

Two-photon transition form factor of $\eta_{c,b}(nS)$ and $\chi_{c0,b0}(nP)$ via relativized mock meson states

Tian-Cheng Ding, Jian Huang, and Muyang Chen^{*}

Department of Physics, *Hunan Normal University*, and Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control of Ministry of Education, Changsha 410081, China



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We construct relativized mock meson states for heavy quarkonium, where the Dirac spinors import kinetic relativistic correction and the dynamic wave functions are the same as those solved from the Schrödinger equation. We find that the relativistic correction imported by Dirac spinors is crucial to study the two-photon transition form factors. Using relativized mock meson states, we give credible predictions for the two-photon transition form factors and decay widths of $\eta_{c,b}(n^1S_0)$ and $\chi_{c,b}(n^3P_0)$ ($n = 1, 2, 3$).

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I. INTRODUCTION

The two-photon transition form factor of the heavy quarkonium system provides a powerful probe into the electromagnetic structure and internal dynamics of these bound states. These transitions, where a quarkonium state decays into two photons, are governed by quantum chromodynamics (QCD) and offer a unique opportunity to study strong interactions. The form factor describing the momentum dependence of the transition amplitude is sensitive to the spatial distribution of the quark-antiquark pair and relativistic corrections. Charmonium and bottomonium systems are particularly well suited for such studies due to their relatively simple structure compared to light quark mesons.

Experimentally, two-photon transitions of charmonium have been extensively studied in CLEO [1], BABAR [2], BESIII [3], and Belle [4]. These experiments have provided high precision measurements of decay widths $\Gamma_{\gamma\gamma}$ and the Q^2 dependence of the form factors. For example, the two-photon decay width of the η_c is measured to be 5.1 ± 0.4 keV, and the form factor is measured up to $Q^2 \approx 50$ GeV [2], while for η_b , two-photon decay is not seen due to significantly smaller decay width. A higher-precision study is expected in the future electron-ion colliders; see Ref. [5].

Theoretically, the two-photon transition form factor has been investigated using a variety of approaches, such

as lattice QCD [6–8], the Dyson-Schwinger equation approach [9], the sum rule [10], nonrelativistic QCD [11–13], perturbative QCD [14,15], the light front approach [16–18], and so on. In potential quark models, the quarkonium wave function is derived by solving the Schrödinger equation with a confining potential; however, the role of relativistic corrections and the precise shape of the quarkonium distribution amplitude are still areas of active research.

In this paper, we focus on the effect of the relativized mock meson state on the two-photon transition form factor of $\eta_{c,b}(nS)$ and $\chi_{c0,b0}(nP)$. This paper is organized as follows. In Sec. II we introduce the model used in this work. In Sec. III we present and comment on our results for the charmonium spectrum, wave functions, and the two-photon transition form factors. Section IV is parallel to Sec. III, but for the bottomonium. A summary and conclusion are given in Sec. V. There are also two Appendixes. Appendix A formulates the derivation of the nonrelativistic limit of the relativized mock meson state, and B formulates the expressions of the two-photon transition form factors of the mesons.

II. QUARK MODEL AND MOCK MESON STATE

In this paper we adopt almost the same quark model as Refs. [19,20] with a few modifications. In the following we first give a brief description of the Hamiltonian we used; then we will focus on the two kinds of representations of the meson and the expressions of the two-photon transition form factor of pseudoscalar and scalar mesons.

The masses and wave functions are obtained by solving the radial Schrödinger equation,

$$(T + V - E)\varphi(r) = 0, \quad (1)$$

^{*}Contact author: muyang@hunnu.edu.cn

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where T is the kinetic energy operator, V is the potential between the quarks, E is the energy of this system, $\varphi(r)$ is the radial wave function, and r is the distance between the quark and antiquark. The relativistic kinetic energy is

$$T = \sqrt{m^2 + \mathbf{k}^2} + \sqrt{\bar{m}^2 + \bar{\mathbf{k}}^2}, \quad (2a)$$

where m and $\bar{m}$ are the masses of the quark and antiquark, $\mathbf{k}$ and $\bar{\mathbf{k}}$ are the 3-momentum of the quark and antiquark. Note that in this paper a bold character stands for a 3-dimensional vector, for example, $\mathbf{k} = \vec{k}$. In the non-relativistic limit, the kinetic energy reduces into

$$T = m + \bar{m} + \frac{\mathbf{k}^2}{2m} + \frac{\bar{\mathbf{k}}^2}{2\bar{m}}. \quad (2b)$$

The potential could be decomposed into

$$V = H^{\text{SI}} + H^{\text{SS}} + H^{\text{T}} + H^{\text{SO}}. \quad (3)$$

H^{SI} is the spin independent part, which is composed of a Coulombic potential and a linear potential,

$$H^{\text{SI}} = -\frac{4\alpha_s(Q^2)}{3r} + br, \quad (4)$$

where b is a constant and $\alpha_s(Q^2)$ is the running coupling of the strong interaction. The other three terms are spin dependent.

$$H^{\text{SS}} = \frac{32\pi\alpha_s(Q^2)}{9m\bar{m}} \tilde{\delta}_\sigma(\mathbf{r}) \mathbf{s} \cdot \bar{\mathbf{s}} \quad (5)$$

is the spin-spin contact hyperfine potential, where $\mathbf{s}$ and $\bar{\mathbf{s}}$ are the spin of the quark and antiquark respectively, and $\tilde{\delta}_\sigma(\mathbf{r}) = (\frac{\sigma}{\sqrt{\pi}})^3 e^{-\sigma^2 r^2}$ with σ being a parameter.

$$H^{\text{T}} = \frac{4\alpha_s(Q^2)}{3m\bar{m}} \frac{1}{r^3} \left(3 \frac{(\mathbf{s} \cdot \mathbf{r})(\bar{\mathbf{s}} \cdot \mathbf{r})}{r^2} - \mathbf{s} \cdot \bar{\mathbf{s}} \right) \quad (6)$$

is the tensor potential. H^{SO} is the spin-orbit interaction potential,

$$H^{\text{SO}} = \frac{\mathbf{S} \cdot \mathbf{L}}{2} \left[\left(\frac{1}{2m^2} + \frac{1}{2\bar{m}^2} \right) \left(\frac{4\alpha_s(Q^2)}{3r^3} - \frac{b}{r} \right) + \frac{8\alpha_s(Q^2)}{3m\bar{m}r^3} \right], \quad (7)$$

where $\mathbf{S} = \mathbf{s} + \bar{\mathbf{s}}$, and $\mathbf{L}$ is the orbital angular momentum of the quark and antiquark system.

In equations (4)–(7), the running coupling takes the following form [21]:

$$\alpha_s(Q^2) = \frac{4\pi}{\beta \log \left(e^{\frac{4\pi}{\beta\alpha_0}} + \frac{Q^2}{\Lambda_{\text{QCD}}^2} \right)}, \quad (8)$$

TABLE I. Parameters used in our calculation.

Flavor	m/GeV	α_0	b/GeV	σ/GeV^2	R_c/fm
c	1.591	1.023	0.144	1.30	0.375
b	4.997	0.875	0.112	2.13	0.156

where Λ_{QCD} is the energy scale below which nonperturbative effects take over, $\beta = 11 - \frac{2}{3}N_f$ with N_f being the flavor number, Q is the momentum transfer, and α_0 is a constant. Equation (8) approaches the one loop running form of QCD at large Q^2 and saturates at low Q^2 .

The potentials containing $\frac{1}{r^3}$, Eqs. (6) and (7) are divergent. A cutoff r_c is introduced to eliminate the divergence, i.e., $\frac{1}{r^3} \rightarrow \frac{1}{r_c^3}$ for $r \leq r_c$. Herein r_c is a parameter to be fixed by observables.

N_f and Λ_{QCD} are chosen according to QCD estimation. In principle, Λ_{QCD} depends on the choice of N_f . However, we found that Λ_{QCD} varying in $[0.2, 0.4]$ GeV induces small changes in the physical quantities ($\lesssim 4\%$) [19]. In this work we fix $\Lambda_{\text{QCD}} = 0.3$ GeV, $N_f = 4$ for charmonium and $N_f = 5$ for bottomium. m_c and m_b are the same as Ref. [19]. So there are four free parameters in all: α_0 , b , σ , and r_c . For charmonium these parameters are tuned to fit the masses of $\eta_c(1S)$, $\eta_c(2S)$, $J/\psi(1S)$, and $\chi_{c0}(1P)$; for bottomonium these parameters are fixed by the masses of $\eta_b(1S)$, $\Upsilon(1S)$, $\Upsilon(2S)$, and $\chi_{b0}(1P)$. The parameters are listed in Table I.

