

Strong spin-gravity coupling in the linear approximation for circular orbits in Schwarzschild's field

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In the context of current research of the properties of gravitational interaction for bodies and particles moving at very high velocities, we consider possible motions of ultrarelativistic spinning particles relative to Schwarzschild's mass. The focus is on nonequatorial circular orbits of such particles. The corresponding solutions of the Mathisson-Papapetrou in the linear spin approximation are investigated. Direct analytical calculations determined necessary and sufficient conditions under which such solutions are admissible. The spin-gravity force which from the point of view of the comoving observer ensures existence of the ultrarelativistic nonequatorial circular orbits in Schwarzschild's field is analyzed. The corresponding illustrations are presented.

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I. INTRODUCTION

In general relativity, important properties of gravity were studied after analysis of solutions of the geodesic equations in different metrics [1,2]. These equations describe the world lines and trajectories of simple test particles (that is, small bodies and particles without taking into account their internal structure) in gravitational field of a massive body. The first step toward considering the effects of the internal structure of test particles on their behavior in a gravitational field was taken by M. Mathisson [3]. He derived the equations for structural particles in the pole-dipole approximation as a generalization of the geodesic equations. These equations opened up the possibility of studying the motions of rotating (spinning) particles in arbitrary metrics, and, like the geodesic equations, for particle velocities up to and including arbitrarily close to the speed of light. Later, similar equations were obtained by A. Papapetrou [4]. However, at the end of this paper one can read an incorrect statement that Mathisson obtained his equations without taking into account the spacetime curvature, that is, only for the trivial case of the absence of a gravitational field. Probably due to the attention paid only to [4] without [3], when describing the equations for spinning particles in book [1], paper [3] is missing from the list of references, only [4] and the later article [5] are present. [Note that

Mathisson died in 1940 at the age 43; he was a famous physicist and mathematician [6].]

The first numerical estimate of the effect of a body's rotation on its trajectory was presented in [7]. It was shown that the correction to the perihelion shift of the planet Mercury is very small compared to the known classical effect that follows from the geodesic equations. In more general consideration, in [1] the two situations are pointed out when the spin-gravity effects can be significant: (1) when rapidly spinning neutron star or a black hole is in the gravitational field of another neutron star or black hole; (2) when an electron is in a strong gravitational field in the late stage of gravitational collapse. Almost simultaneously, in [8] some assumption was made that endowing the test body with intrinsic spin may lead to surprises in general relativity. However, the corresponding calculations and analysis were not presented in [8]. Nevertheless, this publication stimulated further research into solutions of equations for spinning particles. As a result, in [9] attention was drawn to a third situation in addition to the two indicated in [1]. These are cases when a spinning particle moves relative to Schwarzschild's source of the gravitational field with an orbital velocity close to the speed of light.

In the case of the radial motion of a spinning particle in Schwarzschild's field the effect of the spin-gravity coupling is equal to 0 for any particle's velocity and orientation of its spin. That is, this motion is exactly geodesic. The deviation from geodesic arises only when the velocity of a spinning particle has nonzero orbital (tangential) component. In this context the analysis of the spin-orbit interaction was presented in [10], in the cases when the particle's velocity is not highly relativistic. Later papers [11–13] were devoted

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to the study of the consequences of the spin-orbit interaction in the highly relativistic region in Schwarzschild's field, as well as in the Schwarzschild-de Sitter background [14,15]. It was shown that from the point of view of an observer comoving with a spinning particle, the physical cause of the significant deviation of the particle's motion from geodesics is the spin reaction to the gravitomagnetic components of the gravitational field. It is important that these components are proportional to the square of the relativistic Lorentz factor, calculated by the velocity of the particle relative to the source of the gravitational field. The definition of the gravitomagnetic field components in terms of the local components of the curvature tensor, introduced in [16], is used.

When studying the properties of possible motions of spinless particles, circular orbits are considered, among others [1,2]. In the Schwarzschild field, such orbits exist only for the radial coordinate r greater than $1.5r_g$, where r_g is the horizon radius. For all circular geodesic orbits, Schwarzschild's mass is in the center of the corresponding circles. Since the Schwarzschild metric is often used in the spherical space coordinates r , θ and φ , in which circular geodesic orbits lie in the plane with $\theta = \pi/2$, they are called equatorial. That is, in this sense, all geodesic circular orbits are equatorial.

It is important that the circular geodesic orbits with $r = 1.5r_g(1 + \delta)$, $0 < \delta \ll 1$ are ultrarelativistic, while for $\delta = 0$ there is so-called photonic circular orbit. The ultrarelativistic circular orbits of electrically charged particles are taken into account when estimating their synchrotron radiation.

It should be noted that the study of the properties of ultrarelativistic gravity was probably initiated in [17]. It was shown that the gravitational field of a fast-moving Schwarzschild's mass bears an increasing resemblance to a gravitational plane-wave field. Later the encounters between black holes were considered in the limit that the approach velocity tends to the speed of light [18,19]. In [20], the impulsive tidal acceleration of a beam of ultrarelativistic particles in the gravitational field were studied. In [21,22], the ultrarelativistic gravitational force is considered. In the development of these studies, it is important to analyze the gravitational ultrarelativistic force that acts on a spinning particle due to the spin-gravity coupling.

