

New formalism for perturbations of massive gravity theories around arbitrary background spacetimes

Kieran Wood^{✉*}

School of Physics and Astronomy, University Of Nottingham, Nottingham NG7 2RD, United Kingdom and Nottingham Centre Of Gravity, University Of Nottingham, Nottingham NG7 2RD, United Kingdom



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We develop a new technique for studying the perturbations of dRGT-type massive gravity theories around arbitrary background spacetimes. Built initially from the vielbein formulation of the theory, but switching back to the metric formulation afterward, our approach bypasses many of the complications that arise in previous metric formulation approaches to linearizing massive gravity around generic backgrounds, naturally elucidates the ghost-free structure of the interactions, and readily generalizes to higher orders in perturbation theory, as well as to multiple interacting metric tensor fields. To demonstrate the power of our technique, we apply our formalism to a number of commonly occurring example backgrounds—proportional, cosmological, and black hole—recovering and extending many known results from the literature at linear order. Lastly, we provide, for the first time, the cubic order multigravity potential around a generic background spacetime.

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I. INTRODUCTION

Recent years have seen something of a resurgence in the study of massive spin-2 fields and their interactions, owing to many interesting theoretical developments, as well as potential applications to a number of outstanding fundamental physics problems, particularly in cosmology (see [1–4] for reviews). The subject has a rich history dating back to the time of Fierz and Pauli (FP), who in 1939 first wrote down the only consistent linear theory of such a field propagating on Minkowski spacetime [5], later extended to also apply for generic Einstein spacetimes [6,7] (i.e., those of constant curvature). The FP theory is essentially linearized general relativity (GR)—the unique theory of a *massless* spin-2 field—supplemented by a mass term containing all possible quadratic contractions of the metric perturbation $h_{\mu\nu}$ i.e.,

$$\mathcal{L}_{\text{FP}} \supset -\frac{m^2}{8} (h_{\mu\nu} h^{\mu\nu} + A h^2), \quad (1)$$

with $h = \bar{g}^{\mu\nu} h_{\mu\nu}$ the trace with respect to the background metric $\bar{g}_{\mu\nu}$. Such a mass term, if A is chosen arbitrarily, generically excites a ghostly scalar mode contained within

$h_{\mu\nu}$ —the so-called Boulware-Deser (BD) ghost—whose presence would render the vacuum of the theory unstable upon coupling $h_{\mu\nu}$ to matter. The magic of the FP choice of $A = -1$ is that it ensures h_{00} appears in the action as a Lagrange multiplier, enforcing a primary constraint. A further secondary constraint arises from the assertion that the primary constraint should be preserved in time, and together the two constraints act to exorcise the ghost mode and its conjugate momentum from the spectrum of the theory, leaving five physical, healthy degrees of freedom in four dimensions.

Extending the linear FP theory to a fully nonlinear theory of *massive gravity* (i.e., GR plus a nonlinear mass term) was long thought to be an impossible task, as most candidate nonlinear mass terms are doomed to resurrect the BD ghost [8,9]. It was only relatively recently, in 2010, that a satisfactory mass term (in fact, the *unique* such term) circumventing Boulware and Deser's apparent no-go result was found, and subsequently proved to be free from the vexatious ghost at the full nonlinear level [10–19]. Built upon groundwork laid earlier in [20–22], the associated theory of gravity now goes by the name *dRGT massive gravity*, after its three original architects: de Rham, Gabadadze and Tolley, although the two principal formulations with which we typically express it today (metric and vielbein) are actually due to Hassan, Rosen and Hinterbichler [12,23].

The nonlinear mass term in dRGT theory is constructed from an interaction between *two* independent metrics: the physical metric of spacetime, and some fiducial, nondynamical reference metric that one inserts by hand (typically taken to be Minkowski, but one is free to be more general if

*Contact author: kieran.wood@nottingham.ac.uk

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they so wish). By providing a kinetic term for the reference metric, thereby promoting it to a second dynamical field, one obtains the theory of *bigravity* [24], which, due to the special structure of the dRGT interactions, is also ghost-free [25,26]. The generalization to multiple interacting metrics came soon after in [23] (see also [27]), although the general *multigravity* theory is only devoid of the BD ghost up to certain conditions [28–31], upon which we shall elaborate in Sec. II when we cover the mathematics of the theory in detail.

dRGT massive gravity and bigravity both represent minimal modifications to GR: granting a small mass to the graviton, or coupling a single additional massive spin-2 field to gravity, require only a small number of new parameters: three for dRGT (the graviton mass, plus two dimensionless interaction parameters), or five for bigravity (add to the above a Planck mass and cosmological constant for the second metric), in four spacetime dimensions [2,3]. Consequently, these simpler theories are often preferred over full multigravity by phenomenologists, as the smaller number of free parameters makes them much easier to constrain—see for instance [32–35]. However, studying full multigravity, beyond constituting a useful exercise in generality, is not without its own physical merit either—there is, for example, a nice application to the hierarchy problem utilizing the so-called *clockwork mechanism* [36–38] that only becomes viable with a long chain of interacting metrics [39,40], and there are many interesting dark matter scenarios involving multiple interacting spin-2 fields, e.g., those considered in [41,42]. From a more theoretical standpoint, multigravity theories admit a larger class of ghost-free interactions than can be captured with just dRGT gravity and bigravity [31,43,44]. It is also often customary to think of multigravity as encoding a suitable truncation of the Kaluza-Klein graviton spectrum of a higher dimensional theory of (massless) gravity, following the paradigm known as dimensional deconstruction [45,46], wherein each metric is thought of as corresponding to a distinguished spatial hypersurface in the extra dimension, and their special interaction structure is inherited from the structure of higher dimensional GR [20,47–51]; remarkably, certain phenomena typically associated with extra dimensions can be understood entirely in terms of the underlying massive spin-2 substructure, a particularly nice example being the Gregory-Laflamme instability of a black string [52–59].

In pure dRGT massive gravity, the fixed nature of the reference metric means that the would-be diffeomorphism invariance of GR is completely broken by the mass term and hence the one propagating spin-2 field in the theory has a mass. However, in bi- and multigravity, the interaction term remains invariant under the diagonal subgroup of diffeomorphisms that transforms every metric in the same way, hence these theories contain a single massless spin-2 field (associated to this diagonal subgroup) as well as a collection of massive spin-2 fields (associated to the broken diffeomorphisms). This is consistent with a powerful no-go

theorem by Boulanger, Damour, Gualtieri and Henneaux stating, up to some mild assumptions,¹ that theories containing multiple *massless* spin-2 fields interacting non-linearly are inconsistent [60]: in any theory of interacting spin-2 fields, all but one of them must be massive.²

As expected, dRGT massive gravity correctly linearizes to the FP theory around an Einstein background. Multigravity theories linearize to a sort of “multi-FP” theory containing a mass *matrix* coupling different metric perturbations at quadratic order (in this sense the metrics themselves are akin to flavor eigenstates) i.e.,

$$\mathcal{L}_{\text{mFP}} \supset -\frac{\mathcal{M}_{ij}^2}{8} [h_{\mu\nu}^{(i)} h^{(j)\mu\nu} - h^{(i)} h^{(j)}], \quad (2)$$

around backgrounds where all metrics are proportional to the same Einstein space, as we will see explicitly in Sec. IV. The matrix $\mathcal{M}_{ij}^2$ always contains precisely one zero eigenvalue, in accordance with the no-go theorem mentioned earlier [59,61,62].

Around more generic backgrounds, the structure of the perturbations is much more abstruse. This is a problem, because many spacetimes of physical relevance are *not* Einstein spaces, and metric perturbations around them can give rise to important physical effects, e.g., around the Friedmann-Lemaître-Robertson-Walker (FLRW) metric of cosmology, metric perturbations seed the growth of structure in the universe. Thus, if one wishes to study such effects in the context of these massive gravity theories, it remains an important task to figure out how to perturb them around arbitrary backgrounds in a systematic manner. Thankfully, we already have a pretty good handle on how to do this in the metric formulation of multigravity up to linear order, owing initially to a series of papers by Bernard, Deffayet, Schmidt-May and von Strauss [63–65] from 2015, followed by the papers [66,67] by Mazuet and Volkov from a couple of years later (see also [68,69]). The procedure of [63–65], in its original form, is formulaic and readily applicable in principle, but the calculations involved are complex, containing many steps that can quickly become tedious around more complicated backgrounds, and that do not easily generalize to higher orders in perturbation theory. The reason for this, as we will see in Sec. II, is that the interaction term in the metric formulation contains a matrix square root, which can be very awkward to handle, especially at the level of the perturbations: even at linear order, to determine the structure of the perturbed mass term one is forced to solve a complicated matrix equation, whose solution can be highly nontrivial [70]. The authors did

¹Namely: locality, compatibility with Poincaré invariance, spacetime dimension $D > 2$ and a Lagrangian that is at most second-order in derivatives.

²Indeed, this is also true of the Kaluza-Klein graviton spectrum arising from the compactification of an extra dimension—there is one massless mode and a tower of massive modes.

recognize this issue, and provided a means to sidestep it in [65], at least for dRGT massive gravity and bigravity, by using some clever field redefinitions. Essentially, by absorbing the *background* square root matrix into the definition of the metric perturbations, it becomes possible to write down the linearized field equations in terms of these new variables without having to solve any complicated matrix equations. They used this approach to demonstrate the existence of the ghost-killing constraints (at linear level) around generic backgrounds in bigravity in a covariant manner. However, the necessary field redefinitions inevitably muddy the kinetic structure of the perturbations (this is really only an aesthetic issue), and while their procedure works a treat for theories with exactly two metrics, where there is only a single interaction, it will no longer work for theories containing more than two interacting metrics (this is a more serious issue)—we will explain why in Appendix A.

The work of [66,67] improved on this situation by utilizing the equivalence between the vielbein and metric formulations of dRGT massive gravity to sidestep the necessity of dealing with the troublesome matrix square roots, instead working directly with the perturbations of the vielbeins, in terms of which the potential is polynomial. Doing so trades off the aforementioned complicated matrix equations for arguably simpler algebraic constraint equations³; importantly, this happens *without* affecting the kinetic structure, so both challenges associated to the initial approach from [63–65] are bypassed. However, the approach of [66,67] comes at the cost of having to parametrize the spin-2 field not with a genuine symmetric metric perturbation $h_{\mu\nu}$, but instead with a nonsymmetric tensor $X_{\mu\nu}$ containing the maximal 16 independent components (in four dimensions). To show that the theory still propagates the correct number of degrees of freedom for a massive spin-2 field, one is forced to do lots of complex calculations to first compute the mass term in terms of this nonsymmetric tensor, then to demonstrate that the equations of motion lead to just the right number of constraints to eliminate the unphysical degrees of freedom. The authors succeed in doing so, to their credit; it is just that the calculation is very complicated and as a result the underlying structure of the theory appears somewhat esoteric—see Appendix A for the details. Again, it is worth stressing that this is still all at only the linear level, and that it does not easily generalize to additional interacting metrics or more dimensions.

The culmination of all this discussion is that we currently have three procedures for linearizing multigravity theories around generic backgrounds, all of which are workable, but there is no one procedure that is totally satisfactory. We would like to improve this situation.

³Unfortunately, the necessity of solving *some* set of constraint equations, whether they be matrix equations or algebraic ones, seems to be inevitable. Indeed, we will find the same to be true for the procedure we are going to develop in Sec. III.

In this work, we are going to develop a new procedure to determine the perturbations of generic multigravity theories around arbitrary backgrounds. It is inspired by the procedure of [66,67], as we too start initially from the vielbein formalism and convert to the metric formalism later to avoid the complications of dealing with the matrix square roots. The key difference that distinguishes our approach from theirs, which is central to its utility, lies in how we relate the vielbein perturbations to the metric perturbations, and in how we compute the mass terms. We will see that our procedure has four main advantages over the previous approaches. First, we argue that it is the simplest approach to date: the vielbein perturbations are split up explicitly into the physical degrees of freedom corresponding to the metric perturbations and unphysical degrees of freedom corresponding to local Lorentz transformations, avoiding use of the aforementioned nonsymmetric tensor $X_{\mu\nu}$, thereby facilitating the expression of the spin-2 mass terms directly in terms of the metric perturbations from the outset. Second, it readily generalizes to higher orders in perturbation theory. Third, it makes abundantly clear the fact that the theory is ghost-free, as it becomes obvious that h_{00} will always survive as a Lagrange multiplier in the action at all orders, irrespective of the background. Fourth, it works in both arbitrary spacetime dimension and for any number of interacting metrics.

The structure of the paper is as follows: in Sec. II, we review the fundamentals of multigravity at the background level, outlining both its metric and vielbein formulations and explaining how to relate them (and when it is possible to do so); in Sec. III, we develop our perturbation procedure and use it to derive the linearized field equations for multigravity theories around generic backgrounds; in Sec. IV, we apply our formalism to recover and extend some results regarding the linearized field equations of massive gravity theories on proportional, cosmological and black hole backgrounds; in Sec. V, we compute, for the first time, the cubic order multigravity potential around a generic background; finally, we conclude in Sec. VI.

We work in natural units $c = \hbar = 1$ throughout, and always use a mostly plus metric signature.

II. REVIEW OF MULTIGRAVITY

As discussed in the introduction, multigravity has two distinct formulations: one is known as the *metric formalism*, where the interaction potential coupling the various metrics is built from those metrics directly *à la* dRGT, and the other is known as the *vielbein formalism*, where it is instead built from wedge products of the various tetrad 1-forms associated to each of the metrics. The two formalisms are only equivalent if an important relation known as the Deser-van Nieuwenhuisen (DvN) symmetric vielbein condition holds, as we will see in the coming section.

A. Metric formalism

In the metric formalism, the multigravity action for N metrics interacting on a D -dimensional spacetime manifold $\mathcal{M}_D$ reads as follows,

$$I = I_K + I_V + I_M[g_{(i)}], \quad (3)$$

$$I_K = \sum_{i=0}^{N-1} \frac{M_i^{D-2}}{2} \int d^D x \sqrt{-\det g_{(i)}} R_{(i)}, \quad (4)$$

$$I_V = - \sum_{i,j} \int d^D x \sqrt{-\det g_{(i)}} \sum_{m=0}^D \beta_m^{(i,j)} e_m(S_{i \rightarrow j}). \quad (5)$$

Each metric $g_{\mu\nu}^{(i)}$ gets its own Einstein-Hilbert kinetic term, and the ghost-free dRGT potential is built by summing up the elementary symmetric polynomials, e_m , of the building-block matrices $S_{i \rightarrow j}$,

$$S_{i \rightarrow j} = \sqrt{g_{(i)}^{-1} g_{(j)}}, \quad (6)$$

$$e_m(S) = \frac{1}{m!} \delta_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_m} S_{\mu_1}^{\nu_1} \dots S_{\mu_m}^{\nu_m}, \quad (7)$$

together with some constant coefficients $\beta_m^{(i,j)} = \beta_m^{(j,i)}$ (of mass dimension D) to characterize the interactions between $g_{\mu\nu}^{(i)}$ and $g_{\mu\nu}^{(j)}$. In Eqs. (6) and (7), $S_{i \rightarrow j}$ is the square root matrix discussed at length in the introduction, defined in the sense that $(S_{i \rightarrow j}^2)_{\nu}^{\mu} = g_{(i)}^{\mu\lambda} g_{(j)}^{\lambda\nu}$, and $\delta_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_m}$ is the *generalized Kronecker delta*, defined by antisymmetrizing the product of m standard Kronecker deltas,

$$\delta_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_m} = m! \delta_{[\nu_1}^{\mu_1} \dots \delta_{\nu_m]}^{\mu_m} = \epsilon^{\mu_1 \dots \mu_m} \epsilon_{\nu_1 \dots \nu_m}, \quad (8)$$

where $\epsilon_{\mu_1 \dots \mu_m}$ is the totally antisymmetric Levi-Civita symbol (i.e., tensor density). The antisymmetry properties of the generalized delta are wholly responsible for the ghost freedom of the dRGT interaction structure, as we will see more explicitly in Sec. III. For now, note that the FP mass term from Eq. (1), with $A = -1$, is nothing more than $\mathcal{L}_{\text{FP}} \sim \delta_{\alpha\beta}^{\mu\nu} h_{\mu}^{\alpha} h_{\nu}^{\beta}$. A nice property that will become very useful later on is,

$$\delta_{\nu_1 \dots \nu_p}^{\mu_1 \dots \mu_p} \delta_{\mu_1}^{\nu_1} \dots \delta_{\mu_m}^{\nu_m} = \frac{(D-p+m)!}{(D-p)!} \delta_{\nu_{m+1} \dots \nu_p}^{\mu_{m+1} \dots \mu_p}. \quad (9)$$