The two-photon transition form factor of pseudoscalar meson is defined as [15]

$$\begin{aligned} & \int d^4x e^{iq_1 \cdot x} \langle 0 | T \{ j^\mu(x) j^\nu(0) \} | M \rangle \\ & = \epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta} F_{P\gamma\gamma}(q_1^2, q_2^2). \end{aligned} \quad (9)$$

The left-hand side is the transition amplitude of the time-ordered currents between the meson state $|M\rangle$ and the vacuum $|0\rangle$. $j^\mu(x) = e_q \bar{\psi}(x) \gamma^\mu \psi(x)$, e_q is the electric charge of the quark, and $\psi(x)$ is the quark field operator. The right-hand side expresses the Lorentz structure of the transition amplitude, $\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita symbol, $q_{1\alpha}$ and $q_{2\beta}$ are the 4-momentum of the photons, and $F_{P\gamma\gamma}(q_1^2, q_2^2)$ is the form factor.

Traditionally the meson is expressed by the mock meson state in the quark model [19–21]

$$\begin{aligned} |M(p)\rangle &= \sqrt{\frac{2E_p}{N_c}} \chi_{s\bar{s}}^{SM_s} \int \frac{d^3k d^3\bar{k}}{(2\pi)^3} \varphi_M(\mathbf{k}_r) \\ &\quad \cdot \delta^{(3)}(\mathbf{k} + \bar{\mathbf{k}} - \mathbf{p}) b_{k\bar{s}}^\dagger d_{\bar{k}s}^\dagger |0\rangle, \end{aligned} \quad (10)$$

where $\mathbf{p}$ and $E_p = \sqrt{M^2 + \mathbf{p}^2}$ are the momentum and energy of the meson, M is the meson mass, and N_c is the

color number. $b_{k\bar{s}}^\dagger$ and $d_{k\bar{s}}^\dagger$ are the creation operator of the quark and antiquark respectively. $\chi_{s\bar{s}}^{SM_S}$ is the spin wave function, with S being the total spin and M_S its z projection. $\varphi_M(\mathbf{k}_r)$ is the wave function in momentum space, and $\mathbf{k}_r = \frac{\bar{m}\mathbf{k} - m\bar{\mathbf{k}}}{m + \bar{m}}$ is the relative momentum between the quark and antiquark. Herein we use the same symbol φ for the space wave function and the radial wave function, which could be distinguished by the arguments. In equation (10) a Clebsch-Gordan coefficient is indicated to combine the spin and orbital angular momentum into a total angular momentum of the meson.

The relativized mock meson state in analogy to Eq. (10) is

$$|M(p)\rangle = \sqrt{\frac{2E_p}{N_c}} \sum_{s,\bar{s}} \int \frac{d^3k d^3\bar{k}}{(2\pi)^6} \frac{1}{\sqrt{2E_k}} \frac{1}{\sqrt{2\bar{E}_{\bar{k}}}} \times \bar{u}(\mathbf{k}, s) \Gamma_M v(\bar{\mathbf{k}}, \bar{s}) b_{k\bar{s}}^\dagger d_{k\bar{s}}^\dagger |0\rangle. \quad (11)$$

$E_k = \sqrt{m^2 + \mathbf{k}^2}$ and $\bar{E}_{\bar{k}} = \sqrt{\bar{m}^2 + \bar{\mathbf{k}}^2}$ are the kinetic energy of the quark and antiquark. $u(\mathbf{k}, s)$ and $v(\bar{\mathbf{k}}, \bar{s})$ are the Dirac spinors of the quark and antiquark respectively. Γ_M is the meson Bethe-Salpeter amplitude. For the pseudoscalar meson it is generally composed of four terms [9]. Herein we only employ the main term for simplicity, i.e.,

$$\Gamma_P = \Gamma_P(k_r, p) = \gamma_5 \phi_P(k_r, p), \quad (12)$$

where k_r and p are the relative and total 4-momentum, γ_5 is Dirac matrix, and $\phi_P(k_r, p)$ is the scalar wave function. Analogically, for scalar meson

$$\Gamma_S = \Gamma_S(k_r, p) = \mathbf{1}_{4 \times 4} \phi_S(k_r, p), \quad (13)$$

where $\mathbf{1}_{4 \times 4}$ is the unit matrix in Dirac space, and $\phi_S(k_r, p)$ is the scalar wave function.

There are several reasons to go from Eqs. (10) to (11)–(13):

- (i) Equation (10) is not Lorentz covariant, so it leads to systematic errors in describing decay constants and form factors [19,20]. Conversely, the meson state described by Eqs. (11)–(13) is formally Lorentz covariant.
- (ii) Compared with the nonrelativistic form, the relativistic current $j^\mu(x) = e_q \bar{\psi}(x) \gamma^\mu \psi(x)$ leads to substantial corrections to the decay constant and single quark form factor [20]. We expect that the relativistic mock meson state also has a significant impact on the form factors.
- (iii) Equations (11)–(13) introduce relativistic corrections from two aspects; one is from the Dirac spinor, and the other is from the wave function. To assess the relativistic correction from the wave function, $\phi_P(k_r, p)$ and $\phi_S(k_r, p)$ should be solved from a

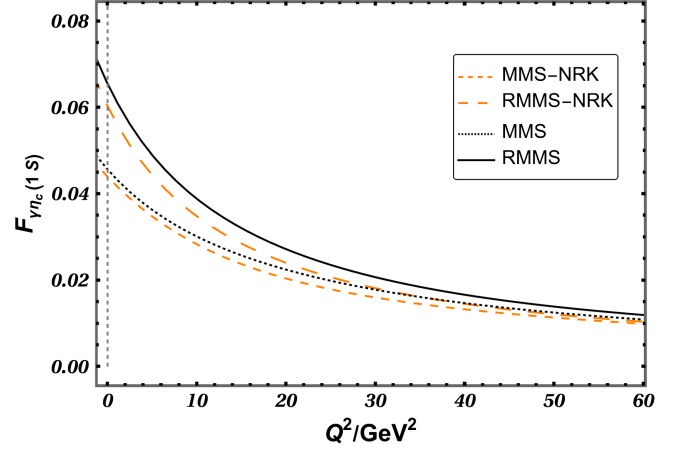


FIG. 1. The two-photon transition form factor of $\eta_c(1S)$. The black solid line (RMMS) is the result with relativized mock meson state, Eq. (11), and the relativistic kinetic energy, Eq. (2a). The orange long-dashed line (RMMS-NRK) is the result with relativized mock meson state, Eq. (11), and the nonrelativistic kinetic (NRK) energy, Eq. (2b). The black dotted line (MMS) is the result with the old mock meson state, Eq. (10), and the relativistic kinetic energy, Eq. (2a). The orange short-dashed line (MMS-NRK) is the result with the old mock meson state, Eq. (10), and the nonrelativistic kinetic energy, Eq. (2b).

covariant dynamical equation, e.g., the Bethe-Salpeter equation, which is technically challenging. However, signs indicate that the relativistic correction from the Dirac spinor is major and that from the wave function is minor. We calculated the effect of the kinetic energy on the wave function, i.e., we compared results from Eqs. (2a) and (2b). From Fig. 1 we see that the relativistic correction from the Dirac spinor is dominant compared to that from the wave function. We expect other corrections to the wave function to be small, though the accurate wave function is still an open question.

Considering the “iii” point, we use the relativistic kinetic energy, Eq. (2a), in the following calculation, and we will not solve $\phi_P(k_r, p)$ and $\phi_S(k_r, p)$ rigorously. We relate $\phi_P(k_r, p)$ and $\phi_S(k_r, p)$ to the nonrelativistic wave functions by requiring that Eq. (11) approaches Eq. (10) in the nonrelativistic limit. The derivations are put in Appendix A, and the results are

$$\phi_P(k_r, p) = \frac{\varphi_P(|\mathbf{k}_r|)}{\sqrt{8\pi}} (2\pi)^3 \delta^{(3)}(\mathbf{k} + \bar{\mathbf{k}} - \mathbf{p}), \quad (14)$$

$$\phi_S(k_r, p) = \frac{2E_k \bar{E}_{\bar{k}}}{E_k + \bar{E}_{\bar{k}}} \frac{\varphi_S(|\mathbf{k}_r|)}{\sqrt{8\pi} |\mathbf{k}_r|} (2\pi)^3 \delta^{(3)}(\mathbf{k} + \bar{\mathbf{k}} - \mathbf{p}). \quad (15)$$

Equations (11)–(15) represent relativized mock meson states, where the Dirac spinors import kinetic relativistic corrections and the dynamic wave functions are the same as Eq. (10). Note that Eqs. (14) and (15) are not Lorentz

invariant, so these relativized mock meson states still show a little frame dependence, but this does not affect our arguments in this paper.

We work in the meson static frame to calculate the two-photon transition form factor, where the 4-momentum of the meson and the two photons are

$$p = (M, \vec{0}), \quad (16)$$

$$q_1 = \left(\frac{M^2 + Q_1^2 - Q_2^2}{2M}, 0, 0, \lambda \right), \quad (17)$$

$$q_2 = \left(\frac{M^2 - Q_1^2 + Q_2^2}{2M}, 0, 0, -\lambda \right), \quad (18)$$

where $\lambda = \frac{1}{2M} \{M^4 + Q_1^4 + Q_2^4 - 2(M^2 Q_1^2 + M^2 Q_2^2 + Q_1^2 Q_2^2)\}^{1/2}$, and $Q_1^2 = q_1^2$, $Q_2^2 = q_2^2$.

