_B f} \| \omega_0^{\mathrm{bulk}}), \\ S_{\mathcal{N}_B}(\omega^{\mathrm{bulk}}_{g,\vartheta_B f} \| \omega^{\mathrm{bulk}}_{g,f}) &= S_{\mathcal{N}_B}(\omega^{\mathrm{bulk}}_{g,-\delta_B f} \| \omega_0^{\mathrm{bulk}}). \end{aligned} \tag{330}$$

The coherent-state relative entropy is exactly quadratic in its difference label. It is therefore unchanged when $\delta_B f$ is replaced by $-\delta_B f$. Equations (327) and (330) give

$$S_{M_B}(\omega_{g,f;B}\|\tilde{\omega}_{g,f;B}) = S_{M_B}(\tilde{\omega}_{g,f;B}\|\omega_{g,f;B}) = \frac{g^2}{2}E^{\mathcal{W}_B}_{\mathrm{can}}(\Psi_{\delta_B f},\Psi_{\delta_B f}). \tag{331}$$

The symmetrized comparison is consequently

$$\mathcal{S}_J(g;B,f) \coloneqq S_{M_B}(\omega_{g,f;B}\|\tilde{\omega}_{g,f;B}) + S_{M_B}(\tilde{\omega}_{g,f;B}\|\omega_{g,f;B}) = g^2 E^{\mathcal{W}_B}_{\mathrm{can}}(\Psi_{\delta_B f},\Psi_{\delta_B f}). \tag{332}$$

The corresponding bilinear BKM form on coherent-state tangents follows by allowing the two coherent states to vary independently. For $f_1, f_2 \in \mathcal{K}^{\mathrm{bulk}}_{\mathbb{R}}$ in the reconstruction sector and $a, b \in \mathbb{R}$, we define

$$f_{a,b} \coloneqq a f_1 - b f_2. \tag{333}$$

Linearity of the causal propagator gives

$$\Psi_{f_{a,b}} = a\Psi_{f_1} - b\Psi_{f_2}. \tag{334}$$

Relative-entropy preservation and the coherent-state difference formula then give

$$S_{M_B}(\omega_{a,f_1;B}\|\omega_{b,f_2;B}) = S(\omega^{\mathrm{bulk}}_{1,f_{a,b}}\|\omega^{\mathrm{bulk}}_0) = \frac{1}{2}E_{\mathrm{can}}(\Psi_{f_{a,b}},\Psi_{f_{a,b}}). \tag{335}$$

For the tangents defined in Eq. (322), the mixed-derivative characterization of the BKM form gives

$$\gamma^{\mathrm{BKM}}_{\omega_B}(v_{B,f_1},v_{B,f_2}) = -\partial_a\partial_b S_{M_B}(\omega_{a,f_1;B}\|\omega_{b,f_2;B})|_{a=b=0} = E^{\mathcal{W}_B}_{\mathrm{can}}(\Psi_{f_1},\Psi_{f_2}). \tag{336}$$

Using Eq. (324), the susceptibility evaluates the BKM form, and hence the canonical energy, on the reflected-difference tangent,

$$\chi_J(B;f) \coloneqq \frac{d^2}{dg^2}\mathcal{S}_J(g;B,f)\Big|_{g=0} = 2\gamma^{\mathrm{BKM}}_{\omega_B}(u_{B,f},u_{B,f}) = 2E^{\mathcal{W}_B}_{\mathrm{can}}(\Psi_{\delta_B f},\Psi_{\delta_B f}). \tag{337}$$

Equations (324), (326), and (337) show that the modular pullback fixes a modular-odd reflected-difference tangent whose susceptibility is exactly twice its AdS-Rindler canonical energy. Because the coherent-state relative entropy is exactly quadratic, Eq. (332) gives the complete dependence on g, rather than only its Hessian at $g = 0$.

