Constructing conformal double field theory through a double copy map

Eric Lescano[1,*] and Jesús A. Rodríguez[2,†]

[1]*Institute for Theoretical Physics (IFT), University of Wroclaw, pl. Maxa Borna 9, 50-204 Wroclaw, Poland*

[2]*Universidad de Buenos Aires, FCEyN, Departamento de Física, Ciudad Universitaria, 1428 Buenos Aires, Argentina*

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We follow the classical double copy (DC) procedure that links Yang-Mills and double field theory (DFT), and apply it on a four-derivative gauge theory which is known to be related to Weyl gravity at the level of the amplitudes. We obtain a perturbative T-duality invariant theory on a double geometry, or conformal double field theory (CDFT), incorporating Weyl gravity plus b-field and dilaton contributions at quadratic order, without the need to impose a gauge-fixing condition. We also extend the formulation to cubic order for the case of vanishing generalized dilaton, which still incorporates Weyl gravity when $\Box h_{\mu\nu} = h = 0$. CDFT, together with ordinary DFT, are examples of T-duality invariant theories constructed through classical DC maps, revealing a promising and deep connection between gauge theories and T-duality invariant models.

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I. INTRODUCTION

The quadratic and cubic contributions to perturbative double field theory (DFT) [1,2] can be constructed by performing an off-shell classical double copy (DC) map [3–14] to the (quadratic and cubic) Yang-Mills Lagrangian [15]. Subsequent studies have delved deeper into the algebraic structures underlying this connection up to quartic order [16–18], providing a strong foundation for constructing quartic DFT as the DC of the quartic contributions coming from the Yang-Mills Lagrangian. These powerful results open up the possibility of extending this method to other gauge theories, leading to new T-duality invariant formulations on the double geometry, beyond DFT. In this context, DFT would be just one particular example of a family of classical T-duality invariant double copies, revealing a deep connection between gauge theories and T-duality invariant theories.

A recent proposal that explores this possibility was presented in [19], where the authors analyzed the higher-derivative gauge model outlined in [20–24]. The DC of this model results in a double gravitational theory defined in terms of a generalized vielbein and a generalized dilaton. This theory is related to quadratic Weyl gravity when the supergravity fields satisfy $g_{\mu\nu} = h_{\mu\nu}$ and $b_{\mu\nu} = \varphi = 0$ (pure gravitational case), and the gauge-fixing condition $\Box h - \partial_\mu \partial_\nu h^{\mu\nu} = 0$ is imposed. The results in [19] suggest that constructing a nonperturbative conformal double field theory (CDFT) is possible, while it can be formulated independently of specific gauge choices.

Remarkably, the reverse engineering of constructing gauge theories from double models through a zeroth and a single copy in the double geometry is possible thanks to the pioneering work of [25], along with further advances in this direction [26–31]. While ordinary DFT has proven to be a powerful framework for studying the classical DC program, the results of this work show the existence of a CDFT, whose single and zeroth copy should be related to the formulation in [20–24]. In other words, our model paves the way toward constructing a nonperturbative CDFT, offering a promising approach to constructing T-duality invariant theories as the classical DC of a given gauge theory.

In this work, we improve upon the model in [19] by including the dynamics of a charged scalar field ϕ_α. The DC contributions from this field are crucial for introducing the missing quadratic terms, which enable us to construct the quadratic CDFT without requiring any gauge-fixing condition. In this way, we demonstrate that it is possible to construct a CDFT through an off-shell and classical DC map, which reduces to quadratic Weyl gravity (plus b-field and dilaton contributions) after imposing the strong constraint. We also go beyond the quadratic order of the CDFT. Although developing the full cubic theory is a challenging endeavor, we focus on the scenario with a vanishing

*Contact author: eric.lescano@uwr.edu.pl
†Contact author: jarodriguez@df.uba.ar

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generalized dilaton. In this simplified context, we construct the cubic Lagrangian by further extending the DC map, fixing all coefficients by comparing the pure gravitational limit, where $\Box h_{\mu\nu} = h = 0$ (stemming from the vanishing generalized dilaton condition), to cubic Weyl gravity under the same conditions. This approach uniquely determines all coefficients in this reduced scenario.

The key findings of this work are as follows: The model presented in [19] is enhanced by incorporating a charged scalar field, which is essential for deriving the complete quadratic CDFT Lagrangian without gauge fixing. The model is further extended to cubic order and vanishing generalized dilaton considering a minimal set of identifications. By leveraging the gauge theory from [20,21] and applying the DC map, the cubic CDFT Lagrangian is constructed. All undetermined coefficients in this Lagrangian are fixed by comparison with cubic Weyl gravity under the constraint $\Box h_{\mu\nu} = h = 0$.