We put Eq. (10) into Eq. (9) and get the pseudoscalar meson two-photon transition form factor corresponding to the old mock meson state. We put Eq. (11) into Eq. (9) and get the expression corresponding to the relativized mock meson state. In this paper, we focus on the single-tag form factor in the spacelike region, $F_{\gamma\gamma}(Q^2) = F_{P\gamma\gamma}(-Q^2, 0)$. The derivation and the expressions of the form factors, and those analogs of the scalar meson are all put in Appendix B.

TABLE II. Mass spectrum of charmonium (in GeV). M_{cc}^{QM} is our result via the quark model, with the parameters listed in Table I. Note that the parameters are tuned to fit the observed masses of $\eta_c(1S)$, $\eta_c(2S)$, $J/\psi(1S)$, and $\chi_{c0}(1P)$. $M_{cc}^{\text{expt.}}$ is the experimental value [23].

$n^{2S+1}L_J$	State	J^{PC}	M_{cc}^{QM}	$M_{cc}^{\text{expt.}}$
1^1S_0	$\eta_c(1S)$	0^{-+}	2.984 (input)	2.984(0.4)
2^1S_0	$\eta_c(2S)$	0^{-+}	3.638 (input)	3.638(1)
3^1S_0	$\eta_c(3S)$	0^{-+}	4.048	...
1^3S_1	$J/\psi(1S)$	1^{--}	3.097 (input)	3.097(0)
2^3S_1	$\psi(2S)$	1^{--}	3.684	3.686(0.1)
3^3S_1	$\psi(4040)$	1^{--}	4.079	4.039(1)
1^3P_0	$\chi_{c0}(1P)$	0^{++}	3.415 (input)	3.415(0.3)
2^3P_0	$\chi_{c0}(2P)$	0^{++}	3.894	...
3^3P_0	$\chi_{c0}(3P)$	0^{++}	4.251	...
1^1P_1	$h_c(1P)$	1^{+-}	3.510	3.525(0.1)
2^1P_1	$h_c(2P)$	1^{+-}	3.933	...
3^1P_1	$h_c(3P)$	1^{+-}	4.271	...
1^3P_1	$\chi_{c1}(1P)$	1^{++}	3.510	3.511(0.1)
2^3P_1	$\chi_{c1}(2P)$	1^{++}	3.943	...
3^3P_1	$\chi_{c1}(3P)$	1^{++}	4.284	...
1^3P_2	$\chi_{c2}(1P)$	2^{++}	3.561	3.556(0.1)
2^3P_2	$\chi_{c2}(3930)$	2^{++}	3.971	3.923(1)
3^3P_2	$\chi_{c2}(3P)$	2^{++}	4.302	...

III. THE CHARMONIUM

A. Charmonium spectrum and wave function

The charmonium spectrum is listed in Table II. Note that the interaction parameters are fitted by the observed masses of $\eta_c(1S)$, $\eta_c(2S)$, $J/\psi(1S)$, and $\chi_{c0}(1P)$, i.e., these four masses are inputs of our model, and all the other masses and the two-photon transition form factors are predictions. For the mesons under the $D\bar{D}$ threshold, the predicted masses agree with the observed values quite well. For those above the $D\bar{D}$ threshold, e.g., $\psi(4040)$ and $\chi_{c2}(3930)$, the deviation is about 40–50 MeV. Coupled-channel effect is needed for a more precise prediction [22]. In this paper, we focus on the $\eta_c(nS)$ and $\chi_{c0}(nP)$ for $n = 1, 2, 3$, among which $\eta_c(3S)$, $\chi_{c0}(2P)$, and $\chi_{c0}(3P)$ are above the $D\bar{D}$ threshold. For a first step study of the effect of the relativized mock meson state, we ignore the coupled-channel effect herein.

The radial wave functions of $\eta_c(nS)$ and $\chi_{c0}(nP)$ are in Fig. 2. The radial wave function is normalized as

$$\int_0^\infty \frac{d|\mathbf{k}|}{(2\pi)^3} \mathbf{k}^2 |\varphi_M(|\mathbf{k}|)|^2 = 1. \quad (19)$$

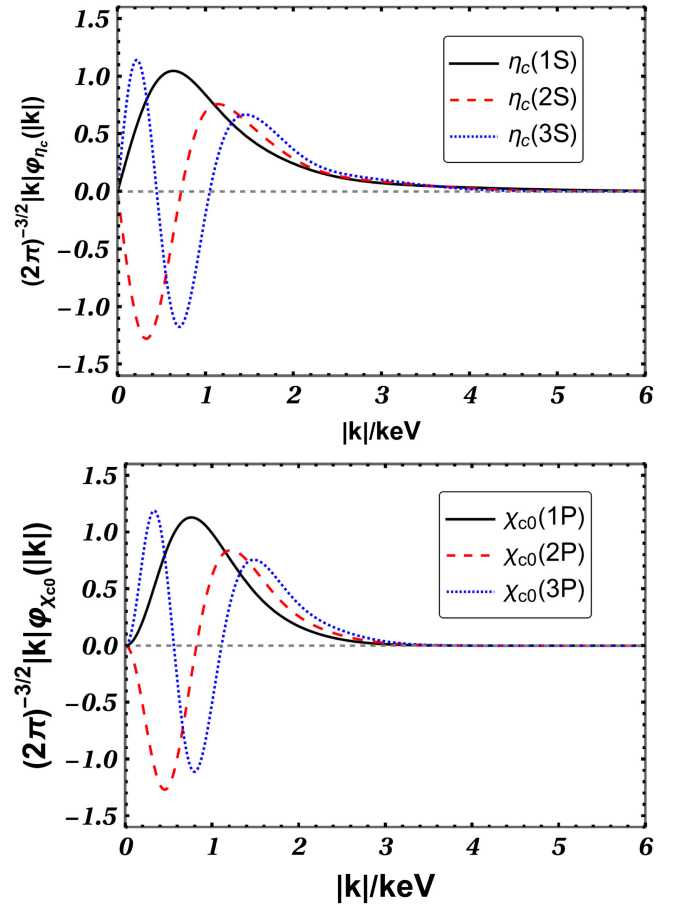


FIG. 2. The radial wave function of $\eta_c(nS)$ (upper) and $\chi_{c0}(nP)$ (lower).

The same wave functions are used both in the old mock meson state, Eq. (10), and the relativized mock meson state, Eqs. (11)–(15). So the relativized mock meson state only takes into account the kinetic relativistic correction

imported by the Dirac spinors, and the dynamic wave function is not modified. Our results in the following show that this kinetic relativistic correction is crucial and dominant in studying the two-photon transition form factors.

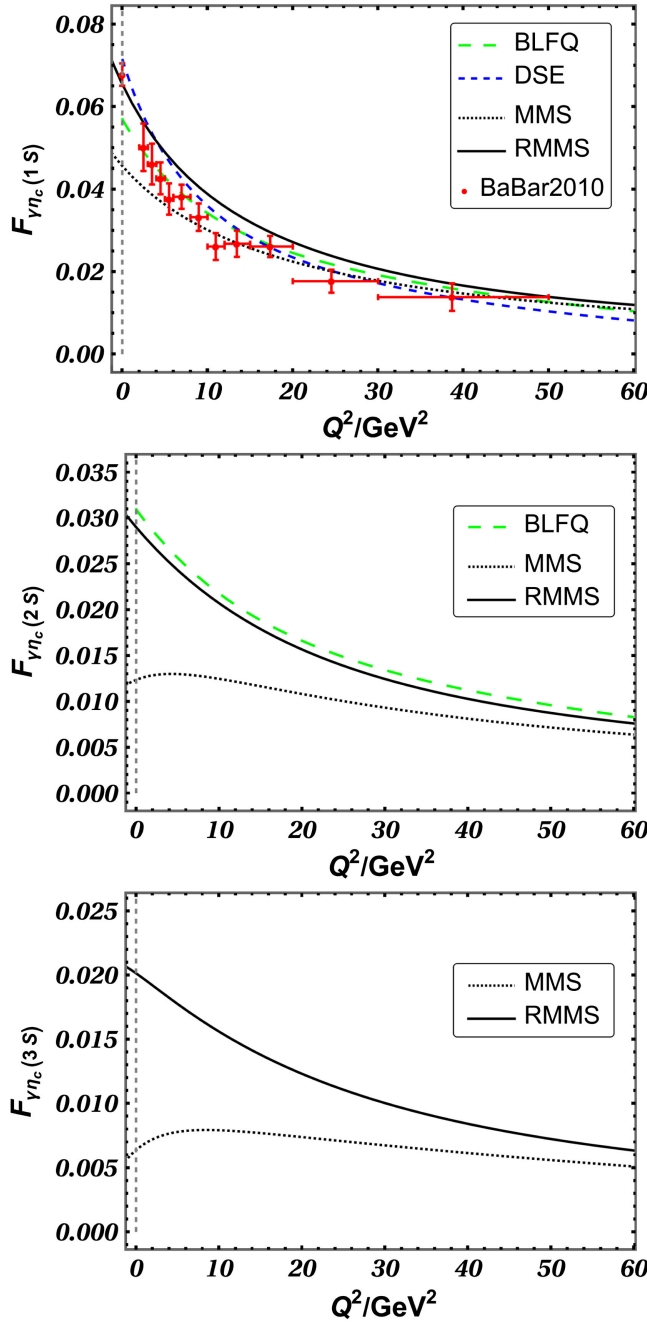


FIG. 3. The two-photon transition form factor of $\eta_c(1S)$ (upper), $\eta_c(2S)$ (middle), and $\eta_c(3S)$ (lower). The black solid lines are our results with relativized mock meson state, Eq. (11). The black dotted lines are our results with the old mock meson state, Eq. (10). The blue short-dashed line is the Dyson-Schwinger equation result [9]. The green long-dashed line is the result of the basis light front quantization approach [18]. The red dots are the *BABAR* results [2].

