

# Exact solution of the Dirac equation in spherical Rindler coordinates

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The exact solution to the Dirac equation corresponding to a bound state in the spherical Rindler metric is derived and its relation to the near-horizon Schwarzschild metric is investigated. It is shown how to represent these solutions with a specially chirped free-electron wave packet. The spherical Rindler frame stands out as being invariant under Lorentz transformations, in contrast with the  $1+1$ D Rindler coordinates, and has been used in the description of the so-called *expanding bubble* spacetime. With heuristic modification of the wave function of the chirped free-electron wave packet quantitative predictions for the quantum properties of the electron crossing the black hole horizon are obtained.

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## I. INTRODUCTION

According to the equivalence principle, the gravitational field can be locally eliminated by an appropriate transformation to a noninertial frame that generally may allow to map electron dynamics in a gravitational field, beyond the effect of the tidal forces, onto that of a free electron in the accelerated frame. In this way the solution to the Dirac equation for a free electron in a constant accelerating frame of Ref. [1], in the so-called 1D Rindler coordinates, describes the electron dynamics in a constant gravitational field, which approximately could fit to a real gravitational setting. However, such a frame is clearly not invariant under general Lorentz transformations, that is, there is a preferred spatial direction along which the linear acceleration takes place. Such a deficiency is removed by a frame which is characterized by uniformly accelerating spherical coordinates, the so called *spherical Rindler* frame.

The spherical Rindler coordinates have been used in the description of an expanding event horizon in flat spacetime [2,3], as well as in the five-dimensional *Witten bubble* spacetime [4–8]. While it was shown in [7] that in such a bubble universe no bound state is possible for massless scalar particles, here we explicitly find a bound state solution for a Dirac fermion.

The results presented in this paper are used in the companion Letter [9] to mimic a gravitational wave function of an electron in the Schwarzschild field by a

free electron wave packet. In particular, the analytical solution of the Dirac equation for the electron dynamics in the near-horizon region of a black hole derived here serves as the benchmark for the results presented in the companion Letter. Note that the general solution of the Dirac equation for the Schwarzschild metric can only be calculated numerically, see, e.g., [10–13]. Further, we find the free electron exact wave function in the specially designed 3D local accelerated frame (3D local Rindler frame with a flat metric), corresponding to the locally constant acceleration along the radial direction. While the latter solution in the form of spherical waves may have applications for the bubble universe [4–8,14], quantum gravity [15], and string theory [16], here it is used to mimic the near-horizon Schwarzschild dynamics of the electron and leads us directly to the design of the specially chirped free electron wave packet.

We discuss the correspondence between free and Rindler solutions, and the connection of the latter to the gravitational near-horizon solution of the Dirac equation. While the general features of the Rindler solution mimic the Schwarzschild one, there are small differences stemming from the time dependence of the 3D Rindler metric, which is absent in the Schwarzschild case. However, we underline an important similarity between Schwarzschild and Rindler solutions that both possess a horizon, albeit it being fictitious in the latter. In fact, the Rindler horizon disappears once the acceleration is turned off. It is this feature that allows us to mimic the near horizon dynamics with the free wave packet upon introducing a cutoff to the momentum distribution.

Applying the mapping technique described above, the near-horizon stationary gravitational solution is emulated by the free electron wave packet. Including the chirping parameter in the free electron wave packet solution, we introduce the decay of the wave function and obtain the

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nonstationary wave function of the electron for the Schwarzschild problem. This wave function allows for the quantitative analysis of the quantum features of the electron dynamics near the horizon and during crossing the horizon.

The structure of the paper is the following. The near-horizon analytical solution for a black hole is derived in Sec. II. The 3D Rindler solution of the Dirac equation is presented in Sec. III. The construction of the chirped free electron wave packet is discussed in Sec. IV. The quantum features of the electron dynamics near the horizon and during crossing the horizon are investigated in Sec. V, and our conclusion and outlook are given in Sec. VI.

## II. QUANTUM DYNAMICS IN A BLACK HOLE

We consider the electron quantum dynamics in a black hole. The recently developed covariant relativistic dynamical inversion (CRDI) [17] technique is used as a general method for constructing exact solutions of the Dirac equation in the presence of gravitational fields and in accelerated frames. The main concept underlying CRDI employs the following most general parametrization of a matrix spinor in Hestenes representation [18–21],

$$\Psi = \sqrt{\rho_s} \exp(i\beta_s/2) \mathcal{R}, \quad (1)$$

which evinces that any spinor  $\Psi$  can be constructed from the one in the rest frame via boost and rotation operations, included in the operator  $\mathcal{R}$  defined as the local Lorentz transformation connecting the observer's frame with the rest frame of the electron. The quantum pseudoscalar parameter  $\beta_s$ , the so-called *chiral angle* or Takabayasi angle [22,23], describes the additional degree of freedom associated with the electron internal spin quantum dynamics. Finally, the parameter  $\rho_s$  is the positive definite scalar density, and  $\mathbf{i} = \gamma_0\gamma_1\gamma_2\gamma_3$  is defined, with the Dirac gamma-matrices  $\gamma_\alpha$  ( $\alpha = 0, 1, 2, 3$ ). The matrix spinor  $\Psi$  satisfies the covariant Hestenes-Dirac equation

$$\gamma^\mu \left( \partial_\mu \ln \sqrt{\rho_s} - \frac{\mathbf{i}}{2} \partial_\mu \beta_s - \frac{1}{2} R_{ij\mu} \sigma^{ij} \right) \Psi \gamma^2 \gamma^1 - \frac{1}{\lambda} \Psi \gamma^0 = 0, \quad (2)$$

with the antisymmetric matrix 4-tensor  $\sigma^{ij}$  [24], the electron Compton wavelength  $\lambda = 1/m$ , and the electron mass  $m$ . Relativistic units with  $\hbar = c = G = 1$  (Planck constant  $\hbar$ , speed of light  $c$ , gravitational constant  $G$ ) are applied throughout. The  $R_{ij\mu}$  tensors are connected with the spinorial connection  $\Omega_{ij\mu}$  as follows

$$\frac{1}{2} \Omega_{ij\mu} \sigma^{ij} - \mathcal{R} \partial_\mu \mathcal{R}^{-1} = -\frac{1}{2} R_{ij\mu} \sigma^{ij}. \quad (3)$$

The spinorial connection is defined by

$$\Omega_{j\mu}^i = e_j^\nu e_\sigma^i (\Lambda_{\nu\mu}^\sigma - e_\mu^\sigma \partial_\nu e_\nu^\sigma), \quad (4)$$

$$\Lambda_{\alpha\nu}^\sigma = \frac{g^{\sigma\rho}}{2} (\partial_\alpha g_{\rho\nu} + \partial_\nu g_{\alpha\rho} - \partial_\rho g_{\alpha\nu}), \quad (5)$$

where  $e_j^\nu$  are the tetrads and  $g_{\mu\nu}$  is the metric tensor.

### A. Schwarzschild solution

Here we derive the analytical solution of the Dirac equation for the electron near the horizon of a Schwarzschild hole. The metric applied for the description of the electron quantum dynamics is,

$$ds^2 = \left( 1 - \frac{1}{r\kappa_*} \right) dt^2 - \frac{dr^2}{1 - \frac{1}{r\kappa_*}} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (6)$$

with  $r\kappa_* > 1$ ,  $\kappa_* = \frac{1}{2M}$ , and  $M$  is the black hole mass. The Hestenes-Dirac equation (2) in this metric is

$$\begin{aligned} & \left( \frac{\gamma^0 \partial_t}{\sqrt{1 - \frac{1}{r\kappa_*}}} + \frac{\gamma^\phi \partial_\phi}{r \sin \theta} + \frac{\gamma^\theta \partial_\theta}{r} - \frac{\gamma^r}{r} \right) \Psi \gamma_2 \gamma_1 \\ & + \gamma^r \left[ \sqrt{1 - \frac{1}{r\kappa_*}} \left( \partial_r + \frac{1}{4r(\kappa_* r - 1)} + \frac{1}{r} \right) \right] \Psi \gamma_2 \gamma_1 \\ & - \frac{1}{\lambda} \Psi \gamma^0 = 0. \end{aligned} \quad (7)$$

where  $\gamma^r = e^{-\gamma_2 \gamma_1 \phi/2} e^{-\gamma_1 \gamma_3 \theta/2} \gamma^3 e^{\gamma_1 \gamma_3 \theta/2} e^{\gamma_2 \gamma_1 \phi/2}$ ,  $\gamma^\theta = e^{-\gamma_2 \gamma_1 \phi/2} e^{-\gamma_1 \gamma_3 \theta/2} \gamma^1 e^{\gamma_1 \gamma_3 \theta/2} e^{\gamma_2 \gamma_1 \phi/2}$ ,  $\gamma^\phi = e^{-\gamma_2 \gamma_1 \phi/2} e^{-\gamma_1 \gamma_3 \theta/2} \times \gamma^2 e^{\gamma_1 \gamma_3 \theta/2} e^{\gamma_2 \gamma_1 \phi/2}$  are the rotated gamma matrices. Aiming at the analytical solution, the goal now is to separate variables in Eq. (7), taking advantage of the spherical symmetry of the metric. Firstly, multiplying Eq. (7) to the right by  $e^{\gamma_3 \gamma_2 \pi/4}$  and defining  $\Phi = (1 - (r\kappa_*)^{-1})^{1/4} \sqrt{\sin \theta} e^{\gamma_1 \gamma_3 \theta/2} \times e^{\gamma_2 \gamma_1 \phi/2} \Psi e^{-\gamma_2 \gamma_1 \phi/2} e^{\gamma_3 \gamma_2 \pi/4}$ , leads to

$$\begin{aligned} & \left( \frac{\gamma^0 \partial_t}{\sqrt{1 - \frac{1}{r\kappa_*}}} + \frac{\gamma^1 \partial_\theta}{r} + \sqrt{1 - \frac{1}{r\kappa_*}} \gamma^3 \right) \Phi \gamma_3 \gamma_1 - \frac{\gamma^2 \Phi}{2r \sin \theta} \\ & - \frac{1}{\lambda} \Phi \gamma^0 = 0. \end{aligned} \quad (8)$$

The separation between the  $\theta$  and radial variables is facilitated by application of the rotation  $e^{\gamma_3 \gamma_2 \pi/4}$ , which is a passive Lorentz transformation that changes the direction of the spin in the observer's frame from the  $z$ - to the  $y$  axis.

In order to separate radial and angular variables we employ the following matrix spinor ansatz

$$\Phi = \eta(t, r) \mathcal{G}(\theta) + \gamma_2 \gamma_1 \eta(t, r) \mathcal{F}(\theta), \quad (9)$$

with the following properties  $\gamma_5 \eta = \eta \gamma_5$ ,  $\gamma_0 \mathcal{G} \gamma_0 = \mathcal{G}$ ,  $\gamma_0 \mathcal{F} \gamma_0 = \mathcal{F}$ ,  $\gamma_3 \gamma_1 \mathcal{G} = \mathcal{G} \gamma_3 \gamma_1$ , and  $\gamma_3 \gamma_1 \mathcal{F} = \mathcal{F} \gamma_3 \gamma_1$ . These

conditions ensure that both  $\mathcal{F}$  and  $\mathcal{G}$  correspond to a rotation on the spin plane (the plane perpendicular to the spin). The  $\mathcal{G}(\theta)$  and  $\mathcal{F}(\theta)$  matrix spinors obey the following equations

$$\partial_\theta \mathcal{F} \gamma_3 \gamma_1 - \frac{\mathcal{G}}{2 \sin \theta} = \kappa \mathcal{F}, \quad \partial_\theta \mathcal{G} \gamma_3 \gamma_1 + \frac{\mathcal{F}}{2 \sin \theta} = -\kappa \mathcal{G}, \quad (10)$$

where  $\kappa$  is the usual quantum number related to the total angular momentum  $j$  [24]:  $\kappa = -(j + 1/2)$ , if  $j = l + 1/2$ , or  $\kappa = j + 1/2$ , if  $j = l - 1/2$ , with the quantum number  $l$  of the orbital angular momentum. The angular equations (10) can be decoupled if one defines  $\mathcal{F} = -\gamma_3 \gamma_1 \omega$  and  $\mathcal{G} = \gamma_0 \omega^\dagger \gamma_0 = \tilde{\omega}$ , leading to the angular equation

$$\partial_\theta \omega + \kappa \omega \gamma_3 \gamma_1 - \frac{\tilde{\omega}}{2 \sin \theta} = 0, \quad (11)$$

which has a solution in the form of the relativistic spherical harmonics with spin, introduced by Hermann Weyl [25],

$$\omega = B \sqrt{\sin \theta} e^{\frac{\gamma_1 \gamma_3 \theta}{2}} \left[ \frac{\gamma_3 \gamma_1}{\kappa} \frac{dP_{\kappa-1}(\cos(\theta))}{d\theta} + P_{\kappa-1}(\cos(\theta)) \right] \quad (12)$$

where  $B$  is a normalization constant, and the  $P_l(x)$  are Legendre polynomials. The separation of variables is accomplished by a series of Lorentz transformations of the matrix spinor with respect to the rotating frame, implementing the so-called Fermi-Walker transport.