Regarding the $S_{i \rightarrow j}$ matrices, there is also some more to say, because in general the matrix square root is not a uniquely defined function. It transpires that the matrix square root that defines $S_{i \rightarrow j}$ from $S_{i \rightarrow j}^2 = g_{(i)}^{-1} g_{(j)}$ must be the *principal* root [71,72], and the light cones of $g_{\mu\nu}^{(i)}$ and $g_{\mu\nu}^{(j)}$ must intersect such that the two metrics share common

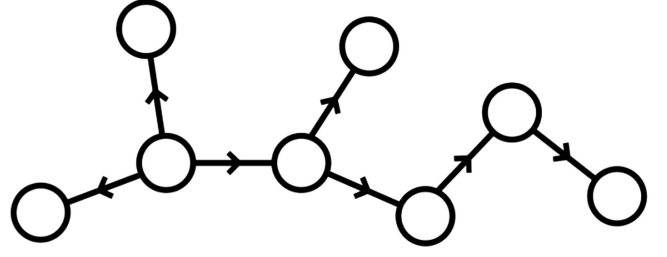


FIG. 1. Directed theory graph representing some generic multimetric theory. The circular nodes represent different metrics, the edges indicate interactions and the arrows point in the direction of positive interaction orientation. Each metric generically has a number of interactions of either orientation, and each edge contributes a term to the field equations of the two metrics it connects; these terms are orientation dependent.

timelike/spacelike directions, so that the action is real and $S_{i \rightarrow j}$ transforms as a $(1, 1)$ -tensor under (diagonal) diffeomorphisms [73]. These two conditions imply a number of additional useful properties, in particular:

- (i) Swapping i and j inverts the matrix, $S_{i \rightarrow j} = S_{j \rightarrow i}^{-1}$.
- (ii) $S_{i \rightarrow j}$ and $S_{j \rightarrow i}$ are equivalent upon lowering an index with the appropriate metric, $g_{\mu\lambda}^{(i)} (S_{i \rightarrow j})_{\nu}^{\lambda} = g_{\mu\lambda}^{(j)} (S_{j \rightarrow i})_{\nu}^{\lambda} \equiv (S_{ij})_{\mu\nu}$.
- (iii) The tensor $(S_{ij})_{\mu\nu}$ with both indices downstairs is symmetric, $(S_{ij})_{\mu\nu} = (S_{ij})_{\nu\mu}$.

The simplest way to view the interaction structure of a given multimetric theory is as a directed graph [62,74], as in Fig. 1; the nodes correspond to metrics and the edges correspond to interactions.

Interactions carry a sense of orientation thanks to property (i): we say that a term in the potential, Eq. (5), that explicitly contains $S_{i \rightarrow j}$ (not $S_{j \rightarrow i}$) is positively oriented with respect to the i th metric and negatively oriented with respect to the j th metric. The orientation of an interaction with respect to a given metric affects the form of that metric's field equations, as we will soon see. It is also simply an artifact of the way one chooses to write down the potential and its interaction coefficients: the following identity holds on the building blocks of the potential,

$$\begin{aligned} & \sqrt{-\det g_{(i)}} \sum_{m=0}^D \beta_m^{(i,j)} e_m(S_{i \rightarrow j}) \\ &= \sqrt{-\det g_{(j)}} \sum_{m=0}^D \beta_{D-m}^{(i,j)} e_m(S_{j \rightarrow i}), \end{aligned} \quad (10)$$

which shows that one can always consider any given positively oriented interaction as a negatively oriented one simply by redefining the interaction coefficients.

⁴To see this, start from $S_{i \rightarrow j}^2 = g_{(i)}^{-1} g_{(j)}$, then multiply by $g_{(i)}$ from the left and $S_{j \rightarrow i} = S_{i \rightarrow j}^{-1}$ from the right.

An important final point is that multimetric interactions of dRGT form are ghost-free so long as there are no cycles present (a cycle is, e.g., $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$, so that the potential is built from *all three* of $S_{1 \rightarrow 2}$, $S_{2 \rightarrow 3}$ and $S_{3 \rightarrow 1}$; in other words, it is a loop in the theory graph) [28–30].

The field equations arising from the action (3) read as follows:

$$M_i^{D-2} G^{(i)\mu}{}_\nu + W^{(i)\mu}{}_\nu = T^{(i)\mu}{}_\nu, \quad (11)$$

where the new term W characterizes the effect of the interactions over and above the standard GR interactions. It is given by,

$$W^{(i)\mu}{}_\nu = \sum_j [W_{i,j}^{(+)}]^\mu{}_\nu + \sum_k [W_{i,k}^{(-)}]^\mu{}_\nu, \quad (12)$$

where (with respect to the i th metric) j denotes positively oriented interactions, k denotes negatively oriented interactions, and we define,⁵

$$[W_{i,j}^{(+)}]^\mu{}_\nu = \sum_{m=0}^D \frac{\beta_m^{(i,j)}}{m!} \delta_{\nu\gamma_1 \dots \gamma_m}^{\mu\lambda_1 \dots \lambda_m} (S_{i \rightarrow j})_{\lambda_1}^{\gamma_1} \dots (S_{i \rightarrow j})_{\lambda_m}^{\gamma_m}, \quad (13)$$

$$[W_{i,k}^{(-)}]^\mu{}_\nu = \sum_{m=0}^D \frac{\beta_{D-m}^{(k,i)}}{m!} \delta_{\nu\gamma_1 \dots \gamma_m}^{\mu\lambda_1 \dots \lambda_m} (S_{i \rightarrow k})_{\lambda_1}^{\gamma_1} \dots (S_{i \rightarrow k})_{\lambda_m}^{\gamma_m}, \quad (14)$$

as the respective contributions from either orientation, whose indices are raised and lowered using the metric corresponding to the first of the two subscript Latin indices, which in this case is $g_{\mu\nu}^{(i)}$. The structure is the same for both contributions; the only real difference is that positively oriented interactions contribute with $\beta_m^{(i,j)}$, while negatively oriented interactions contribute with $\beta_{D-m}^{(k,i)}$. One can show that the two contributions satisfy the following algebraic identity [75],

$$\begin{aligned} & \sqrt{-\det g_{(i)}} [W_{i,j}^{(+)}]^\mu{}_\nu + \sqrt{-\det g_{(j)}} [W_{j,i}^{(-)}]^\mu{}_\nu \\ &= \sqrt{-\det g_{(i)}} \delta_\nu^\mu \sum_{m=0}^D \beta_m^{(i,j)} e_m(S_{i \rightarrow j}), \end{aligned} \quad (15)$$

and thanks to property (iii) of the $S_{i \rightarrow j}$ matrices, the W tensors with both indices downstairs are symmetric,

⁵A comment is in order here: typically, in most massive gravity literature, the W -tensors are expressed in terms of a matrix $Y_{(m)}(S) = \sum_{n=0}^m (-1)^n S^{m-n} e_n(S)$ as (e.g., for the positively oriented contributions) $W_{i,j}^{(+)} = \sum_{m=0}^D (-1)^m \beta_m^{(i,j)} Y_{(m)}(S_{i \rightarrow j})$; one can check that $Y_{(m)\nu}^\mu(S) = \frac{(-1)^m}{m!} \delta_{\nu\gamma_1 \dots \gamma_m}^{\mu\lambda_1 \dots \lambda_m} S_{\lambda_1}^{\gamma_1} \dots S_{\lambda_m}^{\gamma_m}$, so this definition coincides with ours.

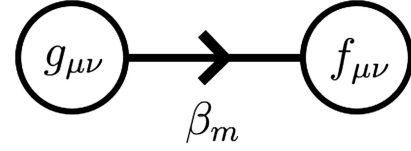


FIG. 2. Directed theory graph for bigravity, with $N = 2$ metrics. Both metrics are treated on the same footing, as one can show using Eq. (10) that the theory is invariant under the simultaneous exchanges $g \leftrightarrow f$, $M_g \leftrightarrow M_f$ and $\beta_m \leftrightarrow \beta_{D-m}$.

$W_{[\mu\nu]}^{(i)} = 0$ (as they must be, given that the Einstein tensors and energy-momentum tensors are also symmetric).

Finally, the Bianchi identities on the Einstein tensors as well as invariance of the matter action I_M under diagonal diffeomorphisms implies the following condition on the W tensors, referred to as the *Bianchi constraint*,

$$\sum_{i=0}^{N-1} \sqrt{-\det g_{(i)}} \nabla_\mu^{(i)} W^{(i)\mu}{}_\nu = 0. \quad (16)$$

If matter couples only to one distinguished metric, or if there is no matter coupling at all (i.e., when one is in vacuum), then this condition is strengthened to:

$$\nabla_\mu^{(i)} W^{(i)\mu}{}_\nu = 0, \quad (17)$$

informing us that there can be no flow of energy-momentum across the interactions between metrics.

As a quick and instructive example to see how this all works, let us consider the simplest possible scenario: bigravity, where one has just $N = 2$ metrics, typically denoted $g_{\mu\nu}^{(0)} \equiv g_{\mu\nu}$ and $g_{\mu\nu}^{(1)} \equiv f_{\mu\nu}$, coupled by a single interaction $S_{g \rightarrow f} \equiv S$ (so $S_{f \rightarrow g} = S^{-1}$) parametrized by one set of coefficients $\beta_m^{(g,f)} \equiv \beta_m$. The theory graph consists of just two nodes adjoined by a single line, as in Fig. 2.

The interaction is positively oriented with respect to $g_{\mu\nu}$ and negatively oriented with respect to $f_{\mu\nu}$, so the field equations are,

$$M_g^{D-2} G^{(g)\mu}{}_\nu + [W_{g,f}^{(+)}]^\mu{}_\nu = T^{(g)\mu}{}_\nu, \quad (18)$$

$$M_f^{D-2} G^{(f)\mu}{}_\nu + [W_{f,g}^{(-)}]^\mu{}_\nu = T^{(f)\mu}{}_\nu, \quad (19)$$

which one can check, using the dictionary from footnote 5 to translate between notations, are precisely the standard bigravity field equations quoted in, e.g., [3,76].

B. Vielbein formalism

In the vielbein formalism, one instead expresses everything given above in the convenient language of differential forms, using the tetrad 1-forms $e^{(i)a} = e_\mu^{(i)a} dx^\mu$ in place of the metrics, with the vielbeins defined in the usual

way through,

$$g_{\mu\nu}^{(i)} = e_\mu^{(i)a} e_\nu^{(i)b} \eta_{ab}. \quad (20)$$

The analog of the multimetric action (3) is given by (see, e.g., [23,40,51]),

$$I = I_K + I_V + I_M[e^{(i)}] \quad (21)$$

$$I_K = \sum_{i=0}^{N-1} \frac{M_i^{D-2}}{2} \int_{\mathcal{M}_D} R_{ab}^{(i)} \wedge \star^{(i)} e^{(i)ab} \quad (22)$$

$$I_V = - \sum_{j_1 \dots j_D=0}^{N-1} \int_{\mathcal{M}_D} \varepsilon_{a_1 \dots a_D} T_{j_1 \dots j_D} e^{(j_1)a_1} \wedge \dots \wedge e^{(j_D)a_D} \quad (23)$$

where the kinetic term is just a rewriting of the standard Einstein-Hilbert term in terms of the curvature 2-form $R_{ab} = \frac{1}{2} R_{abcd} e^c e^d$, using the shorthand $e^{ab\dots} \equiv e^a \wedge e^b \wedge \dots$, and the potential is now built from the wedge products of the various tetrads, with some new symmetric coefficients $T_{j_1 \dots j_D} = T_{(j_1 \dots j_D)}$ (again of mass dimension D) to characterize the interactions.

These coefficients are analogous to the $\beta_m^{(i,j)}$ of the metric formalism, but they are not necessarily equivalent; indeed, as alluded to at the start of this section, not all multigravity theories described by the multivielbein action (21) can be equivalently expressed in the multimetric language of Eq. (3). In fact, this happens only when the DvN condition,

$$\eta_{ab} e_{[\mu}^{(i)a} e_{\nu]}^{(j)b} = 0, \quad (24)$$

is satisfied, which allows one to trade off products of vielbeins for the $S_{i \rightarrow j}$ matrices of the metric formalism [77]. To understand why this is the case, and how it works in practice, it helps to first have a short discussion regarding the degree of freedom count in multigravity theories, which will also prove to be important in Sec. III when we develop our formalism for studying perturbations.

To begin, note that a generic D -dimensional vielbein is an invertible matrix without any symmetry, so contains D^2 independent components. The metric to which it is associated via Eq. (20) is symmetric, containing only $\frac{1}{2}D(D+1)$ components; the residual $\frac{1}{2}D(D-1)$ may be parametrized by a local Lorentz transformation on the vielbein, $e_\mu^{(i)a} \rightarrow \Lambda^{(i)a}_b e_\mu^{(i)b}$, as such transformations leave the metric invariant (since $\Lambda^T \eta \Lambda = \eta$). In the multivielbein action (21), the kinetic terms and matter couplings are fully Lorentz invariant, so all dependence on $\Lambda^{(i)a}_b$ drops out of both I_K and I_M . However, the interaction potential breaks full Lorentz invariance to just the diagonal subgroup transforming all vielbeins in the same way, hence the Lorentz fields remain present in I_V . They are thus auxiliary

fields, which can (in principle) be eliminated in favor of the metric degrees of freedom.

To be more precise, one may always parametrize a generic vielbein as [44,78],

$$e_\mu^{(i)a} = \Lambda^{(i)a}_b E_\mu^{(i)b}, \quad (25)$$

where $\Lambda^{(i)a}_b$ are the local Lorentz matrices and $E_\mu^{(i)a}$ is a restricted vielbein containing only the metric degrees of freedom. The $\Lambda^{(i)a}_b$ matrices may be expressed in terms of another antisymmetric matrix $\omega_{ab}^{(i)} = -\omega_{ba}^{(i)}$, whose $\frac{1}{2}D(D-1)$ components are the aforementioned auxiliary Lorentz fields, via the Cayley transform [43],

$$\Lambda^{(i)a}_b = [(\eta + \omega^{(i)})^{-1}]^{ac} [\eta - \omega^{(i)}]_{bc}. \quad (26)$$

The equations of motion for these Lorentz fields, $\delta I_V / \delta \omega_{ab}^{(i)} = 0$, give the *Lorentz constraints*, which one can show are entirely equivalent to the antisymmetric part of the vielbein field equations, $W_{[\mu\nu]}^{(i)} = 0$, since [44,79],

$$2 \frac{\delta I_V}{e^{(i)}_{[\mu} \eta_{ab} e_{\nu]}^{(i)b}} = [\eta + \omega^{(i)}]_{ac} e_\mu^{(i)c} \frac{\delta I_V}{\delta \omega_{ab}^{(i)}} [\eta + \omega^{(i)}]_{bd} e_\nu^{(i)d}. \quad (27)$$

Explicitly, the W tensors in vielbein form read as follows [58]:

$$W^{(i)\mu}_\nu = e^{(i)\mu}_{\lambda_2 \dots \lambda_D} e_{ab_2 \dots b_D}^{\lambda_2 \dots \lambda_D} \times \sum_{j_2 \dots j_D} \mathcal{P}(i) T_{ij_2 \dots j_D} e_\nu^{(i)a} e_{\lambda_2}^{(j_2)b_2} \dots e_{\lambda_D}^{(j_D)b_D}, \quad (28)$$

defining the mixed index version of the generalized Kronecker delta,

$$e^{(i)\mu}_{a_1 \dots a_D} e_{\nu_1 \dots \nu_D}^{(i)} = e^{(i)\mu}_{a_1} e_{\nu_1}^{(i)} \dots e^{(i)\mu}_{a_D} e_{\nu_D}^{(i)} \delta_{\nu_1 \dots \nu_D}^{\mu_1 \dots \mu_D}, \quad (29)$$

and where $\mathcal{P}(i)$ counts the number of times the index i appears in the interaction coefficients [i.e., a term with $T_{ij_2 \dots j_D}$ has $\mathcal{P}(i) = 1$, a term with $T_{iij_3 \dots j_D}$ has $\mathcal{P}(i) = 2$, and so on]. Enforcing $W_{[\mu\nu]}^{(i)} = 0$ therefore leads to the following set of $N \times \frac{1}{2}D(D-1)$ algebraic constraint equations,

$$g_{\lambda[\mu}^{(i)} u^{(i)\lambda}_{a} e_{\nu]}^{(i)a} = 0 \Rightarrow \eta_{ab} u_{[\mu}^{(i)a} e_{\nu]}^{(i)b} = 0, \quad (30)$$

where we have defined,

$$u^{(i)\mu}_a = e^{(i)\mu}_{\lambda_2 \dots \lambda_D} e_{ab_2 \dots b_D}^{\lambda_2 \dots \lambda_D} \sum_{j_2 \dots j_D} \mathcal{P}(i) T_{ij_2 \dots j_D} e_{\lambda_2}^{(j_2)b_2} \dots e_{\lambda_D}^{(j_D)b_D}; \quad (31)$$

Eq. (30) are the Lorentz constraints. These constraints are not all linearly independent: the weighted sum corresponding to $\eta_{ab} u_{\mu}^{(i)a} u_{\nu}^{(i)b}$ clearly vanishes, so they actually only eliminate $(N-1) \times \frac{1}{2} D(D-1)$ degrees of freedom. A further $\frac{1}{2} D(D-1)$ must be eliminated in some other way to ensure that only the metric degrees of freedom remain dynamical; this is the role played by the single surviving diagonal copy of Lorentz invariance, which ensures that precisely this many local Lorentz fields drop out of the action upon fixing a gauge.