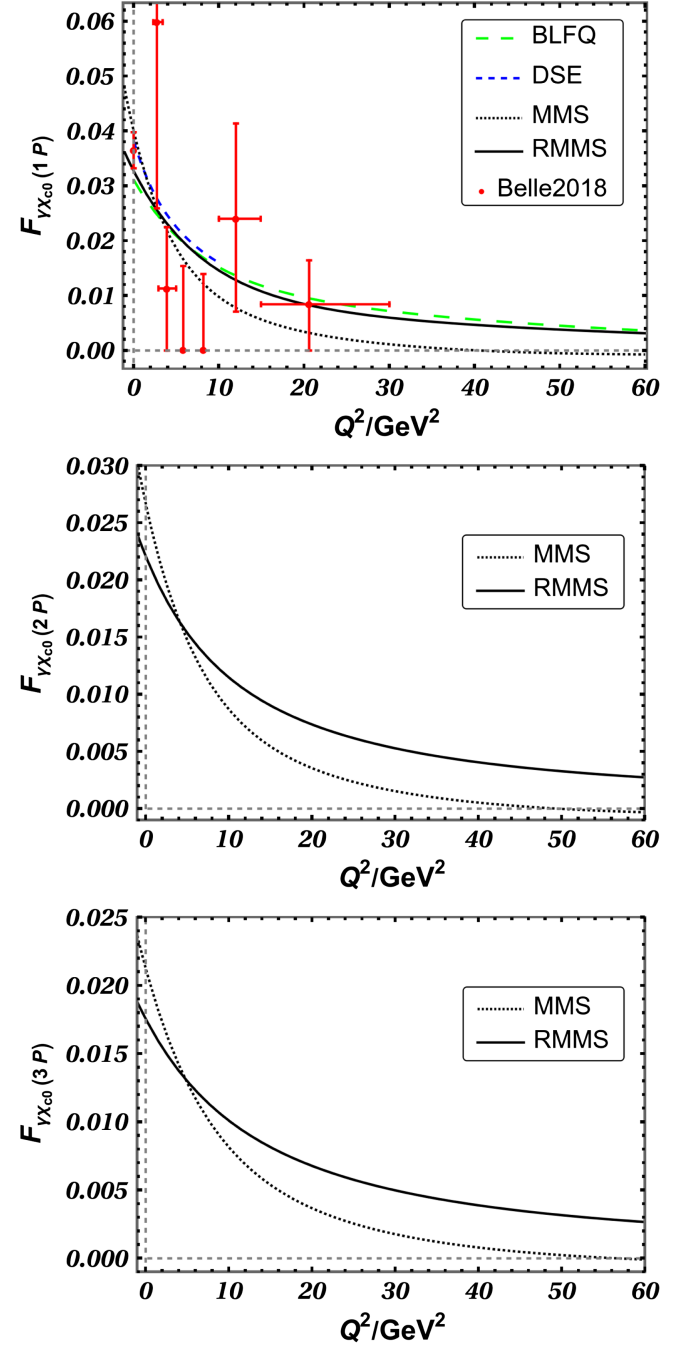


FIG. 4. The two-photon transition form factor of $\chi_{c0}(1P)$ (upper), $\chi_{c0}(2P)$ (middle), and $\chi_{c0}(3P)$ (lower). The black solid lines are our results with relativized mock meson state, Eq. (11). The black dotted lines are our results with the old mock meson state, Eq. (10). The blue short-dashed line is the Dyson-Schwinger equation result [9]. The green long-dashed line is the result of the basis light front quantization approach [18]. The red dots are the preliminary Belle results [4].

TABLE III. Two-photon decay width of charmonium (in keV). $\Gamma_{\gamma\gamma}^{\text{MMS}}$ is our result with the old mock state, and $\Gamma_{\gamma\gamma}^{\text{RMMS}}$ is our result with relativized mock state. $\Gamma_{\gamma\gamma}^{\text{expt.}}$ is the experimental value [23]. $\Gamma_{\gamma\gamma}^{\text{BLFQ}}$ is the basis light-front quantization result [18]. $\Gamma_{\gamma\gamma}^{\text{DSE}}$ is the Dyson-Schwinger equation result [9]. $\Gamma_{\gamma\gamma}^{\text{LQCD}}$ are the lattice QCD results [6,8]. $\Gamma_{\gamma\gamma}^{\text{NRQM}}$ is a nonrelativistic quark model result [20]. $\Gamma_{\gamma\gamma}^{\text{RQM}}$ is a relativistic quark model result [24].

	$\Gamma_{\gamma\gamma}^{\text{MMS}}$	$\Gamma_{\gamma\gamma}^{\text{RMMS}}$	$\Gamma_{\gamma\gamma}^{\text{expt.}}$ [23]	$\Gamma_{\gamma\gamma}^{\text{BLFQ}}$ [18]	$\Gamma_{\gamma\gamma}^{\text{DSE}}$ [9]	$\Gamma_{\gamma\gamma}^{\text{LQCD}}$ [6]	$\Gamma_{\gamma\gamma}^{\text{LQCD}}$ [8]	$\Gamma_{\gamma\gamma}^{\text{NRQM}}$ [20]	$\Gamma_{\gamma\gamma}^{\text{RQM}}$ [24]
$\eta_c(1S)$	2.35	4.84	5.1 ± 0.4	3.7(6)	6.39	2.65(99)	6.67(17)	7.18	5.5
$\eta_c(2S)$	0.32	1.68	2.1 ± 1.4	1.9(4)				1.71	1.8
$\eta_c(3S)$	0.11	1.13	...					1.21	
$\chi_{c0}(1P)$	2.84	1.92	2.2 ± 0.2	1.7(4)	2.39	2.41(1.04)		3.28	2.9
$\chi_{c0}(2P)$	1.80	1.25	...	0.68(22)					1.9
$\chi_{c0}(3P)$	1.42	1.00	...						

B. The two-photon transition form factor and decay width

The prediction of two-photon transition form factors of $\eta_c(nS)$ is in Fig. 3, and those of $\chi_{c0}(nP)$ are in Fig. 4. The results are easily readable from the figures. In the following we will comment on them systematically.

First, let us focus on $F_{\gamma\eta_c}(1S)$, the upper panel of Fig. 3. The black dotted line is our result with the old mock meson state (MMS), and the black solid line is our result with the relativized mock meson state (RMMS). They approach each other in the ultraviolet region, and deviate remarkably in the infrared region. In the lower Q^2 region ($Q^2 \lesssim 10 \text{ GeV}$), the RMMS result is more consistent with the

TABLE IV. Mass spectrum of bottomonium (in GeV). M_{bb}^{QM} is our result via the quark model, with the parameters listed in Table I. Note that the parameters are tuned to fit the masses of $\eta_b(1S)$, $\Upsilon(1S)$, $\Upsilon(2S)$, and $\chi_{b0}(1P)$. $M_{bb}^{\text{expt.}}$ is the experimental value [23].

$n^{2S+1}L_J$	State	J^{PC}	M_{bb}^{QM}	$M_{bb}^{\text{expt.}}$
1^1S_0	$\eta_b(1S)$	0^{-+}	9.399 (input)	9.399(2)
2^1S_0	$\eta_b(2S)$	0^{-+}	10.005	9.999(4)
3^1S_0	$\eta_b(3S)$	0^{-+}	10.321	...
1^3S_1	$\Upsilon(1S)$	1^{--}	9.460 (input)	9.460(0.3)
2^3S_1	$\Upsilon(2S)$	1^{--}	10.023 (input)	10.023(0.3)
3^3S_1	$\Upsilon(3S)$	1^{--}	10.333	10.355(1)
4^3S_1	$\Upsilon(4S)$	1^{--}	10.566	10.579(1)
1^3P_0	$\chi_{b0}(1P)$	0^{++}	9.859 (input)	9.859(1)
2^3P_0	$\chi_{b0}(2P)$	0^{++}	10.216	10.233(1)
3^3P_0	$\chi_{b0}(3P)$	0^{++}	10.472	...
1^1P_1	$h_b(1P)$	1^{+-}	9.915	9.899(1)
2^1P_1	$h_b(2P)$	1^{+-}	10.245	10.260(1)
3^1P_1	$h_b(3P)$	1^{+-}	10.491	...
1^3P_1	$\chi_{b1}(1P)$	1^{++}	9.907	9.893(1)
2^3P_1	$\chi_{b1}(2P)$	1^{++}	10.242	10.255(1)
3^3P_1	$\chi_{b1}(3P)$	1^{++}	10.490	10.513(1)
1^3P_2	$\chi_{b2}(1P)$	2^{++}	9.938	9.912(1)
2^3P_2	$\chi_{b2}(2P)$	2^{++}	10.261	10.269(1)
3^3P_2	$\chi_{b2}(3P)$	2^{++}	10.503	10.524(1)
1^3D_2	$\Upsilon_2(1D)$	2^{--}	10.144	10.164(1)

Dyson-Schwinger equation (DSE) result [9], the blue short-dashed line, and the *BABAR* experiment results, the red dots with error bars [4]. Although the result of the basis light front quantization (BLFQ) approach [18], the green long-dashed line is also consistent with the *BABAR* result, the error at $Q^2 = 0$ is considerable. The RMMS and DSE results match the experimental value at $Q^2 = 0$ much better.

