

Hawking-Page transition in anti-de Sitter massive gravity with noncompact spatial boundary

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In gauge-gravity duality, the Hawking-Page (HP) process plays the important role of representing a confined-deconfined phase transition. For spaces that are asymptotically anti-de Sitter with a planar spatial boundary, there is no HP transition, unless some infrared energy scale is introduced. Massive gravity theories have been applied in the context of holography in order to represent systems with momentum dissipation. In this work, we study the HP transition for massive gravity in an asymptotically anti-de Sitter space with $\mathbb{R}^n \times S^1$ boundary. It is shown that the graviton mass term produces an HP transition at a finite critical temperature.

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I. INTRODUCTION

The Hawking-Page (HP) transition was proposed in [1], using a semiclassical approach to the thermal behavior of Euclidean 4D spaces with negative cosmological constant and spherical symmetry. There are two solutions for Einstein's equations under these conditions: the thermal anti-de Sitter (AdS) and the black hole anti-de Sitter (AdS-BH) spacetimes. At low temperatures, the black hole solution is unstable, so the system is in the AdS phase. For temperatures higher than some critical value, the situation changes, and the system is dominated by the AdS-BH solution.

In the context of gauge-string duality, it was proposed by Witten [2] that the HP transition in AdS_5 space corresponds to a confinement-deconfinement transition in the 4D gauge theory side of the AdS/CFT correspondence. The transition analyzed in [2] occurs only in the case when the AdS_5 space has a compact boundary. For the AdS_5 space with a planar boundary, more appropriate to represent our physical world, there is no HP transition and the black hole is the stable configuration at any temperature—unless some modification is made in order to introduce some dimensional parameter in the background, in general, associated with an infrared cutoff, as is done in [3–9].

Massive gravity theories, meaning theories where the graviton is massive, have been studied for quite a long time

(see, for example, [10,11]). Observational limits for the graviton mass are presented in [12], which sets the upper bound at an order of 10^{-24} eV.

The de Rham–Gabadadze–Tolley [13,14] is a theory of massive gravity in which the Boulware-Deser ghost is absent [15]. This is an important achievement, as it makes it possible to define a unitary theory (see, e.g., [10,11,16]). Furthermore, the issue of negative energy states has also been addressed in [17], where the authors argue that the negative energy sector is suppressed, rendering the theory stable.

In the gauge-gravity dualities, they are interpreted as representing systems with broken translational symmetry and, consequently, momentum dissipation [18]. The HP transition in the case of AdS space with a spherical boundary with the addition of a graviton mass term was studied in [19]. It was found that the presence of the mass term in the gravitational action lowers the critical temperature with respect to the nonmassive case. Another interesting analysis of the HP transition in massive gravity is presented in [20]. See [21] for an interesting analysis of HP transition in other modified theories of relativity.

In this work, we consider AdS space with a planar boundary. As mentioned before, in this case there is no HP transition if one considers just the standard Einstein action for this space. We will introduce a graviton mass term in the action and then analyze the thermodynamics of the resulting system. A phase transition at a finite temperature, related to the mass scale of the graviton, will show up.

The paper is organized as follows. In Sec. II, we review some illustrating examples of Hawking-Page transitions. In Sec. III, we present some aspects of massive gravity that are relevant to this work. In Sec. IV, we discuss the stability of the black hole solution in the massive AdS space and then examine the HP transition between the spaces with and

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without a black hole, focusing on the system's thermodynamics. Finally, in Sec. V, our conclusions and final remarks are presented. The detailed computations for the metrics used in this work are provided in the Appendix.

II. HAWKING-PAGE TRANSITION IN ADS SPACE

We begin our discussion by revisiting the concept of Hawking-Page transition, originally proposed for Euclidean anti-de Sitter space with a periodic time coordinate in [1]. The main idea is that in a quantum theory of gravity the more relevant contributions to the path integral, that sums over all possible metric configurations that are asymptotically AdS, comes from the corresponding classical solutions of Einstein equations. There are two solutions, one with a black hole and the other without. The dominant (or stable) is the one with the smallest action integral. This condition mirrors the thermodynamic stability criterion of minimizing the free energy.

A. The AdS_D space with $S^{D-2} \times S^1$ boundary

Let us extend the case of spacetime dimension $D = 4$ considered in [1] to arbitrary dimensions. The Einstein Hilbert action has the form

$$\mathcal{I} = -\frac{1}{2\kappa^2} \int d^D x \sqrt{|g|} (\mathcal{R} - \Lambda) \quad (1)$$

where $\kappa^2 = 8\pi G_D$ with G_D the D -dimensional Newton's constant with dimension $E^{-(D-2)}$ and $\mathcal{R}$ and Λ denote, respectively, the Ricci scalar and cosmological constant. For AdS spaces with radius L , they depend on the spacetime dimension D as follows:

$$\mathcal{R} = -\frac{D(D-1)}{L^2} \quad \text{and} \quad \Lambda = -\frac{(D-1)(D-2)}{L^2}. \quad (2)$$

The black hole metric with spherical symmetry that emerges as a solution of the action (1) with imaginary time is

$$ds^2 = g_{pq} dx^p dx^q = V(r) dt^2 + V(r)^{-1} dr^2 + r^2 d\Omega_n^2, \quad (3)$$

where $n = D - 2$ is the number of spatial dimensions in the boundary and $d\Omega_n^2$ represents the metric of the n -dimensional sphere. The function $V(r)$ takes the form

$$V(r) = 1 - \frac{M}{r^{n-1}} + \frac{r^2}{L^2} \quad (4)$$

with M being proportional to the black hole mass. The horizon position r_h is determined by $V(r_h) = 0$, and the Hawking temperature is obtained by requiring that there is no conical singularity in the vicinity of the horizon. The result is

$$T = \frac{(n-1)L^2 + (n+1)r_h^2}{4\pi L^2 r_h}. \quad (5)$$

For the thermal AdS space, the metric is in the same form of (3), but with $M = 0$, i.e.,

$$V_0(r) = 1 + \frac{r^2}{L^2}. \quad (6)$$

From Eqs. (1) and (2), one obtains the actions for the black hole AdS:

$$\mathcal{I}^{(\text{BH})} = \frac{\Omega_n}{\kappa^2} \frac{\beta}{L^2} (r_b^{n+1} - r_h^{n+1}) \quad (7)$$

and for the thermal AdS:

$$\mathcal{I}^{(\text{AdS})} = \frac{\Omega_n}{\kappa^2} \frac{\beta_0}{L^2} r_b^{n+1}. \quad (8)$$

Here, r_b denotes a regulator for the boundary position, and Ω_n is the area of the n sphere, while β and β_0 originate from the temporal coordinate integration. These quantities can be related by requiring that the two geometries have the same temporal length at the boundary, yielding

$$\begin{aligned} \beta_0 &= \left(\frac{V(r_b)}{V'(r_b)} \right)^{1/2} \beta \\ &\approx \left(1 - \frac{L^2 M}{2r_b^{n+1}} \right) \beta, \end{aligned} \quad (9)$$

where the approximation holds for large r_b .