The general reflected-difference result reduces to the standard ball-region formula for a wedge-localized label. Let f_{loc} satisfy

$$W(f_{\mathrm{loc}}) \in \mathcal{N}_B, \qquad W(\vartheta_B f_{\mathrm{loc}}) \in \mathcal{N}'_B.$$

The commutant displacement in $W(g\delta_B f_{\mathrm{loc}})$ does not change the state restricted to $\mathcal{N}_B$. Hence,

$$\omega^{\mathrm{bulk}}_{g,\delta_B f_{\mathrm{loc}}} = \omega^{\mathrm{bulk}}_{g,f_{\mathrm{loc}}} \quad \text{on } \mathcal{N}_B. \tag{338}$$

It follows that

$$E^{\mathcal{W}_B}_{\mathrm{can}}(\Psi_{\delta_B f_{\mathrm{loc}}},\Psi_{\delta_B f_{\mathrm{loc}}}) = E^{\mathcal{W}_B}_{\mathrm{can}}(\Psi_{f_{\mathrm{loc}}},\Psi_{f_{\mathrm{loc}}}), \qquad \chi_J(B;f_{\mathrm{loc}}) = 2E^{\mathcal{W}_B}_{\mathrm{can}}(\Psi_{f_{\mathrm{loc}}},\Psi_{f_{\mathrm{loc}}}). \tag{339}$$

Thus, the localized case recovers the canonical energy of the original wedge perturbation, whereas an ambient perturbation is evaluated on the reflected-difference solution $\Psi_f - \mathcal{J}^{\mathrm{cl}}_B\Psi_f$.

E. Modular-odd decomposition and boundary sources

We decompose the bulk test-function class into its modular-even and modular-odd parts, as follows:

$$f^{(B)}_{\pm} \coloneqq \frac{1}{2}(f \pm \vartheta_B f). \tag{340}$$

These classes satisfy

$$\vartheta_B f^{(B)}_{\pm} = \pm f^{(B)}_{\pm}, \qquad \delta_B f = 2f^{(B)}_{-}. \tag{341}$$

Substitution into Eq. (337) gives

$$\chi_J(B;f) = 8E^{\mathcal{W}_B}_{\mathrm{can}}(\Psi_{f^{(B)}_-},\Psi_{f^{(B)}_-}). \tag{342}$$

The modular-even component of f therefore makes no contribution to the susceptibility. In particular, $\chi_J(B;f) = 0$ if $\vartheta_B f = f$. For a modular-odd test-function class f_{odd}, $\vartheta_B f_{\mathrm{odd}} = -f_{\mathrm{odd}}$, one has

$$\chi_J(B;f_{\mathrm{odd}}) = 8E^{\mathcal{W}_B}_{\mathrm{can}}(\Psi_{f_{\mathrm{odd}}},\Psi_{f_{\mathrm{odd}}}). \tag{343}$$

The same decomposition can be expressed directly in terms of the boundary preparation. Let $f_\eta \in \mathcal{K}^{\mathrm{bulk}}_{\mathbb{R}}$ denote the bulk test-function class produced by a boundary source η. Assume that the preparation map

$$\eta \mapsto f_\eta \tag{344}$$

is real linear and equivariant under modular reflection,

$$f_{\eta^{J_B}} = \vartheta_B f_\eta. \tag{345}$$

Here, η^{J_B} denotes the reflected boundary preparation. For a Lorentzian unitary preparation, it is defined at the operator level by Eq. (303). For a Euclidean preparation, it is defined

by the corresponding reflected contour and operator insertions.