In the higher Q^2 region ($40 \text{ GeV} \lesssim Q^2 \lesssim 60 \text{ GeV}$), the MMS, RMMS, and BLFQ results approach each other,

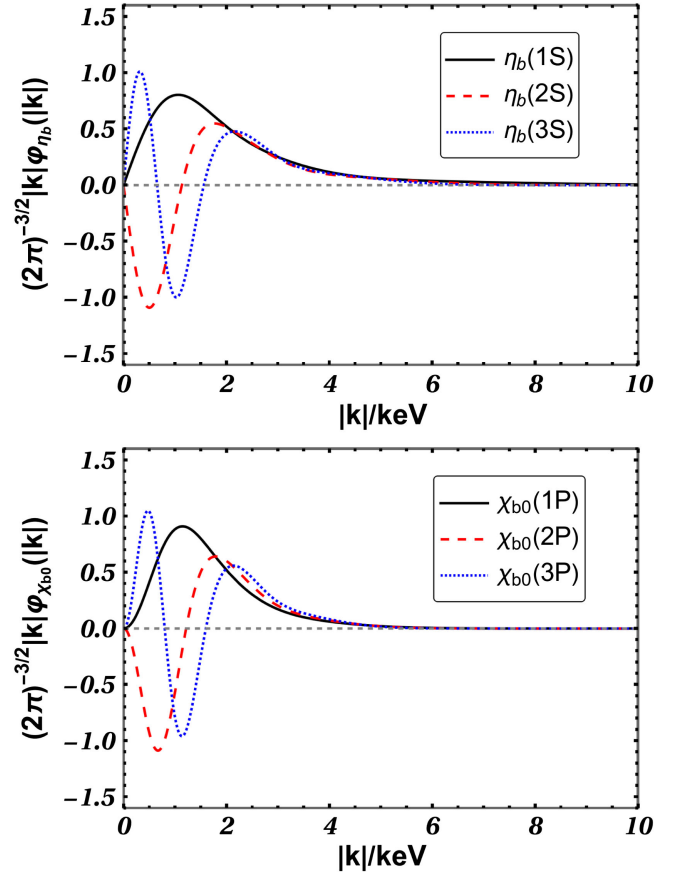


FIG. 5. The radial wave function of $\eta_b(nS)$ (upper) and $\chi_{b0}(nP)$ (lower).

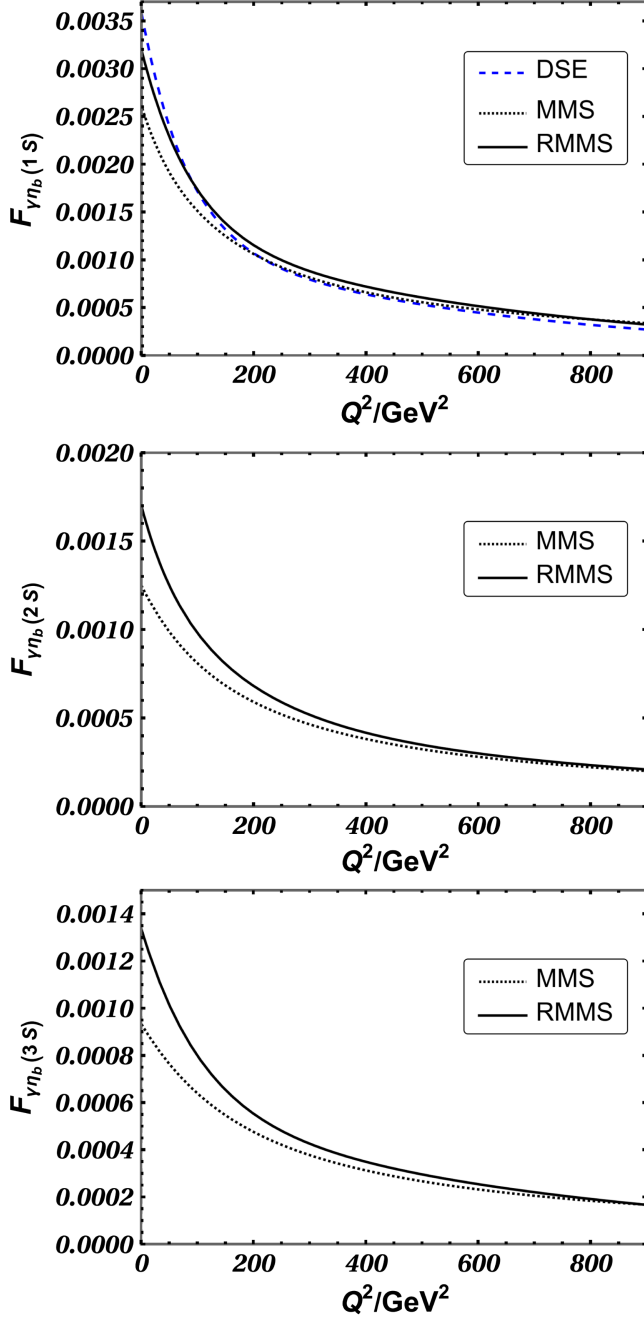


FIG. 6. The two-photon transition form factor of $\eta_b(1S)$ (upper), $\eta_b(2S)$ (middle), and $\eta_b(3S)$ (lower). The black solid lines are our results with relativized mock meson state, Eq. (11). The black dotted lines are our results with the old mock meson state, Eq. (10). The blue short-dashed line is the Dyson-Schwinger equation result [9].

and they are higher than the DSE result. This indicates that the quark model results approach a different $Q^2 \rightarrow \infty$ asymptotic value compared with the DSE result, which is manifestly visible from Fig. 7 of Ref. [18]. As a low energy effective theory, the asymptotic behavior of quark model is not strictly constrained, so the deviation in the higher Q^2

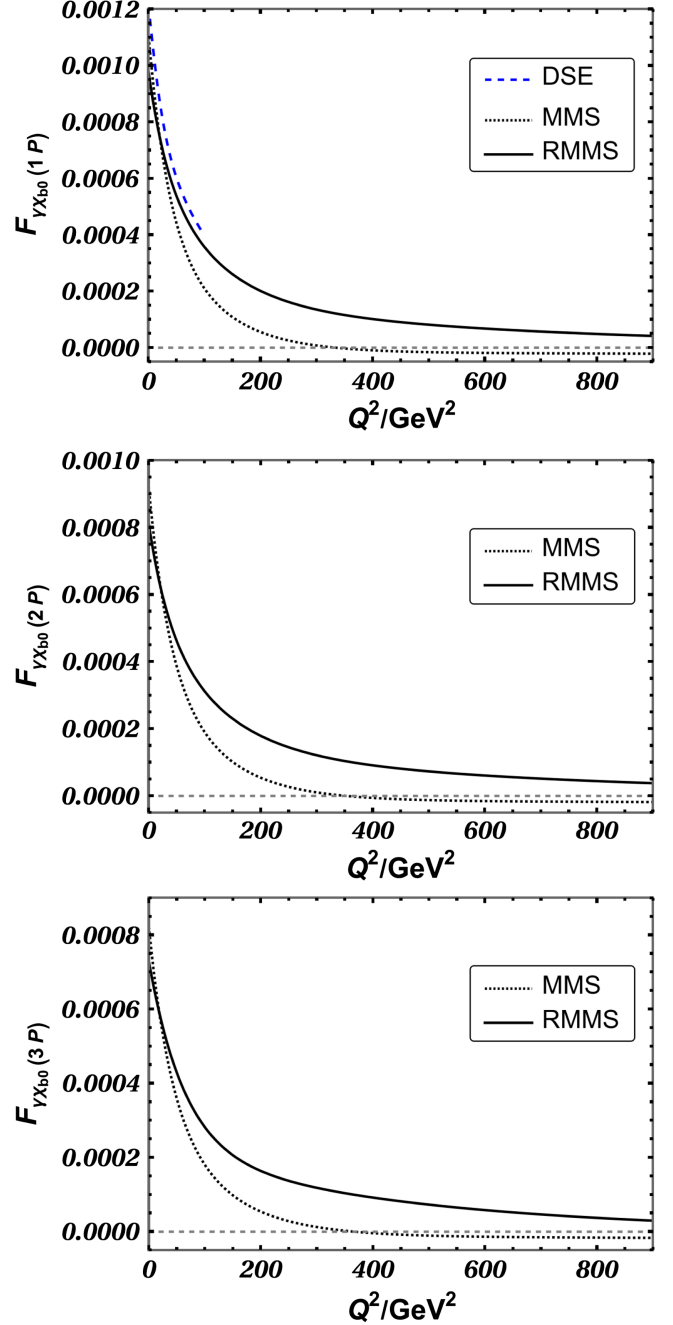


FIG. 7. The two-photon transition form factor of $\chi_{b0}(1P)$ (upper), $\chi_{b0}(2P)$ (middle), and $\chi_{b0}(3P)$ (lower). The black solid lines are our results with relativized mock meson state, Eq. (11). The black dotted lines are our results with the old mock meson state, Eq. (10). The blue short-dashed line is the Dyson-Schwinger equation result [9].

region should not be taken too seriously. In the following, we will focus on the lower Q^2 region.