The radial equation now reads

$$\frac{\gamma^0 \partial_r \eta \gamma_3 \gamma_1}{\sqrt{1 - \frac{1}{\kappa_* r}}} + \sqrt{1 - \frac{1}{\kappa_* r}} \gamma^3 \partial_r \eta \gamma_3 \gamma_1 - \frac{\kappa \gamma^1 \eta}{r} - \frac{1}{\lambda} \eta \gamma^0 = 0. \quad (13)$$

The general solution to Eq. (13) for the energy eigenstate with the eigenvalue  $\varepsilon$  can be written in the following form

$$\eta(t, r) = A[f(r) + \gamma^0 \gamma^1 g(r)] e^{-\gamma_3 \gamma_1 \varepsilon t}, \quad (14)$$

from which it follows

$$\begin{aligned} \frac{f(r) \left( -\sqrt{1 - \frac{1}{\kappa_* r}} + \frac{\varepsilon}{m} \right)}{\lambda \left( 1 - \frac{1}{\kappa_* r} \right)} + g'(r) - \frac{\kappa g(r)}{r \sqrt{1 - \frac{1}{\kappa_* r}}} &= 0, \\ -\frac{g(r) \left( \frac{\varepsilon}{m} + \sqrt{1 - \frac{1}{\kappa_* r}} \right)}{\lambda \left( 1 - \frac{1}{\kappa_* r} \right)} + f'(r) + \frac{\kappa f(r)}{r \sqrt{1 - \frac{1}{\kappa_* r}}} &= 0. \end{aligned} \quad (15)$$

The latter equation in the general case can only be solved numerically, see, e.g., [10–13]. However, we find an analytical solution in the near-horizon approximation, which requires a suitable change of coordinates, as Eq. (15) diverges approaching the horizon  $\kappa_* r \rightarrow 1$ .

Summarizing the applied mathematical technique, starting from Eq. (2) a series of coordinate and Lorentz transformations are applied to the matrix spinor and the Hestenes-Dirac equation in order to be able to separate variables. The final result, Eq. (13), is such that there is no relative rotation or boost between the frame of the observer and the rest frame of the electron; thus all angular dependence of the spinor has been removed, leaving only the total angular momentum quantum number  $\kappa$  behind. Once the radial spinor is found, the passive rotation applied to the observer's frame is undone thus leading to a relative rotation with respect to the rest frame of the electron.

## B. The near-horizon limit

For the application of the near-horizon approximation, we use a modified version of the known Kruskal-Szekeres coordinates, see Appendix A, since they eliminate the singularity at the horizon. We consider the following change of coordinates

$$T = \rho \sinh(\kappa_* \tau), \quad R = \rho \cosh(\kappa_* \tau),$$

where the new radial coordinate  $\rho$  is connected with the original radial coordinate  $r$  by the expression  $\kappa_*^2 \rho^2 = (\kappa_* r - 1) e^{\kappa_* r}$ . In such a coordinate system, which we dub Rindler-Kruskal coordinates (RKC), the Hestenes-Dirac equation is given by

$$\begin{aligned} \left[ \frac{\sqrt{r(\rho)} e^{\frac{1}{2} \kappa_* r(\rho)} \gamma^0 \partial_\tau}{2 \sqrt{\kappa_*} \rho} + \frac{\gamma^1 \partial_\theta}{r(\rho)} + \frac{1}{2} e^{\frac{1}{2} \kappa_* r(\rho)} \sqrt{\kappa_* r(\rho)} \gamma^3 \right. \\ \left. \times \left( \partial_\rho + \frac{1}{2\rho} \right) \right] \Phi \gamma_3 \gamma_1 - \frac{\gamma^2 \Phi}{2r(\rho) \sin \theta} - \frac{1}{\lambda} \Phi \gamma^0 = 0, \end{aligned} \quad (16)$$

where

$$\begin{aligned} \Phi &= \frac{1}{2} e^{-\frac{1}{4} \kappa_* r(\rho)} (\kappa_* r(\rho))^{3/4} \sqrt{\sin \theta} e^{\gamma_1 \gamma_3 \theta/2} e^{\gamma_2 \gamma_1 \phi/2} \\ &\quad \times \Psi e^{-\gamma_2 \gamma_1 \phi/2} e^{\gamma_3 \gamma_2 \pi/4}, \end{aligned}$$

from which we extract the following equation for  $\eta$

$$\begin{aligned} \frac{\sqrt{e^{\kappa_* r(\rho)} \kappa_* r(\rho)}}{2} \left[ \frac{\gamma^0 \partial_\tau \eta \gamma_3 \gamma_1}{\kappa_* \rho} + \gamma^3 \left( \partial_\rho + \frac{1}{2\rho} \right) \eta \gamma_3 \gamma_1 \right] \\ - \frac{\kappa \gamma^1 \eta}{r(\rho)} - \frac{1}{\lambda} \eta \gamma^0 = 0. \end{aligned} \quad (17)$$

We look for the general solution to Eq. (17) via the following ansatz

$$\eta(\tau, \rho) = \frac{A}{\sqrt{\kappa_*} \rho} [f(\rho) + \gamma^0 \gamma^1 g(\rho)] e^{-\gamma_3 \gamma_1 \varepsilon \tau}, \quad (18)$$

where the functions  $f(\rho)$  and  $g(\rho)$  fulfill

$$\begin{aligned}
f'(\rho) - \frac{2g(\rho)}{\lambda\sqrt{\kappa_*r(\rho)e^{\kappa_*r(\rho)}}} + \frac{2\kappa f(\rho)}{r(\rho)\sqrt{\kappa_*r(\rho)e^{\kappa_*r(\rho)}}} &= \frac{\varepsilon g(\rho)}{\kappa_*\rho}, \\
g'(\rho) - \frac{2f(\rho)}{\lambda\sqrt{\kappa_*r(\rho)e^{\kappa_*r(\rho)}}} - \frac{2\kappa g(\rho)}{r(\rho)\sqrt{\kappa_*r(\rho)e^{\kappa_*r(\rho)}}} &= -\frac{\varepsilon f(\rho)}{\kappa_*\rho}.
\end{aligned} \quad (19)$$

These equations have a convenient form without singularity at the horizon to apply the near-horizon limit. In the near-horizon approximation  $\kappa_*\rho \ll 1$ , the metric reads,

$$ds^2 = 4e^{-1}(\rho^2\kappa_*^2 d\tau^2 - d\rho^2) - \kappa_*^{-2}(\sin^2\theta d\phi^2 + d\theta^2), \quad (20)$$

where we have used also  $r \approx \kappa_*^{-1}$ . In this limit Eq. (19) for the radial functions  $f(\rho)$  and  $g(\rho)$  is simplified to the near horizon form

$$\begin{aligned}
g(\rho)\left(\frac{2m}{\sqrt{e}} + \frac{\varepsilon}{\kappa_*\rho}\right) + f'(\rho) - \frac{2\kappa_*f(\rho)}{\sqrt{e}} &= 0, \\
-f(\rho)\left(-\frac{2m}{\sqrt{e}} + \frac{\varepsilon}{\kappa_*\rho}\right) + g'(\rho) + \frac{2\kappa_*g(\rho)}{\sqrt{e}} &= 0.
\end{aligned} \quad (21)$$

In the limit  $\kappa_*\rho \ll 1$ , it is justified to neglect the term  $\frac{2m}{\sqrt{e}}$  with respect to  $\frac{\varepsilon}{\kappa_*\rho}$  in the equations, which allows us to find the analytical solution of the Dirac equation for the near-horizon of a black hole,

$$\begin{aligned}
\frac{f(\rho)}{C_1\sqrt{\kappa_*\rho}} &= K_{\frac{1}{2} - \frac{i\varepsilon}{\kappa_*}}\left(\frac{2|\kappa_*\rho|}{\sqrt{e}}\right) - K_{\frac{1}{2} + \frac{i\varepsilon}{\kappa_*}}\left(\frac{2|\kappa_*\rho|}{\sqrt{e}}\right), \\
\frac{ig(\rho)}{C_1\sqrt{\kappa_*\rho}} &= K_{\frac{1}{2} - \frac{i\varepsilon}{\kappa_*}}\left(\frac{2|\kappa_*\rho|}{\sqrt{e}}\right) + K_{\frac{1}{2} + \frac{i\varepsilon}{\kappa_*}}\left(\frac{2|\kappa_*\rho|}{\sqrt{e}}\right),
\end{aligned} \quad (22)$$

where  $C_1$  is a normalization constant. The other independent solution of Eq. (21) in terms of the Bessel  $I_\nu(z)$  function is discarded because it diverges at infinity. The solution of Eq. (22) is valid only near the horizon, which does not allow us to derive the energy eigenvalues  $\varepsilon$ , as for the latter imposing the boundary conditions at the origin and at infinity is required.

For the case with  $\kappa = -1$ , corresponding to the spin-up electron with the orbital angular momentum  $l = 0$ , the Dirac spinor  $\psi$ , which is the leftmost column of the matrix spinor  $\Psi$ , is given by

$$\psi = \frac{C_1 e^{-i\varepsilon\tau}}{\sqrt{\kappa_*\rho}} \begin{pmatrix} f(\rho) \\ 0 \\ ig(\rho)\cos(\theta) \\ ig(\rho)\sin(\theta)e^{i\phi} \end{pmatrix}. \quad (23)$$

In Fig. 1 the probability density for the near-horizon spinor (23) is shown, the number of fringes indicating a bound state with a high principal quantum number  $n$ .

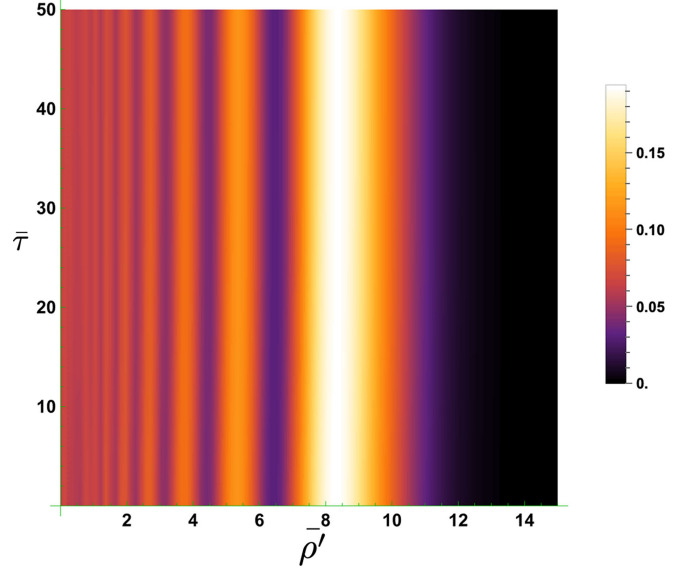


FIG. 1. Spacetime profile of the electron density as a function of  $(\rho', \bar{\tau})$ , with  $\rho' = \rho'/\lambda$  and  $\bar{\tau} = \tau/\lambda$ , for the electron wave function in the near-horizon RKC with  $\varepsilon/\kappa_* = 10$  and  $\lambda\kappa_* = 0.1$ .