The purpose of this paper is to investigate possible nonequatorial ultrarelativistic circular orbits of spinning particles in Schwarzschild's field using the linear spin approximation. The conditions for the existence of such orbits are analyzed in detail.

The paper is organized as follows. A brief overview of the Mathisson-Papapetrou equations is provided in Sec. II. An important partial solution of these equations in Schwarzschild's metric, which describes the equatorial ultrarelativistic circular orbit with $r = 1.5r_g$ is described in Sec. III. Section IV is devoted to analytical calculations

to establish necessary and sufficient conditions under which the MP equations in the linear spin approximation have solutions describing nonequatorial circular orbits in the Schwarzschild field. Section V illustrates an example of such orbit compared to the motion of a spinless particle with the same initial coordinates and velocity. Estimates of the spin-gravity force, which, from the point of view of an observer comoving with the spinning particle, deviates its motion from the geodesic one, are given in Sec. VI. We conclude in Sec. VII.

II. MATHISSON-PAPAPETROU EQUATIONS: EXACT AND LINEAR IN SPIN

We write the equations for a rotating (spinning) test particle in general relativity according to their representation in Mathisson's paper [3]

$$\frac{D}{d\tau} \left(mu^\lambda + u_\mu \frac{DS^{\lambda\mu}}{d\tau} \right) = -\frac{1}{2} u^\pi S^{\rho\sigma} R_{\pi\rho\sigma}^\lambda, \quad (1)$$

$$\frac{DS^{\mu\nu}}{d\tau} + u^\mu u_\sigma \frac{DS^{\nu\sigma}}{d\tau} - u^\nu u_\sigma \frac{DS^{\mu\sigma}}{d\tau} = 0, \quad (2)$$

$$S^{\lambda\nu} u_\nu = 0 \quad (3)$$

where $u^\lambda \equiv dx^\lambda/d\tau$ is the particle's 4-velocity; $S^{\mu\nu}$ is the antisymmetric tensor of spin; m is the constant of the particle's mass and $D/d\tau$ is the covariant derivative along u^λ ; units $c = G = 1$ are chosen. For a spinless particle, when $S^{\mu\nu} = 0$, these equations reduce to the geodesic equations $Du^\lambda/d\tau = 0$.

To evaluate specific spin-gravity effects, instead of (1) and (2), their linear spin approximation

$$\frac{D}{d\tau} u^\lambda = -\frac{1}{2} u^\pi S^{\rho\sigma} R_{\pi\rho\sigma}^\lambda, \quad (4)$$

$$\frac{DS^{\mu\nu}}{d\tau} = 0 \quad (5)$$

is often used. In particular, the known effect of Schiff's precession [23] follows directly from Eq. (5).

Beginning from publications [24,25], Eq. (3) is often replaced by

$$S^{\lambda\nu} P_\nu = 0, \quad (6)$$

where

$$P^\nu = mu^\nu + u_\lambda \frac{DS^{\nu\lambda}}{d\tau} \quad (7)$$

is the particle 4-momentum. The possibility of using relations (3) or (6) is associated with different ways of describing the position of particle's mass [25,26]. Equations (1) and (2),

together with (3) or (7), are called the Mathisson-Papapetrou (MP) or Mathisson-Papapetrou-Dixon (MPD) equations.

If relation (3) is used, then in the linear spin approximation the momentum P^ν is parallel to the velocity u^ν and relations (3) and (7) coincide.

III. AN EQUATORIAL CIRCULAR ORBIT OF A SPINNING PARTICLE WITH $r=3M$ IN SCHWARZSCHILD'S FIELD

We use the Schwarzschild metric in the standard coordinates $x^1 = r, x^2 = \theta, x^3 = \varphi, x^4 = t$. Then the nonzero components of the metric tensor $g_{\mu\nu}$ are

$$\begin{aligned} g_{rr} &= -(1 - 2Mr^{-1})^{-1}, & g_{\theta\theta} &= -r^2, \\ g_{\varphi\varphi} &= -r^2 \sin^2 \theta, & g_{tt} &= 1 - 2Mr^{-1}, \end{aligned} \quad (8)$$

where M is Schwarzschild's mass.

The MP equations in Schwarzschild's metric have the exact partial solution which describes the circular orbit of a spinning particle in the equatorial plane of this metric with the value of the radial coordinate r equal to $3M$ (the standard Schwarzschild coordinates are used) [13,27]. There is the remarkable feature of this solution: it is the same both for the exact MP equations and their linear spin approximation. For realization of the circular orbit with $r = 3M$ the spinning particle must have the orbital 4-velocity $u_\perp \equiv rd\varphi/d\tau$ which is determined by

$$u_\perp = \pm 3^{1/4} \varepsilon^{-1/2}, \quad (9)$$

where

$$\varepsilon \equiv S/mM, \quad (10)$$

and S is the absolute value of the particle's spin, m is the particle's mass.