The figure of $F_{\gamma\eta_c(1S)}$ indicates that the kinetic relativistic correction imported by the Dirac spinors, in Eq. (11), is crucial and dominant in studying the two-photon transition form factors, especially in the infrared region.

TABLE V. Two-photon decay width of bottomonium (in keV). $\Gamma_{\gamma\gamma}^{\text{MMS}}$ is our result with the old mock state, and $\Gamma_{\gamma\gamma}^{\text{RMMS}}$ is our result with the relativized mock state. $\Gamma_{\gamma\gamma}^{\text{DSE}}$ is the Dyson-Schwinger equation result [9]. $\Gamma_{\gamma\gamma}^{\text{NRQM}}$ is a nonrelativistic quark model result [20]. $\Gamma_{\gamma\gamma}^{\text{RQM}}$ is a relativistic quark model result [24]. $\Gamma_{\gamma\gamma}^{\text{HQSS}}$ is an estimation from heavy-quark spin symmetry [25].

	$\Gamma_{\gamma\gamma}^{\text{MMS}}$	$\Gamma_{\gamma\gamma}^{\text{RMMS}}$	$\Gamma_{\gamma\gamma}^{\text{DSE}}$ [9]	$\Gamma_{\gamma\gamma}^{\text{NRQM}}$ [20]	$\Gamma_{\gamma\gamma}^{\text{RQM}}$ [24]	$\Gamma_{\gamma\gamma}^{\text{HQSS}}$ [25]
$\eta_b(1S)$	0.237	0.354	0.469	0.23	0.35	0.560
$\eta_b(2S)$	0.065	0.121		0.07	0.15	0.269
$\eta_b(3S)$	0.040	0.083		0.04	0.10	0.208
$\chi_{b0}(1P)$	0.049	0.039	0.060	0.075	0.038	
$\chi_{b0}(2P)$	0.038	0.029			0.029	
$\chi_{b0}(3P)$	0.032	0.025				

This point is further supported by $F_{\gamma\eta_c}(2S)$, the middle panel of Fig. 3, and $F_{\gamma\chi_{c0}}(1P)$, the upper panel of Fig. 4. For $F_{\gamma\eta_c}(2S)$, the RMMS result is consistent with the BLFQ result, while the MMS result declines significantly in the infrared region. For $F_{\gamma\chi_{c0}}(1P)$, the RMMS result is consistent with the DSE and BLFQ result, while the MMS result decreases faster and declines to a negative value in the ultraviolet region. The RMMS result also matches the experimental value $F_{\gamma\chi_{c0}}(1P)(0)$ better than the MMS result. A preliminary experiment result of $F_{\gamma\chi_{c0}}(1P)(Q^2)$ [4] is also plotted, but it is senseless to be compared with due to the large errors. Overall, the RMMS results are consistent with other theoretical predictions and the available experimental values. The form factor $F_{\gamma\eta_c}(3S)$ is predicted in Fig. 3, and $F_{\gamma\chi_{c0}}(2P)$ and $F_{\gamma\chi_{c0}}(3P)$ are predicted in Fig. 4.

Our predictions of two-photon decay width of $F_{\gamma\eta_c}(nS)$ and $F_{\gamma\chi_{c0}}(nP)$ are listed in Table III. In general, the RMMS result matches the experimental values, BLFQ result, and DSE result much better than the MMS result. The lattice QCD result varies from one another [6,8]. $\Gamma_{\gamma\gamma}^{\text{NRQM}}$ [20] is the result from the combination of a nonrelativistic quark model and an effective Lagrange. $\Gamma_{\gamma\gamma}^{\text{RQM}}$ [24], the result from a relativistic quark model is also consistent with our RMMS result.

IV. THE BOTTOMONIUM

A. Bottomonium spectrum and wave function

In this section we display and comment on the results of bottomonium, parallel to the charmonium.

The bottomonium spectrum is listed in Table IV. The interaction parameters are fitted by the masses of $\eta_b(1S)$, $\Upsilon(1S)$, $\Upsilon(2S)$, and $\chi_{b0}(1P)$, so these four masses are inputs of our model, and all the other masses and the two-photon transition form factors are predictions. All the mesons in Table IV are under the $B\bar{B}$ threshold, and the predicted masses agree with the observed values quite well, with an average error of 12 MeV. The radial wave functions of $\eta_b(nS)$ and $\chi_{b0}(nP)$ are in Fig. 5.

B. The two-photon transition form factor and decay width

The prediction of two-photon transition form factors of $\eta_b(nS)$ is in Fig. 6, and those of $\chi_{b0}(nP)$ are in Fig. 7. There is no experimental value for these quantities. The DSE results, which are proved to be credible, are available for $\eta_b(1S)$ and $\chi_{b0}(1P)$. In both cases, RMMS results are consistent with the DSE result, and MMS results deviate from the RMMS results.

Our predictions of two-photon decay width of $F_{\gamma\eta_b}(nS)$ and $F_{\gamma\chi_{b0}}(nP)$ are listed in Table V. There is no experimental value for these quantities presently. The RMMS results are more consistent with the DSE result, $\Gamma_{\gamma\gamma}^{\text{DSE}}$ [9], and relativistic quark model result, $\Gamma_{\gamma\gamma}^{\text{RQM}}$ [24]. The results from a combination of a nonrelativistic quark model and an effective Lagrange, $\Gamma_{\gamma\gamma}^{\text{NRQM}}$ [20], and those from the heavy-quark spin-symmetry approach, $\Gamma_{\gamma\gamma}^{\text{HQSS}}$ [25], are also listed to compare with.

V. SUMMARY AND CONCLUSION

In this work, we construct a relativized mock meson state, where Dirac spinors import kinetic relativistic correction, and the dynamic wave function is the same as that solved from the Schrödinger equation. Two-photon transition form factors of $\eta_{c,b}(n^1S_0)$ and $\chi_{c,b}(n^3P_0)$ ($n = 1, 2, 3$) are calculated with both the old mock meson state and the relativized mock meson state. We compare our results with the experiment results, Dyson-Schwinger equation results, the basis light front quantization results, and some other theoretical prediction. It turns out that the kinetic relativistic correction imported by Dirac spinors is crucial to study the two-photon transition form factor in a constituent quark model. The RMMS results provide a credible prediction for the two-photon transition form factors and decay widths of $\eta_{c,b}(n^1S_0)$ and $\chi_{c,b}(n^3P_0)$ ($n = 1, 2, 3$).

In the next step, the effects of the relativized mock meson state on single quark form factors, two-photon transition form factors of higher spin mesons, and the hadron transition form factors are to be studied.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: NONRELATIVISTIC LIMIT OF THE RELATIVIZED MOCK MESON STATE

1. The pseudoscalar meson

The pseudoscalar relativized mock meson state is

$$|P(p)\rangle = \sqrt{\frac{2E_p}{N_c}} \sum_{s,\bar{s}} \int \frac{d^3k d^3\bar{k}}{(2\pi)^6} \frac{1}{\sqrt{2E_k}} \frac{1}{\sqrt{2E_{\bar{k}}}} \times \phi_P(k_r, p) \bar{u}(\mathbf{k}, s) \gamma_5 v(\bar{\mathbf{k}}, \bar{s}) b_{k\bar{s}}^\dagger d_{\bar{k}\bar{s}}^\dagger |0\rangle. \quad (\text{A1})$$