In general, solutions to Eq. (30) are not known, so as of yet most multivielbein theories have no known metric formulation.⁶ However, for theories exhibiting exclusively pairwise interactions between neighboring vielbeins, where the $T_{j_1 \dots j_D}$ are restricted to only terms of the form $T_{iiii\dots}$, $T_{jjjj\dots}$, $T_{jjii\dots}$ etc., Eq. (30) reduces to the DvN condition

(24) for each pair of interacting vielbeins [80]. As we stated earlier, whenever the DvN condition holds, the multivielbein theory described above becomes equivalent to the multimetric theory described in the previous section. To see how this works, note that in matrix notation the DvN condition reads $e_{(i)}^T \eta e_{(j)} = e_{(j)}^T \eta e_{(i)}$, from which one finds $e_{(j)} e_{(i)}^{-1} = \eta^{-1} (e_{(i)}^T)^{-1} e_{(j)}^T \eta$, and hence,

$$\begin{aligned} S_{i \rightarrow j}^2 &= g_{(i)}^{-1} g_{(j)} \\ &= (e_{(i)}^T \eta e_{(i)})^{-1} (e_{(j)}^T \eta e_{(j)}) \\ &= e_{(i)}^{-1} \eta^{-1} (e_{(i)}^T)^{-1} e_{(j)}^T \eta e_{(j)} \\ &= (e_{(i)}^{-1} e_{(j)})^2. \end{aligned} \quad (32)$$

Taking the principal square root, one arrives at,

$$(S_{i \rightarrow j})_{\nu}^{\mu} = e^{(i)\mu}_a e_{\nu}^{(j)a}. \quad (33)$$

The DvN condition is thus equivalent to the statement from the metric formalism that $(S_{ij})_{\mu\nu} = (S_{ij})_{\nu\mu}$, which was likewise necessary to ensure $W_{[\mu\nu]}^{(i)} = 0$.

Armed with a means of relating the vielbeins to the metrics for pairwise interactions, one may now consider the subset of the multivielbein potential (23) for which the DvN condition is known to apply, i.e., interactions that couple together m $e^{(j)}$'s and $(D-m)$ $e^{(i)}$'s. Such terms are characterized by interaction coefficients that are of the form $T_{\{j\}^m \{i\}^{D-m}}$, and owing to the symmetry of these coefficients, there are $\binom{D}{m}$ such terms in total. Expanding this particular subset of the potential out into components, one gets [23],

$$\begin{aligned} I_V &\supset - \sum_{i,j} \int_{\mathcal{M}_D} T_{\{j\}^m \{i\}^{D-m}} \varepsilon_{a_1 \dots a_D} e^{(j)a_1} \wedge \dots \wedge e^{(j)a_m} \wedge e^{(i)a_{m+1}} \wedge \dots \wedge e^{(i)a_D} \\ &= - \int d^D x \sqrt{-\det g_{(i)}} \binom{D}{m} T_{\{j\}^m \{i\}^{D-m}} e^{(i)\mu_1 \dots \mu_{D-m}}_{a_1 \dots a_{D-m}} e^{(j)a_1}_{\mu_1} \dots e^{(j)a_m}_{\mu_m} e^{(i)a_{m+1}}_{\mu_{m+1}} \dots e^{(i)a_D}_{\mu_D} \\ &= - \int d^D x \sqrt{-\det g_{(i)}} (D-m)! \binom{D}{m} T_{\{j\}^m \{i\}^{D-m}} e^{(i)\mu_1 \dots \mu_m}_{a_1 \dots a_m} e^{(j)a_1}_{\mu_1} \dots e^{(j)a_m}_{\mu_m} \\ &= - \int d^D x \sqrt{-\det g_{(i)}} D! T_{\{j\}^m \{i\}^{D-m}} \frac{1}{m!} \delta^{\mu_1 \dots \mu_m}_{\nu_1 \dots \nu_m} (S_{i \rightarrow j})^{\nu_1}_{\mu_1} \dots (S_{i \rightarrow j})^{\nu_m}_{\mu_m} \\ &= - \int d^D x \sqrt{-\det g_{(i)}} D! T_{\{j\}^m \{i\}^{D-m}} e_m(S_{i \rightarrow j}), \end{aligned} \quad (34)$$

where on the third line we used Eq. (9), on the fourth line we used Eq. (33), and on the final line we used Eq. (7). The full multivielbein potential (23) is then obtained by summing up all the different contributions of the type (34), coming from every possible value of m , for

all combinations of i and j , i.e.,

$$I_V = - \sum_{i,j} \int d^D x \sqrt{-\det g_{(i)}} \sum_{m=0}^D D! T_{\{j\}^m \{i\}^{D-m}} e_m(S_{i \rightarrow j}). \quad (35)$$

⁶It is also worth noting that not all multivielbein theories are ghost-free—it seems that only a small subset of them remain so, namely: those with pairwise interactions permitting a clean metric formalism as we will describe, and those in which the $T_{j_1 \dots j_D}$ factorize as $T_{j_1} \dots T_{j_D}$ (whose metric formulation is not yet known) [31,44,80].

Comparing against the multimetric potential (5), one sees that, for the subclass of pairwise interactions where the DvN condition holds, the $T_{j_1 \dots j_D}$ of the vielbein formalism are related to the $\beta_m^{(i,j)}$ of the metric formalism by,

$$D!T_{iiii\dots i} = \sum_j \beta_0^{(i,j)} + \sum_k \beta_D^{(k,i)}, \quad (36)$$

$$D!T_{\{j\}^m\{i\}^{D-m}} = \beta_m^{(i,j)}, \quad (37)$$

where, as in Sec. II A, j and k refer, respectively, to positively and negatively oriented interactions with respect to $g_{\mu\nu}^{(i)}$. The sense of interaction orientation from the metric formalism is hence encoded in the vielbein formalism within the structure of $T_{iiii\dots i}$.

In a similar vein, one may show that Eq. (28) for the vielbein W tensor is equivalent to Eq. (12) from the metric formalism whenever the interactions are strictly pairwise, by substituting in Eqs. (36) and (37) for the $T_{j_1\dots j_D}$, then using Eqs. (9) and (33) to rewrite everything in terms of the building-block matrices $S_{i\rightarrow j}$; one eventually identifies the terms appearing in Eqs. (13) and (14).

III. MULTIGRAVITY PERTURBATIONS: GENERAL FORMALISM

We now possess the necessary technology required to develop our formalism for studying perturbations. We begin with the multivielbein action (21), and perturb the vielbeins directly, $e_\mu^{(i)a} = \bar{e}_\mu^{(i)a} + \delta e_\mu^{(i)a}$. We would like to eventually relate the vielbein perturbations to the corresponding metric perturbations and the $S_{i\rightarrow j}$ matrices; any such relation must also account for the Lorentz constraints, as per the discussion in the previous section. Thankfully, an all-orders expansion of a generic vielbein in terms of the metric perturbations, $\delta g^{(i)\mu}_\nu$, and local Lorentz perturbations $\omega^{(i)\mu}_\nu = \bar{e}^{(i)\mu}_a \omega^{(i)a}_b \bar{e}_\nu^{(i)b}$, has already been developed in [43]; it reads as follows⁷:

$$e_\mu^{(i)a} = \bar{e}_\mu^{(i)a} + \bar{e}_\nu^{(i)a} \sum_{n=1}^{\infty} \kappa_i^n \left[\left(\frac{1}{n} \right) (\delta g_{(i)}^n)^\nu_\mu + 2 \sum_{m=1}^n \left(\frac{1}{n-m} \right) (\omega_{(i)}^m)^\nu_\lambda (\delta g_{(i)}^{n-m})^\lambda_\mu \right], \quad (38)$$

where the fractional binomial coefficient is defined by,

$$\left(\frac{1}{n} \right) = \begin{cases} 1 & n = 0 \\ \frac{(-1)^{n-1}}{2^{2n-1}n} \binom{2n-2}{n-1} & n > 0, \end{cases} \quad (39)$$

⁷If one wishes to compare against the vielbein expansion used in [66,67], one should note that the nonsymmetric tensor $X_{\mu\nu}$ they use to parametrize their spin-2 field, which we mentioned in the introduction, is defined by $X^\mu_\nu \equiv \bar{e}^\mu_a \delta e_\nu^a$; our expression (38) simply divides this tensor explicitly into the components coming from the metric perturbations, which are physical, and the components coming from the local Lorentz perturbations, which are unphysical and will be later eliminated by constraints. We discuss this in more detail in Appendix A 3.

and we include factors of $\kappa_i = 1/M_i^{(D-2)/2}$ to ensure canonical normalization of the metric perturbations. One should keep in mind that the local Lorentz fields $\omega^{(i)\mu}_\nu$ are ultimately nondynamical and should eventually be able to be eliminated in favor of $\delta g^{(i)\mu}_\nu$ through the antisymmetric part of the perturbed field equations.

One may rewrite the expansion (38) as a differential form expression, expanding the tetrad 1-forms in the following way:

$$e^{(i)a} = \bar{e}^{(i)a} + \kappa_i \delta e_{(1)}^{(i)a} + \kappa_i^2 \delta e_{(2)}^{(i)a} + \kappa_i^3 \delta e_{(3)}^{(i)a} + \dots, \quad (40)$$

where we identify the 1-form perturbations to each order as,

$$\delta e_{(1)}^{(i)a} = \frac{1}{2} \bar{e}_\nu^{(i)a} [\delta g^{(i)\nu}_\mu - 4\omega^{(i)\nu}_\mu] dx^\mu, \quad (41)$$

$$\begin{aligned} \delta e_{(2)}^{(i)a} = & -\frac{1}{8} \bar{e}_\nu^{(i)a} [\delta g^{(i)\nu}_\lambda \delta g^{(i)\lambda}_\mu + 8\delta g^{(i)\nu}_\lambda \omega^{(i)\lambda}_\mu \\ & - 16\omega^{(i)\nu}_\lambda \omega^{(i)\lambda}_\mu] dx^\mu, \end{aligned} \quad (42)$$

$$\begin{aligned} \delta e_{(3)}^{(i)a} = & \frac{1}{16} \bar{e}_\nu^{(i)a} [\delta g^{(i)\nu}_\lambda \delta g^{(i)\lambda}_\rho \delta g^{(i)\rho}_\mu - 4\delta g^{(i)\nu}_\lambda \delta g^{(i)\lambda}_\rho \omega^{(i)\rho}_\mu \\ & + 16\delta g^{(i)\nu}_\lambda \omega^{(i)\lambda}_\rho \omega^{(i)\rho}_\mu + 32\omega^{(i)\nu}_\lambda \omega^{(i)\lambda}_\rho \omega^{(i)\rho}_\mu] dx^\mu, \end{aligned} \quad (43)$$

One may then simply substitute these expressions into the multivielbein action up to the desired order in perturbation theory. Let us work to quadratic order for now, so that we may determine the linearized multigravity field equations; we will compute the cubic order potential later on in Sec. V.

A. Quadratic action and linearized field equations

The second-order variation of the Einstein-Hilbert term is well known and reads as follows:

$$\begin{aligned} I_K^{(2)} = & \sum_{i=0}^{N-1} \int d^D x \sqrt{-\det \bar{g}_{(i)}} \left[-\frac{1}{4} \delta g^{(i)\nu}_\mu \bar{\mathcal{E}}^{(i)\mu}_\nu{}^\alpha{}_\beta \delta g^{(i)\beta}_\alpha \right. \\ & - \frac{1}{8} \delta g^{(i)\mu\nu} (\bar{g}_{\mu\nu} \bar{R}^{(i)\alpha\beta} - \bar{R}^{(i)\alpha} \delta_\mu^\beta) \delta g_{\alpha\beta}^{(i)} \\ & \left. + \frac{1}{2} \bar{G}_{\alpha\beta}^{(i)} \left(\delta g^{(i)\alpha}_\lambda \delta g^{(i)\lambda\beta} - \frac{1}{4} \delta g^{(i)} \delta g^{(i)\alpha\beta} \right) \right]. \end{aligned} \quad (44)$$

The first term in this expression is the standard FP kinetic term, where the curved spacetime *Lichnerowicz operator* is defined via:

$$\begin{aligned} \bar{\mathcal{E}}^{(i)\mu}_\nu{}^\alpha{}_\beta \delta g^{(i)\beta}_\alpha = & \frac{1}{2} [-\bar{\square}^{(i)} \delta g^{(i)\mu}_\nu + \bar{\nabla}_\lambda^{(i)} \bar{\nabla}^{(i)\mu} \delta g^{(i)\lambda}_\nu \\ & + \bar{\nabla}^{(i)\lambda} \bar{\nabla}_\nu^{(i)} \delta g^{(i)\mu}_\lambda - \bar{\nabla}^{(i)\mu} \bar{\nabla}_\nu^{(i)} \delta g^{(i)\lambda}_\lambda \\ & + \delta_\nu^\mu \bar{\square}^{(i)} \delta g^{(i)} - \delta_\nu^\mu \bar{\nabla}^{(i)\lambda} \bar{\nabla}_\gamma^{(i)} \delta g^{(i)\gamma}_\lambda] \end{aligned} \quad (45)$$

while the remaining two terms encode the explicit contributions to the action coming from the background curvature (which of course vanish around flat spacetime). As expected, the local Lorentz fields $\omega^{(i)\mu}{}_\nu$ are not present in the kinetic term at all, since the Einstein-Hilbert action is Lorentz invariant.