On the real vector space of source preparations, we define

$$\eta_{\pm}^{(B)} := \frac{1}{2}(\eta \pm \eta^{J_B}). \tag{346}$$

Real linearity and reflection equivariance imply

$$\begin{aligned} f_\eta &= f_{\eta_+^{(B)}} + f_{\eta_-^{(B)}}, \\ \vartheta_B f_{\eta_\pm^{(B)}} &= \pm f_{\eta_\pm^{(B)}}, \\ \delta_B f_\eta &= 2 f_{\eta_-^{(B)}}. \end{aligned} \tag{347}$$

Writing $\chi_J(B;\eta) := \chi_J(B;f_\eta)$, the source-prepared susceptibility is therefore

$$\chi_J(B;\eta) = 8E_{\mathrm{can}}^{\mathcal{W}_B}(\Psi_{f_{\eta_-^{(B)}}}, \Psi_{f_{\eta_-^{(B)}}}). \tag{348}$$

Thus, only the modular-odd part of the boundary preparation generates the reflected-difference bulk solution on which the canonical energy is evaluated.

For a neutral Hermitian scalar primary prepared on a reflection-compatible Euclidean contour $\mathcal{C}$, let $\mathcal{R}_B^{\mathcal{C}}$ denote the contour involution induced by the boundary reflection r_B. The reflected source is defined exactly by equality of the corresponding smeared operators. In reflection-adapted coordinates, after the conformal weight and Jacobian have been absorbed into the source profile, this relation takes the form

$$\eta^{J_B}(x) = \epsilon_{\mathcal{O}} \eta^*(\mathcal{R}_B^{\mathcal{C}} x), \qquad x \in \mathcal{C}, \tag{349}$$

where $\epsilon_{\mathcal{O}} = \pm 1$ is the CPT or internal parity of the scalar operator. For a real source coupled to a CPT-even scalar,

$$\eta^{J_B}(x) = \eta(\mathcal{R}_B^{\mathcal{C}} x). \tag{350}$$

For the Lorentzian unitary preparation, the sign required by antiunitarity is instead included in Eq. (303).

F. Semiclassical extension to matter and gravity

For the linear bulk field under exact complementary recovery, the coherent-state identities in Secs. VII C and VII D are exact. For coupled matter and metric perturbations, we work to quadratic order in the perturbation within the semiclassical approximation. At this order, we assume the reconstruction-sector bulk-boundary modular-intertwining relation and reflection equivariance of the linearized boundary-source-to-bulk-solution map. The boundary reflected-difference tangent is then represented by the corresponding reflected-difference solution of the linearized bulk equations. The JLMS relation, the bulk constraints, and the gauge-invariant Hollands-Wald prescription identify its BKM quadratic form with the bulk canonical energy.

Let $\delta\mathcal{X}$ denote a combined linearized matter and metric perturbation satisfying the bulk constraints and the required asymptotic and extremal-surface boundary conditions. The real-linear modular reflection $\mathcal{J}_B^{\mathrm{cl}}$ acts geometrically on the matter and metric fields, including the tensor, spin, and CPT actions appropriate to each field. Since the reflection preserves the linearized equations and boundary conditions, it maps an admissible perturbation to another admissible perturbation. The reflected-difference solution is

$$\delta_J\mathcal{X} := \delta\mathcal{X} - \mathcal{J}_B^{\mathrm{cl}}\delta\mathcal{X}. \tag{351}$$

Let $E_{\mathrm{can}}^{\mathcal{W}_B}$ denote the Hollands-Wald canonical-energy form on the constrained perturbation space. The ball-region BKM form is represented by this canonical energy [25,26,28]. Consequently, at the retained semiclassical order, the modular-reflection susceptibility is

$$\chi_J(B) = 2E_{\mathrm{can}}^{\mathcal{W}_B}(\delta_J\mathcal{X}, \delta_J\mathcal{X}). \tag{352}$$

The factor of 2 is the same factor relating the symmetrized relative-entropy Hessian to the BKM form in the exact linear-field construction.

For a scalar perturbation on the fixed AdS background, Eq. (352) reduces to the scalar canonical-energy expression in Eq. (328). For a purely gravitational perturbation, it becomes

$$\chi_J^{\mathrm{grav}}(B) = 2\int_{\Sigma_B} \omega_{\mathrm{grav}}(\delta_J h, \mathcal{L}_{\xi_B}\delta_J h), \tag{353}$$

where $\delta_J h$ is the metric component of $\delta_J\mathcal{X}$, and ω_{grav} is the covariant gravitational symplectic current in the Hollands-Wald convention.