The known general necessary condition for the spinning test particle must be taken into account [10]

$$S/mr \ll 1. \quad (11)$$

Then in our case with $r = 3M$ we write $\varepsilon \ll 1$. According to (9) and (10) we have $u_\perp^2 \gg 1$. From these relations follow the estimation of the relativistic Lorentz γ -factor for the spinning particle's velocity in the circular orbit with $r = 3M$ by an observer which is at rest in the position $r = 3M$. We use the known general expression for the γ -factor

$$\gamma = (1 - v^2)^{-1/2}, \quad (12)$$

where v^2 is the second power of the particle's 3-velocity relative to the observer. In the case of the diagonal metric, according to the expression for the 3-velocity component v^i

we have [28]

$$v^i = \frac{dx^i}{\sqrt{g_{44}} d\tau} \quad (13)$$

(here and below Latin indices run 1, 2, 3 in contrast to Greek indices that run 1, 2, 3, 4) Then for v^2 we write

$$v^2 = v_i v^i = \gamma_{ik} v^i v^k, \quad (14)$$

where γ_{ik} is the 3-space metric tensor, with the relation between γ_{ik} and $g_{\mu\nu}$: $\gamma_{ik} = -g_{ik}$. It follows from (12) and (14) with $u_\mu u^\mu = 1$ that

$$\gamma = \sqrt{u_i u^i}. \quad (15)$$

In Schwarzschild's metric (8) for the circular orbit we have $u_t u^t = 1 + u_\perp^2$. Then according to (9), (11) and (15) we write $\gamma^2 \gg 1$. That is, for an observer which is at rest in the position $r = 3M$ the necessary particle's velocity for its motion in the circular orbit with $r = 3M$ is ultrarelativistic.

It is appropriate to compare the motion of the spinning particle in the circular orbit for $3M$ with the geodesic motion of the spinless particle which starts in the tangential direction from the same position having the same orbital velocity. It is known from the geodesic equations that in this case the spinless particle falls to the horizon surface within a finite proper time. While the spinning particle remains permanently in the circular orbit. It means that in this case the spin-gravity coupling acts as a strong repulsive force which compensates the usual ("geodesic") attraction. Note that the absolute value of the acceleration $|a|$ in the circular orbit with $r = 3M$ is equal to $3^{-3/2} M^{-1}$. It is useful to compare the numerical value of this acceleration with the classical acceleration in the Newtonian theory of gravitation $a_{\text{class}} = M/r^2$. Then for $r = 3M$ we write $|a| = \sqrt{3} a_{\text{class}}$.

IV. NONEQUATORIAL CIRCULAR ORBITS OF A SPINNING PARTICLE IN SCHWARZSCHILD'S FIELD

Let us check whether the MP equations in the linear spin approximation have solutions that describe nonequatorial circular orbits of the spinning particles in Schwarzschild's field with $r = \text{const}$. After direct analytical calculation we obtain that these equations have solutions with constant nonzero values of the 3-vector spin components S_r and S_θ , with the relation between them

$$S_r \left(1 - \frac{3M}{r} \right) + S_\theta \frac{\cos \theta}{r \sin \theta} = 0. \quad (16)$$

The nonzero constant components of the particle 4-velocity u^φ and u^t are determined by the equations

$$m(\Gamma_{\varphi\varphi}^r u^\varphi u^\varphi + \Gamma_{tt}^r u^t u^t) = -\frac{3M}{r^3} u^\varphi S_\theta \sin \theta, \quad (17)$$

$$m\Gamma_{\varphi\varphi}^\theta u^\varphi u^\varphi = -\frac{3M}{r^3} u^\varphi S_r \sin \theta, \quad (18)$$

where $\Gamma_{\varphi\varphi}^r$, Γ_{tt}^r , and $\Gamma_{\varphi\varphi}^\theta$ are the corresponding Christoffel symbols. For metric (8) there are

$$\Gamma_{\varphi\varphi}^r = (-r + 2M) \sin^2 \theta, \quad (19)$$

$$\Gamma_{tt}^r = Mr^{-2}(1 - 2Mr^{-1}), \quad \Gamma_{\varphi\varphi}^\theta = -\sin \theta \cos \theta. \quad (20)$$

It follows from (18)

$$u^\varphi = \frac{3MS_r}{mr^3 \cos \theta}. \quad (21)$$

For the component u^t we have the expression

$$(u^t)^2 = \frac{9MS_r^2}{m^2 r^3} (2 - 5Mr^{-1})(1 - 2Mr^{-1})^{-1} \tan^2 \theta. \quad (22)$$

It follows from the general relation for the 4-velocity $u_\mu u^\mu = 1$ that by (21) and (22)

$$\frac{18M}{r} \frac{S_r^2}{m^2 r^2} (1 - 3Mr^{-1}) \tan^2 \theta = 1. \quad (23)$$

Now we take into account the general expression for the second power value of the particle's spin

$$S^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}. \quad (24)$$

According to the MP equations S is a constant of motion for any metric and for all possible particle's trajectories.

There are the general relations between the tensor of the particle's spin and its 3-vector

$$S^{ik} = |g|^{-1/2} \varepsilon^{ikl} S_l, \quad S^{it} = \frac{u_k}{u_t} S^{ki}, \quad (25)$$

where $|g|$ is the absolute value of the determinant of the metric tensor and ε^{ikl} is the Levi-Civita symbol.