### III. SOLUTION TO THE DIRAC EQUATION IN SPHERICALLY SYMMETRIC RINDLER COORDINATES

Here we show that the near-horizon Schwarzschild solution can be reproduced by the solution to the Dirac equation in a local accelerated frame described by the flat metric in spherical Rindler coordinates.

First, let us consider more closely the metric of Eq. (20). After redefining the radial variable as  $\rho' = 2\rho/e^{1/2}$ , it reads

$$ds^2 = (\rho'^2\kappa_*^2 d\tau^2 - d\rho'^2) - \kappa_*^{-2}(\sin^2\theta d\phi^2 + d\theta^2). \quad (24)$$

One sees that it has the same form as the Rindler metric [26] at  $\theta = \phi = \text{const}$ , i.e., considering only radial motion. The exact solution to the Dirac equation in the 1 + 1D Rindler coordinates was given in Ref. [1]. Here we derive the solution of the Dirac equation in spherical Rindler coordinates, which corresponds to a locally constant acceleration along the radial direction at each fixed angular coordinates  $(\theta, \phi)$ . The 3 + 1D Rindler coordinates are related to the Minkowski coordinates as

$$\begin{aligned}
t &= \xi \sinh(\kappa\tau), \\
x &= \xi \cosh(\kappa\tau) \sin\theta \sin\phi, \\
y &= \xi \cosh(\kappa\tau) \sin\theta \cos\phi, \\
z &= \xi \cosh(\kappa\tau) \cos\theta,
\end{aligned} \quad (25)$$

leading to the metric

$$ds^2 = \xi^2\kappa^2 d\tau^2 - d\xi^2 - \xi^2 \cosh^2(\kappa\tau)(\sin^2\theta d\phi^2 + d\theta^2), \quad (26)$$

where  $\kappa > 0$  is proportional to the norm of the electron's acceleration four-vector. The metric (26) is the generalization of the Rindler metric to 3 + 1-dimensional space-time, and it is, apart from the time dependent term  $\cosh(\kappa\tau)^2$  in the angular part, similar to the near-horizon metric Eq. (24). The Hestenes-Dirac equation in the spherical Rindler metric reads

$$\left(\frac{\gamma^0 \partial_\tau \cosh(\kappa\tau)}{\kappa} + \frac{\gamma^\phi \partial_\phi}{\sin \theta} + \gamma^\theta \partial_\theta - \gamma^\xi\right) \Psi \gamma_2 \gamma_1 + \xi \cosh(\kappa\tau) \gamma^\xi \left(\partial_\xi + \frac{3}{2\xi}\right) \Psi \gamma_2 \gamma_1 - \frac{\xi \cosh(\kappa\tau)}{\lambda} \Psi \gamma^0 = 0, \quad (27)$$

where  $\gamma^\xi = e^{-\gamma_2 \gamma_1 \phi/2} e^{-\gamma_1 \gamma_3 \theta/2} \gamma^3 e^{\gamma_1 \gamma_3 \theta/2} e^{\gamma_2 \gamma_1 \phi/2}$ ,  $\gamma^\theta = e^{-\gamma_2 \gamma_1 \phi/2} e^{-\gamma_1 \gamma_3 \theta/2} \gamma^1 e^{\gamma_1 \gamma_3 \theta/2} e^{\gamma_2 \gamma_1 \phi/2}$ ,  $\gamma^\phi = e^{-\gamma_2 \gamma_1 \phi/2} e^{-\gamma_1 \gamma_3 \theta/2} \times \gamma^2 e^{\gamma_1 \gamma_3 \theta/2} e^{\gamma_2 \gamma_1 \phi/2}$ .

The variables in Eq. (27) are separated similarly to the method used in Sec. II, see Appendix B. The radial part of the Rindler wave function fulfills the equation,

$$\frac{\gamma^0 \partial_\tau \eta_R}{\kappa \xi} \gamma_3 \gamma_1 + \gamma^3 \left(\partial_\xi + \frac{1}{2\xi}\right) \eta_R \gamma_3 \gamma_1 + \frac{\kappa \gamma^1 \eta_R}{\xi \cosh(\kappa\tau)} - \frac{1}{\lambda} \eta_R \gamma_0 = 0, \quad (28)$$

where the variables  $\xi$  and  $\tau$  cannot be separated due to the term proportional to  $\kappa$  ( $\kappa \neq 0$  for a 1/2-spin particle). Nevertheless, we derive the solution for Eq. (28) in the Appendix B, see Eq. (B6) which in the limit  $\kappa\tau \gg 1$  takes on the simplified form

$$\eta_R \approx \frac{(2 + 2i) \sqrt{4\zeta^2 + 1} c_1}{\pi} [F(\xi) + \gamma^0 \gamma^1 G(\xi)] e^{-\gamma_3 \gamma_1 \zeta \kappa \tau},$$

$$F(\xi) = \left| \sqrt{2} (i K_{i\zeta+\frac{1}{2}}(\xi/\lambda) - K_{i\zeta-\frac{1}{2}}(\xi/\lambda)) \right|,$$

$$G(\xi) = (K_{i\zeta+\frac{1}{2}}(\xi/\lambda) - i K_{i\zeta-\frac{1}{2}}(\xi/\lambda)), \quad (29)$$

with the real normalization constant  $c_1$ . It is the eigenstate in the 3D Rindler metric with the eigenvalue  $\varepsilon = \kappa\zeta$ . The Dirac spinor in the Lab frame (the leftmost column of the matrix spinor) has the form

$$\psi = \frac{4(-1)^{1/4} e^{-i\zeta\kappa\tau} \text{sech}(\kappa\tau)}{\pi \xi} \begin{pmatrix} \psi_1 \\ 0 \\ \psi_3 \cos \theta \\ \psi_3 e^{i\phi} \sin \theta \end{pmatrix}, \quad (30)$$

where

$$\begin{aligned} \psi_1 &= -K_{i\zeta+\frac{1}{2}}(\xi/\lambda) (2\zeta - i \tanh(\kappa\tau) + i \text{sech}(\kappa\tau)) \\ &\quad - K_{i\zeta-\frac{1}{2}}(\xi/\lambda) (2i\zeta + \tanh(\kappa\tau) + \text{sech}(\kappa\tau)), \\ \psi_3 &= K_{i\zeta-\frac{1}{2}}(\xi/\lambda) (-2i\zeta - \tanh(\kappa\tau) + \text{sech}(\kappa\tau)) \\ &\quad + K_{i\zeta+\frac{1}{2}}(\xi/\lambda) (2\zeta - i \tanh(\kappa\tau) - i \text{sech}(\kappa\tau)). \end{aligned}$$

The behavior of the wave function in the lab frame can be illustrated inspecting the electron probability density

$$\begin{aligned} \frac{\sqrt{-g} \text{Tr}[\Psi \Psi^\dagger]}{4} &= \frac{64c_1^2}{\pi^2} \left[ (4\zeta^2 + 1) K_{i\zeta-\frac{1}{2}}(\xi/\lambda) K_{i\zeta+\frac{1}{2}}(\xi/\lambda) \right. \\ &\quad + 2\zeta \text{sech}(\kappa\tau) \left( K_{i\zeta-\frac{1}{2}}(\xi/\lambda)^2 \right. \\ &\quad \left. \left. + K_{i\zeta+\frac{1}{2}}(\xi/\lambda)^2 \right) \right] \kappa \xi \sin(\theta), \quad (31) \end{aligned}$$

where  $g$  is the determinant of the metric tensor. Here,  $\zeta$  is the real valued variable separation constant which is the analog of the energy parameter  $\varepsilon/\kappa_*$  of the Schwarzschild solution. The density of Eq. (31) has a slow time dependence via  $\text{sech}(\kappa\tau)$  which emerges from the time dependence of the metric Eq. (26) in contrast to the gravitational near-horizon one. However, this slow time dependence disappears at  $\tau \rightarrow \infty$  and does not disturb much the analogy between the Rindler and gravitational solutions. We can better understand the meaning of the limit  $\kappa\tau \rightarrow \infty$  by recalling the relationship between  $(\tau, \xi)$  and  $(t, r)$  and noting that  $\kappa\tau = \tanh^{-1}(t/r)$ . Hence,  $\kappa\tau \gg 1$  implies that  $r \approx t$ , i.e., tending toward the light cone.

### A. Connection with Rindler-Kruskal near-horizon solution

There is an equivalence between the Rindler matrix spinor and the near-horizon RKC matrix spinor. One can see this by considering the electron probability density. In the case of large times  $\kappa\tau \gg 1$  the probability density (31) leads to

$$\frac{64}{\pi^2} \sin(\theta) \kappa \xi c_1^2 (4\zeta^2 + 1) K_{-i\zeta+\frac{1}{2}}(\xi/\lambda) K_{i\zeta+\frac{1}{2}}(\xi/\lambda).$$

Let us compare the latter with the electron probability density given by the Rindler-Kruskal near-horizon solution,

$$\frac{64}{\kappa_*} |C_1|^2 \sin(\theta) \rho' K_{-\frac{i\varepsilon}{\kappa_*}+\frac{1}{2}}(\kappa_* \rho') K_{\frac{i\varepsilon}{\kappa_*}+\frac{1}{2}}(\kappa_* \rho'). \quad (32)$$

We see that both probability densities are proportional to each other, suggesting the following identifications of the Rindler wave function parameters:

$$\xi = \lambda \kappa_* \rho', \quad (33)$$

$$\zeta = \frac{\varepsilon}{\kappa_*}. \quad (34)$$

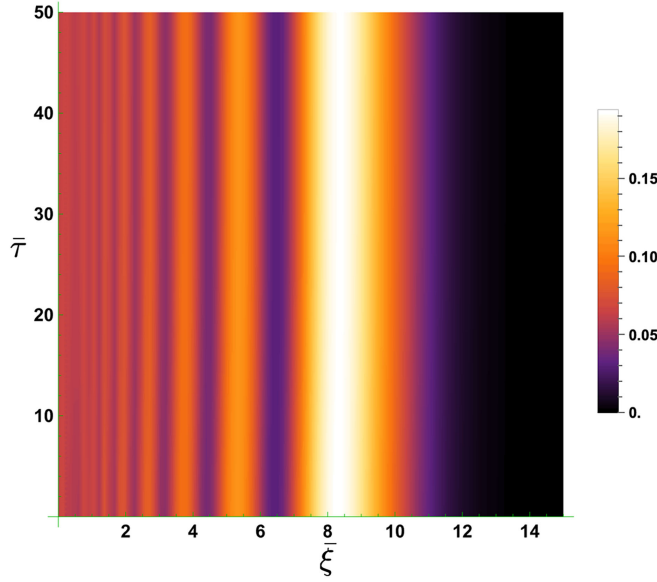


FIG. 2. Spacetime profile of the electron density as a function of  $(\bar{\xi}, \bar{\tau})$  with  $\bar{\xi} = \xi/\lambda$  and  $\bar{\tau} = \tau/\lambda$ , for the electron wave function in the Rindler frame (31) with  $\zeta = 10$  and  $\kappa\lambda = 0.1$ .

The electron probability density of the spherical Rindler solution is shown in Fig. 2 for the parameters analog to the Schwarzschild one in Fig. 1. One sees the same qualitative behavior regarding the fringes as in the Schwarzschild solution. For instance, the number of fringes is the same, and there is no decay of the probability. The slow time dependence of the Rindler density is not visible in this figure.