The second-order variation of the potential term, using Eq. (40) and including only the terms quadratic in the metric perturbations, has two contributions,

$$I_V^{(2)} = - \sum_{ijk_3 \dots k_D} \int \kappa_i \kappa_j \binom{D}{2} T_{ijk_3 \dots k_D} \varepsilon_{abc_3 \dots c_D} \delta e_{(1)}^{(i)a} \wedge \delta e_{(1)}^{(j)b} \wedge \bar{e}^{(k_3)c_3} \wedge \dots \wedge \bar{e}^{(k_D)c_D} \\ - \sum_{ik_2 \dots k_D} \int \kappa_i^2 \binom{D}{1} T_{ik_2 \dots k_D} \varepsilon_{ac_2 \dots c_D} \delta e_{(2)}^{(i)a} \wedge \bar{e}^{(k_2)c_2} \wedge \dots \wedge \bar{e}^{(k_D)c_D}, \quad (46)$$

where we have exploited the symmetry of the $T_{j_1 \dots j_D}$ coefficients, as well as the antisymmetry of the ε tensor and wedge products to bring all of the perturbation 1-forms to the front. Expanding out into components with Eqs. (41)–(43), we explicitly have,

$$I_V^{(2)} = - \sum_{ijk} \int d^D x \sqrt{-\det \bar{g}_{(i)}} \left[\frac{\kappa_i \kappa_j}{4} \binom{D}{2} T_{ijk_3 \dots k_D} \bar{e}^{(i)\mu\lambda_3 \dots \lambda_D}_{abc_3 \dots c_D} (\delta g^{(i)} - 4\omega^{(i)})^\alpha_\mu (\delta g^{(j)} - 4\omega^{(j)})^\beta_\nu \right. \\ \times \bar{e}^{(i)a}_\alpha \bar{e}^{(j)b}_\beta \bar{e}^{(k_3)c_3}_{\lambda_3} \dots \bar{e}^{(k_D)c_D}_{\lambda_D} - \frac{\kappa_i^2}{8} \binom{D}{1} T_{ik_2 \dots k_D} \bar{e}^{(i)\mu\lambda_2 \dots \lambda_D}_{ac_2 \dots c_D} \\ \left. \times (\delta g^{(i)\alpha}_\beta \delta g^{(i)\beta}_\mu - 8g^{(i)\alpha}_\beta \omega^{(i)\beta}_\mu + 16\omega^{(i)\alpha}_\beta \omega^{(i)\beta}_\mu) \bar{e}^{(i)a}_\alpha \bar{e}^{(k_2)c_2}_{\lambda_2} \dots \bar{e}^{(k_D)c_D}_{\lambda_D} \right]. \quad (47)$$

Now let us restrict to only pairwise interactions, so that the DvN condition (24) holds and the background vielbeins can be related to the $\bar{S}_{i \rightarrow j}$ matrices of the metric formalism through Eq. (33). Proceeding in a similar manner to how we did in Eq. (34), using the symmetry of the $T_{j_1 \dots j_D}$ coefficients, their relation to the $\beta_m^{(i,j)}$ of the metric formalism as given by Eqs. (36) and (37), as well as Eqs. (9) and (29) to evaluate any products of generalized deltas arising, we find that the metric formalism version of the above expression reads as follows:

$$I_V^{(2)} = \sum_{i=0}^{N-1} \int d^D x \sqrt{-\det \bar{g}_{(i)}} \left[\frac{\kappa_i^2}{8} (\delta g^{(i)\nu}_\sigma \delta g^{(i)\sigma}_\mu - 8\delta g^{(i)\nu}_\sigma \omega^{(i)\sigma}_\mu + 16\omega^{(i)\nu}_\sigma \omega^{(i)\sigma}_\mu) \bar{W}^{(i)\mu}_\nu \right. \\ - \frac{\kappa_i^2}{8} \mathcal{A}^{(i)\mu}_\alpha{}^\nu_\beta (\delta g^{(i)} - 4\omega^{(i)})^\alpha_\mu (\delta g^{(i)} - 4\omega^{(i)})^\beta_\nu - \sum_j \frac{\kappa_i \kappa_j}{4} [\mathcal{B}_{i,j}^{(+)}]^\mu_\alpha{}^\nu_\beta (\bar{S}_{i \rightarrow j})^\beta_\sigma \\ \times (\delta g^{(i)} - 4\omega^{(i)})^\alpha_\mu (\delta g^{(j)} - 4\omega^{(j)})^\sigma_\nu - \sum_k \frac{\kappa_i \kappa_k}{4} [\mathcal{B}_{i,k}^{(-)}]^\mu_\alpha{}^\nu_\beta (\bar{S}_{i \rightarrow k})^\beta_\sigma (\delta g^{(i)} - 4\omega^{(i)})^\alpha_\mu (\delta g^{(k)} - 4\omega^{(k)})^\sigma_\nu \left. \right], \quad (48)$$

where we recall that the background W tensor is given by Eq. (12), and we have defined the following three new tensors:

$$\mathcal{A}^{(i)\mu}_\alpha{}^\nu_\beta = \sum_j \sum_{m=0}^D \frac{\beta_m^{(i,j)}}{m!} \delta^{\mu\nu\lambda_1 \dots \lambda_m}_{\alpha\beta\gamma_1 \dots \gamma_m} (\bar{S}_{i \rightarrow j})^{\gamma_1}_{\lambda_1} \dots (\bar{S}_{i \rightarrow j})^{\gamma_m}_{\lambda_m} + \sum_k \sum_{m=0}^D \frac{\beta_{D-m}^{(k,i)}}{m!} \delta^{\mu\nu\lambda_1 \dots \lambda_m}_{\alpha\beta\gamma_1 \dots \gamma_m} (\bar{S}_{i \rightarrow k})^{\gamma_1}_{\lambda_1} \dots (\bar{S}_{i \rightarrow k})^{\gamma_m}_{\lambda_m}, \quad (49)$$

$$[\mathcal{B}_{i,j}^{(+)}]^\mu_\alpha{}^\nu_\beta = \sum_{m=0}^D \frac{\beta_m^{(i,j)}}{(m-1)!} \delta^{\mu\nu\lambda_1 \dots \lambda_{m-1}}_{\alpha\beta\gamma_1 \dots \gamma_{m-1}} (\bar{S}_{i \rightarrow j})^{\gamma_1}_{\lambda_1} \dots (\bar{S}_{i \rightarrow j})^{\gamma_{m-1}}_{\lambda_{m-1}}, \quad (50)$$

$$[\mathcal{B}_{i,k}^{(-)}]^\mu_\alpha{}^\nu_\beta = \sum_{m=0}^D \frac{\beta_{D-m}^{(k,i)}}{(m-1)!} \delta^{\mu\nu\lambda_1 \dots \lambda_{m-1}}_{\alpha\beta\gamma_1 \dots \gamma_{m-1}} (\bar{S}_{i \rightarrow k})^{\gamma_1}_{\lambda_1} \dots (\bar{S}_{i \rightarrow k})^{\gamma_{m-1}}_{\lambda_{m-1}}. \quad (51)$$

It is possible to show that the $\mathcal{A}$, $\mathcal{B}$ and $\bar{W}$ -tensors satisfy the following relationship when contracted with some generic tensor $\chi^\mu{}_\nu$ in a particular manner:

$$\begin{aligned} \mathcal{A}^{(i)\mu}{}_{\nu}{}^{\alpha}{}_{\beta}\chi^{\beta}{}_{\alpha} + \sum_j [\mathcal{B}_{i,j}^{(+)}]{}^{\mu}{}_{\nu}{}^{\alpha}{}_{\beta}\chi^{\beta}{}_{\alpha}(\bar{S}_{i\rightarrow j})^{\beta}{}_{\lambda} \\ + \sum_k [\mathcal{B}_{i,k}^{(-)}]{}^{\mu}{}_{\nu}{}^{\alpha}{}_{\beta}\chi^{\beta}{}_{\alpha}(\bar{S}_{i\rightarrow k})^{\beta}{}_{\lambda} = \delta^{\mu\alpha}{}_{\lambda\beta}\chi^{\beta}{}_{\alpha}\bar{W}^{(i)\lambda}{}_{\nu}. \end{aligned} \quad (52)$$

This will allow us to eliminate $\mathcal{A}$ from the action in favor of $\mathcal{B}$ and $\bar{W}$.

The linearized field equations for the metric perturbations now follow readily from the second-order action by varying Eqs. (44) and (48) with respect to $\delta g^{(i)\nu}{}_{\mu}$ and using Eq. (52) to eliminate $\mathcal{A}$. They read as follows:

$$\begin{aligned} \bar{\mathcal{E}}^{(i)\mu}{}_{\nu}{}^{\alpha}{}_{\beta}\delta g^{(i)\beta}{}_{\alpha} + \mathcal{R}^{(i)\mu}{}_{\nu}{}^{\alpha}{}_{\beta}\delta g^{(i)\beta}{}_{\alpha} \\ - \frac{\kappa_i^2}{2} [\bar{W}^{(i)\mu}{}_{\lambda}\delta g^{(i)\lambda}{}_{\nu} + \bar{W}^{(i)\lambda}{}_{\nu}\delta g^{(i)\mu}{}_{\lambda} - \bar{W}^{(i)\mu}{}_{\nu}\delta g^{(i)}] \\ + \frac{\kappa_i}{2} \sum_j [\mathcal{B}_{i,j}^{(+)}]{}^{\mu}{}_{\nu}{}^{\alpha}{}_{\beta}(\bar{S}_{i\rightarrow j})^{\beta}{}_{\lambda} \{[\delta g^{\lambda}{}_{\alpha}]_{i,j} - 4[\omega^{\lambda}{}_{\alpha}]_{i,j}\} \\ + \frac{\kappa_i}{2} \sum_k [\mathcal{B}_{i,k}^{(-)}]{}^{\mu}{}_{\nu}{}^{\alpha}{}_{\beta}(\bar{S}_{i\rightarrow k})^{\beta}{}_{\lambda} \{[\delta g^{\lambda}{}_{\alpha}]_{i,k} - 4[\omega^{\lambda}{}_{\alpha}]_{i,k}\} \\ = \kappa_i T^{(i)\mu}{}_{\nu}, \end{aligned} \quad (53)$$

where the background curvature piece is,

$$\begin{aligned} \mathcal{R}^{(i)\mu}{}_{\nu}{}^{\alpha}{}_{\beta}\delta g^{(i)\beta}{}_{\alpha} \\ = \frac{1}{4} [\delta^{\mu}_{\nu}\bar{R}^{(i)\alpha}{}_{\beta}\delta g^{(i)\beta}{}_{\alpha} + \bar{R}^{(i)\mu}{}_{\nu}\delta g^{(i)} - 2\bar{R}^{(i)}\delta g^{(i)\mu}{}_{\nu}] \\ - [\bar{G}^{(i)\mu}{}_{\lambda}\delta g^{(i)\lambda}{}_{\nu} + \bar{G}^{(i)\lambda}{}_{\nu}\delta g^{(i)\mu}{}_{\lambda} \\ - \frac{1}{4}(\delta^{\mu}_{\nu}\bar{G}^{(i)\alpha}{}_{\beta}\delta g^{(i)\beta}{}_{\alpha} + \bar{G}^{(i)\mu}{}_{\nu}\delta g^{(i)})], \end{aligned} \quad (54)$$

and we have introduced the following notation for brevity:

$$[x]_{i,j} = \kappa_j x_{(j)} - \kappa_i x_{(i)}. \quad (55)$$

The first three terms in the linearized equations (53) are all related to the background curvature, and are essentially the same as they are in GR, if one thinks of the background W -tensor as an effective background source (in particular, around proportional solutions, it behaves like a cosmological constant, as we will see in Sec. IV), while the final two are the terms genuinely arising due to the spin-2 interactions. This is reflected by the fact they are the only terms containing the Lorentz perturbations. Thus, the $\mathcal{B}$ tensors encode the effective masses⁸ of the corresponding spin-2 fields on arbitrary backgrounds.

⁸Here, we are defining an “effective mass” term as an interaction term appearing in the action that is quadratic in the metric perturbations. This is a slight abuse of language: in general, the background spacetime may not be an Einstein space, in which case the structure of the perturbations will not be of Fierz-Pauli form, so it is not clear that one can really identify these terms with the *bona fide* spin-2 masses. Nevertheless, the nomenclature will suffice.

As discussed in Sec. II B, the antisymmetric part of the field equations, after lowering an index with $g_{\mu\nu}^{(i)}$, should give the linearized Lorentz constraints. The only terms in Eq. (53) that are not manifestly symmetric are the $\mathcal{B}$ -tensor terms; setting their antisymmetric part to zero results in the following set of algebraic constraints:

$$(S_{ij})_{\lambda[\mu}\{[\delta g^{\lambda}{}_{\nu}]\}_{i,j} - 4[\omega^{\lambda}{}_{\nu}]_{i,j}\} = 0, \quad (56)$$

or, written more explicitly,

$$\begin{aligned} \kappa_j (S_{ij})_{\mu\lambda}(\delta g^{(j)} - 4\omega^{(j)})^{\lambda}{}_{\nu} + \kappa_i (S_{ij})_{\nu\lambda}(\delta g^{(i)} - 4\omega^{(i)})^{\lambda}{}_{\mu} \\ = [\mu \leftrightarrow \nu], \end{aligned} \quad (57)$$

for every pair of interacting metrics. Equation (57) is of course consistent with the linearization of the DvN condition (24), using our vielbein expansion (38). It also agrees with the analogous vielbein expression derived in [43] [their Eq. (3.22)], as well as the set of algebraic constraints derived in terms of the nonsymmetric $X_{\mu\nu}$ tensor⁹ in [66,67] (Eq. (4.3) in [67]). If one is able to solve Eq. (57) for the Lorentz perturbations, then one is able to express the linearized field Eq. (53) in terms of only the metric perturbations, which is the ultimate goal.

B. Comparison with previous approaches to linearizing multigravity

It is worth pausing at this stage to ask the important question: what benefit is there to constructing the linearized field equations in this manner, as opposed to simply using the matrix technology that was previously derived in [63–65], or using the nonsymmetric tensor technology of [66,67] (both of which we recap in Appendix A), since many of the expressions we have derived throughout this section appear quite complicated?

Regarding the matrix approach, to an extent one could argue, at least at linear level, that which formalism one uses is simply a matter of taste: as mentioned in the introduction, the potentially debilitating stage in the matrix calculation is that one needs to solve a challenging matrix equation in order to determine the linear perturbations to $S_{i\rightarrow j}$; in our formalism, this stage of the calculation is replaced by the requirement that one needs to solve the Lorentz constraints (57), which one might imagine could prove equally challenging around an arbitrary background. True as this might be, one should note that it is not necessary to solve for every $\omega^{(i)}$ individually—just the differences $[\omega^{\mu}{}_{\nu}]_{i,j}$ suffice, as these are the combinations that appear in the dynamical equations for the metric perturbations. As we will see in the following section, around many physically relevant backgrounds, computing these differences is actually quite a

⁹To first order in perturbation theory, this tensor is equivalent to $X_{\mu\nu}^{(1)} = \frac{\kappa}{2}[\delta g_{\mu\nu} - 4\omega_{\mu\nu}]$ —see Appendix A 3.

simple task, certainly simpler than solving the analogous matrix equations [c.f. Eq. (A11)]. In fact, as we will demonstrate, on backgrounds with sufficient symmetry, it can be the case that the differences $[\omega^\mu]_{i,j}$ drop out of the field equations entirely, in which case simply knowing the form of the background $\bar{S}_{i \rightarrow j}$ matrices is all one needs to determine the perturbation structure—our approach then becomes significantly simpler than the matrix approach.

Of course, the same could also be said of the approach from [66,67] using the nonsymmetric tensor $X_{\mu\nu}$, since there the difficulties associated with the matrix equations are also circumvented by working in terms of the vielbein perturbations. Indeed, in spirit, our approach and the one from [66,67] are actually one and the same. However, there are two key practical differences between our formalisms that, at least in our opinion, make our approach the more useful of the two: first, we are able to explicitly parametrize the $X_{\mu\nu}$ tensor in terms of the metric perturbations and local Lorentz perturbations via Eq. (38), which makes eliminating the unphysical degrees of freedom [through the Lorentz constraints (57)] more straightforward; second, we are able to easily compute the effective mass terms by simply contracting the relevant number of background $\bar{S}_{i \rightarrow j}$ matrices with generalized Kronecker deltas, where their approach instead involved solving a number of complicated algebraic equations to express the background $\bar{S}_{i \rightarrow j}$ matrices in terms of the background curvature quantities and $\beta_m^{(i,j)}$ parameters—see Appendix A 3 for the details. Using their approach, it quickly becomes a challenge to compute these effective mass terms once additional dynamical metrics or extra dimensions are involved, whereas with our formalism this is a straightforward task, as we have demonstrated by doing all our calculations to this point in complete generality. Moreover, it is obvious in our formalism how to determine corrections beyond the linear order: one should simply include more terms from Eq. (38) in the expansion of the vielbeins, which will lead to additional tensor structures emerging, taking a similar form to Eqs. (49)–(51), only containing more free indices (we will see this explicitly for the cubic terms in Sec. V).