The gravitational expression is evaluated in Hollands-Wald gauge, or with the equivalent extremal-surface boundary-and-corner completion that makes the subregion canonical energy gauge invariant [28,42–44]. Proper gauge transformations preserving the prescribed boundary conditions are quotiented out. Transformations carrying nonzero boundary or edge charges are not null directions of the completed symplectic form. The gravitational extension therefore preserves the modular selection of the reflected-difference perturbation while replacing the fixed-background scalar quadratic form by the gauge-invariant canonical energy of the coupled linearized solution.

VIII. DISCUSSION AND OUTLOOK

The central conclusion of this work is a fixed-point principle for state comparison that extends from finite-dimensional type I systems to local type III algebras. In the type I setting, the coincidence point is the equality of a

density matrix with its modularly reflected state. In the local quantum-field-theoretic setting, it is the equality of a local state with the modular pullback of its commutant restriction. In finite dimensions, the antiunitary reflection acts within a matrix algebra. In the local theory, the vacuum modular conjugation carries the algebra to its commutant. The pullback is therefore the operator-algebraic operation that makes the comparison intrinsic to the original local algebra.

The fixed representative alone is not the main result. For a vacuum standard pair, vacuum coincidence follows from invariance of the vacuum under modular conjugation. More generally, every normal positive functional has a unique natural-cone representative fixed by the standard-form J. The present construction instead holds fixed the reference standard pair and the ambient preparation. These data determine the commutant restriction, its pullback, and the reflected-difference tangent. Choosing a new canonical representative for each local state would remove the ambient comparison measured by the susceptibility.

At coincidence, the symmetrized relative entropy has no linear term because the two compared states agree and relative entropy is non-negative. Its leading nontrivial coefficient is governed in finite dimensions by BKM metric evaluated on the reflected-difference tangent. For local type III algebras, the corresponding coefficient is the local BKM bilinear form obtained from the second-order expansion of Araki relative entropy at the diagonal. The resulting susceptibility is attached to the tangent selected by the modular pairing rather than to an arbitrary curve of states. It also depends on the localization region because changing the region changes the local algebra, its modular conjugation, and the tangent tested by the BKM form.

The coherent-state examples realize this structure exactly. For the free scalar field on a wedge, the susceptibility is a positive boost-energy quadratic form, equivalently a stress-tensor integral, of the reflected-difference profile. Translations of the wedge change the algebra and its modular reflection, so the localization parameter enters the response intrinsically. For the chiral $U(1)$ current, the construction reduces to a half-line quadratic form. In the thermal half-line theory, the vacuum geometric reflection is replaced by the Borchers-Yngvason standard-subspace reflection. These examples show that the fixed-point susceptibility is well defined in local type III quantum field theory and is not merely a finite-dimensional analogy.

The vacuum-ball construction has a related equilibrium interpretation. The Casini-Huerta-Myers conformal transformation maps the ball vacuum to a KMS state on the hyperbolic cylinder, while the bulk geometry becomes the AdS-Rindler exterior [24]. For regular exponential source deformations that admit a Kubo-Mori score representation, the BKM geometry has the corresponding equilibrium-response form [30,31]. The ball vacuum is therefore a geometrically selected equilibrium state, and the reflected-difference tangent measures a departure from its modular reflection symmetry.

For the vacuum ball and a chosen ambient perturbation, modular conjugation determines its reflected comparison and hence the bulk perturbation on which the canonical energy is evaluated. In the encoded linear coherent family, this perturbation is the difference between a bulk excitation and its AdS-Rindler modular reflection. The symmetrized Araki relative entropy is exactly quadratic, and its Hessian equals twice the canonical energy of this reflected-difference perturbation. At the retained semiclassical order, the same prescription extends to admissible matter and metric perturbations. Unlike ordinary ball-region Fisher information, the modular-reflection susceptibility uses B and J_B to determine the perturbation tested by the quadratic form.