II. FROM GAUGE THEORIES TO T-DUALITY INVARIANT MODELS: STATE OF THE ART

In [15], the authors derived a classical and off-shell DC map to construct the quadratic and cubic perturbative formulations of DFT[1] from the Yang-Mills Lagrangian. This process can be summarized schematically as follows:

$$[\mathrm{YM}(A_\mu{}^a) \overset{DC}{\to} \mathrm{DFT}(e_{\mu\bar{\nu}}, \Phi) \overset{SC}{\to} \mathrm{SuGRA}]^{(2,3)},$$

where (2,3) represents the quadratic and cubic contributions of these theories. SuGRA refers to the Neveu Schwarz-Neveu Schwarz sector of supergravity, which is obtained from DFT after applying the strong constraint (SC). The notation (DC) stands for the DC procedure, which we will explain in detail below.

The method involves expressing the quadratic (cubic) contributions of Yang-Mills theory in momentum space. Following this, the gauge fields, metrics, and structure constants are identified with gravitational objects in a double geometry by promoting the color indices of the gauge group to a second set of gravitational indices. For instance, we have the identifications

$$A_\mu{}^a(k) \to e_{\mu\bar{\nu}}(k, \bar{k}), \tag{1}$$

$$\kappa_{ab} \to \frac{1}{2}\left(\eta^{\bar{\mu}\bar{\nu}} - \frac{\bar{k}^{\bar{\mu}}\bar{k}^{\bar{\nu}}}{k^2}\right), \tag{2}$$

for the gauge field and the Cartan-Killing metric. The notation for the indices in the diagram, and in the rest of the paper, is as follows:

(i) $a, b, \ldots$: gauge color indices (adjoint representation),
(ii) $\mu, \nu, \ldots$: space-time indices,
(iii) $\bar{\mu}, \bar{\nu}, \ldots$: space-time indices.

[1]For reviews, see Refs. [32–34] and the second lecture of [35].

The dependence of the fields is now doubled to include the *dual* momenta $\bar{k}$ in the same footing as the regular momenta k, giving rise to a double geometric structure such as in DFT. Returning to configuration space, the resulting theory has the exact form of DFT. More precisely, at quadratic order it has the structure of the gauge invariant DFT, while at cubic order it is the CDFT action subject to a gauge condition that originates from Siegel gauge in string field theory [15].

With this strategy in mind, the authors in [19] presented a procedure to introduce higher-derivative corrections in a way that the obtained double theory is related to conformal symmetry. The starting point is the higher-derivative gauge Lagrangian

$$\mathcal{L} = \frac{1}{2}\kappa_{ab} D_\mu F^{\mu\nu a} D_\rho F^\rho{}_\nu{}^b, \tag{3}$$

on which the authors apply the same procedure as in [15]. The DC map of the gauge Lagrangian (3) gives rise to a higher-order, double geometrical theory

$$[\mathrm{DFDF}(A_\mu{}^a) \overset{DC}{\to} \mathrm{CDFT}(e_{\mu\bar{\nu}}, \Phi)]^{(2,3)},$$

depending on the fundamental fields of perturbative DFT. This theory further reduces to quadratic Weyl gravity in the already mentioned pure gravitational case[2] (pg: $g_{\mu\nu} = h_{\mu\nu}$ and $b_{\mu\nu} = \varphi = 0$),

$$[\mathrm{CDFT}(e_{\mu\bar{\nu}}, \Phi) \overset{pg}{\to} \text{gauge-fixed Weyl gravity } (h_{\mu\nu})]^{(2)},$$

subject to the gauge-fixing condition

$$\Box h - \partial_\mu \partial_\nu h^{\mu\nu} = 0. \tag{4}$$

The necessity of imposing the previous gauge-fixing condition implies that it is not possible to generate all the terms in the quadratic Weyl Lagrangian. This limitation highlights a significant problem: the lack of a gauge-independent formulation of the theory restricts our ability to construct a fully nonperturbative version of the model.

In order to address this limitation, our approach enhances the model in [19] by including a charged scalar field, ϕ_α. The gauge model that we propose is given by

$$\begin{aligned}\mathcal{L} = {} & a_1 \kappa_{ab} D_\mu F^{\mu\nu a} D_\rho F^\rho{}_\nu{}^b + a_2 \kappa^{\alpha\beta} D_\mu \phi_\alpha D^\mu \phi_\beta \\ & + a_3 f_{abc} F_\mu{}^{\nu a} F_\nu{}^{\lambda b} F_\lambda{}^{\mu c} + a_4 C^\alpha{}_{ab} \phi_\alpha F_{\mu\nu}{}^a F^{\mu\nu b} \\ & + a_5 d^{\alpha\beta\gamma} \phi_\alpha \phi_\beta \phi_\gamma,\end{aligned} \tag{5}$$

where the a_i are real coefficients to be determined. The scalar field transforms under the action of the same gauge

[2]In general, when the Kalb-Ramond field and the dilaton are nonvanishing, matter contributions must be considered.

group but in the real representation, denoted by color indices $\alpha, \beta, \ldots$, and it is allowed to interact with the gauge field ${A_\mu}^a$. This model is inspired in the construction [20,21], and it is related to the DC of Weyl gravity at the level of scattering amplitudes.