Finally, we would like to stress that the ghost freedom of the theory, which we know holds at the full nonlinear level, is clearly manifest in our formalism at the level of the perturbed action. Due to their structure in terms of the generalized deltas, the effective mass terms (involving the $\mathcal{B}$ tensors) will only contribute to the quadratic action a term that is *linear* in $\delta g^{(i)0}_0$: any term involving $(\delta g^{(i)0}_0)^2$ would require a delta with multiple 0 indices on one of its rows, which automatically vanishes by antisymmetry. Since the GR-like terms also only contribute a linear term in $\delta g^{(i)0}_0$ (as it is the linear analog of the lapse function, which in standard GR acts as a Lagrange multiplier), the full quadratic multigravity action also remains linear in $\delta g^{(i)0}_0$. Thus, $\delta g^{(i)0}_0$ enforces a primary constraint. On the other hand,

while the GR-like terms are linear in $\delta g^{(i)j}_0$ (the linear analog of the shift vector), the effective mass terms are quadratic in these variables, so they do not directly give rise to constraints. Instead, $\delta g^{(i)j}_0$ behave as auxiliary fields, which can be uniquely determined in terms of the genuinely dynamical variables and can hence be eliminated from the action¹⁰; the resulting Hamiltonian after eliminating the auxiliary fields is second class. Consequently, when one enforces the constancy of the primary constraint in time, it will inevitably lead to a further secondary, second-class constraint, in complete analogy with what happens in the usual FP story. This is how we see using our formalism that the perturbations of dRGT-type theories remain ghost-free, even around more generic backgrounds.

IV. MULTIGRAVITY PERTURBATIONS: EXAMPLES

Let us now use the formalism we have developed to compute the structure of the multigravity perturbations around some commonly occurring example background spacetimes. We will begin by reproducing the perturbation structure around the so-called *proportional solutions*, the simplest vacuum solutions of multigravity, before moving on to look at the perturbation structure around cosmological and black hole backgrounds.

A. Proportional backgrounds

Let us work in vacuum, with $I_M = 0$ for simplicity. As the name suggests, the proportional solutions are those where all of the various metrics are proportional to one another [61,62],

$$g_{\mu\nu}^{(i)} = a_i^2 \bar{g}_{\mu\nu}, \quad (58)$$

where $\bar{g}_{\mu\nu}$ solves the vacuum field equations of GR with cosmological constant $\bar{\Lambda}$, and a_i are a set of conformal factors that the Bianchi constraint forces to be constant [61].

With this ansatz, the relevant background curvature tensors are,

$$\begin{aligned} \bar{R}^{(i)\mu}{}_\nu &= \frac{\bar{R}^\mu{}_\nu}{a_i^2}, & \bar{R}^{(i)} &= \frac{\bar{R}}{a_i^2}, & \bar{G}^{(i)\mu}{}_\nu &= \frac{\bar{G}^\mu{}_\nu}{a_i^2} \\ \bar{R}_{\mu\nu} &= \frac{2\bar{\Lambda}}{D-2} \bar{g}_{\mu\nu}, & \bar{R} &= \frac{2D\bar{\Lambda}}{D-2}, & \bar{G}_{\mu\nu} &= -\bar{\Lambda} \bar{g}_{\mu\nu}, \end{aligned} \quad (59)$$

¹⁰Really, one can only solve for $N - 1$ of these, since they always appear in the effective mass terms via the difference combinations $[\delta g^j_{0i}]_{i,j}$ (of which there are $N - 1$ independent pairings, assuming no cycles—see Fig. 1). This is of course consistent with the fact that one of the spin-2 fields remains massless; the remaining $\delta g^{(i)j}_0$ is a gauge mode of the surviving diagonal diffeomorphism invariance. The same is true of the $\delta g^{(i)0}_0$: $N - 1$ of them are Lagrange multipliers, and one is pure gauge.

and the building-block matrices are simply $(S_{i \rightarrow j})^\mu_\nu = (a_j/a_i)\delta^\mu_\nu$, leading to the following form for the W -tensor components upon substitution into Eq. (12),

$$W^{(i)\mu}_\nu = \delta^\mu_\nu \left[\sum_j \sum_{m=0}^D \beta_m^{(i,j)} \binom{D-1}{m} \left(\frac{a_j}{a_i}\right)^m + \sum_k \sum_{m=0}^D \beta_{D-m}^{(k,i)} \binom{D-1}{m} \left(\frac{a_k}{a_i}\right)^m \right]. \quad (60)$$

Consequently, the multigravity vacuum equations $M_i^{D-2} G^{(i)\mu}_\nu + W^{(i)\mu}_\nu = 0$ become the following N algebraic, nonlinear simultaneous equations,

$$\frac{\bar{\Lambda} M_i^{D-2}}{a_i^2} = \sum_j \sum_{m=0}^D \beta_m^{(i,j)} \binom{D-1}{m} \left(\frac{a_j}{a_i}\right)^m + \sum_k \sum_{m=0}^D \beta_{D-m}^{(k,i)} \binom{D-1}{m} \left(\frac{a_k}{a_i}\right)^m, \quad (61)$$

which, after fixing one of the a_i via coordinate rescaling, may be solved for $\bar{\Lambda}$ and the remaining $N-1$ conformal factors, the physical solutions being those with real $\bar{\Lambda}$ and a_i . In this way, multigravity naturally admits de Sitter (dS), anti-de Sitter (AdS) and Minkowski vacua, where the interactions between metrics manifest themselves as an effective cosmological constant.

Turning to the perturbations, let us first consider the linearized Lorentz constraints (57). The fact that $\bar{S}_{i \rightarrow j}$ is proportional to the identity matrix hugely simplifies things here, as it implies that S and δg commute, so the metric perturbations drop out of Eq. (57).¹¹ Thus, the Lorentz constraints are trivially solved by,

$$\kappa_i \omega^{(i)\mu}_\nu = \kappa_j \omega^{(j)\mu}_\nu \Rightarrow [\omega^\mu_\nu]_{i,j} = 0, \quad (62)$$

for all pairs i, j . Consequently, the local Lorentz fields drop out of the field equations (53) for the metric perturbations entirely—this solution for the $\omega^{(i)}$'s corresponds to performing an overall Lorentz transformation on all vielbeins, which is a symmetry of the theory.

Moving on to the metric equations (53), using the background curvature tensors from Eq. (59), the two kinetic terms become,

$$\begin{aligned} & \bar{\mathcal{E}}^{(i)\mu}_\nu{}^\alpha{}_\beta \delta g^{(i)\beta}_\alpha + \mathcal{R}^{(i)\mu}_\nu{}^\alpha{}_\beta \delta g^{(i)\beta}_\alpha \\ &= \frac{\bar{\mathcal{E}}^{(i)\mu}_\nu{}^\alpha{}_\beta \delta g^{(i)\beta}_\alpha}{a_i^2} - \frac{\bar{\Lambda}}{a_i^2(D-2)} (D \delta g^{(i)\mu}_\nu - \delta^\mu_\nu \delta g^{(i)}) \\ &+ \frac{\bar{\Lambda}}{a_i^2} \left(2 \delta g^{(i)\mu}_\nu - \frac{1}{2} \delta^\mu_\nu \delta g^{(i)} \right). \end{aligned} \quad (63)$$

¹¹In fact, this can *only* happen around proportional backgrounds, as only when $S \propto \mathbb{1}$ does S commute with a generic δg .

Furthermore, from the background field equations (61) we have $\bar{W}^{(i)\mu}_\nu = \delta^\mu_\nu M_i^{D-2} \bar{\Lambda}/a_i^2$, and hence,

$$\begin{aligned} & \frac{\kappa_i^2}{2} [\bar{W}^{(i)\mu}_\lambda \delta g^{(i)\lambda}_\nu + \bar{W}^{(i)\lambda}_\nu \delta g^{(i)\mu}_\lambda - \bar{W}^{(i)\mu}_\nu \delta g^{(i)}] \\ &= \frac{\bar{\Lambda}}{a_i^2} \left[\delta g^{(i)\mu}_\nu - \frac{1}{2} \delta^\mu_\nu \delta g^{(i)} \right]. \end{aligned} \quad (64)$$

Regarding the mass terms, the fact that $S_{i \rightarrow j} \propto \mathbb{1}$ again simplifies things significantly, as all contractions with the generalized deltas in the $\mathcal{B}$ tensors may be evaluated using Eq. (9) to yield,

$$[\mathcal{B}_{i,j}^{(+)}]^\mu{}_\nu{}^\alpha{}_\beta = \sigma_{i,j}^{(+)} \delta^{\mu\alpha}_{\nu\beta}, \quad (65)$$

$$[\mathcal{B}_{i,k}^{(-)}]^\mu{}_\nu{}^\alpha{}_\beta = \sigma_{i,k}^{(-)} \delta^{\mu\alpha}_{\nu\beta}, \quad (66)$$

where we have defined the following combinations of interaction coefficients and conformal factors:

$$\sigma_{i,j}^{(+)} = \sum_{m=0}^D \beta_m^{(i,j)} \binom{D-2}{m-1} \left(\frac{a_j}{a_i}\right)^{m-1}, \quad (67)$$

$$\sigma_{i,k}^{(-)} = \sum_{m=0}^D \beta_{D-m}^{(k,i)} \binom{D-2}{m-1} \left(\frac{a_k}{a_i}\right)^{m-1}. \quad (68)$$

These two parameters appear ubiquitously in multigravity—we will see them appearing in cosmological and black hole perturbations later on as well. The plus and minus variants associated to any given pair of metrics are related to one another by,

$$\sigma_{j,i}^{(-)} = \left(\frac{a_i}{a_j}\right)^{D-2} \sigma_{i,j}^{(+)}, \quad (69)$$

informing us that the $\sigma_{i,j}^{(\pm)}$ parameters live on the interaction links coupling $g_{\mu\nu}^{(i)}$ and $g_{\mu\nu}^{(j)}$.

Substituting Eqs. (63)–(66) into Eq. (53), one finds that the linearized field equations around a proportional background are,

$$\begin{aligned} & \frac{\bar{\mathcal{E}}^{(i)\mu}_\nu{}^\alpha{}_\beta \delta g^{(i)\beta}_\alpha}{a_i^2} - \frac{2\bar{\Lambda}}{a_i^2(D-2)} \left(\delta g^{(i)\mu}_\nu - \frac{1}{2} \delta^\mu_\nu \delta g^{(i)} \right) \\ &+ \frac{1}{2} \left\{ \sum_j \kappa_i \frac{a_j}{a_i} \sigma_{i,j}^{(+)} (\delta^\mu_\nu [\delta g]_{i,j} - [\delta g^\mu_\nu]_{i,j}) \right. \\ &+ \left. \sum_k \kappa_i \frac{a_k}{a_i} \sigma_{i,k}^{(-)} (\delta^\mu_\nu [\delta g]_{i,k} - [\delta g^\mu_\nu]_{i,k}) \right\} = 0. \end{aligned} \quad (70)$$

We can rewrite this so that all tensorial quantities behave as if they lived in the *common* background of $\bar{g}_{\mu\nu}$, by defining,

$$\delta \tilde{g}^{(i)\mu}{}_\nu \equiv a_i^2 \delta g^{(i)\mu}{}_\nu = \bar{g}^{\mu\lambda} \delta g_{\lambda\nu}^{(i)}, \quad (71)$$

which has its indices manipulated with $\bar{g}_{\mu\nu}$, rather than $g_{\mu\nu}^{(i)}$. Rewriting the above set of field equations in terms of the tilded metric perturbations, then lowering an index using $\bar{g}_{\mu\nu}$, one arrives at,

$$\begin{aligned} \bar{\mathcal{E}}_{\mu\nu}{}^\alpha{}_\beta \delta \tilde{g}^{(i)\beta}{}_\alpha - \frac{2\bar{\Lambda}}{D-2} \left(\delta \tilde{g}_{\mu\nu}^{(i)} - \frac{1}{2} \bar{g}_{\mu\nu} \delta \tilde{g}^{(i)} \right) \\ + \frac{a_i^2}{2} \left\{ \sum_j \kappa_i \sigma_{i,j}^{(+)} \left[\kappa_i \frac{a_j}{a_i} (\delta \tilde{g}_{\mu\nu}^{(i)} - \bar{g}_{\mu\nu} \delta \tilde{g}^{(i)}) \right. \right. \\ \left. \left. - \kappa_j \frac{a_i}{a_j} (\delta \tilde{g}_{\mu\nu}^{(j)} - \bar{g}_{\mu\nu} \delta \tilde{g}^{(j)}) \right] \right. \\ \left. + \sum_k \kappa_i \sigma_{i,k}^{(-)} \left[\kappa_i \frac{a_k}{a_i} (\delta \tilde{g}_{\mu\nu}^{(i)} - \bar{g}_{\mu\nu} \delta \tilde{g}^{(i)}) \right. \right. \\ \left. \left. - \kappa_k \frac{a_i}{a_k} (\delta \tilde{g}_{\mu\nu}^{(k)} - \bar{g}_{\mu\nu} \delta \tilde{g}^{(k)}) \right] \right\} = 0. \quad (72) \end{aligned}$$

One can check that these equations agree with the linearized multimetric field equations given in [62], for the two classes of theory graph they consider (“star-type” and “chain-type” interactions); our Eq. (72) constitute the appropriate generalization to arbitrary interaction structures with any number of positively and negatively oriented interactions per metric.

A final step we can take, given that we have already rewritten everything with respect to the common background metric $\bar{g}_{\mu\nu}$, is to ensure that the perturbations are all canonically normalized with respect to $\bar{g}_{\mu\nu}$. This requires that we take,

$$\delta \tilde{g}_{\mu\nu}^{(i)} = a_i h_{\mu\nu}^{(i)}. \quad (73)$$

With this choice, the linearized field equations (72) become,

$$\begin{aligned} \bar{\mathcal{E}}_{\mu\nu}{}^\alpha{}_\beta h^{(i)\beta}{}_\alpha - \frac{2\bar{\Lambda}}{D-2} \left(h_{\mu\nu}^{(i)} - \frac{1}{2} \bar{g}_{\mu\nu} h^{(i)} \right) \\ + \frac{\mathcal{M}_{ij}^2}{2} (h_{\mu\nu}^{(j)} - \bar{g}_{\mu\nu} h^{(j)}) = 0, \quad (74) \end{aligned}$$

which is precisely in the multi-FP form alluded to by Eq. (2), where the mass matrix has components,

$$\mathcal{M}_{ii}^2 = \kappa_i^2 a_i^2 \left(\sum_j \frac{a_j}{a_i} \sigma_{i,j}^{(+)} + \sum_k \frac{a_k}{a_i} \sigma_{i,k}^{(-)} \right), \quad (75)$$

$$\mathcal{M}_{ji}^2 = \left(\frac{a_j}{a_i} \right)^{4-D} \mathcal{M}_{ij}^2 = -\kappa_i \kappa_j a_i^2 \sigma_{i,j}^{(+)}, \quad (76)$$

and all tensorial quantities behave as if they lived in the common background of $\bar{g}_{\mu\nu}$. It is a simple check to show

that there always exists a single massless eigenvector (i.e., with eigenvalue $m_0^2 = 0$) scaling as $O^{j0} \propto a_j/\kappa_j$, irrespective of any choices for $\beta_m^{(i,j)}$, corresponding to the surviving diagonal copy of diffeomorphism invariance in the theory. This is in accordance with the no-go theorem mentioned in the introduction forbidding theories of multiple interacting massless spin-2 fields [60].