In the case of circular orbits in Schwarzschild's field with $u^r = 0$ and $u^\theta = 0$ it follows from (24), (25), and (8) the nonzero values for $S^{\mu\nu}$

$$S^{r\varphi} = |g|^{-1/2} S_\theta, \quad S^{\theta\varphi} = |g|^{-1/2} S_r, \\ S^{rt} = \frac{u_3}{u_4} |g|^{-1/2} S_\theta, \quad S^{\theta t} = -\frac{u_3}{u_4} |g|^{-1/2} S_r. \quad (26)$$

Using (26) in (24) we obtain

$$S^2 = (u_t u^t)^{-1} [S_r^2 + r^{-2} (1 - 2mr^{-1})^{-1} S_\theta^2]. \quad (27)$$

Substituting the expression for S_θ^2 through S_r^2 which follows from (16) according to (27) we write

$$S^2 = S_r^2 [(1 - 3Mr^{-1})^2 \sin^2 \theta + (1 - 2Mr^{-1}) \cos^2 \theta] \\ \times (1 - 2Mr^{-1})^{-2} (u^t)^{-2} \cos^{-2} \theta. \quad (28)$$

[It follows from (16) that for $S_r = 0$ it is necessary $\cos \theta = 0$.] From (22) and (28) we obtain

$$\frac{S^2}{m^2 r^2} = \frac{r}{9M} [(1 - 3Mr^{-1})^2 \sin^2 \theta + (1 - 2Mr^{-1}) \cos^2 \theta] \\ \times (1 - 2Mr^{-1})^{-1} (2 - 5Mr^{-1})^{-1} \sin^{-2} \theta. \quad (29)$$

According to the last expression, the condition for the spinning test particle (11) is satisfied if and only if the constant value $1 - 3Mr^{-1}$ is much less than 1 (by (23) it is necessary $r > 3M$) and the value $|\cos \theta|$ is much less than 1 as well. In the following we use the notation

$$\delta_1 \equiv 1 - 3Mr^{-1}, \quad \delta_2 \equiv |\cos \theta|, \quad (30)$$

with the conditions

$$0 < \delta_1 \ll 1, \quad 0 < \delta_2 \ll 1. \quad (31)$$

After (30) and (31) we write

$$r = 3M[1 + \delta_1 + O(\delta_1^2)], \quad \sin^2 \theta = 1 - \delta_2^2. \quad (32)$$

Substituting (32) into (29) we have

$$\frac{S^2}{m^2 r^2} = 3\delta_1^2 + \delta_2^2, \quad (33)$$

where the right-hand side is written in the main approximation, i.e., terms that are significantly smaller due to conditions (31) are omitted.

Using notation (31) we write the expressions for the components of the particle's 4-velocity

$$(u^\varphi)^2 = \frac{1}{54M^2 \delta_1}, \quad (u^t)^2 = \frac{1}{2\delta_1}. \quad (34)$$

From these expressions the relation follows:

$$(u^t)^2 \gg 1, \quad (35)$$

that is, the constant velocity of the spinning particle in the nonequatorial circular orbits is ultrarelativistic. In this sense, here the situation is similar to that for the circular equatorial orbit with $r = 3M$ described above in Sec. III.

Therefore, Eqs. (29)–(34) determine the conditions for existence of the nonequatorial circular orbits.

A. On the limiting transition

It is easy to verify that at the limiting transitions $\delta_1 \rightarrow 0$, $\delta_2 \rightarrow 0$ and $S_r \rightarrow 0$ the nonequatorial circular orbits considered above tend to the equatorial circular orbit with $r = 3M$, which is presented in Sec. III. To do this, instead of the expression (21) for u^φ , where at $S_r \rightarrow 0$ and $\delta_2 \rightarrow 0$ the right-hand side tends to the mathematical uncertainty $0/0$, it is necessary to evaluate the expression for u^φ , which follows from equation (17). Substituting the expressions from (19), (20) into Eq. (17) and using the relation $u_\mu u^\mu = 1$, we obtain the quadratic equation for $u_\perp = ru^\varphi$

$$u_\perp^2 (1 - 3Mr^{-1}) \sin^2 \theta - 3u_\perp MS_\theta r^{-3} m^{-1} \sin \theta - Mr^{-1} = 0. \quad (36)$$

There are two roots of this equation

$$u_\perp = 3MS_\theta r^{-3} m^{-1} \pm [9M^2 S_\theta^2 r^{-6} m^{-2} + 4Mr^{-1} \times (1 - 3Mr^{-1}) \sin^2 \theta]^{1/2} [2(1 - 3Mr^{-1}) \sin^2 \theta]^{-1}. \quad (37)$$

In the limiting case $r \rightarrow 3M$ (i.e., $\delta_1 \rightarrow 0$), Eq. (37) with the sign “−” yields a single finite root

$$u_\perp = -\frac{mr^2}{3S_\theta}. \quad (38)$$

[The second root with the sign “+” means that $|u_\perp|$ tends to infinity when $r \rightarrow 3M$.]