As we are interested in constructing a double theory that includes conformal gravity in the appropriate limit, our strategy for determining the coefficients in the action will be to expand the Lagrangian (5) to quadratic and cubic order, and to compare the reduction in a convenient limit, the pure gravity limit, with/without generalized dilaton contributions for the quadratic/cubic case, respectively. In the cubic case, we will also consider the condition $\Box h_{\mu\nu} = h = 0$, which is fully compatible with the vanishing of the generalized dilaton. The next two sections are devoted to presenting this construction.

III. QUADRATIC CONFORMAL DOUBLE FIELD THEORY

Starting from the theory proposed in (5), the only source of quadratic contributions comes from the kinetic terms of the two fundamental fields. Specifically, we have

$$\mathcal{L}^{(2)} = 4a_1\kappa_{ab}\partial_\mu\partial^{[\mu}A^{\nu]a}\partial^\rho\partial_{[\rho}{A_{\nu]}}^b + a_2\kappa^{\alpha\beta}\partial_\mu\phi_\alpha\partial^\mu\phi_\beta. \tag{6}$$

Following the method outlined in [15,19], we integrate by parts and then transform this action into momentum space, yielding

$$S^{(2)}_{\mathrm{DC}} = -\int d^Dk[a_1k^4\kappa_{ab}\Pi^{\mu\nu}(k){A_\mu}^a(k)A^b_\nu(-k) + a_2k^2\kappa^{\alpha\beta}\phi_\alpha(k)\phi_\beta(-k)], \tag{7}$$

where $\Pi^{\mu\nu}(k) = \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$ suggests the identification (2) for the Cartan-Killing metric, which complements the identification (1) for the gauge field. At this stage, it is necessary to identify the objects associated with the kinetic term of the scalar field to complete the DC prescription for our model. This leads to

$$\phi_\alpha(k) \to k_\mu e^{\mu\bar\nu}(k,\bar k) + 2\bar k^{\bar\nu}\Phi(k,\bar k), \tag{8}$$

$$\kappa^{\alpha\beta} \to \frac{\bar k_{\bar\mu}\bar k_{\bar\nu}}{k^2}, \tag{9}$$

where $\Phi(k,\bar k)$ is the DFT generalized dilaton in momentum space. The identification of the scalar field is strongly indicated by the missing quadratic terms noted in [19], which are essential to completing the quadratic contributions of conformal gravity without relying on a specific gauge.

The quadratic DC Lagrangian, after performing all the identifications, is given by

$$S^{(2)}_{\mathrm{DC}} = -\frac{1}{2}\int d^Dx d^D\bar x[a_1(\Box e^{\mu\bar\nu}\Box e_{\mu\bar\nu} - \Box e^{\mu\bar\nu}\partial_\mu\partial^\rho e_{\rho\bar\nu} - \Box e^{\mu\bar\nu}\bar\partial_{\bar\nu}\bar\partial^{\bar\sigma}e_{\mu\bar\sigma} + \partial^\mu\bar\partial^{\bar\nu}e_{\mu\bar\nu}\partial^\rho\bar\partial^{\bar\sigma}e_{\rho\bar\sigma}) - 2a_2(\partial_\mu\partial_{\bar\nu}e^{\mu\bar\nu} + 2\Box\Phi)^2]. \tag{10}$$

This action reproduces the action obtained in [19] but includes new contributions coming from the DC of the scalar field ϕ_α.

Now we consider the pure gravity limit, which demands that we consider only one set of coordinates x or $\bar x$. In this context, this is equivalent to solving the strong constraint in DFT (see Appendix A), together with $b_{\mu\nu} = \varphi = 0$. The previous action becomes

$$S^{(2)}_{\mathrm{DC}} = -\frac{1}{2}\int d^Dx[a_1(\Box h^{\mu\nu}\Box h_{\mu\nu} - 2\Box h^{\mu\nu}\partial_\mu\partial^\rho h_{\rho\nu} + \partial^\mu\partial^\nu h_{\mu\nu}\partial^\rho\partial^\lambda h_{\rho\lambda}) - 2a_2(\Box h - \partial_\mu\partial_\nu h^{\mu\nu})^2],$$

which presents the same structure as the quadratic contributions of the Weyl gravity action (see Appendix B). In fact, if we demand both actions to be the same, the coefficients a_1, a_2 are fixed as

$$a_1 = -2\left(\frac{D-3}{D-2}\right), \tag{11}$$

$$a_2 = -\frac{1}{(D-1)}\left(\frac{D-3}{D-2}\right). \tag{12}$$

It is notable, though expected, that the action vanishes in $D = 3$, as in three dimensions, and the Weyl tensor is identically zero. The terms associated with the coefficient a_1 in (10) were already obtained in [19]. The new result presented here is the emergence of terms with a nonzero coefficient a_2, which arises from the scalar field dynamics. Although these terms were absent in [19], the authors in that work compensated for the absence through the gauge-fixing condition (4) acting on the conformal gravity quadratic action.