As we did in Sec. II, as a sanity check, let us compare the results of this section against known results in bigravity, where the mass matrix for $N = 2$ metrics is simple enough to diagonalize explicitly. Adopting the standard bigravity notation $g_{\mu\nu}^{(0)} \equiv g_{\mu\nu}$ and $g_{\mu\nu}^{(1)} \equiv f_{\mu\nu}$, rescaling the coordinates to fix $a_g = 1$, then denoting $a_f = c$, directly from the theory graph in Fig. 2 one sees that only $\sigma_{g,f}^{(+)}$ and $\sigma_{f,g}^{(-)}$ are present in the mass matrix, and by Eq. (69) these two parameters are related by $\sigma_{f,g}^{(-)} = c^{2-D} \sigma_{g,f}^{(+)}$. Therefore, the mass matrix explicitly reads,

$$\mathcal{M}^2 = \begin{bmatrix} \kappa_g^2 c \sigma_{g,f}^{(+)} & -\kappa_g \kappa_f \sigma_{g,f}^{(+)} \\ -\kappa_g \kappa_f c^{4-D} \sigma_{g,f}^{(+)} & \kappa_f^2 c^{3-D} \sigma_{g,f}^{(+)} \end{bmatrix}, \quad (77)$$

which one can check has the following eigenvalues:

$$m_0^2 = 0, \quad m_1^2 = \frac{c \sigma_{g,f}^{(+)}}{M_g^{D-2}} \left[\frac{1 + (\alpha c)^{D-2}}{(\alpha c)^{D-2}} \right], \quad (78)$$

with the corresponding eigenvectors being,

$$H_{\mu\nu}^{(0)} = \frac{1}{\sqrt{1 + c^2 \alpha^{D-2}}} (h_{\mu\nu}^{(g)} + c \alpha^{\frac{D-2}{2}} h_{\mu\nu}^{(f)}), \quad (79)$$

$$H_{\mu\nu}^{(1)} = \frac{1}{\sqrt{1 + c^{2(D-3)} \alpha^{D-2}}} (h_{\mu\nu}^{(f)} - c^{D-3} \alpha^{\frac{D-2}{2}} h_{\mu\nu}^{(g)}), \quad (80)$$

defining $\alpha = M_f/M_g$ as the ratio of the gravitational couplings. The equations of motion for the mass modes, as well as the corresponding heavy graviton mass m_1 , agree precisely with the standard bigravity results [61,76], as they should (though in [76] the perturbations were not explicitly canonically normalized with respect to $\bar{g}_{\mu\nu}$, so the precise structure of their $H_{\mu\nu}^{(i)}$ in terms of the $h_{\mu\nu}^{(i)}$ is slightly different to here, following instead from the noncanonical field equations (72)—the masses are the same though, as they should be).

B. Cosmological backgrounds

Now let us turn to a background in which the perturbation structure is somewhat more involved, namely, that of an expanding FLRW universe, where the most general homogeneous and isotropic ansatz one can make for the metrics at background level (in $D = 4$ dimensions) is,

$$g_{\mu\nu}^{(i)} dx^\mu dx^\nu = -c_i^2(t) dt^2 + a_i^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2 \right], \quad (81)$$

where $c_i(t)$ are the *lapses*, $a_i(t)$ are the *scale factors*, $d\Omega_2^2$ is the line element for a 2-sphere, and it can be shown that the Bianchi constraints (17) force the spatial curvature k to be the same for all metrics [81]. One of the lapses, say, on the distinguished metric $g_{\mu\nu}^{(i_*)}$, can be fixed to $c_{i_*}(t) = 1$ throughout all time by choosing a preferred time coordinate; ordinarily one would choose $g_{\mu\nu}^{(i_*)}$ to be the metric to which matter couples, so that t then corresponds to the cosmic time.

The background cosmology of multigravity, with the ansatz (81) for the metrics, is discussed in detail in [4], following a breadth of earlier work regarding the cosmology of massive gravity and bigravity [82–97]. Essentially, solutions split into three branches depending on how the Bianchi constraints are satisfied, of which two are unstable to nonlinear ghosts and growing tensor modes [69,98,99], but the third (which requires $c_i = \dot{a}_i/\dot{a}_{i_*}$) is stable and has particularly interesting (and potentially observationally relevant [33,34,100–103]) phenomenology, e.g., there is a dynamical effective dark energy component arising due to the W tensors that has a phantom equation of state in the distant past but which later relaxes to an effective cosmological constant once all external matter has diluted away.

For our purposes, all we wish to do is to use our perturbation formalism to compute the effective mass terms around this FLRW background. To this end, immediately from Eq. (81) one sees that the building-block matrices $S_{i \rightarrow j}$ take the simple form,

$$S_{i \rightarrow j} = \text{diag} \left(\frac{c_j}{c_i}, \frac{a_j}{a_i}, \frac{a_j}{a_i}, \frac{a_j}{a_i} \right). \quad (82)$$

Substituting into Eq. (57) leads to the following Lorentz constraints:

$$[\omega^m_n]_{i,j} = 0, \quad (83)$$

$$4 \left(\frac{a_j}{a_i} + \frac{c_j}{c_i} \right) [\omega^0_m]_{i,j} = - \left(\frac{a_j}{a_i} - \frac{c_j}{c_i} \right) [\delta g^0_m]_{i,j}, \quad (84)$$

which are naturally split in a $3 + 1$ manner owing to the presence of the lapses. The spatial components, Eq. (83), are solved trivially and hence drop out of the metric field equations, thanks to the proportionality of the spatial metrics. However, the components mixing space and time indices, Eq. (84), are nontrivially solved, and will hence contribute to the metric equations for $\delta g^{(i)0}_m$.

Regarding said metric equations, by substituting Eq. (82) into Eqs. (50) and (51), one finds the following structure for the effective spin-2 mass terms:

$$[\mathcal{B}_{i,j}^{(+)}]_0^{\alpha\beta} (\bar{S}_{i \rightarrow j})^\beta_{\lambda\chi} \chi^\lambda_\alpha = \frac{a_j}{a_i} \sigma_{i,j}^{(+)}(t) (\delta^0_0 \chi - \chi^0_0), \quad (85)$$

$$[\mathcal{B}_{i,j}^{(+)}]_n^{\alpha\beta} (\bar{S}_{i \rightarrow j})^\beta_{\lambda\chi} \chi^\lambda_\alpha = \frac{c_j}{c_i} \sigma_{i,j}^{(+)}(t) \chi^0_n \delta_n^m + \frac{a_j}{a_i} \varsigma_{i,j}^{(+)}(t) (\delta_n^m \chi^k_k - \chi^m_n), \quad (86)$$

$$[\mathcal{B}_{i,j}^{(+)}]_m^{\alpha\beta} (\bar{S}_{i \rightarrow j})^\beta_{\lambda\chi} \chi^\lambda_\alpha = -\frac{c_j}{c_i} \sigma_{i,j}^{(+)}(t) \chi^0_m, \quad (87)$$

$$[\mathcal{B}_{i,j}^{(+)}]_0^{\alpha\beta} (\bar{S}_{i \rightarrow j})^\beta_{\lambda\chi} \chi^\lambda_\alpha = -\frac{a_j}{a_i} \sigma_{i,j}^{(+)}(t) \chi^m_0, \quad (88)$$

for the positively oriented interactions, and,

$$[\mathcal{B}_{i,k}^{(-)}]_0^{\alpha\beta} (\bar{S}_{i \rightarrow k})^\beta_{\lambda\chi} \chi^\lambda_\alpha = \frac{a_k}{a_i} \sigma_{i,k}^{(-)}(t) (\delta^0_0 \chi - \chi^0_0), \quad (89)$$

$$[\mathcal{B}_{i,k}^{(-)}]_n^{\alpha\beta} (\bar{S}_{i \rightarrow k})^\beta_{\lambda\chi} \chi^\lambda_\alpha = \frac{c_k}{c_i} \sigma_{i,k}^{(-)}(t) \chi^0_n \delta_n^m + \frac{a_k}{a_i} \varsigma_{i,k}^{(-)}(t) (\delta_n^m \chi^k_k - \chi^m_n), \quad (90)$$

$$[\mathcal{B}_{i,k}^{(-)}]_m^{\alpha\beta} (\bar{S}_{i \rightarrow k})^\beta_{\lambda\chi} \chi^\lambda_\alpha = -\frac{c_k}{c_i} \sigma_{i,k}^{(-)}(t) \chi^0_m, \quad (91)$$

$$[\mathcal{B}_{i,k}^{(-)}]_0^{\alpha\beta} (\bar{S}_{i \rightarrow k})^\beta_{\lambda\chi} \chi^\lambda_\alpha = -\frac{a_k}{a_i} \sigma_{i,k}^{(-)}(t) \chi^m_0, \quad (92)$$

for the negatively oriented interactions, where we have defined the following time-dependent parameters:

$$\sigma_{i,j}^{(+)}(t) = \beta_1^{(i,j)} + 2\beta_2^{(i,j)} \frac{a_j}{a_i} + \beta_3^{(i,j)} \left(\frac{a_j}{a_i} \right)^2, \quad (93)$$

$$\sigma_{i,k}^{(-)}(t) = \beta_3^{(k,i)} + 2\beta_2^{(k,i)} \frac{a_k}{a_i} + \beta_1^{(k,i)} \left(\frac{a_k}{a_i} \right)^2, \quad (94)$$

$$\varsigma_{i,j}^{(+)}(t) = \beta_1^{(i,j)} + \beta_2^{(i,j)} \left(\frac{c_j}{c_i} + \frac{a_j}{a_i} \right) + \beta_3^{(i,j)} \frac{c_j}{c_i} \frac{a_j}{a_i}, \quad (95)$$

$$\varsigma_{i,k}^{(-)}(t) = \beta_3^{(k,i)} + \beta_2^{(k,i)} \left(\frac{c_k}{c_i} + \frac{a_k}{a_i} \right) + \beta_1^{(k,i)} \frac{c_k}{c_i} \frac{a_k}{a_i}, \quad (96)$$

which are related by,

$$\sigma_{i,j}^{(+)}(t) = \left(\frac{a_j}{a_i} \right)^2 \sigma_{j,i}^{(-)}(t), \quad (97)$$

$$\varsigma_{i,j}^{(+)}(t) = \frac{c_j}{c_i} \frac{a_j}{a_i} \varsigma_{j,i}^{(-)}(t). \quad (98)$$

We note that the $\sigma_{i,j}^{(\pm)}(t)$ parameters are simply time-dependent analogues of the constant $\sigma_{i,j}^{(\pm)}$ parameters we introduced in Eqs. (67) and (68) to encode the spin-2 mass

matrix around a proportional background, where the constant conformal factors have been replaced by the time-dependent scale factors. The $\varsigma_{i,j}^{(\pm)}$ parameters have no such analog in general, but note that if $a_i = c_i$ (which happens at late times after all external matter has diluted away [33]) then the two metrics are actually proportional; in this case, $\sigma_{i,j}^{(\pm)}(t)$ and $\varsigma_{i,j}^{(\pm)}(t)$ coincide, and the perturbation structure of Eqs. (85)–(92) reduces down to that of Eqs. (65) and (66) around a proportional solution, as it should do [of course, the nontrivial Lorentz constraints (84) also become trivial in this scenario, as the expression on the right-hand side vanishes].

One can check that the effective mass terms in Eqs. (85)–(92) agree with those appearing in the linearized cosmological field equations of bigravity—see, e.g., [104,105]—upon splitting up the components of $\delta g^{(i)\mu}{}_\nu$ into their scalar, vector and tensor parts (the remaining terms in the linearized field equations are all GR-like terms coming from the kinetic sector, whose computation is well understood—see any classic cosmology textbook for details, e.g., [106]—so we omit them here). In particular, in the $\delta g^{(i)0}{}_m$ equations, for which the Lorentz constraints are nontrivial, the effective mass term (written explicitly for a positively oriented interaction) is,

$$\begin{aligned} [\mathcal{B}_{i,j}^{(+)}]_m^{\alpha}{}_{\beta} (\bar{S}_{i \rightarrow j})_{\lambda}^{\beta} \{ [\delta g^{\lambda}{}_{\alpha}]_{i,j} - 4[\omega^{\lambda}{}_{\alpha}]_{i,j} \} \\ = -2 \frac{\frac{a_j}{a_i} \frac{c_j}{c_i}}{\frac{a_j}{a_i} + \frac{c_j}{c_i}} \sigma_{i,j}^{(+)}(t) [\delta g^0{}_m]_{i,j}, \end{aligned} \quad (99)$$

which one can check agrees with Eq. (A.4) in [105] upon parametrizing the metric perturbations accordingly.

The increased generality of our approach provides a number of benefits over previous matrix approaches, for example: a nice application of Eqs. (85)–(92) is that one can derive the Higuchi bound [6,7,107–109] on cosmological spacetimes in a very neat and clean manner, as well as see very clearly why certain branches of cosmological solutions in multigravity are unstable—see [4] for details.

C. Black hole backgrounds

The final example we would like to give concerns the structure of the effective mass terms around black hole backgrounds. Such calculations are important when one wishes to compute, e.g., the quasinormal spectrum of a black hole in massive gravity, with a view to comparison against gravitational wave measurements of black hole binaries. To this end, a complete catalog of the known black hole solutions in generic multimetric theories was constructed in [58,59], again following a lot of good work in this direction for massive gravity and bigravity [55,56,110–117]; all such solutions that may be written down analytically can be cast into Kerr-Schild form,

$$g_{\mu\nu}^{(i)} = a_i^2 [g_{\mu\nu}^{(\Lambda)} + 2\phi_i l_{\mu} l_{\nu}], \quad (100)$$

where a_i are again constant conformal factors, $g_{\mu\nu}^{(\Lambda)}$ is the metric of AdS, Minkowski or dS space ($\Lambda < 0$, $\Lambda = 0$, $\Lambda > 0$, respectively) in a particular coordinate system, ϕ_i are scalar functions containing the Schwarzschild radii for each metric (which are in principle independent at this stage) and l is a vector tangent to a null-geodesic congruence on $g_{\mu\nu}^{(\Lambda)}$ —see [58] for the explicit expressions. The ansatz (100) is able to encompass black holes of arbitrary dimension that can rotate in multiple planes; in $D = 4$, or without any rotation, it can also account for electric charge.

As in the cosmological case, there are three distinct branches of solutions (this time depending on whether the metrics are simultaneously diagonalizable) and as before two of these are expected to be pathological: they are analogous to the two pathological branches of cosmological solutions in multiple ways, and we in fact believe the underlying source of the pathology to be the same in both cases—see [59] for a discussion. However, the pathology has not yet been confirmed explicitly, and it is likely that one will need to go to nonlinear order in perturbation theory to see it—the cubic terms we are going to compute in Sec. V should help here.

In any case, with the multigravity metrics given in the Kerr-Schild form (100), the fact that l is null ($l_{\mu} l^{\mu} = 0$) makes it surprisingly simple to calculate the form of the $S_{i \rightarrow j}$ matrices, as there is an early truncation in the expansion of the matrix square root [115],

$$(S_{i \rightarrow j})^{\mu}{}_{\nu} = \frac{a_j}{a_i} [\delta_{\nu}^{\mu} - (\phi_i - \phi_j) l^{\mu} l_{\nu}]. \quad (101)$$

Substituting into Eqs. (50) and (51) and again utilizing the null character of l , one finds the following perturbation structure at linear order,

$$\begin{aligned} [\mathcal{B}_{i,j}^{(+)}]_{\nu}^{\mu}{}_{\alpha}{}^{\beta} = \sigma_{i,j}^{(+)} \delta_{\nu\beta}^{\mu\alpha} + 2 \frac{a_j}{a_i} (\phi_i - \phi_j) \left(\beta_2^{(i,j)} + \frac{a_j}{a_i} \beta_3^{(i,j)} \right) \\ \times (\delta_{\nu}^{\mu} l^{\alpha} l_{\beta} + \delta_{\beta}^{\alpha} l^{\mu} l_{\nu}), \end{aligned} \quad (102)$$

$$\begin{aligned} [\mathcal{B}_{i,k}^{(-)}]_{\nu}^{\mu}{}_{\alpha}{}^{\beta} = \sigma_{i,k}^{(-)} \delta_{\nu\beta}^{\mu\alpha} + 2 \frac{a_k}{a_i} (\phi_i - \phi_k) \left(\beta_2^{(k,i)} + \frac{a_k}{a_i} \beta_1^{(k,i)} \right) \\ \times (\delta_{\nu}^{\mu} l^{\alpha} l_{\beta} + \delta_{\beta}^{\alpha} l^{\mu} l_{\nu}), \end{aligned} \quad (103)$$

as well as the following set of Lorentz constraints:

$$4[\omega_{\mu\nu}]_{i,j} = (\phi_i - \phi_j) \{ [\delta g_{\lambda[\mu}]_{i,j} l^{\lambda} l_{\nu]} - 4[\omega_{\lambda[\mu}]_{i,j} l^{\lambda} l_{\nu]} \}. \quad (104)$$

If all the ϕ_i are equal, then clearly all the metrics in Eq. (100) are proportional, and Eqs. (102) and (103) recover the corresponding Eqs. (65) and (66) for the $\mathcal{B}$ tensors around a proportional solution, as they should. Likewise, the

linearized Lorentz constraints (104) reduce to $[\omega_{\mu\nu}]_{i,j} = 0$, as befitting a proportional solution. When $\phi_i \neq \phi_j$, the perturbation structure and linearized Lorentz constraints are much more involved: in fact, the linearized field equations around nonproportional black hole solutions in multigravity have only previously been computed for a background that is four-dimensional and nonrotating [56,59,118]. We will show in Appendix B how to recover those results from our Eqs. (102) and (103), but one should note that our expressions are far more general: they hold in arbitrary dimension, and constitute the first time the effective mass term around a nonproportional *rotating* black hole background in multigravity (and by extension massive gravity and bigravity) has

been computed, demonstrating the power of our formalism for determining the perturbations.