To compare (38) and (9), we consider expression (27). From (27) at $S_r \rightarrow 0$ we have the relation between the spin component S_θ and the absolute value of the particle’s spin

$$S_\theta^2 m^{-2} r^{-4} = (1 + u_\perp^2)(1 - 2Mr^{-1}) S^2 m^{-2} r^{-2}. \quad (39)$$

At the limiting transition $r \rightarrow 3M$, using (38) and notation (10), from (39) we obtain the quadratic equation for $u_\perp$

$$u_\perp^4 + u_\perp^2 - 3\varepsilon^{-2} = 0. \quad (40)$$

The positive root of this equation in the main approximation is $u_\perp^2 = \varepsilon^{-1} \sqrt{3}$, then

$$|u_\perp| = 3^{1/4} \varepsilon^{-1/2}. \quad (41)$$

This expression coincides with (9) (due to (38) the sign of $u_\perp$ depends on the sign of S_θ). Therefore, there is a correct limiting transition from nonequatorial circular orbits to an equatorial circular orbit with $r = 3M$ when $r \rightarrow 3M$ and $S_r \rightarrow 0$.

Together with Eq. (17), the properties of the nonequatorial circular orbits of a spinning particle in Schwarzschild’s field determine Eq. (18). Note that Eq. (18) is satisfied at the limiting transitions $S_r \rightarrow 0$ and $\delta_2 \rightarrow 0$, i.e., at the transition from nonequatorial circular orbits to an equatorial circular orbit with $r = 3M$.

V. ILLUSTRATIONS

It follows from Eqs. (17) and (18) that the ultrarelativistic nonequatorial circular orbits are possible both above and below the plane $\theta = \pi/2$ in Schwarzschild’s metric, depending on the spin orientation and the direction of the particle’s orbital velocity. For example, the corresponding orbits exist when $\theta < \pi/2$, $u_\perp > 0$, $S_r > 0$ and $S_\theta < 0$.

We present some illustrations for the typical case with the numerical values from (30) $\delta_1 = 10^{-2}$ and $\delta_2 = 5 \times 10^{-2}$. Then by (30) the angle θ is equal approximately to 87° , that is 3° above the equatorial plane. Figure 1 shows the difference in the dependence of the coordinate r on proper time for the spinning particle and the spinless particle which begin motion with the same position having the same initial velocity. The spinning particle is held at the constant value r , whereas the spinless particle falls to the horizon surface for the short time. According to Figs. 2 and 3, the graphs of the dependence of the angles θ and φ on proper time for the spinless particle end at the time when this particle reaches the horizon surface. Figure 4 shows the trajectory of the spinless particle falling to the horizon, while the trajectory of the spinning particle remains circular.

Further description of the example of a nonequatorial ultrarelativistic circular orbit with $\delta_1 = 10^{-2}$ and $\delta_2 = 5 \times 10^{-2}$ is given in the next section.

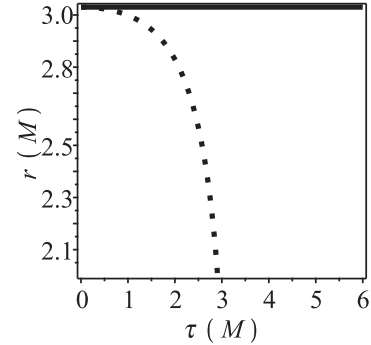


FIG. 1. Radial coordinate vs proper time for the spinning (solid line) and spinless (dotted line) particles.

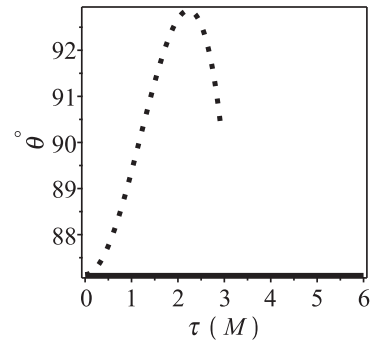


FIG. 2. Angle θ vs proper time for the spinning (solid line) and spinless (dotted line) particles.

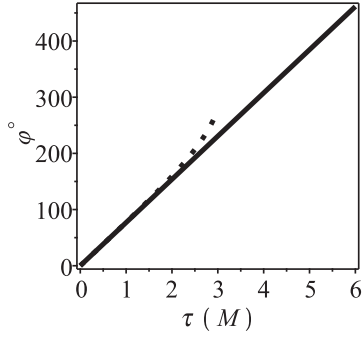


FIG. 3. Angle φ vs proper time for the spinning (solid line) and spinless (dotted line) particles.

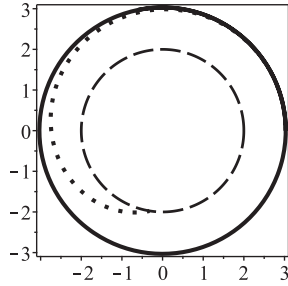


FIG. 4. Trajectories of the spinning (solid line) and spinless (dotted line) particles. The dashed circle corresponds to the horizon surface.

VI. THE FORCE AND ENERGY OF THE SPIN-GRAVITY COUPLING FOR A PARTICLE IN THE NONEQUATORIAL CIRCULAR ORBITS

In addition to the MP equations in the linear spin approximation (4) and (5), we take into account their consequence in terms of the local tetrads values $\lambda_{(\nu)}^\mu$ which describe the frame of reference comoving with the spinning particle, when $\lambda_{(4)}^\mu = u^\mu$ [13]. [Here, and in the following, the local indices are placed in the parenthesis.] We orient the first space axis (1) of the local basis along the particle's spin, and the second space axis (2) is oriented along the direction of its orbital velocity relative to the Schwarzschild's mass. In further calculations we use the general relation between the metric tensor $g^{\pi\rho}$ and $\lambda_{(\nu)}^\mu$

$$\lambda_{(\mu)}^\pi \lambda_{(\nu)}^\rho \eta^{(\mu)(\nu)} = g^{\pi\rho}, \quad (42)$$

where $\eta^{(\mu)(\nu)} = \text{diag}(-1, -1, -1, 1)$ is the Minkowski tensor.