IV. CUBIC CONFORMAL DOUBLE FIELD THEORY WITH $\Phi = 0$

A. Constructing the cubic double copy

Once the quadratic contributions of our model have been obtained and the coefficients a_1 and a_2 have been calculated, ensuring that, in the pure gravity limit, the model reduces to Weyl gravity, we are ready to proceed with the calculation of the cubic theory. Now we will use the full Lagrangian introduced in (5):

$$\mathcal{L} = a_1\kappa_{ab}D_\mu F^{\mu\nu a}D_\rho F^{\rho}{}_{\nu}{}^{b} + a_2\kappa^{\alpha\beta}D_\mu\phi_\alpha D^\mu\phi_\beta + a_3 f_{abc}F_\mu{}^{\nu a}F_\nu{}^{\lambda b}F_\lambda{}^{\mu c} + a_4 C^{\alpha}{}_{ab}\phi_\alpha F_{\mu\nu}{}^{a}F^{\mu\nu b} + a_5 d^{\alpha\beta\gamma}\phi_\alpha\phi_\beta\phi_\gamma. \tag{13}$$

In this Lagrangian, f_{abc} are the structure constants of the gauge algebra, corresponding to the adjoint representation in which the gauge field $A_\mu{}^a$ transforms. The scalar field ϕ^α is charged under a real representation R of the gauge group, whose generators are denoted by $(T_R^a)^{\alpha\beta}$ and are assumed to be real and antisymmetric in its last two indices.[3]

The interaction terms involving the scalar field introduce additional group-theoretic tensors. The coefficients $C_{\alpha ab}$ play the role of Clebsch-Gordan coefficients, mediating the coupling between the scalar and the gauge sector, while $d_{\alpha\beta\gamma}$ is a totally symmetric tensor. These objects are inplicitly defined through the relations

$$C^{\alpha ab}C^{\alpha cd} = f^{ace}f^{ebd} + f^{ade}f^{ecb},$$
$$C^{\alpha ab}d^{\alpha\beta\gamma} = (T_R^a)^{\beta\alpha}(T_R^b)^{\alpha\gamma} + C^{\beta ac}C^{\gamma cb} + (a \leftrightarrow b),$$

which ensure consistency with the gauge symmetry and determine the structure of the interactions in terms of the underlying group theory.

The gauge covariant derivatives are defined as

$$D_\rho F_{\mu\nu}{}^{a} = \partial_\rho F_{\mu\nu}{}^{a} + g f^{a}{}_{bc}A_\rho{}^{b}F_{\mu\nu}{}^{c},$$
$$D_\mu\phi^\alpha = \partial_\mu\phi^\alpha - ig(T_R^a)^{\alpha\beta}A_{\mu a}\phi_\beta,$$

with the Yang-Mills curvature defined in the standard way $F_{\mu\nu}{}^{a} = 2\partial_{[\mu}A_{\nu]}{}^{a} + g f^{a}{}_{bc}A_\mu{}^{b}A_\nu{}^{c}$.

Our goal in this section is to construct the gravitational cubic contributions arising from the DC prescription of this model, determining the remaining coefficients a_3, a_4, and a_5 in the same way as we did for the quadratic terms. While analyzing the full Lagrangian is a very ambitious task, we will focus on the case with vanishing generalized dilaton [15]

$$\Phi(k,\bar{k}) = \frac{1}{k^2}k^\mu\bar{k}_{\bar{\nu}}e_\mu{}^{\bar{\nu}} = 0. \tag{14}$$

To construct a minimal proposal for the cubic Lagrangian of CDFT, we identify the following quantities:

$$f_{abc} \to i\frac{a_6}{a_1}(\eta^{\bar{\mu}\bar{\nu}}\bar{k}_{12}^{\rho} + \eta^{\bar{\nu}\bar{\rho}}\bar{k}_{23}{}^{\bar{\mu}} + \eta^{\bar{\rho}\bar{\mu}}\bar{k}_{31}^{\bar{\nu}}), \tag{15}$$

$$C_{\alpha ab} \to i\bar{k}_1^{\bar{\mu}}\eta^{\bar{\nu}\bar{\rho}}, \tag{16}$$

$$(T_R^a)^{\alpha\beta}A_{\mu a} \to i\omega_{\mu\bar{\nu}\bar{\rho}}(k,\bar{k}), \tag{17}$$

$$d_{\alpha\beta\gamma} \to \frac{i}{\bar{k}^2}\bar{k}_1^{(\bar{\mu}}\bar{k}_2^{\bar{\nu}}k_3^{\bar{\rho})}, \tag{18}$$

where $k_{ij} = k_i - k_j$. In (17), $\omega_{\mu\bar{\nu}\bar{\rho}}$ illustrates a natural correspondence between a gauge connection and a spin connection for the generalized vielbein. In these identifications, the only object requiring an additional coefficient to fix its cubic contributions is f_{abc}. In this case, the coefficient a_1 is inherited from the cubic contribution of the $DFDF$ term, which needs to be canceled. The identifications (15)–(18) can be further extended to a more general form in agreement with the index constraints that every object carries, which would be convenient in a more general setup thanks to extra coefficients. For our proposes, we will show that these identifications are enough to reproduce Weyl gravity in a particular (pure gravitational) limit.