#### IV. CORRESPONDENCE BETWEEN THE RINDLER SOLUTION AND THE FREE ELECTRON WAVE PACKET

After establishing the mapping of the Rindler spinor to the Schwarzschild near-horizon solution, we now show how the Rindler spinor can be constructed by a suitable superposition of free electron spherical waves. The free electron Dirac equation in flat spacetime is derived from Eq. (13) applying the limit  $(\kappa_* r)^{-1} \ll 1$ ,

$$\gamma^0 \partial_t \eta \gamma_3 \gamma_1 + \gamma^3 \partial_r \eta \gamma_3 \gamma_1 - \frac{\kappa \gamma^1 \eta}{r} - \frac{1}{\lambda} \eta \gamma^0 = 0. \quad (35)$$

To establish a formal connection between the free Dirac equation in spherical coordinates and the Dirac equation in the spherical Rindler frame, we perform the following change of coordinates

$$\begin{aligned} t &= \xi \sinh(\kappa \tau), \\ r &= \xi \cosh(\kappa \tau), \end{aligned} \quad (36)$$

in the radial equation for the free electron in Minkowski spacetime of Eq. (35), that leads to

$$e^{\gamma^0 \gamma^3 \kappa \tau} \left( \frac{\gamma^0 \partial_\tau \eta}{\kappa \xi} + \gamma^3 \partial_\xi \eta \right) \gamma_3 \gamma_1 - \frac{\kappa \gamma^1 \eta}{\xi \cosh(\kappa \tau)} - \frac{1}{\lambda} \eta \gamma^0 = 0.$$

Using the mapping

$$\eta = e^{\gamma^0 \gamma^3 \frac{\kappa \tau}{2}} \eta_R, \quad (37)$$

we arrive at Eq. (28) for the radial part of the Dirac equation in the 3D Rindler case. Thus the mapping relationship Eq. (37) establishes the formal correspondence between the free and the Rindler solutions.

For developing the free electron counterpart of the Rindler solution of Eq. (30), which is expressed via the Bessel  $K_{i\zeta \pm 1/2}$  functions, we recall their integral representation, see Eq. (C5); it then follows that the free electron counterpart of the Rindler solution for the radial wave function is representable by a positive energy free electron wave packet,

$$\hat{\Psi}_\alpha = \int \frac{d^3 p}{(2\pi)^{3/2}} \sqrt{\rho_s} \Psi_p e^{-\gamma^2 \gamma^1 [E_p t + ab(p)]}, \quad (38)$$

built from the free electron spherical Dirac waves  $\Psi_p$ , with  $\kappa = -1$ , see Eq. (C4) in Appendix C. Here,  $E_p = \sqrt{\mathbf{p}^2 + m^2}$  is the electron energy,  $b(p) = \tanh^{-1}(p/E_p)$  the rapidity, and the parameter  $\alpha$  the chirping. The chirping follows from the imaginary index of the Bessel function in the wave function of Eq. (30). By choosing  $\alpha = \varepsilon/\kappa_*$  and  $\rho_s = \text{const}$  one obtains a spinor similar to the stationary Rindler solution of Eq. (30).

A finite momentum width of the wave packet is introduced by the envelop function  $\sqrt{\rho_s} = e^{-\beta E_p/m}$ , with the parameter  $\beta$ . For the case  $\beta > 0$  the stationary wave packet decays, thus allowing the description by the wave packet Eq. (38) of an electron crossing the horizon, similarly to the decay of the stationary state in the gravitational setting.

The constructed free electron wave packet has the following spacetime form in the lab frame (the standard Dirac spinor  $\hat{\psi}_\alpha$  is the leftmost column of the matrix spinor  $\hat{\Psi}_\alpha$ ), see Appendix C:

$$\hat{\psi}_\alpha = \frac{ie^{\pi\alpha} \mathcal{N} \left( \frac{t'-r}{t'+r} \right)^{-\frac{1}{4} + \frac{i\alpha}{2}}}{\pi^{3/2} \lambda r \sqrt{r^2 - t'^2}} \begin{pmatrix} \psi_1 \\ 0 \\ \frac{\psi_3 \cos \theta}{r} \\ \frac{\psi_3 e^{i\phi} \sin \theta}{r} \end{pmatrix}, \quad (39)$$

where

$$\begin{aligned} \psi_1 &= \frac{K_{i\alpha - \frac{1}{2}}(\bar{\xi})}{\xi' \lambda^2} [2rr_- + \lambda(2\alpha + i)r_+] \\ &\quad + iK_{i\alpha + \frac{1}{2}}(\bar{\xi}) \left[ \frac{2r}{\lambda} + (2\alpha - i) \frac{r_-}{r_+} \right], \end{aligned} \quad (40)$$

$$\psi_3 = \frac{iK_{i\alpha+\frac{1}{2}}(\frac{\bar{\xi}'}{\lambda})}{\xi'\lambda^2} [-2r^2r_- + \lambda r_+ [(2\alpha + i)r + 2ir_-]] - K_{i\alpha+\frac{1}{2}}(\frac{\bar{\xi}'}{\lambda}) \left[ \frac{2r^2}{\lambda} - \frac{r_- [(2\alpha - i)r - 2ir_+]}{r_+} \right] \quad (41)$$

with  $\bar{\xi}' = \sqrt{r^2 - t'^2}/\lambda$ ,  $r_{\pm} \equiv r \pm t'$ , and  $t' = t - i\beta/m$ . Note that the complex part of  $t'$  comes from multiplying the spinor by the decaying envelope function  $e^{-\beta E_p/m}$ . This is quite natural, as the parameter  $\beta$  induces the decay of the wave function, or in other words, the transition from the bound state to the free falling state through the classically forbidden region, which is described as the imaginary time propagation [27].

In the Rindler frame one has,

$$\psi_\beta = \Omega^{-1} e^{-\gamma^0 \gamma^3 \frac{\kappa \tau}{2}} (\eta \tilde{\omega} - \gamma_2 \gamma_1 \eta \gamma_3 \gamma_1 \omega) e^{-\frac{\pi \gamma_3 \gamma_2}{4}} \mathbf{u}, \quad (42)$$

$$\mathcal{H}_\beta = e^{-\gamma^0 \gamma^3 \frac{\kappa \tau}{2}} \eta e^{-\pi \gamma_3 \gamma_2 / 4} \mathbf{u}, \quad (43)$$

where  $\mathbf{u} = (1, 0, 0, 0)^T$ , the spinor in the rotating frame  $\psi_\beta$  is constructed using the spinors  $\hat{\psi}_\alpha$ ,  $\mathcal{F} = -\gamma_3 \gamma_1 \omega$  and  $\mathcal{G} = \gamma_0 \omega^\dagger \gamma_0 = \tilde{\omega}$  [see Eq. (12)]. The column spinor (43) in the nonrotating (Fermi-Walker transported) frame is given by

$$\mathcal{H}_\beta = \delta \begin{pmatrix} -\mathcal{A} a_1 K_{i\alpha+\frac{1}{2}}(\frac{\xi_\beta}{\lambda}) + b_1 K_{\frac{1}{2}-i\alpha}(\frac{\xi_\beta}{\lambda}) \\ \mathcal{A} a_2 K_{i\alpha+\frac{1}{2}}(\frac{\xi_\beta}{\lambda}) + b_2 K_{\frac{1}{2}-i\alpha}(\frac{\xi_\beta}{\lambda}) \\ -\mathcal{A} a_3 K_{i\alpha+\frac{1}{2}}(\frac{\xi_\beta}{\lambda}) + b_3 K_{\frac{1}{2}-i\alpha}(\frac{\xi_\beta}{\lambda}) \\ \mathcal{A} a_4 K_{i\alpha+\frac{1}{2}}(\frac{\xi_\beta}{\lambda}) + b_4 K_{\frac{1}{2}-i\alpha}(\frac{\xi_\beta}{\lambda}) \end{pmatrix} \quad (44)$$

where  $\xi_\beta = \sqrt{\beta^2 + 2i\xi \sinh(\kappa\tau)\beta + \xi^2}$ ,  $\mathcal{A} = \frac{(\beta e^{\kappa\tau} - i\xi)}{\xi_\beta}$ ,  $\delta = \frac{(1+i)e^{\frac{\kappa\tau}{2}} \cosh(\kappa\tau) (\xi e^{-\kappa\tau} + i\beta)^{\frac{5}{4}+i\alpha}}{\sqrt{2\pi^{3/2}} \xi (e^{2\kappa\tau} + 1)^2 (\xi e^{\kappa\tau} - i\beta)^{3/4} \xi_\beta^{2+i\alpha}}$ , and the parameters  $a_1, a_2, a_3, a_4$  are given in Appendix D. It is noteworthy that, in the near horizon approximation, the probability distribution  $\psi_\beta^\dagger \psi_\beta$  coincides with the gravitational solution.

The electron probability density in the Rindler frame is shown in Fig. 3 for the cases of  $\beta = 0$  and for  $\beta = 0.5$ , respectively. In the case  $\beta = 0$ , one sees a similarity of the wave packet fringes with that of the near-horizon gravitational counterpart of Fig. 1 for  $\bar{\tau} \sim 15$ . At times before the fringe breach however, there is a compression of the fringes due to the time-dependent term in the metric in Eq. (26). However, they still maintain the same topological structure as in the gravitation case. Note that Fig. 3(a) describes the density of the free electron *wave packet*, mimicking the gravitational solution. It should not be confused with Fig. 2, which represents the density of the bound *wave function* in the spherical Rindler space of Eq. (30). The difference between these figures comes from the fact that the spinor components of the wave packet for  $\beta = 0$  retains its time

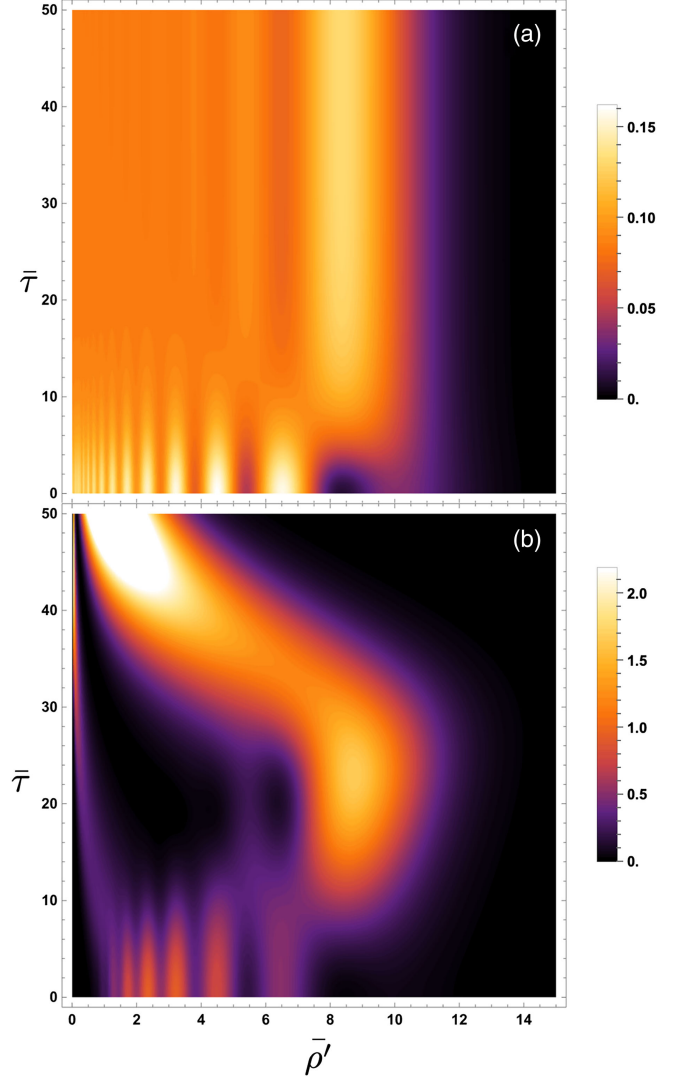


FIG. 3. Spacetime profile of the electron density  $\sqrt{-g}\psi_\beta^\dagger\psi_\beta$  ( $g$  is the determinant of the metric) as a function of  $(\bar{\rho}', \bar{\tau})$ , with  $\bar{\rho}' = \rho'/\lambda$  and  $\bar{\tau} = \tau/\lambda$ , for the free electron modulated wave packet of Eq. (44) with  $\alpha = 10$ ,  $\lambda\kappa = 0.1$  and: (a)  $\beta = 0$ , (b)  $\beta = 0.25$ .

dependence other than just a overall phase in the limit  $\kappa\tau \gg 1$  in contrast to the bound wave function of the eigenstate.

When  $\beta > 0$  the behavior of fringes is significantly modified as can be seen in Fig. 3. In the gravitational case, due to the approximation applied for the analytical near-horizon solution, we have a stationary state without decay. However, one sees in Fig. 3 a time-dependent picture. In Fig. 3 an important slow time dependence can be seen when those fringes corresponding to the bound state dynamics gradually begin to merge into one fringe that later crosses the horizon. Such behavior of the wave function density is a consequence of applying a finite momentum width for the free electron wave packet, described by the parameter  $\beta$  in Eq. (38).