It follows from Eqs. (4) that [13]

$$ma_{(1)} + SR_{(1)(4)(2)(3)} = 0, \quad (43)$$

$$ma_{(2)} + SR_{(2)(4)(2)(3)} = 0, \quad (44)$$

$$ma_{(3)} + SR_{(3)(4)(2)(3)} = 0, \quad (45)$$

where $a_{(i)}$ is the acceleration of a spinning particle relative to the free motion of a spinless particle, and $R_{(i)(4)(2)(3)}$ are the corresponding local components of the Riemann tensor. We use general relations (43)–(45) to estimate the quantities $a_{(i)}$ in our specific case of the motion of a spinning particle in the ultrarelativistic nonequatorial circular orbits in Schwarzschild's field. It is not difficult to verify that according to (8) and (42) the nonzero tetrad components can be presented as

$$\begin{aligned} \lambda_{(1)}^r &= -g^{rr}(-g_{\theta\theta})^{1/2} S_r S_\theta^{-1} (1 + g^{rr} g_{\theta\theta} S_r^2 S_\theta^{-2})^{-1/2}, \\ \lambda_{(1)}^\theta &= -g^{\theta\theta}(-g_{\theta\theta})^{1/2} (1 + g^{rr} g_{\theta\theta} S_r^2 S_\theta^{-2})^{-1/2}, \\ \lambda_{(2)}^\varphi &= u^\varphi u^t \sqrt{\frac{g_{tt}}{-u_\varphi u^\varphi}}, \quad \lambda_{(2)}^t = \sqrt{\frac{-u_\varphi u^\varphi}{g_{tt}}}, \\ \lambda_{(3)}^r &= u^\varphi \sqrt{\frac{g^{rr} g_{\varphi\varphi}}{-u_\varphi u^\varphi}} (1 + g^{rr} g_{\theta\theta} S_r^2 S_\theta^{-2})^{-1/2}, \\ \lambda_{(3)}^\theta &= -S_r S_\theta^{-1} \lambda_{(3)}^r \lambda_{(4)}^\theta = u^\varphi, \quad \lambda_{(4)}^t = u^t. \end{aligned} \quad (46)$$

We take into account the expressions for the components of Riemann's tensor $R_{\alpha\beta\gamma\delta}$ for metric (8)

$$\begin{aligned} R_{r\theta r\theta} &= M(r-2m)^{-1}, \quad R_{r\varphi r\varphi} = M(r-2m)^{-1} \sin^2 \theta, \\ R_{\theta\varphi\theta\varphi} &= -2Mr \sin^2 \theta, \quad R_{rt rt} = 2Mr^{-3}, \\ R_{\theta t \theta t} &= -Mr^{-3}(r-2M), \\ R_{\varphi t \varphi t} &= -Mr^{-3}(r-2M) \sin^2 \theta. \end{aligned} \quad (47)$$

Using the general relation between the local and global components of this tensor

$$R_{(\pi)(\rho)(\sigma)(\tau)} = \lambda_{(\pi)}^\alpha \lambda_{(\rho)}^\beta \lambda_{(\sigma)}^\gamma \lambda_{(\tau)}^\delta R_{\alpha\beta\gamma\delta}, \quad (48)$$

by (47) we obtain

$$\begin{aligned} R_{(1)(4)(2)(3)} &= -Mr^{-2} u^\varphi u^t \cos \theta, \quad R_{(2)(4)(2)(3)} = 0, \\ R_{(3)(4)(2)(3)} &= Mr^{-2} u^\varphi u^t (4 - 9Mr^{-1}) \\ &\quad \times (1 - 2Mr^{-1})^{-1/2} \sin \theta. \end{aligned} \quad (49)$$

Then according to (43)–(45) and (46)–(49) we obtain the expressions for the components of the spin-gravity force $F^{(i)} \equiv ma^{(i)}$ which from the point of view of a comoving observer acts on the spinning particle

$$F^{(1)} = \frac{m\delta_2}{M\delta_1 3\sqrt{3}} (3 + \delta_1^{-2} \delta_2^2)^{-1/2}, \quad F^{(2)} = 0, \quad (50)$$

$$F^{(3)} = -\frac{m}{18M} (3 + \delta_1^{-2} \delta_2^2)^{-1/2} (3 - \delta_1^{-2} \delta_2^2). \quad (51)$$

By Eq. (50) the component of the spin-gravity force $F^{(2)}$ in the direction of the angle φ is equal to 0. Before describing

the orientation of the other components of this force, let us consider the inclination of the particle's spin to the radial coordinate line.