We observe that the Weyl tensor is nontrivial in $D > 3$ and therefore a_1^{-1} is well defined. The identification (15) follows the same structure as in [15], but the inclusion of the $\frac{a_1}{a_6}$ coefficients introduces a global factor in this identification, which is crucial for reproducing Weyl gravity in arbitrary dimensions, as we will demonstrate in the next part of this section.

The minimal cubic DC Lagrangian with $\Phi = 0$ is given by

$$\begin{aligned}
S_{\mathrm{DC}}^{(3)}|_{\Phi=0} = \int d^Dx d^D\bar{x}\left(-\frac{a_6}{2}\right)[&\Box\partial^\rho\overline{\partial}^{\bar{\sigma}}e_{\mu\bar{\nu}}e^{\mu\bar{\nu}}e_{\rho\bar{\sigma}} - \Box\overline{\partial}^{\bar{\sigma}}e_{\mu\bar{\nu}}\partial^\rho e^{\mu\bar{\nu}}e_{\rho\bar{\sigma}} + \Box\overline{\partial}^{\bar{\sigma}}e_{\mu\bar{\nu}}\partial^\mu e^{\rho\bar{\nu}}e_{\rho\bar{\sigma}} - \Box\overline{\partial}^{\bar{\sigma}}e_{\mu\bar{\nu}}e^{\rho\bar{\nu}}\partial^\mu e_{\rho\bar{\sigma}}\\
&- \Box e_{\mu\bar{\nu}}\partial^\mu\overline{\partial}^{\bar{\sigma}}e^{\rho\bar{\nu}}e_{\rho\bar{\sigma}} + \Box e_{\mu\bar{\nu}}\overline{\partial}^{\bar{\sigma}}e^{\rho\bar{\nu}}\partial^\mu e_{\rho\bar{\sigma}} - \partial^\mu\partial^\rho\partial^\lambda\overline{\partial}^{\bar{\sigma}}e_{\mu\bar{\nu}}e_\rho{}^{\bar{\nu}}e_{\lambda\bar{\sigma}} + \partial^\mu\partial^\rho\overline{\partial}^{\bar{\sigma}}e_{\mu\bar{\nu}}\partial^\lambda e_\rho{}^{\bar{\nu}}e_{\lambda\bar{\sigma}}\\
&- \partial^\rho\partial^\lambda\overline{\partial}^{\bar{\sigma}}e_{\mu\bar{\nu}}\partial^\mu e_\rho{}^{\bar{\nu}}e_{\lambda\bar{\sigma}} + \partial^\rho\partial^\lambda\overline{\partial}^{\bar{\sigma}}e_{\mu\bar{\nu}}e_\rho{}^{\bar{\nu}}\partial^\mu e_{\lambda\bar{\sigma}} + \partial^\rho\partial^\lambda e_{\mu\bar{\nu}}\partial^\mu\overline{\partial}^{\bar{\sigma}}e_\rho{}^{\bar{\nu}}e_{\lambda\bar{\sigma}} - \partial^\rho\partial^\lambda e_{\mu\bar{\nu}}\overline{\partial}^{\bar{\sigma}}e_\rho{}^{\bar{\nu}}\partial^\mu e_{\lambda\bar{\sigma}}\\
&- \overline{\partial}^{\bar{\nu}}\overline{\partial}^{\bar{\sigma}}\overline{\partial}^{\bar{\kappa}}\partial^\rho e_{\mu\bar{\nu}}e^\mu{}_{\bar{\sigma}}e_{\rho\bar{\kappa}} + \overline{\partial}^{\bar{\nu}}\overline{\partial}^{\bar{\sigma}}\overline{\partial}^{\bar{\kappa}}e_{\mu\bar{\nu}}\partial^\rho e^\mu{}_{\bar{\sigma}}e_{\rho\bar{\kappa}} + \overline{\partial}^{\bar{\nu}}\overline{\partial}^{\bar{\sigma}}e_{\mu\bar{\nu}}\overline{\partial}^{\bar{\kappa}}\partial^\mu e^\rho{}_{\bar{\sigma}}e_{\rho\bar{\kappa}} - \overline{\partial}^{\bar{\nu}}\overline{\partial}^{\bar{\sigma}}e_{\mu\bar{\nu}}\overline{\partial}^{\bar{\kappa}}e^\rho{}_{\bar{\sigma}}\partial^\mu e_{\rho\bar{\kappa}}]\\
&+ a_2\partial^\sigma\partial_\lambda e^{\lambda\bar{\epsilon}}\omega^{(1)}_{\sigma\bar{\epsilon}\bar{\tau}}\partial_\epsilon e^{\epsilon\bar{\tau}} + 3a_3[\partial_\mu e^{\nu\bar{\rho}}\partial_{\bar{\rho}}\partial_\nu e^\lambda{}_{\bar{\epsilon}}\partial_\lambda e^{\mu\bar{\epsilon}} - \partial_\mu e^{\nu\bar{\rho}}\partial_\nu e^{\lambda}{}_{\bar{\epsilon}}\partial_{\bar{\rho}}\partial_\lambda e^{\mu\bar{\epsilon}}]\\
&- 3a_3[\partial_\mu\partial_{\bar{\epsilon}}e^{\nu\bar{\rho}}\partial_\nu e^\lambda{}_{\bar{\rho}}\partial^\mu e_\lambda{}^{\bar{\epsilon}} - \partial_\mu e^{\nu\bar{\rho}}\partial_{\bar{\epsilon}}\partial_\nu e^\lambda{}_{\bar{\rho}}\partial^\mu e_\lambda{}^{\bar{\epsilon}}] - 3a_3[\partial_\mu e^{\nu\bar{\rho}}\partial_\nu\partial_{\bar{\rho}}e^{\lambda\bar{\sigma}}\partial^\mu e_{\lambda\bar{\sigma}} - \partial_\mu e^{\nu\bar{\rho}}\partial_\nu e^{\lambda\bar{\sigma}}\partial^\mu\partial_{\bar{\rho}}e_{\lambda\bar{\sigma}}]\\
&- 3a_3[\partial_\mu e^{\nu\bar{\rho}}\partial_\nu e^{\lambda\bar{\epsilon}}\partial^\mu\partial_{\bar{\epsilon}}e_{\lambda\bar{\rho}} - \partial_\mu\partial_{\bar{\epsilon}}e^{\nu\bar{\rho}}\partial_\nu e^{\lambda\bar{\epsilon}}\partial^\mu e_{\lambda\bar{\rho}}] + a_4\partial_{\bar{\mu}}\partial_\epsilon e^{\epsilon\bar{\mu}}\partial_{[\sigma}e_{\lambda]}{}^{\bar{\nu}}\partial^\lambda e^\sigma{}_{\bar{\nu}},
\end{aligned} \tag{19}$$