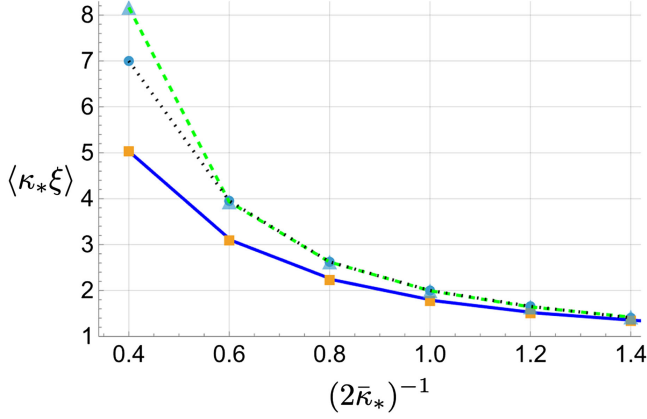


FIG. 4. Coordinate expectation values for the quasistationary state  $1S_{1/2}$  vs half of the Schwarzschild radius. Comparison of the results of the free electron model (orange squares) with those of the numerical calculations of Ref. [10] (blue cycles). The parameter  $\beta$  in the free electron model is derived from the fit to the quasiclassical infall time in the case of (squares), while in the case of (triangles)  $\beta$  is the inverse of the imaginary part of the energy given in Ref. [10].

However, the parameter  $\beta$  is arbitrary in our heuristic free electron model. The question arises of how we can tune the parameter  $\beta$  to correspond to the real physical situation of the gravitational problem. If this is realized, we can recover the gravitational bound state's decay rate and the electron's full wave function in this setting, which is not available in the near-horizon gravitational solution. This parameter is deduced in our model from the fit to the quasiclassical infall time (i.e. by fitting the classical trajectory of a freely falling particle near the black-hole with the dominant fringe of our wave function), see the companion Letter [9] and Eq. (E11). The correctness of the choice of the parameter  $\beta$  in our model we test, comparing the expectation value of the coordinate of the quasistationary bound state provided by our model with that of the gravitational wave function calculated numerically in Ref. [10], see Fig. 4. The figure shows that determining the parameter  $\beta$  via the quasiclassical infall time, our model of the free electron wave packet allows for the prediction of the coordinate expectation value for the  $1S_{1/2}$  quasistationary state in the case of the quasiclassical regime  $r_s \gtrsim \bar{\lambda}$  ( $2\bar{\kappa}_* \lesssim 1$ ), with the Schwarzschild radius  $r_s$ . The free electron model becomes even more accurate when we use  $\beta$  parameter as the inverse of the imaginary part of the energy derived numerically. We underline that the considered  $1S_{1/2}$  quasistationary state in Fig. 4 is the ground state at the black hole, which is a fully quantum state. Nevertheless the decay of this state can be approximately described quasiclassically via the infall time when  $r_s \gtrsim \bar{\lambda}$ . This is because after crossing the horizon the electron becomes quickly relativistic with the characteristic de-Broglie wavelength of the order of the Compton wavelength  $\bar{\lambda}$ .

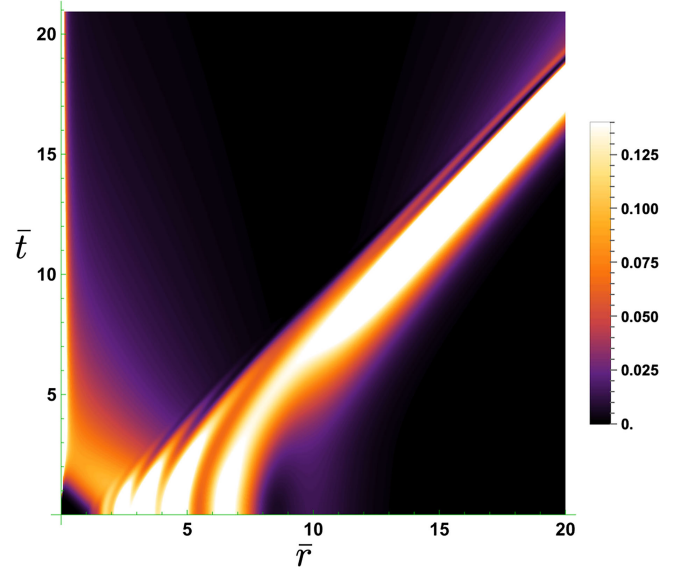


FIG. 5. Spacetime profile of the electron density  $\sqrt{-g}\hat{\psi}_\alpha^\dagger\hat{\psi}_\alpha$  as a function of  $(\bar{r}, \bar{t})$ , with  $\bar{t} = t/\bar{\lambda}$  and  $\bar{r} = r/\bar{\lambda}$ , for the free electron modulated wave packet of Eq. (44) with  $\alpha = 10$ ,  $\beta = 0.25$ ,  $\bar{\lambda}\kappa = 0.1$  in the lab frame.

Before proceeding, note that Fig. 3 shows the electron density in the Rindler frame, while the density structure in the lab frame is illustrated in Fig. 5.

## V. QUANTUM DYNAMICS OF THE ELECTRON WHEN CROSSING THE HORIZON

### A. Chiral angle dynamics

In this section, the parameters used in the figures are the same as in the accompanying paper. Using the near-horizon stationary gravitational solution, we have constructed the Rindler solution, and found its mapping into the free electron wave packet. Including the  $\beta$  parameter in the free electron wave packet solution, we introduced the decay of the wave function. In this way, we obtain the nonstationary wave function of the electron for the Schwarzschild problem, which contains much more information about quantum dynamics than just the decay rate discussed in the accompanying paper. In particular, this wave function can provide information on the electron spin evolution when approaching and crossing the horizon.

We focus on the subtle quantum information encoded in the chiral angle  $\beta_s$  of the Dirac spinor Eq. (1). The chiral angle in terms of the Dirac spinor is defined as [22,23]

$$\beta_s = \tan^{-1} \left( \frac{\psi_\beta^\dagger \gamma_0 \mathbf{i} \psi_\beta}{\psi_\beta^\dagger \gamma_0 \psi_\beta} \right). \quad (45)$$

The chiral angle is a space-dependent function. When calculating different observables, it generates the observable spin dependence (for instance, the energy level splitting due

to spin-orbit coupling). Therefore, all spin properties of an electron crossing the black hole horizon will be determined by the corresponding behavior of the chiral angle.

With the obtained wave function, we investigate the dynamics of the chiral angle near the horizon of a Schwarzschild black hole. The chiral angle is depicted in Fig. 6(a) for the parameters  $\alpha = 15$ ,  $\lambda\kappa = 0.1$ , and  $\bar{\beta} = 0.25$  used in Fig. 3(a). The chiral angle dynamics are separated into three characteristic regions similar to the density in Fig. 3(a):  $\bar{\tau} < 10$ ,  $10 < \bar{\tau} < 22$ , and  $\bar{\tau} > 22$ . In the bound state ( $\bar{\tau} < 12$ ), the chiral angle  $\beta_s$  shows a typical behavior of reversing the sign near the edges of the interference fringes, or when crossing the node of the wave function, see Fig. 6(b). In the considered case of the dynamics near the black hole, the common oscillatory behavior of the chiral angle at the nodes is transformed, demonstrating an elliptical pattern for the chiral angle jumping surface in the  $(\rho', \tau)$  plane. It is interesting to note that in the region  $\bar{\tau} > 17$  following the classical trajectory, the chiral angle is not flat, but shows a jump of sign at the edges around the trajectory [Fig. 6(c)].

Moreover, the topology of the jumping surfaces of the chiral angle is transformed when moving between the characteristic regions mentioned above, mostly following the corresponding change in the fringe structure. Thus, in the consolidation region ( $10 < \bar{\tau} < 22$ ), we can see several elliptical structures with smaller eccentricity than those of the bound state, which again correspond to the fringes in Fig. 3(a) (in the latter the side fringes are damped and not visible). They exist all the way up to the time when the wave packet begins to plunge into the abyss at  $\rho' = 0$ . Once the electron begins to fall into the black hole, the ellipses merge and become a line.

The dynamics of the chiral angle has a close connection to the spin dynamics. In the Fermi-Walker rotated frame in which the spinor  $\mathcal{H}_\beta$  [Eq. (44)] is defined, both the zeroth component of the spin 4-vector  $S_0$  and  $\beta_s$  are exactly zero. Hence the average spin vector is simply  $\langle \mathbf{S} \rangle = (0, 1, 0)$ . Upon changing to the rotating frame, both  $\beta_s$  and  $S_0$  are locally not zero, but cancel out when integrating over  $\theta$  in the interval  $(0, \pi)$ . This must be the case since  $\beta_s$  is a pseudoscalar, hence being invariant under Lorentz transformations. This connection between  $S_0$  and  $\beta_s$  points to an intimate relationship between the chiral angle and spin dynamics. The latter we see in Fig. 6(d), where the spacetime profile of the zeroth component of the spin 4-vector  $S_0 = \psi_\beta^\dagger \gamma_5 \psi / \bar{\psi}_\beta \psi_\beta$  is shown. The topological structure of  $S_0$  is very similar to that of the chiral angle  $\beta_s$  in that we again see discontinuous changes in sign associated with the fringes which, for the case of  $S_0$ , correspond to the helicity flip.

In Fig. 7 we show the dynamics of the radial component of the electron's spin 4-vector as a function of  $(\rho', \theta)$  for  $\bar{\tau} = 0, 10, 20, 50$ . The vertical lines correspond to the change in direction of  $S_r$  and coincide with the fringes seen in Fig. 6.

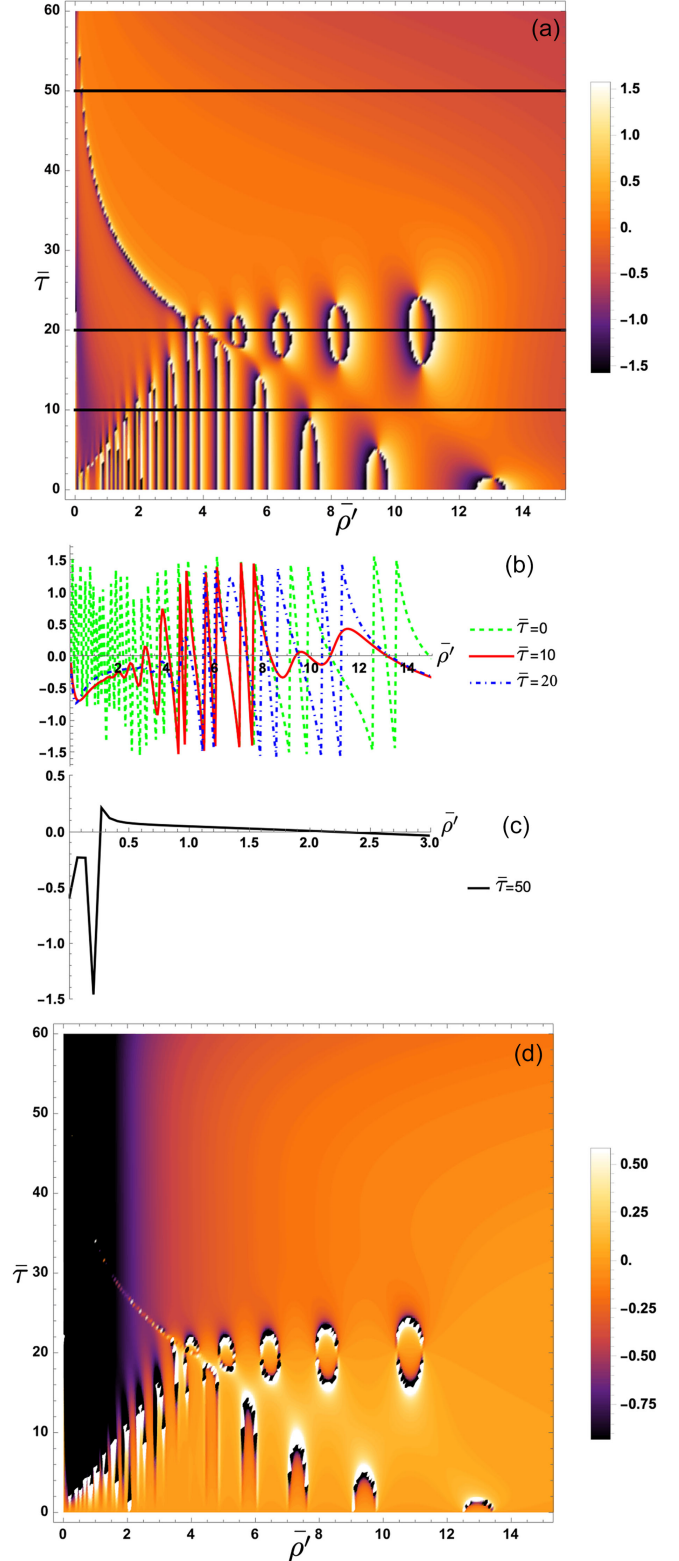


FIG. 6. Spacetime profile of (a)  $\beta_s$ , where (b) and (c) depict the spatial profiles of  $\beta_s$  for the specific times highlighted in (a) and (d)  $S_0 = \psi_\beta^\dagger \gamma_5 \psi / \bar{\psi}_\beta \psi_\beta$ . The employed parameters are  $\lambda\kappa = 0.1$ ,  $\alpha = 15$ ,  $\bar{\beta} = 0.25$ .