To remove the divergences in the limit $r_b \rightarrow \infty$, one can define the renormalized action difference as

$$\begin{aligned} \Delta\mathcal{I} &\equiv \mathcal{I}^{(\text{BH})} - \mathcal{I}^{(\text{AdS})} \\ &= \frac{\Omega_n}{2\kappa^2} \beta \left(r_h^{n-1} - \frac{r_h^{n+1}}{L^2} \right), \end{aligned} \quad (10)$$

where the condition $V(r_h) = 0$ was used.

As explained at the beginning of this section, the stable space configuration corresponds to the lowest action. Therefore, for $\Delta\mathcal{I} > 0$, the thermal AdS space is stable; for $\Delta\mathcal{I} < 0$, the black hole solution becomes the stable one. The phase transition occurs at $\Delta\mathcal{I} = 0$, where both configurations have equal action values. This condition determines the critical temperature for the Hawking-Page transition as

$$T_c = \frac{n}{2\pi L}. \quad (11)$$

B. The AdS_D space with $R^{D-2} \times S^1$ boundary

It was proposed in [2], in the context of gauge-gravity duality, that the Hawking-Page transition in the gravitation side corresponds to the confinement-deconfinement phase transition in the gauge theory side.

In order to represent a gauge theory living in a non-compact space, one considers a flat R^{D-2} spatial boundary geometry. The metric of the black hole in Poincaré coordinates becomes

$$ds^2 = \frac{L^2}{z^2} (f(z) dt^2 + f(z)^{-1} dz^2 + d\vec{x}_n^2), \quad (12)$$

where the horizon function $f(z)$ is given by

$$f(z) = 1 - \frac{z^{n+1}}{z_h^{n+1}}. \quad (13)$$

The condition of absence of conical singularity leads to the Hawking temperature

$$T = \frac{(n+1)}{4\pi} \frac{1}{z_h}. \quad (14)$$

The metric of the thermal AdS is obtained using $f(z) = 1$. Following the same procedure as in the case of compact spatial boundary, one obtains the action difference

$$\Delta\mathcal{I} = -\frac{L^n}{2\kappa^2} \beta V_n \frac{1}{z_h^{n+1}}, \quad (15)$$

where V_n is the n -dimensional spatial volume.

Equation (15) is zero only when $z_h \rightarrow \infty$ and the temperature goes to zero by Eq. (14). For any nonzero temperature, $\Delta\mathcal{I} < 0$. So, the black hole geometry is always the stable one. In other words, there is no HP transition.

The zero critical temperature is a consequence of the absence of any dimensionful parameter in the gauge-string duality in this case. Important physical systems governed by gauge theories, like the quark gluon plasma, typically exhibit a finite critical temperature. In order to model holographically such systems, various approaches have been proposed that modify the AdS geometry in order to break conformal invariance in the gauge theory side, thereby generating nonzero critical temperatures. Some examples can be found in [3–9]. In the following section, we will study a different situation where an energy parameter is naturally present: the case when the gravitons are massive.

III. ADS SPACE WITH MASSIVE GRAVITY

The formulation of massive gravity that we use in this work was proposed in [13–15] and used in [19] to analyze the HP transition for the case of a spherical spatial boundary. One introduces a graviton mass term in the action of the form

$$\mathcal{I}_{\text{grav}} = -\frac{1}{2\kappa^2} \int d^{n+2}x \sqrt{|g|} m^2 \sum_{i=1}^{n+2} c_i \mathcal{U}_i, \quad (16)$$

where m is the graviton mass, c_i are dimensionless constants, and $\mathcal{U}_i$ are the symmetric polynomial eigenvalues of the matrix $\mathcal{K}$. We consider only the first two terms of the sum:

$$\mathcal{U}_1 = \text{Tr } \mathcal{K}, \quad (17)$$

$$\mathcal{U}_2 = (\text{Tr } \mathcal{K})^2 - \text{Tr } \mathcal{K}^2. \quad (18)$$

The $(n+2) \times (n+2)$ matrix $\mathcal{K}$ is defined in terms of the dynamical metric and of a reference metric $f_{\mu\nu}$ as follows:

$$\mathcal{K}^\mu{}_\rho \mathcal{K}^\rho{}_\nu = g^{\mu\sigma} f_{\sigma\nu}. \quad (19)$$

Following [18,22], we consider the case of a flat reference metric:

$$f_{\mu\nu} = \sum_j \delta_\mu^j \delta_\nu^j, \quad (20)$$

where the sum over j runs over the spatial boundary coordinates.

This choice breaks the diffeomorphism invariance in the spatial coordinates of the boundary, leading to nonconservation of spatial momentum in the boundary theory. For more details about massive gravity theory, see the reviews [10,11].

Considering that only c_1 and c_2 are nonzero in the graviton action (16), one can write an effective action as

$$\begin{aligned} \mathcal{I} = & -\frac{1}{2\kappa^2} \int d^{n+2}x \sqrt{|g|} (\mathcal{R} - \Lambda) \\ & -\frac{1}{2\kappa^2} \int d^{n+2}x \sqrt{|g|} (\lambda_1 \text{Tr } \mathcal{K} + \lambda_2 [(\text{Tr } \mathcal{K})^2 - \text{Tr } \mathcal{K}^2]) \\ & -\frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^{n+1}x \sqrt{|h|} K, \end{aligned} \quad (21)$$

where we have defined $\lambda_1 \equiv c_1 m^2$ and $\lambda_2 \equiv c_2 m^2$ for simplicity, both parameters having dimensions of energy squared. The last term in the action is the boundary term (Gibbons-Hawking-York term), where K is the trace of the extrinsic curvature of the boundary and h is the determinant of the induced metric $h_{\mu\nu}$. Note that if one sets the graviton

mass m to zero, Eq. (21) reduces to the Einstein-Hilbert action.