[3]We use the same conventions as in [20] for the gauge real and adjoint representations of the theory.

where the notation $\omega^{(1)}_{\mu\bar{\mu}\bar{\nu}}$ refers to the first-order perturbation of the connection. It is possible to see that the Lagrangian (5) produces terms proportional to the generalized dilaton associated with the coefficient a_5, and therefore this coefficient does not appear in this Lagrangian. To determine the remaining coefficients, we will compare (19) with the cubic contributions of Weyl gravity.

We will perform this analysis under the assumption of a vanishing wave equation $\Box h_{\mu\nu} = 0$ and the traceless condition $h = 0$. Given these conditions, the generalized dilaton will not contribute to the gravitational terms after parametrization. Furthermore, our method cannot fix the coefficient a_5, as its contributions depend solely on the generalized dilaton, which vanishes in this case.

In the next section, we will demonstrate that this method is sufficiently robust to uniquely determine the coefficients a_3, a_4, and a_6.

B. Fixing the coefficients when $\Phi = 0$

The cubic contributions for Weyl gravity are given by

$$S^{(3)}_{\mathrm{CG}} = \int d^D x \left[\frac{1}{2} h C^{(1)}_{\mu\rho\lambda} C^{(1)\mu\nu\rho\lambda} - 4 h^{\mu\nu} C^{(1)}_{\mu\rho\lambda\sigma} C^{(1)\rho\lambda\sigma}_{\nu} + 2 C^{(2)}_{\mu\nu\rho\lambda} C^{(1)\mu\nu\rho\lambda}\right]. \tag{20}$$

Imposing the conditions

$$\Box h_{\mu\nu} = h = 0, \tag{21}$$

which are fully compatible with $\Phi = 0$, and performing a series of integrations by parts with the help of Cadabra [36],[4] the cubic Lagrangian can be reduced to the following two contributions:

$$L^{(3)} = f(D)\partial_\delta\partial_\lambda h^{\epsilon\lambda}\partial^\nu h_{\nu\sigma}\partial_\epsilon h^{\sigma\delta}, + g(D)\partial_\mu\partial_\sigma h^\mu{}_\epsilon \partial_\delta\partial_\lambda h^{\sigma\epsilon} h^{\delta\lambda}, \tag{22}$$

where

$$f(D) = \frac{-2(D-3)(11D+15)}{5(D-2)(D-1)}, \qquad g(D) = \frac{(D-3)(34D+5)}{5(D-2)(D-1)}. \tag{23}$$

As before, we find that $D = 3$ is the only physically meaningful root of both $f(D)$ and $g(D)$, consistent with the physical interpretation of the Weyl tensor. When the double conformal model (19) is parametrized and restricted to the pure gravitational limit, it takes the following form:

$$L^{(3)}_{DC} = -\left(\frac{a_6}{4} + 3a_3\right)\partial_\delta\partial_\lambda h^{\epsilon\lambda}\partial^\nu h_{\nu\sigma}\partial_\epsilon h^{\sigma\delta} + \frac{15}{4} a_6 \partial_\mu\partial_\sigma h^\mu{}_\epsilon \partial_\delta\partial_\lambda h^{\sigma\epsilon} h^{\delta\lambda} + \left(-a_6 + \frac{a_4}{2}\right)\partial_\mu\partial_\epsilon h^{\mu\epsilon}\partial_\sigma h_{\lambda\nu}\partial^\nu h^{\sigma\lambda}). \tag{24}$$

Therefore, the coefficients are uniquely fixed as

$$a_3 = -\frac{1}{45} g(D) - \frac{1}{3} f(D), \tag{25}$$

$$a_4 = 2a_6 = \frac{8}{15} g(D). \tag{26}$$

The preceding identifications determine the double action (19), which represents the full cubic CDFT with a vanishing generalized dilaton. In (26), we observe the necessity of the a_6 coefficient in identifying the structure constants. Additionally, we note that both the F^3 term and the interaction term between the gauge field and scalar field are essential for reproducing Weyl gravity in this limit.