This construction also differs from holographic entanglement asymmetry [29]. An internal-symmetry twirl and the present modular pair can both produce a BKM coefficient represented by a bulk quadratic form, but they select different tangent directions. The present susceptibility isolates the component odd under the AdS-Rindler modular/CPT reflection rather than the component that breaks an internal symmetry. The coherent scalar construction in Secs. VII C–VII E gives an explicit realization of this selection. Section VII F extends the same modular selection principle to admissible coupled matter and metric perturbations at the retained second-order semiclassical level.

The main technical limitation is the assumed differentiability of Araki relative entropy at the diagonal. The BKM interpretation applies to deformations for which the required second-order expansion exists, and the explicit quantum-field-theoretic calculations in this paper use coherent-state deformations. An important extension is to determine broader classes of faithful normal deformations with the same local BKM Hessian. These may include perturbations not generated by Weyl translations, perturbations produced by interacting dynamics, and families whose reflected tangent has no elementary description.

Localization provides a second direction for further work. In the examples considered here, it appears through boost weights, half-line weights, thermal standard-subspace reflections, and AdS-Rindler entanglement-wedge reflections. A more intrinsic treatment could relate this dependence to modular inclusions, translated standard pairs, and variations of local algebras under changes of region. Such a formulation may clarify the relation between the present susceptibility and other local response quantities, including modular energy variations, relative-entropy inequalities, and holographic canonical-energy constraints.

The vacuum-ball analysis relies on its geometric modular flow. For more general regions or excited reference states, modular flow is typically nonlocal, and no simple bulk Killing description analogous to the AdS-Rindler flow is available. Extending the construction would require control

of the relevant standard pair, the corresponding bulk-boundary modular data, and the domain of the reflected perturbation.

It would also be useful to compare the intrinsic type III coefficient with regulated type I approximations. Cutoff density matrices should be treated as approximations to the Araki comparison rather than as definitions of the local quantity. Higher-order terms in the symmetrized Araki functional may define additional local response coefficients. In holographic settings, they may also encode nonlinear gravitational response or backreaction associated with the modularly reflected bulk comparison.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the author.

APPENDIX A: RELATIVE MODULAR OPERATORS AND ARAKI RELATIVE ENTROPY

This Appendix fixes the relative modular-operator convention used in the main text. Let $\mathfrak{M}$ be a von Neumann algebra in standard form on a Hilbert space $\mathcal{H}$, with modular conjugation J and natural cone $\mathcal{P}^\natural \subset \mathcal{H}$. Standard-form theory implies that each faithful normal state ω on $\mathfrak{M}$ is represented by a unique vector $\xi_\omega \in \mathcal{P}^\natural$ [31,38]. Its defining relation is

$$\omega(A) = \langle \xi_\omega, A\xi_\omega \rangle, \qquad A \in \mathfrak{M}. \tag{A1}$$

One of the defining standard-form properties is that the natural cone is fixed pointwise by J, as follows:

$$J\xi = \xi, \qquad \xi \in \mathcal{P}^\natural. \tag{A2}$$

In particular, $J\xi_\omega = \xi_\omega$. The vector ξ_ω is therefore the canonical J-symmetric standard-form representative of ω. This statement concerns the canonical representative of a state on $\mathfrak{M}$; it does not identify an arbitrary ambient vector implementing the same local state with that representative. This representation is intrinsic to the standard form and does not require a density matrix or a trace on $\mathfrak{M}$.