The black hole metric solution of the action (21) that is asymptotically AdS and has planar symmetry in the spatial coordinates has the form

$$ds^2 = \frac{L^2}{z^2} (f(z) dt^2 + f(z)^{-1} dz^2 + d\vec{x}_n^2), \quad (22)$$

with a modified horizon function given by

$$f(z) = 1 + \frac{\lambda_1 L}{n} z + \lambda_2 z^2 - \frac{z^{n+1}}{z_h^{n+1}} \left(1 + \frac{\lambda_1 L}{n} z_h + \lambda_2 z_h^2 \right). \quad (23)$$

The derivation of this solution is presented in the Appendix. The Hawking temperature is obtained by requiring regularity in imaginary time, giving

$$T = \frac{1}{4\pi z_h} ((n+1) + \lambda_1 L z_h + (n-1)\lambda_2 z_h^2), \quad (24)$$

remembering that z_h denotes the horizon position. The solution without a black hole corresponds to the limit $z_h \rightarrow \infty$, which is equivalent to setting the black hole mass to zero. This leads to

$$f_0(z) = 1 + \frac{\lambda_1 L}{n} z + \lambda_2 z^2. \quad (25)$$

IV. HP TRANSITION IN MASSIVE ADS SPACE

A. Regularized action

As in the cases presented in Sec. II, we obtain the regularized action using the subtraction scheme defined as

$$\Delta\mathcal{I} = \lim_{\varepsilon \rightarrow 0} (\mathcal{I}^{(\text{BH})} - \mathcal{I}^{(\text{AdS})}). \quad (26)$$

Since the total action (21) is composed of two parts, bulk and surface (boundary), Eq. (26) can be rewritten as

$$\Delta\mathcal{I} = \Delta\mathcal{I}_{\text{Bulk}} + \Delta\mathcal{I}_{\text{Surf}}. \quad (27)$$

Thus, let us compute the contribution of the surface term to the regularized action. The trace of the extrinsic curvature to the black hole solution is obtained as

$$\begin{aligned} K = \nabla_a n^a &= \frac{1}{\sqrt{|g|}} \frac{d}{dz} (\sqrt{|g|} n^z) \\ &= \frac{(n+1)}{L} f(z)^{1/2} - \frac{1}{2} \frac{z}{L} f(z)^{-1/2} f'(z), \end{aligned} \quad (28)$$

where the normal vector

$$n^a = \left(0, -\frac{z}{L} f(z)^{1/2}, 0, \dots \right)$$

was used. Using the horizon function (23), we obtain

$$\begin{aligned} K &= \frac{(n+1)}{L} f(z)^{1/2} \\ &\quad - \frac{z}{L} \left(\frac{\lambda_1 L}{2n} z + \lambda_2 z^2 - \frac{(n-1)}{2} M z^{n+1} \right) f(z)^{-1/2}, \end{aligned} \quad (29)$$

with

$$M = \frac{1}{z_h^{n+1}} \left(1 + \frac{\lambda_1 L z_h}{n} + \lambda_2 z_h^2 \right).$$

Inserting this result into the surface term of the action (21), along with $\sqrt{|h|} = z^{n+1} f(z)/L^{n+1}$, yields

$$\begin{aligned} \mathcal{I}_{\text{Surf}}^{(\text{BH})} &= -\frac{V_n L^n}{\kappa^2} \beta \frac{1}{z_h^{n+1}} \left[(n+1) + \frac{(2n+1)\lambda_1 L}{2n} z \right. \\ &\quad \left. + n\lambda_2 z^2 - \frac{(n+1)}{2} M z^{n+1} \right]. \end{aligned} \quad (30)$$

Repeating the steps using the $f_0(z)$ function given by Eq. (25), the surface term for the thermal solution is obtained as

$$\mathcal{I}_{\text{Surf}}^{(\text{AdS})} = -\frac{V_n L^n}{\kappa^2} \beta_0 \frac{1}{z_h^{n+1}} \left[(n+1) + \frac{(2n+1)\lambda_1 L}{2n} z + n\lambda_2 z^2 \right]. \quad (31)$$

As in the spherically symmetric AdS case, one can relate the periods β and β_0 of the time coordinates by matching the asymptotic limits of the two geometries on the boundary:

$$\begin{aligned} \beta_0 &= \left(\frac{f(\varepsilon)}{f_0(\varepsilon)} \right)^{1/2} \beta \\ &\approx \left(1 - \frac{\varepsilon^{n+1}}{2z_h^{n+1}} - \frac{\lambda_1 L}{2n} \frac{\varepsilon^{n+1}}{z_h^n} - \frac{\lambda_2}{2} \frac{\varepsilon^{n+1}}{z_h^{n-1}} \right) \beta \\ &\approx \left(1 - \frac{1}{2} M \varepsilon^{m+1} \right) \beta, \end{aligned} \quad (32)$$

where the expansion about $\varepsilon \rightarrow 0$ was used. With this relation, one can write the subtraction of the boundary terms as

$$\begin{aligned}
\Delta \mathcal{I}_{\text{Surf}} &= \lim_{\varepsilon \rightarrow 0} (\mathcal{I}_{\text{Surf}}^{(\text{BH})} - \mathcal{I}_{\text{Surf}}^{(\text{AdS})}) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{V_n L^n}{\kappa^2} \beta \left[\frac{(n+1)M}{2} - \frac{\lambda_1 L M}{2n} \varepsilon \right. \\
&\quad \left. - \frac{n\lambda_2 M}{2} \varepsilon^2 - \frac{(n+1)M}{2} \right] \\
&= \lim_{\varepsilon \rightarrow 0} \frac{V_n L^n}{\kappa^2} \beta \left[-\frac{\lambda_1 L M}{2n} \varepsilon - \frac{n\lambda_2 M}{2} \varepsilon^2 \right] \\
&= 0.
\end{aligned} \tag{33}$$

Therefore, the regularized action through the subtraction scheme is simply the difference between the bulk actions, $\Delta \mathcal{I}_{\text{bulk}}$. The reason behind the vanishing $\Delta \mathcal{I}_{\text{Surf}}$ is due to the fact that, far from the horizon, on the boundary, the geometry of the black hole solution is asymptotically identical to the geometry of the space without the black hole, at least up to leading orders. This lies on the main idea of the Hawking-Page transitions, comparing spaces that are asymptotically the same.