V. CUBIC POTENTIAL

Finally, we are going to determine the multigravity potential to cubic order around an arbitrary background; previously, this calculation has only been performed around a proportional background (see Sec. IV) in $D = 4$ [41], so here we again chart new territory.

The cubic order variation of the potential, considering all terms from the expansion (40) that are third-order in the metric perturbations, has three contributions,

$$\begin{aligned} I_V^{(3)} = & - \sum_{ijkl \dots l_D} \int \kappa_i \kappa_j \kappa_k \binom{D}{3} T_{ijkl \dots l_D} \epsilon_{abcd \dots d_D} \delta e_{(1)}^{(i)a} \wedge \delta e_{(1)}^{(j)b} \wedge \delta e_{(1)}^{(k)c} \wedge \bar{e}^{(l_4)d_4} \wedge \dots \wedge \bar{e}^{(l_D)d_D} \\ & - \sum_{ijl_3 \dots l_D} \int \kappa_i^2 \kappa_j \binom{D}{2} T_{ijl_3 \dots l_D} \epsilon_{abd_3 \dots d_D} \delta e_{(2)}^{(i)a} \wedge \delta e_{(1)}^{(j)b} \wedge \bar{e}^{(l_3)d_3} \wedge \dots \wedge \bar{e}^{(l_D)d_D} \\ & - \sum_{il_2 \dots l_D} \int \kappa_i^3 \binom{D}{1} T_{il_2 \dots l_D} \epsilon_{ad_2 \dots d_D} \delta e_{(3)}^{(i)a} \wedge \bar{e}^{(l_2)d_2} \wedge \dots \wedge \bar{e}^{(l_D)d_D}, \end{aligned} \quad (105)$$

which can be expanded out using Eqs. (41)–(43) to yield,

$$\begin{aligned} I_V^{(3)} = & - \sum_{ijkl} \int d^D x \sqrt{-\det \bar{g}_{(i)}} \left[\frac{\kappa_i \kappa_j \kappa_k}{8} \binom{D}{3} T_{ijkl \dots l_D} \bar{e}^{(i)\mu\nu\rho\lambda_4 \dots \lambda_D}_{abcd_4 \dots d_D} \right. \\ & \times (\delta g^{(i)} - 4\omega^{(i)})^\alpha_\mu (\delta g^{(j)} - 4\omega^{(j)})^\beta_\nu (\delta g^{(k)} - 4\omega^{(k)})^\gamma_\rho \bar{e}^{(i)a}_\alpha \bar{e}^{(j)b}_\beta \bar{e}^{(k)c}_\gamma \bar{e}^{(l_4)d_4} \dots \bar{e}^{(l_D)d_D} \\ & - \frac{\kappa_i^2 \kappa_j}{16} \binom{D}{2} T_{ijl_3 \dots l_D} \bar{e}^{(i)\mu\nu\lambda_3 \dots \lambda_D}_{abd_3 \dots d_D} (\delta g^{(i)2} + 8\delta g^{(i)} \omega^{(i)} - 16\omega^{(i)2})^\alpha_\mu (\delta g^{(j)} - 4\omega^{(j)})^\beta_\nu \bar{e}^{(i)a}_\alpha \bar{e}^{(j)b}_\beta \bar{e}^{(l_3)d_3} \dots \bar{e}^{(l_D)d_D} \\ & \left. - \frac{\kappa_i^3}{16} \binom{D}{1} T_{il_2 \dots l_D} \bar{e}^{(i)\mu\lambda_2 \dots \lambda_D}_{ad_2 \dots d_D} (\delta g^{(i)3} - 4\delta g^{(i)2} \omega^{(i)} + 16\delta g^{(i)} \omega^{(i)2} + 32\omega^{(i)3})^\alpha_\mu \bar{e}^{(i)a}_\alpha \bar{e}^{(l_2)d_2} \dots \bar{e}^{(l_D)d_D} \right], \end{aligned} \quad (106)$$

where, e.g., $(\delta g^{(i)2})^\alpha_\mu$ is shorthand for $\delta g^{(i)\alpha}_\lambda \delta g^{(i)\lambda}_\mu$.

Restricting to pairwise interactions where the DvN condition (24) holds, using the symmetry of the $T_{j_1 \dots j_D}$ coefficients, Eqs. (36) and (37) to relate these coefficients to the metric $\beta_m^{(i,j)}$, and Eqs. (9) and (29) to evaluate products of generalized deltas, one finds the following cubic potential:

$$\begin{aligned} I_V^{(3)} = & \sum_{i=0}^{N-1} \int d^D x \sqrt{-\det \bar{g}_{(i)}} \\ & \times \left\{ -\frac{\kappa_i^3}{16} (\delta g^{(i)3} - 4\delta g^{(i)2} \omega^{(i)} + 16\delta g^{(i)} \omega^{(i)2} + 32\omega^{(i)3})^\alpha_\mu \bar{W}^{(i)\mu}_\alpha \right. \\ & + \frac{\kappa_i^3}{32} \mathcal{A}^{(i)\mu}_\alpha{}^\nu{}_\beta (\delta g^{(i)2} + 8\delta g^{(i)} \omega^{(i)} - 16\omega^{(i)2})^\alpha_\mu (\delta g^{(i)} - 4\omega^{(i)})^\beta_\nu \\ & - \frac{\kappa_i^3}{48} \mathcal{Z}^{(i)\mu}_\alpha{}^\nu{}_\beta{}^\rho (\delta g^{(i)} - 4\omega^{(i)})^\alpha_\mu (\delta g^{(i)} - 4\omega^{(i)})^\beta_\nu (\delta g^{(i)} - 4\omega^{(i)})^\rho_\rho \\ & \left. + \sum_j \left[\frac{\kappa_i^2 \kappa_j}{16} [\mathcal{B}^{(+)}_{i,j}]^\mu_\alpha{}^\nu{}_\beta (\bar{S}_{i \rightarrow j})^\beta_\sigma (\delta g^{(i)2} + 8\delta g^{(i)} \omega^{(i)} - 16\omega^{(i)2})^\alpha_\mu (\delta g^{(j)} - 4\omega^{(j)})^\sigma_\nu \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{\kappa_j^2 \kappa_i}{16} [\mathcal{B}_{i,j}^{(+)}]^\mu{}_\alpha{}^\nu{}_\beta (\bar{S}_{i \rightarrow j})^\beta{}_\sigma (\delta g^{(j)2} + 8\delta g^{(j)} \omega^{(j)} - 16\omega^{(j)2})^\alpha{}_\mu (\delta g^{(i)} - 4\omega^{(i)})^\sigma{}_\nu \\
& - \frac{\kappa_i^2 \kappa_j}{16} [\mathcal{X}_{i,j}^{(+)}]^\mu{}_\alpha{}^\nu{}_\beta (\bar{S}_{i \rightarrow j})^\gamma{}_\sigma (\delta g^{(i)} - 4\omega^{(i)})^\alpha{}_\mu (\delta g^{(i)} - 4\omega^{(i)})^\beta{}_\nu (\delta g^{(j)} - 4\omega^{(j)})^\sigma{}_\rho \\
& - \frac{\kappa_j^2 \kappa_i}{16} [\mathcal{Y}_{i,j}^{(+)}]^\mu{}_\alpha{}^\nu{}_\beta (\bar{S}_{i \rightarrow j})^\beta{}_\sigma (\bar{S}_{i \rightarrow j})^\gamma{}_\lambda (\delta g^{(i)} - 4\omega^{(i)})^\alpha{}_\mu (\delta g^{(j)} - 4\omega^{(j)})^\sigma{}_\nu (\delta g^{(j)} - 4\omega^{(j)})^\lambda{}_\rho \Big] \\
& + \sum_k [\text{negative orientation terms with } j \rightarrow k, \mathcal{B}^{(+)}, \mathcal{X}^{(+)}, \mathcal{Y}^{(+)} \rightarrow \mathcal{B}^{(-)}, \mathcal{X}^{(-)}, \mathcal{Y}^{(-)}] \Big\}, \quad (107)
\end{aligned}$$

where $\mathcal{A}$ and $\mathcal{B}$ are as in Eqs. (49)–(51), and we have defined the new six-index tensors,

$$\begin{aligned}
& \mathcal{Z}^{(i)\mu}{}_\alpha{}^\nu{}_\beta{}^\rho{}_\gamma \\
& = \sum_j \sum_{m=0}^D \frac{\beta_m^{(i,j)}}{m!} \delta_{\alpha\beta\gamma\gamma_1 \dots \gamma_m}^{\mu\nu\rho\lambda_1 \dots \lambda_m} (\bar{S}_{i \rightarrow j})_{\lambda_1}^{\gamma_1} \dots (\bar{S}_{i \rightarrow j})_{\lambda_m}^{\gamma_m} \\
& + \sum_k \sum_{m=0}^D \frac{\beta_m^{(k,i)}}{m!} \delta_{\alpha\beta\gamma\gamma_1 \dots \gamma_m}^{\mu\nu\rho\lambda_1 \dots \lambda_m} (\bar{S}_{i \rightarrow k})_{\lambda_1}^{\gamma_1} \dots (\bar{S}_{i \rightarrow k})_{\lambda_m}^{\gamma_m}, \quad (108)
\end{aligned}$$

$$\begin{aligned}
& [\mathcal{X}_{i,j}^{(+)}]^\mu{}_\alpha{}^\nu{}_\beta{}^\rho{}_\gamma \\
& = \sum_{m=0}^D \frac{\beta_m^{(i,j)}}{(m-1)!} \delta_{\alpha\beta\gamma\gamma_1 \dots \gamma_{m-1}}^{\mu\nu\rho\lambda_1 \dots \lambda_{m-1}} (\bar{S}_{i \rightarrow j})_{\lambda_1}^{\gamma_1} \dots (\bar{S}_{i \rightarrow j})_{\lambda_{m-1}}^{\gamma_{m-1}}, \quad (109)
\end{aligned}$$

$$\begin{aligned}
& [\mathcal{Y}_{i,j}^{(+)}]^\mu{}_\alpha{}^\nu{}_\beta{}^\rho{}_\gamma \\
& = \sum_{m=0}^D \frac{\beta_m^{(i,j)}}{(m-2)!} \delta_{\alpha\beta\gamma\gamma_1 \dots \gamma_{m-2}}^{\mu\nu\rho\lambda_1 \dots \lambda_{m-2}} (\bar{S}_{i \rightarrow j})_{\lambda_1}^{\gamma_1} \dots (\bar{S}_{i \rightarrow j})_{\lambda_{m-2}}^{\gamma_{m-2}}, \quad (110)
\end{aligned}$$

as well as analogous negative orientation terms for $\mathcal{X}$ and $\mathcal{Y}$ given by the simultaneous exchanges $j \rightarrow k$, and $\beta_m \rightarrow \beta_{D-m}$. As in the quadratic case, $\mathcal{A}$ may be eliminated in favor of $\bar{W}$ and $\mathcal{B}$ using Eq. (52). A similar relation exists that will allow one to eliminate $\mathcal{Z}$ in favor of $\bar{W}$, $\mathcal{X}$ and $\mathcal{Y}$, namely¹²:

¹²There are some interesting combinatorics at play here: schematically, at quadratic order, the relation (52) is $\delta_{(2)}W = \mathcal{A} + \mathcal{B}S$, while at cubic order the relation (111) is $\delta_{(3)}W = \mathcal{Z} + 2\mathcal{X}S + \mathcal{Y}SS$; the coefficients of the terms on the right-hand sides of these expressions form the second and third rows of Pascal's triangle. Presumably, at quartic order, there will then exist a similar relation involving four new eight-index tensor structures on the right-hand side of $\delta_{(4)}W = (\dots)$ appearing with coefficients 1, 3, 3, 1.

$$\begin{aligned}
& \mathcal{Z}^{(i)\mu}{}_\alpha{}^\nu{}_\beta{}^\rho{}_\gamma \mathcal{X}^\beta{}_\nu \mathcal{X}^\gamma{}_\rho \\
& + \sum_j \left[2[\mathcal{X}_{i,j}^{(+)}]^\mu{}_\alpha{}^\nu{}_\beta{}^\rho{}_\gamma \mathcal{X}^\beta{}_\nu \mathcal{X}^\sigma{}_\rho (\bar{S}_{i \rightarrow j})^\gamma{}_\sigma \right. \\
& + [\mathcal{Y}_{i,j}^{(+)}]^\mu{}_\alpha{}^\nu{}_\beta{}^\rho{}_\gamma \mathcal{X}^\sigma{}_\nu \mathcal{X}^\lambda{}_\rho (\bar{S}_{i \rightarrow j})^\beta{}_\sigma (\bar{S}_{i \rightarrow j})^\gamma{}_\lambda \\
& + \sum_k \left[2[\mathcal{X}_{i,k}^{(-)}]^\mu{}_\alpha{}^\nu{}_\beta{}^\rho{}_\gamma \mathcal{X}^\beta{}_\nu \mathcal{X}^\sigma{}_\rho (\bar{S}_{i \rightarrow k})^\gamma{}_\sigma \right. \\
& + [\mathcal{Y}_{i,k}^{(-)}]^\mu{}_\alpha{}^\nu{}_\beta{}^\rho{}_\gamma \mathcal{X}^\sigma{}_\nu \mathcal{X}^\lambda{}_\rho (\bar{S}_{i \rightarrow k})^\beta{}_\sigma (\bar{S}_{i \rightarrow k})^\gamma{}_\lambda \Big] \\
& = \delta_{\lambda\beta\gamma}^{\mu\nu\rho} \mathcal{X}^\beta{}_\nu \mathcal{X}^\gamma{}_\rho \bar{W}^{(i)\lambda}{}_\alpha. \quad (111)
\end{aligned}$$

Again the structure of the interaction terms ($\mathcal{X}$ and $\mathcal{Y}$) in terms of the generalized deltas ensures that $\delta g^{(i)0}{}_0$ will always appear in the action as a Lagrange multiplier, and $\delta g^{(i)j}{}_0$ will always appear as an auxiliary field, leading to the necessary constraints that guarantee the removal of the BD ghost in the standard way.