Given two faithful normal states ω and ϕ, with natural cone representatives ξ_ω and ξ_ϕ, the relative Tomita operator is defined on the dense domain $\mathfrak{M}\xi_\phi$ by

$$S_{\omega|\phi}(A\xi_\phi) = A^*\xi_\omega, \qquad A \in \mathfrak{M}. \tag{A3}$$

This antilinear operator is closable. Its closure, denoted again by $S_{\omega|\phi}$, determines the relative modular operator

$$\Delta_{\omega|\phi} := S^*_{\omega|\phi} S_{\omega|\phi}. \tag{A4}$$

The operator $\Delta_{\omega|\phi}$ is positive and self-adjoint. When $\omega = \phi$, this construction reduces to the ordinary modular operator of the state ω.

With these conventions, the Araki relative entropy is [9,10]

$$S_{\mathfrak{M}}(\omega\|\phi) := -\langle \xi_\omega, \log \Delta_{\phi|\omega}\xi_\omega \rangle. \tag{A5}$$

The order of the relative modular operator in Eq. (A5) is part of the convention: the entropy $S_{\mathfrak{M}}(\omega\|\phi)$ is expressed in terms of $\Delta_{\phi|\omega}$. The logarithm is understood through the spectral calculus, with the usual extended-value convention. This definition applies directly to type III local algebras, where there is no intrinsic trace and no canonical local density matrix.

In the finite-dimensional type I case, Eq. (A5) reduces to the usual Umegaki relative entropy. Let $\mathfrak{M} = \mathcal{B}(\mathcal{K})$, and let

$$\omega(A) = \mathrm{Tr}(\rho A), \qquad \phi(A) = \mathrm{Tr}(\sigma A), \qquad A \in \mathcal{B}(\mathcal{K}), \tag{A6}$$

where ρ and σ are faithful density matrices on $\mathcal{K}$. In the Hilbert-Schmidt standard form, the natural cone representatives are $\rho^{1/2}$ and $\sigma^{1/2}$, and left and right multiplication define

$$L_\sigma(X) = \sigma X, \qquad R_\rho(X) = X\rho. \tag{A7}$$

The relative modular operator appearing in Eq. (A5) is then

$$\Delta_{\phi|\omega} = L_\sigma R_\rho^{-1}. \tag{A8}$$

Since L_σ and R_ρ commute, one obtains

$$\log \Delta_{\phi|\omega} = L_{\log\sigma} - R_{\log\rho}. \tag{A9}$$

Substitution into Eq. (A5) gives

$$\begin{aligned} S_{\mathfrak{M}}(\omega\|\phi) &= -\mathrm{Tr}[\rho^{1/2}(\log\sigma\rho^{1/2} - \rho^{1/2}\log\rho)] \\ &= \mathrm{Tr}\rho(\log\rho - \log\sigma). \end{aligned} \tag{A10}$$

Thus, the Araki expression used for local algebras is the operator-algebraic extension of the finite-dimensional relative entropy used in the type I part of the paper [33].

The local construction uses the following pullback fact. Let $(\mathfrak{M}, \Omega)$ be a standard pair, and let J be its modular conjugation. Tomita-Takesaki theory gives

$$J\mathfrak{M}J = \mathfrak{M}'. \tag{A11}$$

We define

$$j(A) \coloneqq JA^*J, \qquad A \in \mathfrak{M}. \tag{A12}$$

Then, j is a linear unital $*$-anti-isomorphism

$$j\colon \mathfrak{M} \to \mathfrak{M}'. \tag{A13}$$

Indeed,

$$j(AB) = j(B)j(A), \quad j(A^*) = j(A)^*, \quad j(\mathbf{1}) = \mathbf{1}. \tag{A14}$$

Since j is a von Neumann algebra anti-isomorphism, it is ultraweakly continuous. Consequently, if ψ is a normal state on $\mathfrak{M}'$, then $\psi \circ j$ is a normal state on $\mathfrak{M}$. Positivity follows from

$$(\psi \circ j)(A^*A) = \psi(j(A^*A)) = \psi(j(A)j(A)^*) \geq 0, \tag{A15}$$

and normalization follows from

$$(\psi \circ j)(1) = \psi(\mathbf{1}) = \mathbf{1}. \tag{A16}$$

This is the abstract mechanism used in Sec. IV: the commutant restriction of an ambient state is pulled back to the original local algebra by the modular anti-isomorphism induced by the vacuum modular conjugation.