It is important to note that, if a different regularization procedure is chosen, such as the holographic regularization, the surface terms are crucial to the analysis (see, e.g., [4,23]).

In order to obtain the regularized action, let us first calculate the gravitational term in action (21). Using the same cosmological constant from Eq. (2), expressed in terms of n , one obtains

$$\mathcal{I}_1 = -\frac{1}{2\kappa^2} \int d^{n+2}x \sqrt{|g|} \left(\mathcal{R} + \frac{n(n+1)}{L^2} \right). \tag{34}$$

The additional terms in the action (21) modify the Ricci scalar, which becomes

$$\mathcal{R} = -\frac{(n+1)(n+2)}{L^2} - \frac{(n+1)\lambda_1 L z}{L^2} - \frac{n(n-1)\lambda_2 z^2}{L^2}. \tag{35}$$

Using this result in Eq. (34) yields

$$\begin{aligned}
\mathcal{I}_1 &= \frac{L^n}{2\kappa^2} \beta V_n \int_{\varepsilon}^{z_h} dz \left(\frac{2(n+1)}{z^{n+2}} + \frac{(n+1)\lambda_1 L}{z^{n+1}} \right. \\
&\quad \left. + \frac{n(n-1)\lambda_2}{z^n} \right),
\end{aligned} \tag{36}$$

where V_n and β originate from the integration over the boundary coordinates and ε is a regularization parameter. Integrating in the holographic coordinate, one finds

$$\begin{aligned}
\mathcal{I}_1 &= \frac{L^n}{\kappa^2} \beta V_n \left(-\frac{1}{z_h^{n+1}} - \frac{(n+1)\lambda_1 L}{2n} \frac{1}{z_h^n} - \frac{n}{2} \frac{\lambda_2}{z_h^{n-1}} + \frac{1}{\varepsilon^{n+1}} \right. \\
&\quad \left. + \frac{(n+1)\lambda_1 L}{2n} \frac{1}{\varepsilon^n} + \frac{n}{2} \frac{\lambda_2}{\varepsilon^{n-1}} \right).
\end{aligned} \tag{37}$$

Now let us consider the term linear in the trace of $\mathcal{K}$ in the action (21):

$$\mathcal{I}_2 = -\frac{1}{2\kappa^2} \int d^{n+2}x \sqrt{|g|} \lambda_1 \text{Tr } \mathcal{K}. \tag{38}$$

Using the definition (19) and the metric (22), the trace reduces to

$$\text{Tr } \mathcal{K} = n \frac{z}{L}. \tag{39}$$

Consequently, the action (38) becomes

$$\begin{aligned}
\mathcal{I}_2 &= -\frac{1}{2\kappa^2} \beta V_n \int_{\varepsilon}^{z_h} dz \frac{L^{n+2}}{z^{n+2}} \lambda_1 n \frac{z}{L} \\
&= \frac{1}{\kappa^2} \beta V_n \left(\frac{\lambda_1 L}{2z_h^n} - \frac{\lambda_1 L}{2\varepsilon^n} \right).
\end{aligned} \tag{40}$$

The last term in the massive gravity action consists of the quadratic combination

$$\mathcal{I}_3 = -\frac{1}{2\kappa^2} \int d^{n+2}x \sqrt{|g|} \lambda_2 [(\text{Tr } \mathcal{K})^2 - \text{Tr } \mathcal{K}^2]. \tag{41}$$

For the diagonal matrix $\mathcal{K}$, this combination simplifies to

$$[(\text{Tr } \mathcal{K})^2 - \text{Tr } \mathcal{K}^2] = n(n-1) \frac{z^2}{L^2}. \tag{42}$$

Inserting this result into (41) yields

$$\begin{aligned}
\mathcal{I}_3 &= -\frac{1}{2\kappa^2} \beta V_n \int_{\varepsilon}^{z_h} \frac{L^{n+2}}{z^{n+2}} \lambda_2 n(n-1) \frac{z^2}{L^2} \\
&= \frac{L^n}{\kappa^2} \beta V_n \left(\frac{n\lambda_2}{2z_h^{n-1}} - \frac{n\lambda_2}{2\varepsilon^{n-1}} \right).
\end{aligned} \tag{43}$$

Summing the three contributions yields the total action for the planar black hole solution in AdS massive gravity:

$$\mathcal{I}^{(\text{BH})} = \frac{L^n}{\kappa^2} \beta V_n \left(-\frac{1}{z_h^{n+1}} - \frac{\lambda_1 L}{2n} \frac{1}{z_h^n} + \frac{1}{\varepsilon^{n+1}} + \frac{\lambda_1 L}{2n} \frac{1}{\varepsilon^n} \right). \tag{44}$$

The action for the space without a black hole is obtained by taking the limit $z_h \rightarrow \infty$:

$$\mathcal{I}^{(\text{AdS})} = \frac{L^n}{\kappa^2} \beta_0 V_n \left(\frac{1}{\epsilon^{n+1}} + \frac{\lambda_1 L}{2n} \frac{1}{\epsilon^n} \right). \quad (45)$$

Using the relation between β_0 and β expressed in Eq. (32), the action (45) is rewritten as

$$\begin{aligned} \mathcal{I}^{(\text{AdS})} = & \frac{L^n}{\kappa^2} \beta V_n \left(\frac{1}{\epsilon^{n+1}} - \frac{1}{2z_h^{n+1}} - \frac{\lambda_1 L}{2n} \frac{1}{z_h^n} \right. \\ & \left. + \frac{\lambda_1 L}{2n} \frac{1}{\epsilon^n} - \frac{\lambda_2}{2z_h^{n-1}} \right) + \mathcal{O}(\epsilon). \end{aligned} \quad (46)$$

The regularized action is obtained through the subtraction scheme as

$$\begin{aligned} \Delta\mathcal{I} = & \lim_{\epsilon \rightarrow 0} (\mathcal{I}^{(\text{BH})} - \mathcal{I}^{(\text{AdS})}) \\ = & -\frac{L^n}{2\kappa^2} \beta V_n \left(\frac{1}{z_h^{n+1}} - \frac{\lambda_2}{z_h^{n-1}} \right). \end{aligned} \quad (47)$$