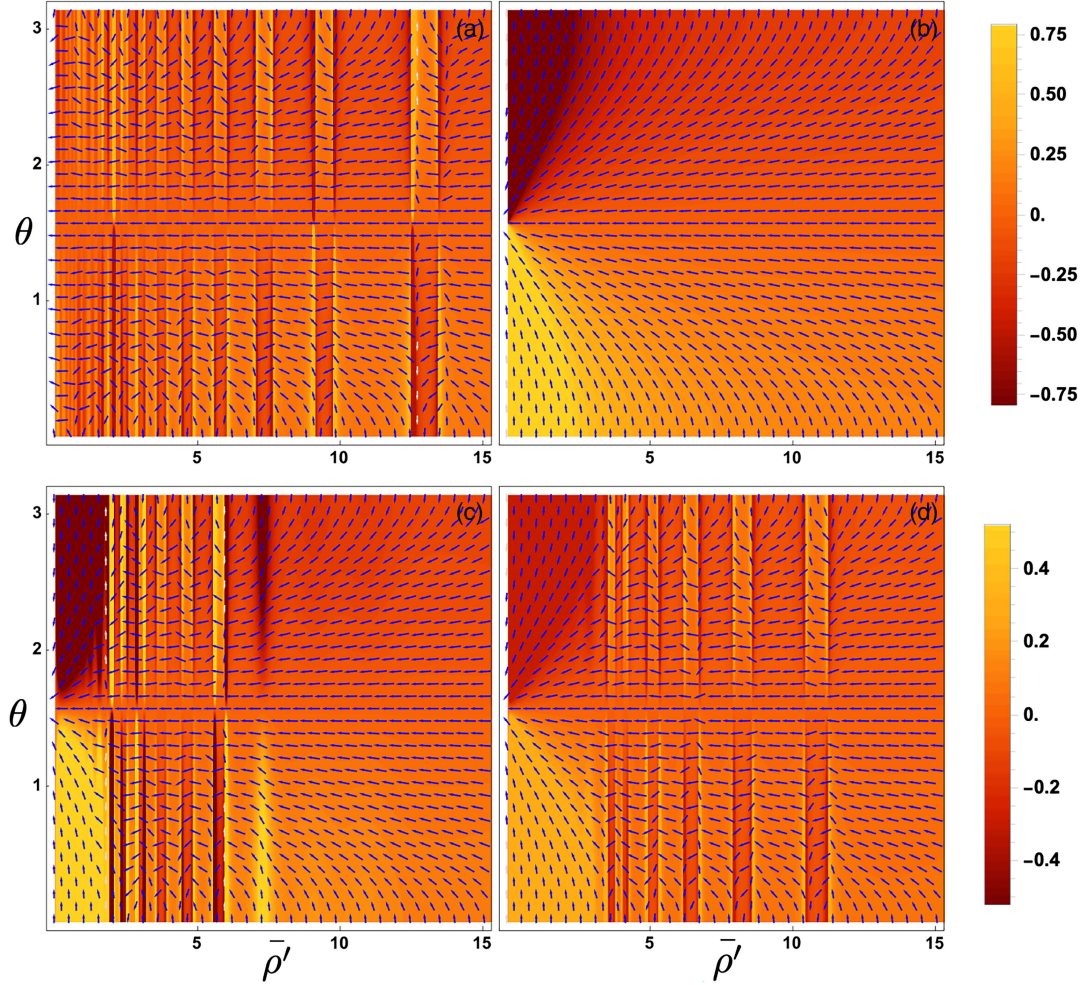


FIG. 7. Snapshot of the spacetime profile for  $S_r$  for (a)  $\bar{\tau} = 0$ , (b)  $\bar{\tau} = 50$ , (c)  $\bar{\tau} = 10$ , and (d)  $\bar{\tau} = 20$ . Superimposed is the vector field  $(S_\theta, S_r)$  for the corresponding values of  $\bar{\tau}$ . The employed parameters are  $\lambda\kappa = 0.1$  and  $\alpha = 15$ , and  $\bar{\beta} = 0.25$ .

In Fig. 7 the vector field  $(S_\theta, S_r)$ , with  $S_\theta = \psi_\beta^\dagger \gamma^5 \gamma^\theta \psi_\beta / \bar{\psi}_\beta \psi_\beta = -\sin \theta$ , as a function of  $(\rho', \theta)$  is given by the blue arrows. One sees that the regions where the  $S_r$  changes direction coincide with the vertical strips in Fig. 7 which are roughly located in between the fringes of the probability density. Notably the vector spin expectation value is constant  $\langle \mathbf{S} \rangle = (-1, 0, 0)$ .

To illustrate the time variation of the space-dependent spin vector field, we present in Fig. 8 the superimposed vector field plots for (a)  $\bar{\tau} = 0$  red arrows,  $\bar{\tau} = 50$  gray arrows and (b)  $\bar{\tau} = 10$  red arrows,  $\bar{\tau} = 20$  gray arrows. One see that the change in the direction of the  $S_r$  component as a function of  $\theta$  coincides with the vertical stripes in Fig. 7. Moreover, as the fringes dissipates, the spin vector field becomes more smooth.

## VI. CONCLUSION

In conclusion, we found the exact bound state solution to the Dirac equation in the spherical Rindler spacetime

describing a uniformly expanding spacetime, which provides the scheme for the construction of a free electron wave packet capable of simulating the electron quantum dynamics near the horizon of a Schwarzschild black hole. The key ingredients are a particular choice of the wave packet chirping and a finite width in momentum space.

We begin with the near-horizon stationary solution of the Dirac equation for an electron in the gravitational field of the Schwarzschild black hole. Given the correspondence between the near-horizon Schwarzschild metric and the spherical Rindler metric, we further mimic the gravitational solution via the exact solution of the Dirac equation for a free electron in a locally accelerated Rindler frame. Once the Rindler-Dirac equation is transformed to the nonrotating frame, it becomes easy to deduce that the solution can be described by a chirped free electron wave packet in Minkowski spacetime. We heuristically introduce the  $\beta$  parameter in the free electron wave packet solution, which induces decay of the wave function. The latter can be adjusted to the real gravitational case with the help of the

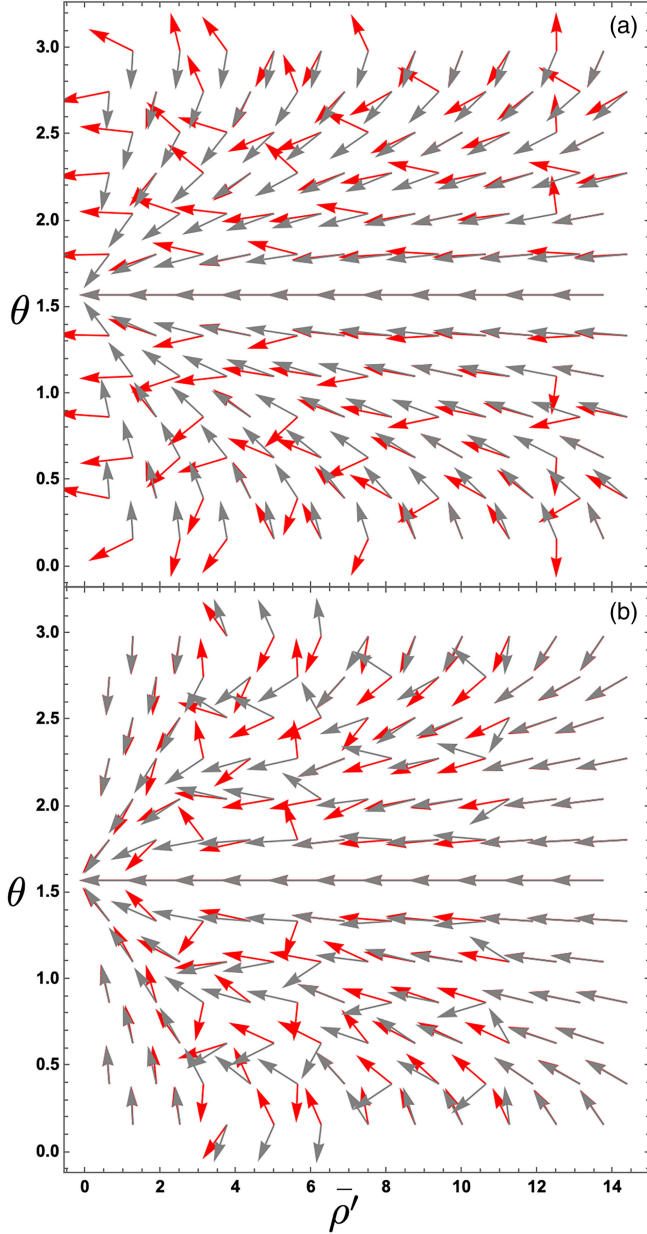


FIG. 8. Superimposed are the vector field plots (a) of Fig. 7(a) red arrows, Fig. 7(b) gray arrows and (b) of Fig. 7(c) red arrows and Fig. 7(d) gray arrows.

quasiclassical trajectory description as indicated in the accompanying paper. This allows us to derive the nonstationary wave function of the electron for the gravitational problem, and to investigate the detailed information on the quantum dynamics contained in it. This information is presented in the most evident way when transforming the final free wave packet to the Rindler frame.

With simulations employing free electron wave packets, the electron quantum dynamics near the horizon of a black hole can be explored. As our analysis indicates, the nontrivial behavior of the chiral angle and the corresponding dynamics of the spin spacetime distribution near the

horizon and when crossing the horizon could exist. This is an example of how experiments with tailored free electron wave packets can enhance our understanding of the quantum properties of electrons in a gravitational field.

### DATA AVAILABILITY

The numerical data used has been published in Ref. [10]. The other numerical calculations were done straightforwardly with *Mathematica* 14.3 using the equations provided in the paper and has not been made available online.

### APPENDIX A: KRUSKAL-SZEKERES COORDINATES

The Kruskal-Szekeres coordinates are introduced to represent the Hestenes-Dirac equation in a similar form to that in the 3D Rindler case. The following change of coordinates is applied for  $r > \kappa_*^{-1}$ , i.e., outside the horizon

$$\begin{aligned} T\kappa_* &= (\kappa_* r - 1)^{1/2} e^{\kappa_* r/2} \sinh(\kappa_* t/2), \\ \kappa_* R &= (\kappa_* r - 1)^{1/2} e^{\kappa_* r/2} \cosh(\kappa_* t/2), \end{aligned} \quad (\text{A1})$$

when the line element takes the form

$$ds^2 = \frac{4e^{-r\kappa_*}}{\kappa_* r} (dT^2 - dR^2) - r(T, R)^2 (\sin^2 \theta d\phi^2 + d\theta^2). \quad (\text{A2})$$

The radial coordinate is related to the  $R$  and  $T$  coordinates as

$$R^2 - T^2 = \frac{(r\kappa_* - 1)}{\kappa_*^2} e^{r\kappa_*}.$$

The corresponding relation for the interior region is the above expression multiplied by  $-1$  on both sides. In Kruskal-Szekeres coordinates, the radial spinor equation is described by Eq. (17).