By comparing with [19], it becomes evident that both the charged scalar field and the cubic self-interactions of the gauge field, $A_\mu{}^a$, are necessary to accurately match Weyl gravity in the pure gravitational scenario.

V. CONCLUSIONS AND OUTLOOK

We have constructed the quadratic Lagrangian for a T-duality invariant theory that arises as the classical double copy of the gauge Lagrangian given in (5). The field content of this theory matches that of DFT, including a generalized frame and a generalized dilaton. Upon parametrization and imposition of the pure gravity limit, the theory reduces to quadratic Weyl gravity. We refer to this framework as conformal double field theory.

The model presented in this paper extends the formulation introduced in [19] by incorporating the dynamics of a charged scalar field, following a proposal outlined in [20,21]. The presence of this scalar proves essential for constructing both the quadratic and cubic sectors of CDFT. In particular, the scalar contributions allow us to reproduce Weyl gravity coupled to matter, after imposing the strong constraint, without the need for gauge fixing. The global coefficients in the seed gauge Lagrangian must be chosen as in (12) to correctly embed Weyl gravity.

We have also constructed the minimal cubic structure of CDFT in the sector with vanishing generalized dilaton, fixing the interaction coefficients by imposing the constraints in (21) and matching to cubic Weyl gravity. The only undetermined coefficient is a_5, which depends solely on the generalized dilaton. Fixing this remaining term would require including the dilaton dynamics and relaxing the condition $h = 0$ in the pure gravity limit. Alternatively,

[4]The Cadabra code used in this work is available at https://github.com/conformal-dft/CDFT-Cadabra.

it could be possible to construct a nonperturbative version of CDFT directly using T-duality multiplets. Although the generalized Riemann tensor in DFT is not fully determined, a careful analysis of the biparametric higher-derivative DFT proposed in [37] might allow one to fix its free parameters so that it reduces to Weyl gravity in the appropriate limit.

Overall, our results reveal a promising and deep connection between gauge theory and T-duality-invariant gravity through classical double-copy constructions. Moreover, this approach provides a natural framework to investigate higher-derivative corrections in string theory. In particular, the following directions stand out for future work:

(i) Including the dynamics of the B-field and the dilaton: A natural next step is to examine whether terms such as RiemH^2, which appear in the α'-corrected bosonic and heterotic supergravity actions, can be reproduced within our framework. Matching such terms will likely require extending the identifications in (15)–(18) to include B-field and dilaton contributions.

(ii) Relaxing the covariant constraints: A key open question is whether relaxing the conditions in (21) to allow for a nonvanishing trace h could provide a consistent way to incorporate the generalized dilaton. While the dilaton does not contribute explicitly to the α' corrections, it can arise through covariant field redefinitions. This could suggest a natural interpretation of dilaton terms within the Weyl gravity structure.

(iii) Extending the double-copy map to higher orders: A longer-term goal is to apply the classical double-copy construction to higher orders in fields. This would open the door to exploring quartic and higher-derivative terms, such as Riem^4 (Riemann tensor) within a manifest T-duality covariant setting. At present, such an extension remains an open technical challenge, but progress in this direction could shed light on the structure of higher-order corrections in bosonic supergravity.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DOUBLE FIELD THEORY

In this paper, we use the original formulation of double field theory, a T-duality invariant reformulation of supergravity in which the internal space is doubled. However, the T-duality transformations act nonlinearly in this approach. It is important to distinguish this from the so-called generalized metric formulation of DFT, where T-duality is manifest and acts linearly on the fields. For the benefit of readers not specialized in this area, we briefly review both formulations in this appendix.

1. Nonmanifest T-duality invariant formulation

Double field theory [1,2] was originally developed as a T-duality invariant reformulation of the low-energy effective action of closed strings. To incorporate the symmetry between momentum and winding modes, the theory is formulated on a double internal space $2d$, with doubled internal coordinates $X^A = (x^a, \bar{x}_a)$, where x^a are the standard internal spacetime coordinates and $\bar{x}_a$ are dual coordinates conjugate to string winding. Fields are in principle allowed to depend on both sets of coordinates, but consistency with string theory imposes the level-matching condition, expressed as [1]

$$D^2 = \bar{D}^2, \tag{A1}$$

which translates into $k^2 = \bar{k}^2$ in momentum space.

The metric and the Kalb-Ramond field are combined into a gravitational background field

$$E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}, \tag{A2}$$

and fluctuations around this background are encoded in the generalized frame field

$$e_{\mu\nu} = h_{\mu\nu} + b_{\mu\nu}, \tag{A3}$$

where the symmetric part corresponds to the graviton and the antisymmetric part to the B field.