B. Stability of the black hole solution

Before we proceed to the Hawking-Page calculations, it is important to analyze the thermodynamic stability of the black hole solution. Its temperature is given by Eq. (24), having two competing terms: one that decreases and another that increases with z_h . The resulting variation of the temperature with z_h , for $n = 2$, is illustrated in Fig. 1(a). The temperature has a minimum when the horizon position z_h has the value

$$z_0 = \left(\frac{n+1}{n-1} \right)^{1/2} \sqrt{\frac{1}{\lambda_2}}. \quad (48)$$

The corresponding minimum temperature that the black hole can reach is

$$T_0 = \frac{1}{2\pi} (n^2 - 1)^{1/2} \sqrt{\lambda_2} + \frac{\lambda_1 L}{4\pi}. \quad (49)$$

Equation (48) requires that λ_2 can take only positive real values. On the other hand, the parameter $\lambda_1 L$ appears only in Eq. (49) as a temperature shift and, in principle, can assume any real value. However, if $\lambda_1 L$ is negative, it is necessary to impose the condition

$$\lambda_1 L \geq -4(n^2 - 1)^{1/2} \sqrt{\lambda_2}, \quad (50)$$

to avoid T_0 from being negative.

The relevant quantities to analyze the black hole thermodynamic stability are the entropy and its derivative, which is related to the heat capacity. The connection with the canonical ensemble thermodynamics is established through the relation between the action and the partition function, $\mathcal{Z} = e^{-\Delta\mathcal{I}}$. Consequently,

$$\ln \mathcal{Z} = -\Delta\mathcal{I} = \frac{L^n}{2\kappa^2} \beta V_n \left(\frac{1}{z_h^{n+1}} - \frac{\lambda_2}{z_h^{n-1}} \right). \quad (51)$$

The entropy can be expressed as

$$S = \beta \langle E \rangle + \ln \mathcal{Z}, \quad (52)$$

where $\langle E \rangle$ is the internal energy, which can be obtained by the relation

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln \mathcal{Z}. \quad (53)$$

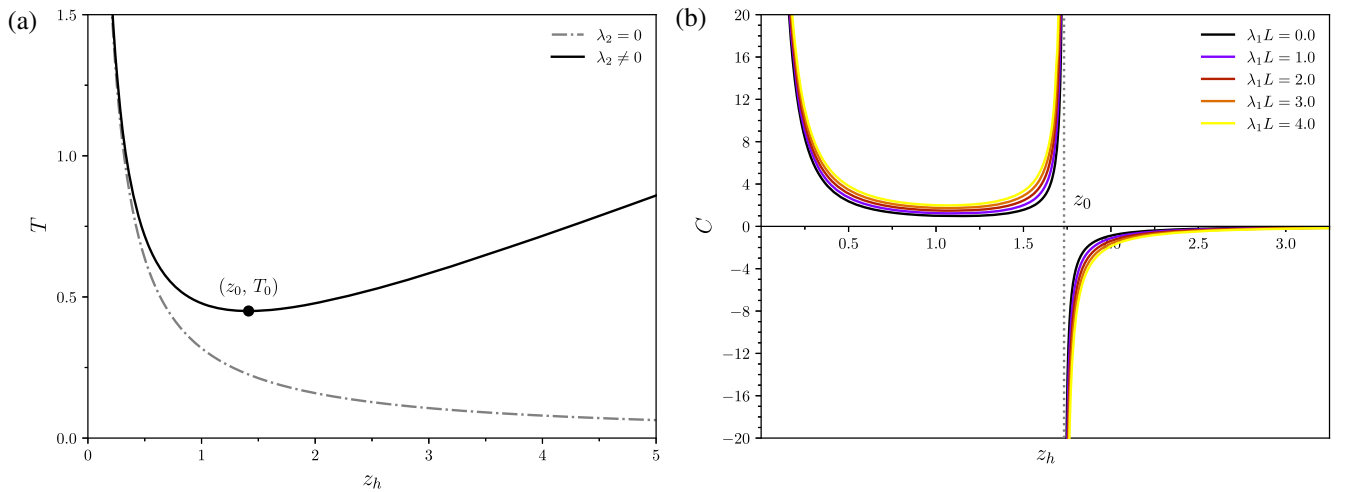


FIG. 1. (a) Schematic plot of the temperature given by Eq. (24). The dashed gray line is the case when $\lambda_2 = 0$, for which temperature of the planar AdS black hole is recovered. (b) Heat capacity for various values of $\lambda_1 L$. The dashed gray line is the z_0 . We set $n = 2$ and $L^2 V_2 / \kappa^2 = 1$ in the plot.

Using Eq. (51), one gets

$$\begin{aligned}\langle E \rangle &= -\frac{L^n}{2\kappa^2} V_n \left[\left(\frac{1}{z_h^{n+1}} - \frac{\lambda_2}{z_h^{n-1}} \right) \right. \\ &\quad \left. + \beta \left(-\frac{(n+1)}{z_h^{n+2}} \frac{\partial z_h}{\partial \beta} + \frac{(n-1)\lambda_2}{z_h^n} \frac{\partial z_h}{\partial \beta} \right) \right] \\ &= \frac{L^n}{2\kappa^2} V_n \left[\frac{n}{z_h^{n+1}} + \frac{n\lambda_2}{z_h^{n-1}} + \frac{\lambda_1 L}{z_h^n} \right],\end{aligned}\quad (54)$$

where the simplification from the first to the second expression is obtained using

$$\frac{\partial z_h}{\partial \beta} = \frac{z_h (n+1) + \lambda_1 L z_h + (n-1)z_h^2}{\beta (n+1) - (n-1)\lambda_2 z_h^2}, \quad (55)$$

derived from (24). Note that this relation expresses the black hole's response to temperature changes. Another interesting observation is that, besides the geometric quantities L and z_h , the parameters λ_1 and λ_2 , and, consequently, the graviton mass, contribute to the system's internal energy in Eq. (54).