### APPENDIX B: THE DIRAC SOLUTION IN THE 3D RINDLER COORDINATES

For the separation of variables in Eq. (27), we firstly define the transformation matrix  $\Omega = \cosh(c\tau\kappa) \sqrt{\sin \theta} e^{\gamma_1 \gamma_3 \theta/2} e^{\gamma_2 \gamma_1 \phi/2} \xi$  and consider the following identities

$$\gamma^3 \Omega \partial_\xi \Omega^{-1} = -\frac{3\gamma^3}{2\xi}, \quad (\text{B1})$$

$$\frac{\gamma^2 \Omega \partial_\phi \Omega^{-1}}{\sin \theta} + \gamma^1 \Omega \partial_\theta \Omega^{-1} = \gamma^3. \quad (\text{B2})$$

We begin by transforming the Hestenes-Dirac matrix spinor in (27) as follows. Multiplying Eq. (27) by  $\Omega$  to the right, this gives

$$\begin{aligned} & \left( \frac{\gamma^0 \Omega \partial_\tau (\cosh(\tau\kappa))}{\kappa} + \frac{\gamma^2 \Omega \partial_\phi}{\sin \theta} + \gamma^1 \Omega \partial_\theta - \gamma^3 \Omega \right) \Omega^{-1} \Omega \Psi \\ & \times \gamma_2 \gamma_1 + \xi \cosh(\tau\kappa) \gamma^3 \Omega \left( \partial_\xi + \frac{3}{2\xi} \right) \Omega^{-1} \Omega \Psi \gamma_2 \gamma_1 \\ & - \frac{\xi \cosh(\tau\kappa)}{\tilde{\lambda}} \Omega \Psi \gamma^0 = 0. \end{aligned}$$

Upon using the relations (B1) we finally get

$$\begin{aligned} & \left( \frac{\gamma^0 \cosh(\tau\kappa) \partial_\tau}{\kappa} + \frac{\gamma^2 \partial_\phi}{\sin \theta} + \gamma^1 \partial_\theta \right) \Omega \Psi \gamma_2 \gamma_1 \\ & + \xi \cosh(\tau\kappa) \gamma^3 \left( \partial_\xi + \frac{1}{2\xi} \right) \Omega \Psi \gamma_2 \gamma_1 - \frac{\xi \cosh(\tau\kappa)}{\tilde{\lambda}} \Omega \Psi \gamma^0 \\ & = 0. \end{aligned} \quad (\text{B3})$$

Multiplying Eq. (B3) to the right by  $e^{\gamma_3 \gamma_2 \pi/4}$  and defining  $\Phi = \Omega \Psi e^{-\gamma_2 \gamma_1 \phi/2} e^{\gamma_3 \gamma_2 \pi/4}$ , leads to

$$\begin{aligned} & \frac{\gamma^0 \cosh(\tau\kappa) \partial_\tau \Phi \gamma_3 \gamma_1}{\kappa} + (\gamma^1 \partial_\theta \Phi + \cosh(\tau\kappa) \xi \gamma^3 \partial_\xi \Phi) \gamma_3 \gamma_1 \\ & - \frac{\gamma^2 \Phi}{2 \sin \theta} + \frac{\gamma^3 \cosh(\tau\kappa) \Phi \gamma^3 \gamma^1}{2} - \frac{\xi \cosh(\tau\kappa)}{\tilde{\lambda}} \Phi \gamma^0 = 0. \end{aligned} \quad (\text{B4})$$

In order to separate radial and angular variables, we consider the following ansatz

$$\Phi = \eta_R(\tau, \xi) \mathcal{G}(\theta) + \gamma_2 \gamma_1 \eta_R(\tau, \xi) \mathcal{F}(\theta),$$

with  $\gamma_5 \eta_R = \eta_R \gamma_5$ ,  $\gamma_0 \mathcal{G} \gamma_0 = \mathcal{G}$ ,  $\gamma_0 f \gamma_0 = f$ ,  $\gamma_3 \gamma_1 \mathcal{G} = \mathcal{G} \gamma_3 \gamma_1$ , and  $\gamma_3 \gamma_1 \mathcal{F} = \mathcal{F} \gamma_3 \gamma_1$ . Plugging it into Eq. (27) we obtain (the subscript R has been omitted here to simplify the notation)

$$\begin{aligned} & \left[ \frac{\gamma_0 \cosh(\tau\kappa)}{\kappa} \left( \partial_\tau (\eta \mathcal{G} + \gamma_2 \gamma_1 \eta \mathcal{F}) \gamma_3 \gamma_1 + \frac{\varepsilon}{\hbar} (\eta \mathcal{G} + \gamma_2 \gamma_1 \eta \mathcal{F}) \right) \right. \\ & \left. - \gamma_1 \eta \left( \partial_\theta \mathcal{G} \gamma_3 \gamma_1 + \frac{\mathcal{F}}{2 \sin \theta} \right) - \gamma_2 \eta \left( \partial_\theta \mathcal{F} \gamma_3 \gamma_1 - \frac{\mathcal{G}}{2 \sin \theta} \right) \right] \\ & - \xi \cosh(\tau\kappa) \left( \gamma_3 (\partial_\xi \eta) \gamma_3 \gamma_1 \mathcal{G} - \gamma_3 \gamma_2 \gamma_1 \partial_\xi \eta \gamma_3 \gamma_1 \mathcal{F} \right. \\ & \left. - \frac{1}{\tilde{\lambda}} [\eta \gamma_0 \mathcal{G} + \gamma_2 \gamma_1 \eta \gamma_0 \mathcal{F}] \right) = 0. \end{aligned} \quad (\text{B5})$$

In order to solve the radial equation (28), consider the following ansatz for the matrix spinor

$$\begin{aligned} \eta_R = & [f_a(\xi) + \gamma^0 \gamma^1 g_a(\xi)] [g(\tau) + \gamma^0 \gamma^3 f(\tau)] e^{-\gamma_3 \gamma_1 \xi \kappa \tau} \\ & + [f_a^*(\xi) + \gamma^0 \gamma^1 g_a^*(\xi)] [g^*(\tau) + \gamma^0 \gamma^3 f^*(\tau)] e^{-\gamma_3 \gamma_1 \xi \kappa \tau}, \end{aligned} \quad (\text{B6})$$

where the matrix spinor with the complex conjugated functions ensures that the function multiplying each of its products of gamma matrices are real. This particular form of the  $\eta_R$  matrix spinor consists of a boost along the directions  $\gamma_1 \gamma_0$  and  $\gamma_3 \gamma_0$  followed by a rotation on the spin plane  $\gamma_1 \gamma_3$ . The general solution is

$$\begin{aligned} f_a(\xi) &= (1+i) a_1 K_{1/2+i\xi}(\xi/\tilde{\lambda}), f(\tau) = \frac{2ic_1}{\pi a_1} \tanh(\kappa\tau), \\ g_a(\xi) &= -(1+i) a_1 K_{1/2-i\xi}(\xi/\tilde{\lambda}), \\ g(\tau) &= \frac{4c_1}{\pi a_1} \left( \zeta - i \frac{\text{sech}(\kappa\tau)}{2} \right), \end{aligned}$$

where  $a_1$  and  $c_1$  are real constants. Here,  $\zeta$  is the separation constant of the  $(\tau, \xi)$  variables in Eq. (28). In the limit  $\kappa\tau \gg 1$ , the matrix spinor (B6) takes on the simplified form of Eq. (29).

### APPENDIX C: FREE ELECTRON SPHERICAL WAVE AND WAVE PACKET IN MINKOWSKI SPACETIME

Here we derive the expression for the spherical waves of a relativistic free electron. The angular part of the Dirac equation is separated via the ansatz of Eq. (9), yielding the radial equation for a free electron Eq. (35). Applying the ansatz (14) to the radial equation yields

$$\begin{aligned} & \frac{f(r)(\frac{\varepsilon}{m} - 1)}{\tilde{\lambda}} + g'(r) - \frac{\kappa g(r)}{r} = 0, \\ & - \frac{g(r)(\frac{\varepsilon}{m} + 1)}{\tilde{\lambda}} + f'(r) + \frac{\kappa f(r)}{r} = 0. \end{aligned} \quad (\text{C1})$$

The solutions of the latter in terms of spherical Bessel functions are

$$\begin{aligned} f(r) &= \mathcal{N} \sqrt{pr} J_{\kappa+\frac{1}{2}}(pr), \\ g(r) &= \mathcal{N} \sqrt{pr} \frac{cp}{E_p + m} J_{\kappa-\frac{1}{2}}(pr), \end{aligned} \quad (\text{C2})$$

where  $\mathcal{N}$  is the normalization constant. The matrix spinor is then

$$\Psi_p = \Omega^{-1} (\eta(r) \mathcal{G}(\theta) + \gamma_2 \gamma_1 \eta(r) \mathcal{F}(\theta)) e^{-\gamma_3 \gamma_2 \pi/4} e^{\gamma_2 \gamma_1 \phi/2}. \quad (\text{C3})$$

In the case with  $\kappa = -1$ , we have the well-known corresponding Dirac spinor, see e.g. [28],

$$\psi_p = \frac{\mathcal{N}}{E_p} \sqrt{\frac{m + E_p}{2m}} \begin{pmatrix} \frac{2\lambda \cos(pr)}{\sqrt{\pi r}} \\ 0 \\ \frac{2ic\lambda \cos(\theta)(pr \sin(pr) + \cos(pr))}{\sqrt{\pi r^2(E_p + m)}} \\ \frac{2ic\lambda \sin(\theta)(pr \sin(pr) + \cos(pr))e^{i\phi}}{\sqrt{\pi r^2(E_p + m)}} \end{pmatrix} \quad (C4)$$

constructed from boosting an electron at rest with spin up.

We calculate the free electron wave packet Eq. (38) as follows. We extend the limits of integration to  $\pm\infty$  in order to be able to represent it by Bessel functions, writing the trigonometric functions in terms of exponentials, and recombine them,

$$\int_{-\infty}^{\infty} \frac{p^2 dp}{E_p} \begin{pmatrix} \frac{\sqrt{\frac{2}{\pi}} \lambda \sqrt{\frac{E_p}{m} + 1}}{r} \\ 0 \\ \frac{i\sqrt{\frac{2}{\pi}} \lambda^2 \cos(\theta)(1 + i pr)}{r^2 \sqrt{\frac{E_p}{m} + 1}} \\ \frac{i\sqrt{\frac{2}{\pi}} \lambda^2 \sin(\theta)(1 + i pr)e^{i\phi}}{r^2 \sqrt{\frac{E_p}{m} + 1}} \end{pmatrix} e^{i pr}.$$

Upon making the substitution  $p = m \sinh b$  and using the identity

$$\int_{-\infty}^{\infty} dt e^{i(z \cosh t + y \sinh t) - \nu t} = 2 \left( \frac{z + y}{z - y} \right)^{\frac{\nu}{2}} K_{\nu} \left( \sqrt{y^2 - z^2} \right), \quad (C5)$$

we finally get Eq. (39).