An affine connection can be constructed in terms of the background field $E_{\mu\nu}$ as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu E_{\lambda\nu} + \partial_\nu E_{\lambda\mu} - \partial_\lambda E_{\mu\nu}), \tag{A4}$$

which includes an antisymmetric component corresponding to H torsion. The associated spin connection can be defined via a vielbein postulate for the internal metric

$$\partial_\mu s_\nu{}^a - \Gamma^{\rho}_{\mu\nu} s_\rho{}^a + \omega_\mu{}^{ab} s_{\nu b} = 0, \tag{A5}$$

where $s_\mu{}^a$ is the internal vielbein satisfying $s_{\mu a}\eta^{ab} s_{\nu b} = g_{\mu\nu}$.

2. Manifest T-duality formulation

In the generalized metric formulation of DFT, T-duality is made manifest by introducing a dynamical generalized metric $\mathcal{H}_{MN}$, which transforms linearly under the group $O(D, D)$ and encodes both the spacetime metric and the B field. The scalar degree of freedom is captured by the

generalized dilaton Φ, which is defined via $e^{-2\Phi} = \sqrt{-g}e^{-2\varphi}$. This combination ensures that the dilaton transforms as a scalar under generalized diffeomorphisms, the generalization of the ordinary diffeomorphism and the B field transformations, and remains invariant under T-duality.

Closure of the generalized diffeomorphism algebra requires the so-called strong constraint

$$\partial_M \partial^M \star = 0, \qquad \partial_M \star \partial^M \star = 0, \tag{A6}$$

for all fields and gauge parameters. These conditions effectively reduce the doubled space to a physical subspace where all fields depend only on half of the coordinates (x or $\bar{x}$). This restriction guarantees compatibility with the low-energy limit of string theory.

A key feature of this formulation is the appearance of a generalized spin connection associated with the doubled frame, defined through the generalized vielbein postulate

$$\partial_M E_N{}^A - \Gamma_{MN}{}^P E_P{}^A + W_M{}^{AB} E_{NB} = 0. \tag{A7}$$

This generalized spin connection plays a role analogous to the usual spin connection in Riemannian geometry, but is adapted to the doubled structure. It enables the construction of covariant derivatives, generalized curvature tensors, and torsionlike objects.

In our framework, the presence of matter fields in a representation of the gauge group leads, via the double copy, to a richer geometric structure on the gravity side. The resulting field content and interactions naturally produce components consistent with the generalized spin connection of DFT. Although our construction is performed in linearized order, this connection serves as a crucial bridge between the gauge-theoretic seed and the higher-derivative gravitational theory.

Although a geometric structure with connections and curvatures is possible in DFT, it is well known that the generalized Riemann tensor is not fully determined. This is due to the fact that the generalized spin connection cannot be completely expressed in terms of the fundamental degrees of freedom. As a result, the direct construction of higher-order terms such as $\mathcal{R}_{MNPQ}\mathcal{R}^{MNPQ}$ is a challenging task. More comprehensive discussions can be found in standard reviews [32–35].

Nevertheless, higher-derivative extensions of DFT do exist. A particularly interesting construction was proposed in Ref. [37], based on a generalization of the Green-Schwarz mechanism for anomaly cancellation. In this approach, four-derivative terms involving the generalized fluxes are included. The resulting theory depends on two parameters whose values correspond to different theories, namely the bosonic and heterotic strings [38,39], as well as the HSZ theory [40,41].

APPENDIX B: QUADRATIC WEYL GRAVITY

The Weyl gravity action is given by

$$S_{\mathrm{CG}} = \int d^D x \sqrt{-g} C_{\mu\nu\rho\lambda} C^{\mu\nu\rho\lambda}, \tag{B1}$$

where the Weyl tensor is defined as

$$C_{\mu\nu\rho\lambda} = R_{\mu\nu\rho\lambda} - \frac{2}{D-2}(g_{\mu[\rho}R_{\lambda]\nu} - g_{\nu[\rho}R_{\lambda]\mu}) + \frac{2}{(D-1)(D-2)} R g_{\mu[\rho}g_{\lambda]\nu}. \tag{B2}$$

We will consider perturbations around the flat space such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\rho}h_\rho{}^\nu + \cdots,$$
$$\sqrt{-g} = 1 + \frac{1}{2}h + \cdots, \qquad h = h_\mu{}^\mu.$$

The quadratic contributions that come from this setup are

$$S^{(2)}_{\mathrm{CG}} = \frac{D-3}{D-2}\int d^D x \Big[(\Box h^{\mu\nu}\Box h_{\mu\nu} - 2\Box h^{\mu\nu}\partial_\mu\partial^\rho h_{\rho\nu} + \partial^\mu\partial^\nu h_{\mu\nu}\partial^\rho\partial^\lambda h_{\rho\lambda}) - \frac{1}{D-1}(\Box h - \partial_\mu\partial_\nu h^{\mu\nu})^2 \Big]. \tag{B3}$$

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