Substituting Eqs. (51) and (54) into the entropy definition (52), one finds

$$\begin{aligned}S &= \frac{L^n}{2\kappa^2} \beta V_n \frac{1}{z_h^{n+1}} [(n+1) + \lambda_1 L z_h + (n-1)\lambda_2 z_h^2] \\ &= \frac{L^n}{4G_D} V_n \frac{1}{z_h^n}.\end{aligned}\quad (56)$$

So, one finds that the entropy is a monotonically decreasing function of the horizon position that would go to zero in the limit $z_h \rightarrow \infty$. However, as shown in Fig. 1(a) and by Eq. (24), for $z_h > z_0$ the temperature increases with z_h . Therefore, in this region one would find that the entropy would decrease with temperature. This would violate the thermodynamic laws and indicates that the black hole solutions are unstable in this region.

In order to illustrate the consequences of the decrease of the entropy with temperature, let us calculate the heat capacity for this system. Using the definition

$$C = T \left(\frac{\partial S}{\partial T} \right) \quad (57)$$

and Eq. (56), one obtains

$$C = \frac{nL^n}{4G_D} \frac{V_n}{z_h^n} \left(\frac{(n+1) + \lambda_1 L z_h + (n-1)\lambda_2 z_h^2}{(n+1) - (n-1)\lambda_2 z_h^2} \right). \quad (58)$$

This expression is plotted in Fig. 1(b). From the denominator in parentheses, one notes that the heat capacity exhibits a discontinuity at $z_h = z_0$ where $C \rightarrow \pm\infty$. For $z_h > z_0$, the quadratic term in the

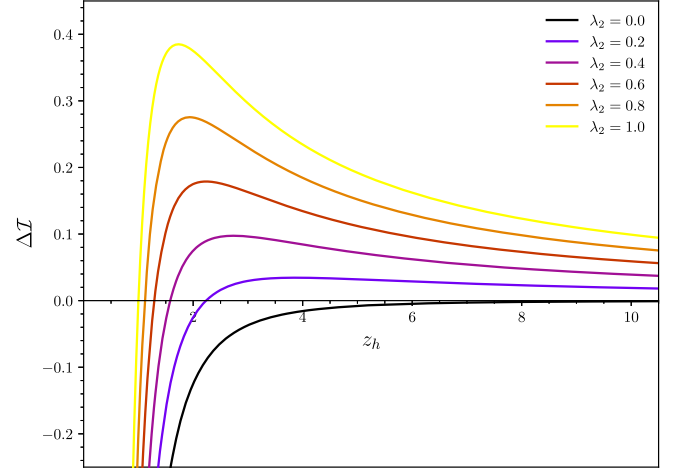


FIG. 2. The action as a function of z_h for various values of λ_2 ; the roots of the functions are the critical horizon positions z_c . We used $n = 2$ and $L^2 V_2 / k^2 = 1$ in the plot.

denominator dominates, so C is negative in this region. This characterizes the instability of black hole solutions in this region. For $z_h < z_0$, the heat capacity is always positive and the BH solutions are stable.

Thus, for a HP transition to occur, the critical horizon position must be smaller than z_0 —or, in terms of the temperature, the critical temperature must be higher than T_0 .

C. Hawking-Page transition

As explained in the previous section, the transition occurs at $\Delta\mathcal{I} = 0$. For $\Delta\mathcal{I} > 0$ the solution without the black hole is the stable phase, while for $\Delta\mathcal{I} < 0$ the BH solution is stable. This condition determines, from Eq. (47) the critical horizon position z_c :

$$z_c = \sqrt{\frac{1}{\lambda_2}}. \quad (59)$$

This result is illustrated in Fig. 2, where $\Delta\mathcal{I}$ is plotted for different values of λ_2 . The critical horizon positions correspond to the values of z_h where the plots cross the horizontal axis. Note that z_c depends only on λ_2 , being independent of both space dimension and the parameter λ_1 . The corresponding critical Hawking-Page temperature is given by

$$T_c = \frac{n}{2\pi} \sqrt{\lambda_2} + \frac{\lambda_1 L}{4\pi}. \quad (60)$$

Note that the critical temperature (60) exhibits the same shift due to the presence of $\lambda_1 L$ as T_0 in (49). The constraint (50) avoids negative values of T_0 . The distribution of the

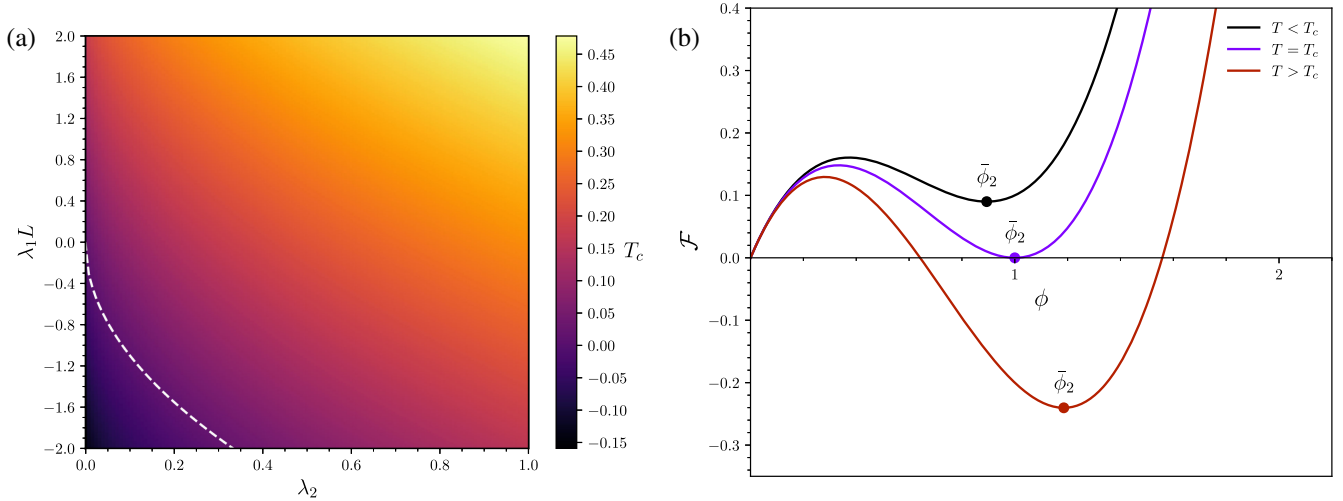


FIG. 3. (a) Critical temperature for $n = 2$ as a function of $\lambda_1 L$ and λ_2 . The white dashed line represents the isothermal curve for the constraint (50) when the equality holds. Values below this line are not allowed. (b) Free energy as function of z_h and T , for the cases $T < T_c$, $T = T_c$, and $T > T_c$, using $n = 2$, $L^2 V_2/k^2 = 1$, $\lambda_1 L = 0$, and $\lambda_2 = 1$ for simplicity.

values of T_c with respect to the parameters $\lambda_1 L$ and λ_2 is shown in Fig. 3(a).