In the Dirac representation, the spinors are

$$u(\mathbf{k}, s) = \sqrt{E_k + m} \begin{pmatrix} 1 \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{k}}{E_k + m} \end{pmatrix} \xi^s, \quad (\text{A2})$$

$$v(\bar{\mathbf{k}}, \bar{s}) = \sqrt{\bar{E}_{\bar{k}} + \bar{m}} \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \bar{\mathbf{k}}}{\bar{E}_{\bar{k}} + \bar{m}} \\ 1 \end{pmatrix} \eta^{\bar{s}}, \quad (\text{A3})$$

where $\boldsymbol{\sigma}$ is the Pauli matrix, and ξ^s and $\eta^{\bar{s}}$ are the two-component spinors for quark and antiquark. In the non-relativistic limit, $\mathbf{k} \rightarrow 0$ and $\bar{\mathbf{k}} \rightarrow 0$, Eq. (A1) reduces to

$$|P(p)\rangle = \sqrt{\frac{2E_p}{N_c}} \int \frac{d^3k d^3\bar{k}}{(2\pi)^6} \phi_P(k_r, p) \sqrt{2} \chi_{s\bar{s}}^{00} b_{k\bar{s}}^\dagger d_{\bar{k}\bar{s}}^\dagger |0\rangle, \quad (\text{A4})$$

where $\chi_{s\bar{s}}^{00} = \frac{1}{\sqrt{2}} (\xi^\dagger \eta^\downarrow - \xi^\downarrow \eta^\dagger)$ is the singlet spin wave function. Comparing Eq. (A4) with Eq. (10), we get

$$\phi_P(k_r, p) = \frac{\varphi_P(\mathbf{k}_r)}{\sqrt{2}} (2\pi)^3 \delta^{(3)}(\mathbf{k} + \bar{\mathbf{k}} - \mathbf{p}). \quad (\text{A5})$$

2. The scalar meson

The scalar relativized mock meson state is

$$|S(p)\rangle = \sqrt{\frac{2E_p}{N_c}} \sum_{s,\bar{s}} \int \frac{d^3k d^3\bar{k}}{(2\pi)^6} \frac{1}{\sqrt{2E_k}} \frac{1}{\sqrt{2E_{\bar{k}}}} \times \phi_S(k_r, p) \bar{u}(\mathbf{k}, s) v(\bar{\mathbf{k}}, \bar{s}) b_{k\bar{s}}^\dagger d_{\bar{k}\bar{s}}^\dagger |0\rangle. \quad (\text{A6})$$

In the nonrelativistic limit, Eq. (A6) reduces to

$$|S(p)\rangle = \sqrt{\frac{2E_p}{N_c}} \sum_{s,\bar{s}} \int \frac{d^3k d^3\bar{k}}{(2\pi)^6} \frac{\sqrt{8\pi}}{2\tilde{m}} |\mathbf{k}_r| \times \phi_S(k_r, p) \chi_{s\bar{s}}^{1m_z} Y^{1-m_z} b_{k\bar{s}}^\dagger d_{\bar{k}\bar{s}}^\dagger |0\rangle, \quad (\text{A7})$$

where $\tilde{m} = \frac{m\bar{m}}{m+\bar{m}}$ is the reduced quark mass, and $\chi_{s\bar{s}}^{1m_z} Y^{1-m_z} = \frac{1}{\sqrt{3}} (\xi^\dagger \eta^\uparrow Y^{1-1} - \frac{\xi^\dagger \eta^\downarrow - \xi^\downarrow \eta^\dagger}{\sqrt{2}} Y^{10} + \xi^\downarrow \eta^\downarrow Y^{1+1})$ is just the triplet spin wave function and the first-order spherical harmonic function combined by the Clebsch-Gordan coefficient. Comparing Eq. (A7) with Eq. (10), we get

$$\phi_S(k_r, p) = \frac{2\tilde{m}\varphi_S(|\mathbf{k}_r|)}{\sqrt{8\pi}|\mathbf{k}_r|} (2\pi)^3 \delta^{(3)}(\mathbf{k} + \bar{\mathbf{k}} - \mathbf{p}). \quad (\text{A8})$$

In order to make sure the single quark elastic form factor is being normalized correctly, $\tilde{m}$ is replaced by $\tilde{E} = \frac{E_k \bar{E}_{\bar{k}}}{E_k + \bar{E}_{\bar{k}}}$.

APPENDIX B: TWO-PHOTON TRANSITION FORM FACTORS

We display the expressions of the two-photon transition form factor in the meson static frame, $p = (M, \vec{0})$. The 4-momentum of the two photons is

$$q_1 = (q_1^t, 0, 0, q_1^z) = \left(\frac{M^2 + Q_1^2 - Q_2^2}{2M}, 0, 0, \lambda \right), \quad (\text{B1})$$

$$q_2 = (q_2^t, 0, 0, q_2^z) = \left(\frac{M^2 - Q_1^2 + Q_2^2}{2M}, 0, 0, -\lambda \right), \quad (\text{B2})$$

where $Q_1^2 = q_1^2$, $Q_2^2 = q_2^2$ are the 4-momentum square and $\lambda = \frac{1}{2M} \{M^4 + Q_1^4 + Q_2^4 - 2(M^2 Q_1^2 + M^2 Q_2^2 + Q_1^2 Q_2^2)\}^{1/2}$.

1. $F_{\gamma\gamma}$ from old mock meson state

The two-photon symmetry form of the form factor is

$$\int d^4x d^4y e^{iq_1 \cdot x + iq_2 \cdot y} \langle 0 | T \{ j^\mu(x) j^\nu(y) \} | M(p) \rangle = \epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta} F_{P\gamma\gamma}(q_1^2, q_2^2) (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p). \quad (\text{B3})$$

The left-hand side,

$$\begin{aligned}
\text{LHS} &= \int d^4x d^4y e^{iq_1 \cdot x + iq_2 \cdot y} \langle 0 | j^\mu(x) j^\nu(y) | M(p) \rangle |_{x^0 > y^0} + \{x \leftrightarrow y\} \\
&= \int d^4x d^4y e^{iq_1 \cdot x} e^{i(q_2 \cdot y - p \cdot y + q_1 \cdot y)} \cdot \langle 0 | j^\mu(x) j^\nu(0) | M(p) \rangle |_{x^0 > 0} + \{q_1, \mu \leftrightarrow q_2, \nu\} \\
&= (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p) \int d^4x e^{iq_1 \cdot x} \cdot \langle 0 | j^\mu(x) j^\nu(0) | M(p) \rangle |_{x^0 > 0} + \{q_1, \mu \leftrightarrow q_2, \nu\},
\end{aligned}$$

so that we get the symmetric form

$$\int d^4x e^{iq_1 \cdot x} \langle 0 | j^\mu(x) j^\nu(0) | M(p) \rangle |_{x^0 > 0} + \{q_1, \mu \leftrightarrow q_2, \nu\} = M^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta} F_{P\gamma\gamma}(q_1^2, q_2^2). \quad (\text{B4})$$

It is worth mentioning that the time integration is operated as, e.g., $\int_0^\infty dx^0 e^{i(q_1' - E_k - E_{k-q_1})x^0} = \int_0^{\infty(1-i\epsilon)} dx^0 e^{i(q_1' - E_k - E_{k-q_1})x^0} = \frac{i}{q_1' - E_k - E_{k-q_1}}$. Let $\{\mu, \nu\} = \{1, 2\}$ or $\{2, 1\}$; after some manipulation of the Dirac matrices, we get the expression of the two-photon transition form factor

$$F_{P\gamma\gamma}(q_1^2, q_2^2) = \frac{e_q^2 \sqrt{2MN_c}}{q_1^t q_2^z - q_2^t q_1^z} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\varphi_P(\mathbf{k})}{E_k E_{k+q_1}} \left(\frac{1}{E_k + E_{k+q_1} - q_1^t} + \frac{1}{E_k + E_{k+q_1} - q_2^t} \right) \sqrt{2m}(2k^z + q_1^z). \quad (\text{B5})$$

The decay width is

$$\Gamma_{P\gamma\gamma} = \frac{\pi\alpha^2 M^3}{4} |F_{P\gamma\gamma}(0, 0)|^2. \quad (\text{B6})$$

2. $F_{\gamma\eta_c}$ from relativized mock meson state

$$\begin{aligned}
F_{P\gamma\gamma}(q_1^2, q_2^2) &= \frac{e_q^2 \sqrt{2MN_c}}{q_1^t q_2^z - q_2^t q_1^z} \left\{ \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\varphi_P(\mathbf{k})}{\sqrt{2}} [A_1 + A_2] \right. \\
&\quad \left. - \{q_1 \leftrightarrow q_2\} \right\}, \quad (\text{B7})
\end{aligned}$$

$$\begin{aligned}
\text{where } A_1 &= \frac{(m-\bar{m})E_{k+q_1}k^z - (m-\bar{m})E_k k^z + mE_k q_1^z + \bar{m}E_k q_1^z}{2E_k \bar{E}_k E_{k+q_1} (E_k + E_{k+q_1} - q_1^t)}, \\
A_2 &= \frac{(m-\bar{m})\bar{E}_{k-q_1}k^z - (m-\bar{m})\bar{E}_k k^z + mE_k q_1^z + \bar{m}E_k q_1^z}{2E_k \bar{E}_k \bar{E}_{k-q_1} (E_k + \bar{E}_{k-q_1} - q_1^t)}.
\end{aligned}$$