#### APPENDIX D: CORRESPONDENCE BETWEEN FREE AND RINDLER SOLUTIONS

The coefficients  $a_1, a_2, a_3, a_4$  and  $b_1, b_2, b_3, b_4$  in Eq. (44) are

$$\begin{aligned} a_1 &= e^{-\frac{\kappa\tau}{2}} \{ -2i\beta e^{\kappa\tau} [\beta(e^{\kappa\tau} - 1) + m\xi^2 \cosh(2\kappa\tau) + m\xi^2] + \xi^2 [2\alpha + (2\alpha - 3i)e^{2\kappa\tau} + 2ie^{\kappa\tau} - i] \\ &\quad + \xi [2\beta + 2e^{2\kappa\tau}(-\beta + (1 + 2i\alpha)\beta \cosh(\kappa\tau) + 2\beta \sinh(\kappa\tau) + m\xi^2 \cosh(2\kappa\tau) + m\xi^2)] \}, \\ a_2 &= 2e^{\frac{\kappa\tau}{2}} \{ \beta [\beta(e^{\kappa\tau} - 1) + m\xi^2 \cosh(2\kappa\tau) + m\xi^2] + \xi^2 [(2 + 2i\alpha) \cosh(\kappa\tau) + \sinh(\kappa\tau) - 1] \\ &\quad + i\xi e^{\kappa\tau} [-\beta + \beta((1 + 2i\alpha) \cosh(\kappa\tau) + 2 \sinh(\kappa\tau) - \sinh(2\kappa\tau) + \cosh(2\kappa\tau)) + m\xi^2 \cosh(2\kappa\tau) + m\xi^2] \}, \\ a_3 &= e^{-\frac{\kappa\tau}{2}} \{ 2i\beta e^{\kappa\tau} [\beta(e^{\kappa\tau} + 1) - m\xi^2 \cosh(2\kappa\tau) - m\xi^2] - 2\xi^2 e^{\kappa\tau} [(2(\alpha - i) \cosh(\kappa\tau) - i(\sinh(\kappa\tau) + 1))] \\ &\quad + \xi [m\xi^2 (e^{2\kappa\tau} + 1)^2 + \beta(2 + e^{\kappa\tau}(-2i\alpha + (-3 - 2i\alpha)e^{2\kappa\tau} - 2e^{\kappa\tau} + 1))] \}, \\ a_4 &= e^{-\frac{\kappa\tau}{2}} \{ 2\beta e^{\kappa\tau} [\beta(e^{\kappa\tau} + 1) - m\xi^2] - 2\beta m\xi^2 e^{\kappa\tau} \cosh(2\kappa\tau) + \xi^2 [2i\alpha + (3 + 2i\alpha)e^{2\kappa\tau} + 2e^{\kappa\tau} + 1] \\ &\quad - i\xi [m\xi^2 (e^{2\kappa\tau} + 1)^2 + \beta(2 + e^{\kappa\tau}(-2i\alpha + (-3 - 2i\alpha)e^{2\kappa\tau} - 2e^{\kappa\tau} + 1))] \}, \\ b_1 &= e^{\frac{\kappa\tau}{2}} \{ i\beta [m\xi^2 (e^{2\kappa\tau} + 1)^2 - 2\beta e^{\kappa\tau} (e^{\kappa\tau} - 1)] + \beta\xi [-2i\alpha + e^{2\kappa\tau}(-2i\alpha + 4 \sinh(\kappa\tau) - 1) + 3] \\ &\quad + \xi^2 e^{\kappa\tau} [2\alpha + (2\alpha + i)e^{2\kappa\tau} - 2ie^{\kappa\tau} + 3i] + 4m\xi^3 e^{\kappa\tau} \cosh^2(\kappa\tau) \}, \\ b_2 &= 2e^{\frac{3\kappa\tau}{2}} \{ \beta e^{\kappa\tau} (\beta(e^{-\kappa\tau} - 1) + m\xi^2 \cosh(2\kappa\tau) + m\xi^2) + \xi^2 e^{\kappa\tau} ((2 - 2i\alpha) \cosh(\kappa\tau) - \sinh(\kappa\tau) - 1) \\ &\quad - i\xi (-\beta + \beta((1 - 2i\alpha) \cosh(\kappa\tau) - 2 \sinh(\kappa\tau) + \sinh(2\kappa\tau) + \cosh(2\kappa\tau)) + m\xi^2 \cosh(2\kappa\tau) + m\xi^2) \}, \\ b_3 &= e^{\frac{3\kappa\tau}{2}} \{ 2i\beta e^{\kappa\tau} [\beta(-e^{-\kappa\tau} - 1) + m\xi^2 \cosh(2\kappa\tau) + m\xi^2] + \xi^2 [2\alpha + (2\alpha + i)e^{2\kappa\tau} + 2ie^{\kappa\tau} + 3i] \\ &\quad - 2c\xi [-\beta + \beta((-1 + 2i\alpha) \cosh(\kappa\tau) + 2 \sinh(\kappa\tau) + \sinh(2\kappa\tau) + \cosh(2\kappa\tau)) + m\xi^2 \cosh(2\kappa\tau) + m\xi^2] \}, \\ b_4 &= 2e^{\frac{3\kappa\tau}{2}} \{ \beta e^{\kappa\tau} [\beta(-e^{-\kappa\tau} - 1) + m\xi^2 \cosh(2\kappa\tau) + m\xi^2] + \xi^2 e^{\kappa\tau} [2i(\alpha + i) \cosh(\kappa\tau) + \sinh(\kappa\tau) - 1] \\ &\quad - i\xi [-\beta + \beta((-1 + 2i\alpha) \cosh(\kappa\tau) + 2 \sinh(\kappa\tau) + \sinh(2\kappa\tau) + \cosh(2\kappa\tau)) + m\xi^2 \cosh(2\kappa\tau) + m\xi^2] \}. \end{aligned}$$

## APPENDIX E: KINEMATICS IN THE SCHWARZSCHILD SPACETIME

In this section we highlight some features of the Schwarzschild spacetime.

Consider the standard Schwarzschild line element

$$ds^2 = \left(1 - \frac{1}{\kappa_* r}\right) dt^2 - \frac{dr^2}{1 - \frac{1}{\kappa_* r}} - r^2 d\Omega^2, \quad (\text{E1})$$

with the arclength  $s$  being the proper time multiplied by the speed of light, and  $d\Omega^2 = \sin^2 \theta d\phi^2 + d\theta^2$ . We introduce the Lagrangian  $\mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ , that is,

$$\mathcal{L} = \left(1 - \frac{1}{\kappa_* r}\right) \dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{1}{\kappa_* r}} - r^2 (\sin^2 \theta \dot{\phi}^2 + \dot{\theta}^2) \quad (\text{E2})$$

where the dot stands for derivative with respect to  $s$ . Since  $\partial \mathcal{L} / \partial t = \partial \mathcal{L} / \partial \phi = 0$  we have two constants of motion arising from the Euler-Lagrange equations for  $t$  and  $\phi$

$$-\frac{d}{ds} \left[ 2 \left(1 - \frac{1}{\kappa_* r}\right) \dot{t} \right] = 0 \rightarrow \left(1 - \frac{1}{\kappa_* r}\right) \dot{t} = \frac{E}{m}, \quad (\text{E3})$$

$$\frac{d}{ds} (-2r^2 \sin^2 \theta \dot{\phi}) = 0 \rightarrow -2r^2 \sin^2 \theta \dot{\phi} = L. \quad (\text{E4})$$

Note that  $E$  is the total energy and  $L$  the orbital angular momentum. If we take  $L = 0$ , this means that  $\dot{\phi} = 0$  independent of the initial value of  $\theta$ . The Lagrangian then takes the form

$$\mathcal{L} = \left(1 - \frac{1}{\kappa_* r}\right) \dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{1}{\kappa_* r}} - r^2 \dot{\theta}^2.$$

Since now  $\partial \mathcal{L} / \partial \theta = 0$ , another constant of motion  $r^2 \dot{\theta} = A$  arises. For particle trajectories we have  $\mathcal{L} = 1$ , leading to the equation of motion

$$\dot{r}^2 = \frac{E^2}{m^2} - \left(1 - \frac{1}{\kappa_* r}\right) \left(1 + \frac{A^2}{r^2}\right). \quad (\text{E5})$$

For the following initial conditions  $r(0) = r_0$ ,  $\dot{t}(0) = a$ ,  $\dot{\theta}(0) = b$ , and  $\dot{r}(0) = v_0/c$  we have

$$\frac{E}{m} = \left(1 - \frac{1}{\kappa_* r_0}\right) a, \quad A = b r_0^2.$$

Equation (E5) gives the equation for the proper time as a function of  $r$

$$\frac{ds}{dr} = - \frac{1}{\sqrt{\frac{E^2}{m^2 c^4} - \left(1 - \frac{1}{\kappa_* r}\right) \left(1 + \frac{A^2}{r^2}\right)}} \quad (\text{E6})$$

where the minus sign is chosen since we are considering infalling observers, while the equation for the coordinate time is

$$\frac{cdt}{dr} = - \frac{1}{\left(1 - \frac{1}{\kappa_* r}\right) \sqrt{\frac{E^2}{m^2} - \left(1 - \frac{1}{\kappa_* r}\right) \left(1 + \frac{A^2}{r^2}\right)}}. \quad (\text{E7})$$

Assume for simplicity that  $b = 0$ . From the equation  $\mathcal{L} = 1$  and the initial conditions we obtain the following

$$\frac{E^2}{m^2} = \left(1 - \frac{1}{\kappa_* r_0}\right) + v_0^2,$$

leading to

$$\frac{ds}{dr} = - \frac{1}{\sqrt{\frac{1}{\kappa_* r} - \frac{1}{\kappa_* r_0} + v_0^2}}, \quad (\text{E8})$$

$$\frac{dt}{dr} = - \frac{1}{\left(1 - \frac{1}{\kappa_* r}\right) \sqrt{\frac{1}{\kappa_* r} - \frac{1}{\kappa_* r_0} + v_0^2}}, \quad (\text{E9})$$

for the proper time and coordinate time, respectively. Integrating Eq. (E8) from  $r_0$  to the singularity  $r = 0$  gives

$$s = \frac{r_0 \sqrt{\kappa_* r_0} (\sin^{-1}(\sqrt{1 - \kappa_* r_0 v_0^2}) - \sqrt{\kappa_* r_0} v_0 \sqrt{1 - \kappa_* r_0 v_0^2})}{(1 - \kappa_* r_0 v_0^2)^{3/2}}, \quad (\text{E10})$$

where  $v_0 < (\kappa_* r_0)^{-1/2}$ . For the case with  $v_0 = 0$ , we obtain

$$s = \frac{r_0 \sqrt{r_0 \kappa_*} \pi}{2}.$$

Moreover, for the case with  $\beta = 0$  the average position in the limit  $\tau \rightarrow \infty$  can be calculated exactly with the wave packet in the frame comoving with the electron and it is given by

$$\langle \xi \rangle = \frac{\lambda(16\alpha^4 + 40\alpha^2 + 9) \tanh(\pi\alpha)}{9\pi\alpha(\alpha^2 + 1)}.$$

Moreover, the normalized electron's velocity can also be calculated exactly and its norm is given by

$$|\mathbf{v}| = \frac{2\sqrt{\pi^2 \alpha^2 (\alpha^2 + 1)^2 + (4\alpha^2 + 1)^2 \tanh^2(\pi\alpha)}}{3\pi\alpha(\alpha^2 + 1)}.$$

In order to get the proper time it takes for the electron to reach the horizon starting from the above  $\langle \xi \rangle$  and with zero velocity it is then given by boosting to the frame in which  $|\mathbf{v}| = 0$ . Hence,

$$s' = \frac{\langle \xi \rangle \sqrt{\langle \xi \rangle \kappa_*} \pi}{2} \gamma, \quad (\text{E11})$$

where  $\gamma = (1 - |\mathbf{v}|^2)^{-1/2}$ .

From classical general relativity, the coordinate time it takes for a particle initially at rest to fall from some initial position  $r_0 > 1/\kappa_*$  to  $1/\kappa_* < r_1 < r_0$  is

$$\tau = \frac{\kappa_* \sqrt{r_1(\kappa_* r_0 - 1)(r_0 - r_1)} + 2 \log \left( \frac{\sqrt{r_1} \sqrt{\kappa_* r_0 - 1} + \sqrt{r_0 - r_1}}{\sqrt{\kappa_* r_0 r_1 - r_0}} \right)}{2 \sqrt{\kappa_* \left( \kappa_* - \frac{1}{r_0} \right)}} + \frac{\sqrt{\kappa_* r_0 - 1} (\kappa_* r_0 + 2) \cos^{-1} \left( \sqrt{\frac{r_1}{r_0}} \right)}{2 \sqrt{\kappa_* \left( \kappa_* - \frac{1}{r_0} \right)}}. \quad (\text{E12})$$

Note that it goes to infinity as  $r_0 \rightarrow \kappa_*^{-1}$ .

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