A. The dependence of spin inclination on the parameters δ_1 and δ_2

Let us estimate the angle between the direction of the particle's spin and the radial line for different values of the parameters δ_1 and δ_2 . By definition, for this angle α we write

$$\cos \alpha = \frac{\vec{e}^{(1)} \vec{e}^r}{|\vec{e}^{(1)}| |\vec{e}^r|} = \frac{\lambda^{r(1)}}{\sqrt{-g^{rr}}}, \quad (52)$$

where $\vec{e}^{(1)}$ and $\vec{e}^r$ are the vectors of the local and global basis, respectively. According to (46) we have

$$\lambda^{r(1)} = -g^{rr} r S_r S_\theta^{-1} (1 - g^{rr} r^2 S_r^2 S_\theta^{-2})^{-1/2}. \quad (53)$$

Taking into account (16) with notation (30) and condition (31), we use in expression (53) the relation

$$\frac{S_r}{S_\theta} = -\frac{\delta_2}{r \delta_1}. \quad (54)$$

Then it follows from (53) and (54)

$$\lambda^{r(1)} = g^{rr} \delta_2 \delta_1^{-1} (1 - g^{rr} \delta_2^2 \delta_1^{-2})^{-1/2}. \quad (55)$$

As a result, by (52) we obtain

$$\cos \alpha = \sqrt{-g^{rr}} \delta_2 \delta_1^{-1} (1 - g^{rr} \delta_2^2 \delta_1^{-2})^{-1/2}. \quad (56)$$

From (56) for $\delta_2 \rightarrow 0$ with the fixed δ_1 it follows $\alpha \rightarrow \pi/2$, and this is the case of the equatorial motions with $S_r = 0$. Then according to (28), in order for the value S remains the fixed constant, it is necessary $S_r \rightarrow 0$. It means that the particle's spin is oriented close to the orthogonal direction to the equatorial plane. At another limiting transition when $\delta_1 \rightarrow 0$ with the fixed δ_2 from (56) we have $\alpha \rightarrow 0$ and then by (54) it means that $S_\theta/S_r \rightarrow 0$, i.e., the orientation of the spin tends to the radial direction. Note that by (15) and (35) when $\delta_1 \rightarrow 0$ the necessary particles velocity tends to the speed of light. For all others ratio δ_2 to δ_1 the values of the angle α are between 0 and $\pi/2$. For example, at the above considered case with $\delta_1 = 10^{-2}$ and $\delta_2 = 5 \times 10^{-2}$ from (56) we have

$$\cos \alpha = \frac{5}{\sqrt{28}} \quad (57)$$

and $\alpha \approx 19^\circ$ (see Fig. 5).

Now we estimate the dependence of the components $F^{(1)}$ and $F^{(3)}$ from (50) and (51) on the parameters δ_1 and δ_2 . By (50) for any fixed small δ_2 with δ_1 tends to zero we have

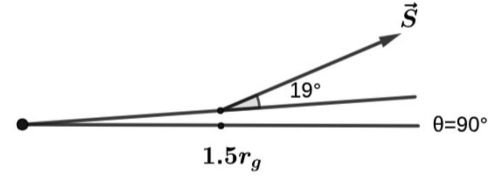


FIG. 5. Schematic illustration of the orientation of the particle's spin for the example when $\delta_1 = 10^{-2}$ and $\delta_2 = 5 \times 10^{-2}$.

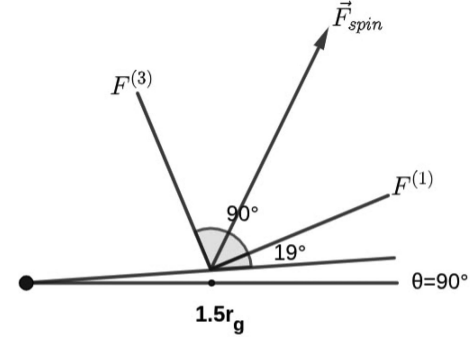


FIG. 6. Schematic illustration of the orientation of the spin-gravity force for the example when $\delta_1 = 10^{-2}$ and $\delta_2 = 5 \times 10^{-2}$.

$$F^{(1)} \rightarrow \frac{m}{M3\sqrt{3}}. \quad (58)$$

At another boundary transition, when δ_1 is fixed and δ_2 tends to zero, it follows from (50) $F^{(1)} \rightarrow 0$. Similarly, according to (51), for any fixed small value δ_2 with δ_1 tends to zero we write

$$F^{(3)} \rightarrow \frac{m \delta_2}{18 M \delta_1}. \quad (59)$$

While for fixed δ_1 , and δ_2 tends to zero, it follows from (51)

$$F^{(3)} \rightarrow -\frac{m}{6 M \sqrt{3}}. \quad (60)$$

Let us estimate the components $F^{(1)}$ and $F^{(3)}$ from (50) and (51) for the example $\delta_1 = 10^{-2}$ and $\delta_2 = 5 \times 10^{-2}$ (see Fig. 6). Then we have

$$F^{(1)} = \frac{5m}{3M\sqrt{84}}, \quad F^{(3)} = \frac{11m}{9M\sqrt{28}}. \quad (61)$$

To estimate how large the values of these components are, we will write down their ratio to the value of the Newtonian force F_{class} , when in the known classical formula we put $r = 3M$, i.e.