APPENDIX B: THERMAL CHIRAL $U(1)$ HALF-LINE

This Appendix formulates the finite-temperature analogue of the chiral half-line formulas of Sec. VI. The modular pullback construction and the coherent-state reduction are the same as in Secs. IV and VI; only the thermal modular reflection and the thermal entropy kernel are different.

Let ω_β be the thermal equilibrium state at inverse temperature $\beta > 0$, and let $(\pi_\beta, \mathcal{H}_\beta, \Omega_\beta)$ be its GNS representation. For a real test function $f \in C_c^\infty(\mathbb{R}, \mathbb{R})$, write

$$W_\beta(f) \coloneqq \pi_\beta(W(f)). \tag{B1}$$

Here, $\mathfrak{A}((-\infty, t))$ denotes the abstract chiral Weyl C^*-algebra of the half-line. The represented thermal half-line von Neumann algebra is

$$\mathcal{A}_\beta((-\infty, t)) \coloneqq \pi_\beta(\mathfrak{A}((-\infty, t)))''. \tag{B2}$$

The pair $(\mathcal{A}_\beta((-\infty, t)), \Omega_\beta)$ is standard. Let J_t^β denote its modular conjugation. Then

$$J_t^\beta \mathcal{A}_\beta((-\infty, t)) J_t^\beta = \mathcal{A}_\beta((-\infty, t))'. \tag{B3}$$

The associated modular pullback map is

$$j_t^\beta(A) \coloneqq J_t^\beta A^* J_t^\beta, \qquad A \in \mathcal{A}_\beta((-\infty, t)). \tag{B4}$$

In the vacuum half-line case, the modular conjugation is implemented by the geometric reflection $x \mapsto 2t - x$. In the thermal representation, this pointwise reflection is replaced by the Borchers-Yngvason standard-subspace modular conjugation [39]. Its induced action on Weyl labels is defined by

$$J_t^\beta W_\beta(f) J_t^\beta = W_\beta(\vartheta_t^\beta f). \tag{B5}$$

The map ϑ_t^β is an involutive antisymplectic reflection on the thermal one-particle space:

$$(\vartheta_t^\beta)^2 = 1, \qquad \sigma(\vartheta_t^\beta f, \vartheta_t^\beta g) = -\sigma(f, g). \tag{B6}$$

Unlike the vacuum reflection $f(x) \mapsto f(2t - x)$, the thermal reflection is not represented here by a pointwise formula on the light ray.

For a real test function h, we define the thermal reflected difference profile

$$\delta_t^\beta h \coloneqq h - \vartheta_t^\beta h. \tag{B7}$$

This is the finite-temperature analog of the vacuum profile $h - h(2t - \cdot)$. Although h is initially taken in $C_c^\infty(\mathbb{R}, \mathbb{R})$, the vectors $\vartheta_t^\beta h$ and $\delta_t^\beta h$ are understood in the thermal one-particle completion.

Let ω_h^β denote the coherent thermal state obtained from ω_β by Weyl displacement:

$$\omega_h^\beta(B) \coloneqq \langle W_\beta(h)\Omega_\beta, BW_\beta(h)\Omega_\beta \rangle, \qquad B \in \mathcal{B}(\mathcal{H}_\beta). \tag{B8}$$

With the normalization conventions of Sec. VI, the thermal half-line relative entropy is [16,39]

$$S_{\mathcal{A}_\beta((-\infty,t))}(\omega_h^\beta \| \omega_\beta) = \int_{-\infty}^{t} (h'(x))^2 \beta (1 - e^{2\pi(x-t)/\beta}) dx. \tag{B9}$$