3. $F_{\gamma\chi_{c0}}$ from old mock meson state

The two-photon transition form factor of scalar meson is defined as [15]

$$\begin{aligned}
M^{\mu\nu} &= \int d^4x e^{iq_1 \cdot x} \langle 0 | T\{j^\mu(x) j^\nu(0)\} | M \rangle \\
&= T_1^{\mu\nu} F_1(q_1^2, q_2^2) + T_2^{\mu\nu} F_2(q_1^2, q_2^2), \quad (\text{B8})
\end{aligned}$$

where

$$T_1^{\mu\nu} = q_1 \cdot q_2 g^{\mu\nu} - q_2^\mu q_1^\nu, \quad (\text{B9})$$

$$T_2^{\mu\nu} = q_1^2 q_2^2 g^{\mu\nu} + q_1 \cdot q_2 q_1^\mu q_2^\nu - q_1^2 q_2^\mu q_2^\nu - q_2^2 q_1^\mu q_1^\nu, \quad (\text{B10})$$

with $g^{\mu\nu} = \text{diag}\{1, -1, -1, -1\}$. In the single-tag case, $q_1^2 = -Q^2$, $q_2^2 = 0$, $F_{\gamma S}(Q^2) = F_1(-Q^2, 0)$ could be calculated with $\{\mu, \nu\} = \{1, 1\}$ or $\{2, 2\}$.

$$F_{\gamma S}(Q^2) = \frac{e_q^2 2\sqrt{2MN_c}}{M^2 + Q^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{-i\varphi_S(|\mathbf{k}|) \cdot A_3}{4E_k E_{k+q_1} (E_k + E_{k+q_1} - q_1^t)}, \quad (\text{B11})$$

where $A_3 = -\frac{2\sqrt{2}|\mathbf{k}|m}{\sqrt{\pi}} - \frac{\sqrt{2}k^z/|\mathbf{k}|}{\sqrt{\pi(E_k+m)}} [(E_k+m)^2(q_1^z+k^z) + q_1^z k^2 - 2q_1^z(k^z)^2 + k^2 k^z - 2(k^z)^3] + \frac{2\sqrt{2}(1-(k^z/|\mathbf{k}|)^2)}{\sqrt{\pi(E_k+m)}} |\mathbf{k}|(k \cdot q_1)(k^z)^2$. The decay width is

$$\Gamma_{S\gamma\gamma} = \frac{\pi\alpha^2 M^3}{4} |F_{\gamma S}(0)|^2. \quad (\text{B12})$$

4. $F_{\gamma\chi_{c0}}$ from relativized mock meson state

$$\begin{aligned}
F_{\gamma S}(Q^2) &= \frac{e_q^2 2\sqrt{2MN_c}}{M^2 + Q^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{2E_k \bar{E}_k}{E_k + \bar{E}_k} \frac{\varphi_S(|\mathbf{k}_r|)}{\sqrt{8\pi}|\mathbf{k}_r|} \\
&\quad \times (-i)[A_4 + A_5] + \{q_1 \leftrightarrow q_2\}, \quad (\text{B13})
\end{aligned}$$

where $A_4 = [2E_k \bar{E}_k E_{k+q_1} (\bar{E}_k + E_{k+q_1} - q_1^t)]^{-1} \times \{m^2 \bar{m} + m[E_{k+q_1} \bar{E}_k - E_k \bar{E}_k + q_1 \cdot k - 2(k^x)^2] - \bar{m}[E_{k+q_1} E_k - (k + q_1) \cdot k + 2(k^x)^2]\}$, $A_5 = [2E_k \bar{E}_k \bar{E}_{k-q_1} (E_k + \bar{E}_{k-q_1} - q_1^t)]^{-1} \times \{m\bar{m}^2 - m[\bar{E}_{k-q_1} \bar{E}_k + (q_1 - k) \cdot k + 2(k^x)^2] + \bar{m}[\bar{E}_{k-q_1} E_k - E_k \bar{E}_k - q_1 \cdot k - 2(k^x)^2]\}$.

- [1] K. M. Ecklund *et al.*, Two-photon widths of the χ_{cJ} states of charmonium, *Phys. Rev. D* **78**, 091501 (2008).
- [2] J. P. Lees *et al.*, Measurement of the $\gamma\gamma^* \rightarrow \eta_c$ transition form factor, *Phys. Rev. D* **81**, 052010 (2010).
- [3] M. Ablikim *et al.*, Two-photon widths of the $\chi_{c0,2}$ states and helicity analysis for $\chi_{c2} \rightarrow \gamma\gamma$, *Phys. Rev. D* **85**, 112008 (2012).
- [4] M. Masuda *et al.*, Study of K_s^0 pair production in single-tag two-photon collisions, *Phys. Rev. D* **97**, 052003 (2018).
- [5] Carlos A. Bertulani, Reinaldo Francener, Victor P. Goncalves, and Juciene T. de Souza, Particle production by γ - γ interactions in future electron-ion colliders, *Phys. Rev. C* **111**, 025201 (2025).
- [6] Jozef J. Dudek and Robert G. Edwards, Two photon decays of charmonia from lattice QCD, *Phys. Rev. Lett.* **97**, 172001 (2006).
- [7] Brian Colquhoun, Laurence J. Cooper, Christine T. H. Davies, and G. Peter Lepage, Precise determination of decay rates for $\eta_c \rightarrow \gamma\gamma$, $J/\psi \rightarrow \gamma\eta_c$, and $J/\psi \rightarrow \eta_c e^+ e^-$ from lattice QCD, *Phys. Rev. D* **108**, 014513 (2023).
- [8] Yu Meng, Xu Feng, Chuan Liu, Teng Wang, and Zuoheng Zou, First-principle calculation of the η_c 2γ decay width from lattice QCD, *Sci. Bull.* **68**, 1880 (2023).
- [9] Jing Chen, Minghui Ding, Lei Chang, and Yu-xin Liu, Two photon transition form factor of $\bar{c}c$ quarkonia, *Phys. Rev. D* **95**, 016010 (2017).
- [10] Wolfgang Lucha and Dmitri Melikhov, The $\gamma^*\gamma^* \rightarrow \eta_c$ transition form factor, *Phys. Rev. D* **86**, 016001 (2012).
- [11] Feng Feng, Yu Jia, and Wen-Long Sang, Can nonrelativistic QCD explain the $\gamma\gamma^* \rightarrow \eta_c$ transition form factor data?, *Phys. Rev. Lett.* **115**, 222001 (2015).
- [12] Wen-Long Sang, Feng Feng, Yu Jia, and Shuang-Ran Liang, Next-to-next-to-leading-order QCD corrections to $\chi_{c0,2} \rightarrow \gamma\gamma$, *Phys. Rev. D* **94**, 111501 (2016).
- [13] Li Yang, Wen-Long Sang, Hong-Fei Zhang, Yu-Dong Zhang, and Ming-Zhen Zhou, Study on $\eta_{c2}(\eta_{b2})$ electromagnetic decay into double photons, *Phys. Rev. D* **103**, 034018 (2021).
- [14] Fu-guang Cao and Tao Huang, Large corrections to asymptotic $F_{\eta_c\gamma}$ and $F_{\eta_b\gamma}$ in the light cone perturbative QCD, *Phys. Rev. D* **59**, 093004 (1999).
- [15] Martin Hoferichter and Peter Stoffer, Asymptotic behavior of meson transition form factors, *J. High Energy Phys.* **05** (2020) 159.
- [16] Hui-Young Ryu, Ho-Meoyng Choi, and Chueng-Ryong Ji, Systematic twist expansion of $(\eta_c, \eta_b) \rightarrow \gamma^*\gamma$ transition form factors in light-front quark model, *Phys. Rev. D* **98**, 034018 (2018).
- [17] Izabela Babiarz, Victor P. Goncalves, Roman Pasechnik, Wolfgang Schäfer, and Antoni Szczurek, $\gamma^*\gamma^* \rightarrow \eta_c(1s, 2s)$ transition form factors for spacelike photons, *Phys. Rev. D* **100**, 054018 (2019).
- [18] Yang Li, Meijian Li, and James P. Vary, Two-photon transitions of charmonia on the light front, *Phys. Rev. D* **105**, L071901 (2022).
- [19] Chao Sun, Ru-Hui Ni, and Muyang Chen, Decay constants of $B_c(nS)$ and $B_c^*(nS)$, *Chin. Phys. C* **47**, 023101 (2023).
- [20] Olga Lakhina and Eric S. Swanson, Dynamic properties of charmonium, *Phys. Rev. D* **74**, 014012 (2006).
- [21] S. Godfrey and Nathan Isgur, Mesons in a relativized quark model with chromodynamics, *Phys. Rev. D* **32**, 189 (1985).
- [22] Qian Deng, Ru-Hui Ni, Qi Li, and Xian-Hui Zhong, Charmonia in an unquenched quark model, *Phys. Rev. D* **110**, 056034 (2024).
- [23] S. Navas *et al.*, Review of particle physics, *Phys. Rev. D* **110**, 030001 (2024).
- [24] D. Ebert, R. N. Faustov, and V. O. Galkin, Two photon decay rates of heavy quarkonia in the relativistic quark model, *Mod. Phys. Lett. A* **18**, 601 (2003).
- [25] J. P. Lansberg and T. N. Pham, Two-photon width of η_b, η'_b and η''_b from heavy-quark spin symmetry, *Phys. Rev. D* **75**, 017501 (2007).