It is important to note that Eqs. (48) and (59) imply that $z_c < z_0$, while Eqs. (49) and (60) imply that $T_0 < T_c$. This result is expected from the analysis of stability.

An interesting point is that, in the limit of high-dimensional spaces (i.e., large n), the prefactors in Eqs. (48) and (49) behave, respectively, as

$$\sqrt{\frac{n+1}{n-1}} = 1 + O(1/n); \quad (61)$$

$$\sqrt{n^2 - 1} = n - O(1/n). \quad (62)$$

In this limit, z_0 and T_0 approach z_c and T_c , respectively.

The cases of more interest are the critical temperatures

$$T_c^{(4)} = \frac{1}{\pi} \sqrt{\lambda_2} + \frac{\lambda_1 L}{4\pi} \quad \text{and} \quad T_c^{(5)} = \frac{3}{2\pi} \sqrt{\lambda_2} + \frac{\lambda_1 L}{4\pi}, \quad (63)$$

corresponding to 4D ($n = 2$) and 5D ($n = 3$) spaces, respectively.

To gain some insight into the thermodynamics of this transition, following [24], let us construct the free energy as

$$\mathcal{F} = \langle E \rangle - TS. \quad (64)$$

Substituting the internal energy (54) and the entropy (56) into the free energy definition (64) yields

$$\mathcal{F} = \frac{L^n}{2\kappa^2} V_n \left[\frac{n}{z_h^{n+1}} + \frac{n\lambda_2}{z_h^{n-1}} + \frac{\lambda_1 L}{z_h^n} - \frac{4\pi}{z_h^n} T \right]. \quad (65)$$

At this point, we consider z_h and T as independent variables. The relation between them will be recovered

by the extremization of $\mathcal{F}$. Given that the free energy in (65) is expressed as a power series in z_h , we can analyze it using Landau's theory of phase transitions. Using the change of variable $\phi = \frac{1}{z_h}$ for simplicity, the free energy is rewritten as

$$\mathcal{F} = \frac{L^n}{2\kappa^2} V_n [n\phi^{n+1} + n\lambda_2\phi^{n-1} + \lambda_1 L\phi^n - 4\pi\phi^n T]. \quad (66)$$

Thermodynamic equilibrium requires that stable states correspond to minima of the free energy. This extremal condition yields

$$\begin{aligned} \frac{\partial}{\partial \phi} \mathcal{F} &= n\bar{\phi}^n [(n+1) + (n-1)\lambda_2\bar{\phi}^{-2} \\ &\quad + \lambda_1 L\bar{\phi}^{-1} - 4\pi T\bar{\phi}^{-1}] = 0, \end{aligned} \quad (67)$$

where the $\bar{\phi}$ denotes the value at a minimum of $\mathcal{F}$. For $n > 1$, we find the trivial solution

$$\bar{\phi}_1 = 0 \quad (68)$$

that represents the space without a black hole (the $z_h \rightarrow \infty$ limit). Using this trivial solution in (66), we find that $\mathcal{F} = 0$ for this space.

The nontrivial solution to (66) is derived from the following equation:

$$(n+1)\bar{\phi}_2^2 + (\lambda_1 L - 4\pi T)\bar{\phi}_2 + (n-1)\lambda_2 = 0. \quad (69)$$

It is important to note that this equation can be directly obtained from (24), such that its solutions correspond to the inverse of the horizon position as a function of the temperature. This solution is given by

$$\bar{\phi}_2 = \frac{-(\lambda_1 L - 4\pi T) \pm \sqrt{(\lambda_1 L - 4\pi T)^2 - 4(n^2 - 1)\lambda_2}}{2(n+1)}. \quad (70)$$

The negative sign in the root square term corresponds to $\bar{\phi}_2 \leq \phi_0 \equiv z_0^{-1}$, representing unstable black hole configurations. The physically relevant stable minimum, describing the black hole space, is the one with positive sign. As it is evident from the first curve in Fig. 3(b), this solution leads to $\mathcal{F} > 0$ for temperatures $T < T_c$, indicating that the dominant minimum corresponds to $\bar{\phi}_1$, the black hole free space. For $T > T_c$, the solution $\bar{\phi}_2$ leads to $\mathcal{F} < 0$, becoming the thermodynamically favored configuration, as shown in the last curve in Fig. 3(b).

This analysis confirms our previous conclusions, obtained from analysis of the regularized actions. The phase transition occurs at $T = T_c$, where the minima of $\bar{\phi}_1$ and $\bar{\phi}_2$ are equal, as seen in the purple line in Fig. 3(b). At this critical temperature, $\bar{\phi}$ exhibits a discontinuous jump, the characteristic behavior of an order parameter in a first-order transition.

D. Case $n = 1$

In a three-dimensional space ($n = 1$), only the polynomial $\mathcal{U}_1$ is nonvanishing when considering matrices in the form (19). Therefore, in this case, we have only a single parameter λ_1 . Thus, the action (21) reduces to

$$\mathcal{I} = -\frac{1}{2\kappa^2} \int d^3x \sqrt{|g|} \left(\mathcal{R} + \frac{2}{L^2} \right) - \frac{1}{2\kappa^2} \int d^3x \sqrt{|g|} \lambda_1 \text{Tr} \mathcal{K}, \quad (71)$$

which leads to a metric of the same form as the AdS black hole, but with the horizon function now expressed as (see the Appendix)

$$f(z) = 1 + \lambda_1 L z - (1 + \lambda_1 L z) \frac{z^2}{z_h^2}, \quad (72)$$

yielding a Hawking temperature given by

$$T = \frac{1}{4\pi z_h} (2 + \lambda_1 L z_h). \quad (73)$$

Note that this temperature exhibits a similar behavior as in the planar AdS case but with a $\frac{\lambda_1 L}{4\pi}$ shift. That is, the temperature decreases continuously, having a minimum value $\lambda_1 L/4\pi$ at infinity. Consequently, the black hole is stable for any finite value of z_h . This can be seen by setting $n = 1$ in (48), which leads to $z_0 \rightarrow \infty$.