$$F_{\text{class}} = \frac{m}{9M}. \quad (62)$$

Taking into account (61) and (62) we write

$$\frac{F^{(1)}}{F_{\text{class}}} = \sqrt{\frac{75}{28}}, \quad \frac{F^{(3)}}{F_{\text{class}}} = \frac{11}{\sqrt{28}}. \quad (63)$$

Then for the absolute value of the spin-gravity force

$$F_{\text{spin}} = \sqrt{(F^{(1)})^2 + (F^{(3)})^2} \quad (64)$$

from (63) we have

$$\frac{F_{\text{spin}}}{F_{\text{class}}} = \sqrt{7}. \quad (65)$$

That is, the value of the spin-gravity force is close in order to the classical Newtonian force calculated for a mass M at the distance $r = 1.5r_g = 3M$. So, in this sense, the spin-gravity coupling is strong.

Note that the presented analysis of the dependence of the components $F^{(1)}$ and $F^{(3)}$ on δ_1 and δ_2 provides a physical explanation of the limiting transition at $\delta_1 \rightarrow 0$ from the nonequatorial circular orbits of a spinning particle to the equatorial circular orbit of a spinless particle with $r \rightarrow 3M$. From the geodesic equations in Schwarzschild's metric, the expression for the orbital velocity $u_{\perp} = ru^{\varphi}$ follows with r close to $3M$ ($r > 3M$)

$$u_{\perp}^2 = \frac{1}{3\delta_1}, \quad \delta_1 \rightarrow 0. \quad (66)$$

From the MP equations according to (34) we have

$$u_{\perp}^2 = \frac{1}{6\delta_1}. \quad (67)$$

That is, the right-hand sides in Eqs. (66) and (67) do not coincide. This difference is due to an obvious physical reason. Indeed, as noted above, when δ_1 is any small fixed nonzero value and $\delta_2 \rightarrow 0$ (the last means that the spinning particle passes into equatorial plane), then $F^{(1)} \rightarrow 0$. At the same time, at the same limiting transition according to Eq. (60), the component $F^{(3)}$ remains finite. That is, at this limiting transition the spin-gravity force does not disappear. Whereas a spinless particle is under the action of only ordinary ("geodesic") gravitational attraction.

B. The energy of the spinning particle in the ultrarelativistic nonequatorial circular orbits

There is the energy E as a constant of motion for the MP equations in the linear spin approximation for Schwarzschild's metric [29,30]

$$E = mu_t + \frac{1}{2}g_{tt,r}S^{rt}. \quad (68)$$

Using (8) and (16) we calculate

$$\frac{1}{2}g_{tt,r}S^{rt} = \frac{Mu_{\varphi}S_{\theta}}{r^4u_t \sin \theta}. \quad (69)$$

Then for the above considered nonequatorial ultrarelativistic circular orbits with (34) we obtain

$$E = E_{\text{geod}}(1 + 3^{-1}\delta_1), \quad (70)$$

where $E_{\text{geod}} = mu_t$ is the energy for the geodesic motion. That is, because by (31) $0 < \delta_1 \ll 1$, the correction of the spin to the particle's energy is small.

VII. CONCLUSIONS

Direct analytical calculations, presented in Secs. IV and VI, reveal the properties of the spin-gravity coupling in the space region near the photonic circular orbit with $r = 3M = 1.5r_g$ in Schwarzschild's field. The corresponding solutions of the MP equations in the linear spin approximation (4) and (5), which describe the ultrarelativistic nonequatorial circular orbits of a spinning particle, are considered. In this approximation, two known supplementary conditions for the MP equations (3) and (6) coincide.

In Sec. IV, the necessary and sufficient conditions for the existence of the nonequatorial circular orbits in Schwarzschild's field are studied. One of these conditions requires that the velocity of the spinning particle relative the field source must be ultrarelativistic, i.e., close to the speed of light. Another condition determines the inclination of the spin to the equatorial plane and to the radial coordinate line.

The spin-gravity force as estimated by the observer co-moving with the spinning particle is investigated in Sec. VI. The dependence of this force on the spin orientation for the particle in the ultrarelativistic nonequatorial circular orbit is analyzed for a concrete example. The corresponding illustrations are presented in Secs. V and VI.

The space region, in which the physical effects considered in the article occur, is limited in the radial coordinate by the condition $1.5r_g < r < 1.5r_g(1 + \delta)$, where $0 < \delta \ll 1$. Nevertheless, in the real case of a massive black hole the value $1.5r_g\delta$ is not small. For example, in the case of a black hole with the mass of 10^6 the Sun's mass, when r_g in the relevant system of units is 3×10^6 kilometers, at $\delta = 10^{-2}$ the value $1.5r_g\delta$ is equal to 4.5×10^4 km.

The obtained results on ultrarelativistic nonequatorial circular orbits of spinning particles can be useful for studying physical processes in thin accretion disks and for estimating synchrotron radiation of charged particles [31].

It is known that there is a correspondence between the general relativistic Dirac equation [32,33] and the MP equations in the linear spin approximation. In this context, the properties of the ultrarelativistic spin-gravity coupling at the classical level are useful for studying the gravity interaction of quantum particles at very high energies [34].

DATA AVAILABILITY

The data that support the findings of this article are openly available [34].

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