The zero-temperature limit agrees with the vacuum half-line result, since

$$\beta(1 - e^{2\pi(x-t)/\beta}) = 2\pi(t - x) + O(\beta^{-1}), \quad \beta \to \infty. \tag{B10}$$

Now take the coherent thermal family

$$\varpi_g^\beta \coloneqq \omega_{gh}^\beta. \tag{B11}$$

Repeating the modular pullback construction of Sec. IV, with J_t replaced by J_t^β, gives

$$\begin{aligned} \omega_{g,t}^\beta &= \omega_{gh}^\beta \restriction_{\mathcal{A}_\beta((-\infty,t))}, \\ \tilde{\omega}_{g,t}^\beta &= \omega_{g\vartheta_t^\beta h}^\beta \restriction_{\mathcal{A}_\beta((-\infty,t))}. \end{aligned} \tag{B12}$$

At $g = 0$, the two local states coincide, as follows:

$$\omega^{\beta}_{0,t} = \tilde{\omega}^{\beta}_{0,t} = \omega_{\beta}\restriction_{\mathcal{A}_{\beta}((-\infty,t))}. \tag{B13}$$

Thus, $g = 0$ is the thermal self-dual point.

The coherent-state difference identity and homogeneity of the relative entropy give [14]

$$\begin{aligned} S_{\beta}(g,t) &\coloneqq S_{\mathcal{A}_{\beta}((-\infty,t))}(\omega^{\beta}_{g,t}\|\tilde{\omega}^{\beta}_{g,t}) + S_{\mathcal{A}_{\beta}((-\infty,t))}(\tilde{\omega}^{\beta}_{g,t}\|\omega^{\beta}_{g,t}) \\ &= 2g^2 S_{\mathcal{A}_{\beta}((-\infty,t))}(\omega^{\beta}_{\delta^{\beta}_t h}\|\omega_{\beta}). \end{aligned} \tag{B14}$$

Comparing with the local BKM expansion gives

$$\begin{aligned} F^{\mathcal{A}}_{\mathrm{BKM},\beta}(t) &= 2S_{\mathcal{A}_{\beta}((-\infty,t))}(\omega^{\beta}_{\delta^{\beta}_t h}\|\omega_{\beta}), \\ I_{A,\beta}(t) &= 2F^{\mathcal{A}}_{\mathrm{BKM},\beta}(t). \end{aligned} \tag{B15}$$

Substituting Eq. (B9) yields

$$F^{\mathcal{A}}_{\mathrm{BKM},\beta}(t) = 2\int_{-\infty}^{t} ((\delta^{\beta}_t h)'(x))^2 \beta(1 - e^{2\pi(x-t)/\beta})dx, \tag{B16}$$

and therefore

$$I_{A,\beta}(t) = 4\int_{-\infty}^{t} ((\delta^{\beta}_t h)'(x))^2 \beta(1 - e^{2\pi(x-t)/\beta})dx. \tag{B17}$$

The derivative in Eqs. (B16) and (B17) is understood in the entropy form domain of the thermal half-line. Equivalently, one may view it as the image under the quadratic-form map

$$\phi_{t,\beta}(f) \coloneqq f'\restriction_{(-\infty,t)} \in L^2((-\infty,t), \beta(1 - e^{2\pi(x-t)/\beta})dx). \tag{B18}$$

Thus, when $f = \delta^{\beta}_t h$, the notation $(\delta^{\beta}_t h)'(x)$ denotes

$$(\delta^{\beta}_t h)'(x) \coloneqq \phi_{t,\beta}(\delta^{\beta}_t h)(x). \tag{B19}$$

For ordinary compactly supported test functions, this agrees with the usual derivative. For the reflected vector $\vartheta^{\beta}_t h$, it denotes the corresponding element of the thermal entropy form domain.

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