Performing the same calculations as in previous sections, we obtain a regularized action expressed as

$$\mathcal{I} = -\frac{V_1 L}{2\kappa^2} \beta \frac{1}{z_h^2}, \quad (74)$$

which is the same as the AdS₃ action in Eq. (15).

The free energy corresponding to action (74) is obtained from (66) using $n = 1$ and $\lambda_2 = 0$. In this case, there is just one minimum,

$$\bar{\phi} = \frac{4\pi T - \lambda_1 L}{2}, \quad (75)$$

that corresponds to the black hole solution. Since only the BH solution is stable, there is no HP transition. From this discussion, one observes that the lowest dimension in which the massive gravity system exhibits a Hawking-Page transition at nonvanishing temperature is $D = 4$ ($n = 2$). For the analysis of Bañados-Teitelboim-Zanelli black holes with phase transitions, see [25].

V. CONCLUSION

In this work, we considered a formulation of massive gravity like the ones studied in [13–15, 19, 26]. Since this formulation breaks the diffeomorphism invariance [18, 22], it is used in order to construct a holographic description of systems without momentum conservation in dc conductivity study.

Reference [19] presents an investigation of the Hawking-Page transition in an AdS₄ space with an $S^2 \times S^1$ boundary, within the massive gravity context. It shows that the higher the graviton mass, the lower the critical temperature.

In most of the applications of holography, the manifolds involved have a planar spatial boundary. This was the main motivation for the present work, where we extended the analysis of [19] to the case of a $\mathbb{R}^n \times S^1$ boundary. Our main result is given in Eq. (60), where we find a non-vanishing critical temperature, proportional to the parameter λ_2 , which encodes the graviton mass. This shows that, in massive gravity, one finds a Hawking-Page transition even in the case of a flat spatial boundary. This result is consistent with Ref. [20], where it was found that the HP transition is present for all the topologies of the boundary.

An interesting point to be remarked is that the presence of a massive graviton makes the black hole unstable within a certain temperature range. This thermodynamical instability was carefully examined, and we obtained the minimum temperature T_0 for the black hole to exist, highlighting the role of dimensionality in this behavior.

We also discussed the particular three-dimensional case, which does not exhibit a Hawking-Page transition. The black hole is always stable. Consequently, we identify $d = 4$ as the lowest critical dimension for the transition to occur.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: EINSTEIN'S EQUATIONS

Taking the variation of the action (21) with respect to the metric g^{mn} gives the following Einstein's equations [18]

$$\mathcal{R}_{mn} - \frac{1}{2}\mathcal{R}g_{mn} + \frac{1}{2}\Lambda g_{mn} + \lambda_1 A_{mn} + \lambda_2 L_{mn} = 0, \quad (\text{A1})$$

where $\mathcal{R}_{mn}$ is the Ricci tensor and

$$A_{mn} = -\frac{1}{2}(g_{mn}\text{Tr}\mathcal{K} - \mathcal{K}_{mn}) \quad (\text{A2})$$

and

$$L_{mn} = -\left(\mathcal{K}_{mn}^2 - \mathcal{K}_{mn}\text{Tr}\mathcal{K} + \frac{1}{2}g_{mn}[(\text{Tr}\mathcal{K})^2 - \text{Tr}\mathcal{K}^2]\right) \quad (\text{A3})$$

are parts of the stress-energy tensor of the graviton. The tensors (A2) and (A3) are obtained using the definition of $\mathcal{K}$ in Eq. (19) and writing the variations:

$$\delta\text{Tr}\mathcal{K} = \frac{\partial}{\partial g^{mn}}(g^{mn}f_{nm})^{1/2}\delta g^{mn} = \frac{1}{2}\mathcal{K}_{mn}\delta g^{mn}, \quad (\text{A4})$$

$$\delta(\text{Tr}\mathcal{K})^2 = \mathcal{K}_{mn}\text{Tr}\mathcal{K}\delta g^{mn}, \quad (\text{A5})$$

$$\delta\text{Tr}\mathcal{K}^2 = \frac{\partial}{\partial g^{mn}}[(g^{ik}f_{kq})^{1/2}(g^{qk}f_{ki})^{1/2}]\delta g^{mn} = \mathcal{K}_{mn}^2\delta g^{mn}. \quad (\text{A6})$$

Choosing metric ansatz with Poincaré AdS form

$$ds^2 = \frac{L^2}{z^2}(-f(z)dt^2 + f(z)^{-1}dz^2 + d\vec{x}^2), \quad (\text{A7})$$

and using (20). Equation (A1) leads to the following linear differential equation for the tt component:

$$zf'(z) - (n+1)f(z) + (n+1) + \lambda_1 Lz + \lambda_2(n-1)z^2 = 0. \quad (\text{A8})$$

Solving for $f(z)$, one finds

$$f(z) = z^{n+1} \left[\int dz \left(-\frac{(n+1)}{z^{n+2}} - \frac{\lambda_1 L}{z^{n+1}} - \frac{\lambda_2(n-1)}{z^n} \right) + C \right]. \quad (\text{A9})$$

At this point, note that, for $n = 1$, the λ_2 term vanishes. This occurs because, for a 3×3 matrix as defined in (19), only the $\mathcal{U}_1 = \text{Tr}\mathcal{K}$ symmetric polynomial is nonzero.

First, let us consider the solution for the case $n > 1$, which reads

$$f(z) = 1 + \frac{\lambda_1 L}{n}z + \lambda_2 z^2 - Mz^{n+1}. \quad (\text{A10})$$

The integration constant C has been redefined as $-M$, which can be determined by requiring that $f(z_h) = 0$. Thus,

$$M = \frac{1}{z_h^{n+1}} \left(1 + \frac{\lambda_1 L z_h}{n} + \lambda_2 z_h^2 \right), \quad (\text{A11})$$

indicating that M is proportional to the black hole's energy density [see (54)]. Substituting this result back into (A10), one finds the horizon function (23).

For the case $n = 1$, the horizon function follows from Eq. (A9):

$$f(z) = 1 + \lambda_1 Lz - (1 + \lambda_1 Lz_h) \frac{z^2}{z_h^2}, \quad (\text{A12})$$

where we already imposed the condition $f(z_h) = 0$. Setting $\lambda_1 = 0$, we recover the usual AdS₃ black hole solution, as can be seen in [27].

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