One can check that around a proportional background Eq. (107) reduces to the cubic order potential given in [41]. We recall that when $g_{\mu\nu}^i = a_i^2 \bar{g}_{\mu\nu}$, one has $(\bar{S}_{i \rightarrow j})^\mu{}_\nu = (a_j/a_i) \delta^\mu{}_\nu$, so the Lorentz constraints become trivial, as δg and $\bar{S}$ commute. Therefore, around proportional backgrounds, all of the local Lorentz fields drop out of the cubic potential, and we only need to care about the terms involving exclusively metric perturbations. Using Eq. (9) to evaluate products of deltas, alongside Eqs. (65) and (66) for the $\mathcal{B}$ tensors, one finds that the relevant new tensor structures dictating the form of the spin-2 interactions are,

$$[\mathcal{X}_{i,j}^{(+)}]^\mu{}_\alpha{}^\nu{}_\beta{}^\rho{}_\gamma = \eta_{i,j}^{(+)} \delta_{\alpha\beta\gamma}^{\mu\nu\rho}, \quad (112)$$

$$[\mathcal{Y}_{i,j}^{(+)}]^\mu{}_\alpha{}^\nu{}_\beta{}^\rho{}_\gamma = \frac{a_i}{a_j} [\sigma_{i,j}^{(+)} - \eta_{i,j}^{(+)}] \delta_{\alpha\beta\gamma}^{\mu\nu\rho}, \quad (113)$$

$$[\mathcal{X}_{i,k}^{(-)}]^\mu{}_\alpha{}^\nu{}_\beta{}^\rho{}_\gamma = \eta_{i,k}^{(-)} \delta_{\alpha\beta\gamma}^{\mu\nu\rho}, \quad (114)$$

$$[\mathcal{Y}_{i,k}^{(-)}]^\mu{}_\alpha{}^\nu{}_\beta{}^\rho{}_\gamma = \frac{a_i}{a_k} [\sigma_{i,k}^{(-)} - \eta_{i,k}^{(-)}] \delta_{\alpha\beta\gamma}^{\mu\nu\rho}, \quad (115)$$

where we have defined,

$$\eta_{i,j}^{(+)} = \sum_{m=0}^D \beta_m^{(i,j)} \binom{D-3}{m-1} \left(\frac{a_j}{a_i}\right)^{m-1}, \quad (116)$$

$$\eta_{i,k}^{(-)} = \sum_{m=0}^D \beta_{D-m}^{(k,i)} \binom{D-3}{m-1} \left(\frac{a_k}{a_i}\right)^{m-1}, \quad (117)$$

related by,

$$\eta_{i,j}^{(+)} = \sigma_{i,j}^{(+)} - \left(\frac{a_j}{a_i}\right)^{D-2} \eta_{j,i}^{(-)}. \quad (118)$$

Using Eqs. (52) and (111) to eliminate $\mathcal{A}$ and $\mathcal{Z}$, and ignoring the terms left multiplying $\bar{W}$ (which are just contributions from the background curvature), one finds that the spin-2 interaction potential at cubic order reads,

$$\begin{aligned} I_V^{(3)} \supset & \sum_{i=0}^{N-1} \int d^D x \sqrt{-\det \bar{g}_{(i)}} \\ & \times \left\{ \frac{\kappa_i^3}{48} (\delta g^{(i)3} - 3\delta g^{(i)} \delta g^{(i)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\mu} + 2\delta g^{(i)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\rho} \delta g^{(i)\rho}{}_{\mu}) \left[\sum_j \frac{a_j}{a_i} \eta_{i,j}^{(+)} + \sum_k \frac{a_k}{a_i} \eta_{i,k}^{(-)} \right] \right. \\ & + \frac{\kappa_i^3}{96} (2\delta g^{(i)3} - 9\delta g^{(i)} \delta g^{(i)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\mu} + 7\delta g^{(i)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\rho} \delta g^{(i)\rho}{}_{\mu}) \left[\sum_j \frac{a_j}{a_i} \sigma_{i,j}^{(+)} + \sum_k \frac{a_k}{a_i} \sigma_{i,k}^{(-)} \right] \\ & - \frac{1}{16} \left[\sum_j \frac{a_j}{a_i} \eta_{i,j}^{(+)} [\kappa_i^2 \kappa_j (\delta g^{(i)2} \delta g^{(j)} - \delta g^{(j)} \delta g^{(i)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\mu} - 2\delta g^{(i)} \delta g^{(i)\mu}{}_{\nu} \delta g^{(j)\nu}{}_{\mu} + 2\delta g^{(i)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\rho} \delta g^{(j)\rho}{}_{\mu}) \right. \\ & - \kappa_i \kappa_j^2 (\delta g^{(j)2} \delta g^{(i)} - \delta g^{(i)} \delta g^{(j)\mu}{}_{\nu} \delta g^{(j)\nu}{}_{\mu} - 2\delta g^{(j)} \delta g^{(j)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\mu} + 2\delta g^{(j)\mu}{}_{\nu} \delta g^{(j)\nu}{}_{\rho} \delta g^{(i)\rho}{}_{\mu}) \\ & + \sum_k \frac{a_k}{a_i} \eta_{i,k}^{(-)} [\kappa_i^2 \kappa_k (\delta g^{(i)2} \delta g^{(k)} - \delta g^{(k)} \delta g^{(i)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\mu} - 2\delta g^{(i)} \delta g^{(i)\mu}{}_{\nu} \delta g^{(k)\nu}{}_{\mu} + 2\delta g^{(i)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\rho} \delta g^{(k)\rho}{}_{\mu}) \\ & - \kappa_i \kappa_k^2 (\delta g^{(k)2} \delta g^{(i)} - \delta g^{(i)} \delta g^{(k)\mu}{}_{\nu} \delta g^{(k)\nu}{}_{\mu} - 2\delta g^{(k)} \delta g^{(k)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\mu} + 2\delta g^{(k)\mu}{}_{\nu} \delta g^{(k)\nu}{}_{\rho} \delta g^{(i)\rho}{}_{\mu})] \\ & + \frac{1}{16} \left[\sum_j \frac{a_j}{a_i} \sigma_{i,j}^{(+)} [\kappa_i^2 \kappa_j (\delta g^{(j)} \delta g^{(i)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\mu} - \delta g^{(i)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\rho} \delta g^{(j)\rho}{}_{\mu}) \right. \\ & + \kappa_i \kappa_j^2 (2\delta g^{(i)} \delta g^{(j)\mu}{}_{\nu} \delta g^{(j)\nu}{}_{\mu} - \delta g^{(j)2} \delta g^{(i)} - 3\delta g^{(j)\mu}{}_{\nu} \delta g^{(j)\nu}{}_{\rho} \delta g^{(i)\rho}{}_{\mu} + 2\delta g^{(j)} \delta g^{(j)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\mu}) \\ & + \sum_k \frac{a_k}{a_i} \sigma_{i,k}^{(-)} [\kappa_i^2 \kappa_k (\delta g^{(k)} \delta g^{(i)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\mu} - \delta g^{(i)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\rho} \delta g^{(k)\rho}{}_{\mu}) \\ & \left. \left. + \kappa_i \kappa_k^2 (2\delta g^{(i)} \delta g^{(k)\mu}{}_{\nu} \delta g^{(k)\nu}{}_{\mu} - \delta g^{(k)2} \delta g^{(i)} - 3\delta g^{(k)\mu}{}_{\nu} \delta g^{(k)\nu}{}_{\rho} \delta g^{(i)\rho}{}_{\mu} + 2\delta g^{(k)} \delta g^{(k)\mu}{}_{\nu} \delta g^{(i)\nu}{}_{\mu})] \right] \right\}. \quad (119) \end{aligned}$$

This expression is quite horrendous, but one can check that upon redefining $\delta g_{\mu\nu}^{(i)} \rightarrow \delta g_{\mu\nu}^{(i)}/\kappa_i$ to match the normalization used in [41], working explicitly in $D = 4$ spacetime dimensions, setting all $a_i = 1$ to match the background they expanded around, and limiting oneself to only $N = 2$ interacting metrics (they denote $\delta g_{\mu\nu}^{(1)} \equiv h_{\mu\nu}$ and $\delta g_{\mu\nu}^{(2)} \equiv l_{\mu\nu}$, with the interaction positively oriented from h to l), Eq. (119) agrees precisely with their cubic potential for bigravity [their Eq. (B.10)].

VI. CONCLUSION

In this work, we have developed a new procedure for computing the structure of metric perturbations around

arbitrary background spacetimes in dRGT-type multigravity theories. Our approach hinges on the equivalence between the metric and vielbein formalisms of multigravity in theories involving exclusively pairwise interactions, as performing the initial perturbation of the action in vielbein form circumvents the need to deal with the cumbersome matrix square root underpinning the interactions in the metric formalism. This is the first real advantage of our approach over previous approaches. The second is that expressing the perturbed interaction terms via the generalized Kronecker delta elucidates their ghost-free nature in a much more transparent manner, as it becomes clear at the level of the action that the ghost-killing constraint will always be present, at any order, regardless of the background spacetime.

The final and most important advantage is the generality of our approach: it works very naturally in arbitrary spacetime dimension, for an arbitrary number of interacting metrics, and crucially, to an arbitrary order in perturbation theory, where previous approaches were limited to only linear order.

To verify that our formalism functions as intended, we used it to reproduce the linearized field equations of multigravity around three commonly occurring example background spacetimes—proportional, cosmological and black hole—in the process also extending some of these results to multiple interacting metrics, and in the black hole case, providing for the first time the structure of the linearized mass term around a nonproportional, *rotating* black hole. To then demonstrate the power of our formalism beyond just the linear order, we computed the cubic order multigravity potential around an arbitrary background, which to our knowledge constitutes the first time this has been done. We verified that our generic cubic potential reduces to the only known cubic potential from bigravity [41] around a proportional background.

This work should prove useful to anybody who wishes to study perturbations in massive gravity theories around complicated background spacetimes and/or to higher than linear order. For example, we mentioned one potential future use earlier: some (nonproportional) black hole solutions in multigravity are expected to be pathological, but the pathology likely only surfaces nonlinearly; using our formalism to determine the perturbation structure around such solutions to cubic order should hopefully shed some light on this issue. Another example from the realm of cosmology is that FLRW solutions in massive gravity and bigravity are subject to a gradient instability at early times, which manifests at linear order in cosmological perturbation theory [97,104,109,119]; it is largely expected that this instability will be screened by a cosmological analog of the Vainshtein mechanism [97,120], but in order to know with certainty one would have to directly compare the linear order cosmological perturbations with the higher order ones. This calculation has yet to be done, as such higher order terms have up to now been inaccessible; they can, however, be computed using our formalism. Furthermore, by going to higher orders in cosmological perturbation theory, one may begin to study various other nonlinear effects, for instance, scalar-induced gravitational waves in massive gravity theories, where the nonlinear interactions between the scalar and tensor metric perturbations that are decoupled at linear level become important, with potentially measurable physical effects. These are just some examples, but the possibilities are of course many. We hope our formalism provides a helpful tool to those wishing to study such interesting questions in the future.

Calculations involving tensor contractions around cosmological and black hole backgrounds were aided by use of the `xAct Mathematica` package suite [121].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DETAILS OF THE PREVIOUS APPROACHES TO LINEAR MULTIGRAVITY PERTURBATIONS

In this Appendix we provide some details of the previous approaches to multigravity perturbations in the metric formalism around arbitrary backgrounds, following [63–65] and later [66,67]. We will explain where the complications arise that motivated our alternate procedure.

1. Matrix equation approach

As mentioned in footnote 5, in matrix form, the W tensor at background level is typically written as,

$$W^{(i)\mu}{}_{\nu} = \sum_j \sum_{m=0}^D (-1)^m \beta_m^{(i,j)} Y_{(m)\nu}^{\mu}(S_{i \rightarrow j}) + \sum_k \sum_{m=0}^D (-1)^m \beta_{D-m}^{(k,i)} Y_{(m)\nu}^{\mu}(S_{i \rightarrow k}), \quad (\text{A1})$$

where the matrices $Y_{(m)}(S)$ are defined by,

$$Y_{(m)}(S) = \sum_{n=0}^m (-1)^n S^{m-n} e_n(S), \quad (\text{A2})$$

with the elementary symmetric polynomials given iteratively in terms of the matrix traces as,

$$e_n(S) = -\frac{1}{n} \sum_{m=1}^n (-1)^m \text{Tr}(S^m) e_{n-m}(S), \quad (\text{A3})$$

starting from $e_0(S) = 1$. Equation (A1) is equivalent to our Eqs. (12) and (A3) is equivalent to our Eq. (7).

To derive the linearized field equations using the approach of [63,64], one considers the first order variation of the W tensor in matrix form,

$$\begin{aligned} \delta W^{(i)\mu}{}_\nu &= \delta g^{(i)\mu}{}_\lambda \bar{W}^{(i)\lambda}{}_\nu \\ &+ \sum_j \sum_{m=0}^D (-1)^m \beta_m^{(i,j)} \delta Y_{(m)\nu}^\mu(S_{i \rightarrow j}) \\ &+ \sum_k \sum_{m=0}^D (-1)^m \beta_{D-m}^{(k,i)} \delta Y_{(m)\nu}^\mu(S_{i \rightarrow k}). \end{aligned} \quad (\text{A4})$$

The variation of the Y 's is given (for any given $i \rightarrow j$ interaction) by,

$$\begin{aligned} \delta Y_{(m)}(S) &= \sum_{k=1}^m (-1)^k \left[S^{m-k} \delta e_k(S) \right. \\ &\quad \left. - e_{k-1}(S) \sum_{n=0}^{m-k} S^n \delta S S^{m-k-n} \right], \end{aligned} \quad (\text{A5})$$

where the variation of the symmetric polynomials is found from Eq. (A3) to be,

$$\delta e_k(S) = - \sum_{n=1}^k (-1)^n \text{Tr}(S^{n-1} \delta S) e_{k-n}(S). \quad (\text{A6})$$

The complication lies in the fact that δS is given by the *matrix* equation,

$$S \delta S + \delta S S = \delta S^2, \quad (\text{A7})$$

and therefore determining its form is not as straightforward as simply starting from $S_{i \rightarrow j}^2 = g_{(i)}^{-1} g_{(j)}$ and Taylor expanding the square root (although it is possible to do this when $S \propto \mathbb{1}$, which is the case for the proportional solutions, and is the reason that they are easier to deal with). Around a generic background solution, this is a *Sylvester matrix equation*, of the form,

$$AX - XB = C, \quad (\text{A8})$$

where A , B and C are given constant matrices and one wishes to solve for the unknown matrix X . The solution to the Sylvester equation is known in the mathematical literature, and is given by the following expression [70]:

$$X = q_B^{-1}(A) \sum_{k=1}^D \sum_{n=0}^{k-1} (-1)^k e_{D-k}(B) A^{k-n-1} C B^n, \quad (\text{A9})$$

where $q_B(A)$ is the unique polynomial in the matrix A whose coefficients are the same as those of the characteristic polynomial of B [$q_B^{-1}(A)$ is then the inverse of this matrix]; that is,

$$q_B(A) = \sum_{m=0}^D (-1)^m e_{D-m}(B) A^m. \quad (\text{A10})$$

In our case, comparison with Eq. (A7) tells us that we have $A = S$, $B = -S$ and $C = \delta S^2$. Therefore, the solution for δS is,

$$\delta S = q_{-S}^{-1}(S) \sum_{k=1}^D \sum_{n=0}^{k-1} (-1)^{n+k} e_{D-k}(-S) S^{k-n-1} \delta S^2 S^n. \quad (\text{A11})$$

One can easily obtain $\delta S_{i \rightarrow j}^2$ in terms of either the metric perturbations of $g_{(i)}$ and $g_{(j)}$, or of their inverses, by starting from $S_{i \rightarrow j}^2 = g_{(i)}^{-1} g_{(j)}$ and substituting in $g_{(i)} = \bar{g}_{(i)} + \delta g_{(i)}$ for the perturbed metrics. The result is,

$$\delta S_{i \rightarrow j}^2 = g_{(i)}^{-1} [\delta g_{(j)} - \delta g_{(i)} S_{i \rightarrow j}^2] \quad (\text{A12})$$

$$= [S_{i \rightarrow j}^2 \delta g_{(i)}^{-1} - \delta g_{(i)}^{-1}] g_{(j)}, \quad (\text{A13})$$

or in components,

$$(\delta S_{i \rightarrow j}^2)^\mu{}_\nu = g_{(i)}^{\mu\lambda} [\delta g_{(j)\lambda\nu} - \delta g_{(i)\lambda\sigma} (S_{i \rightarrow j}^2)^\sigma{}_\nu] \quad (\text{A14})$$

$$= [(S_{i \rightarrow j}^2)^\mu{}_\lambda \delta g_{(j)}^{\lambda\sigma} - \delta g_{(i)}^{\mu\sigma}] g_{(j)\sigma\nu}. \quad (\text{A15})$$

Substituting either of these expressions into Eq. (A11) determines δS , which one can then substitute into Eq. (A5) to get the Y variations, and lastly substitute these into Eq. (A4) to determine the linearized W tensors. However, the difficulty with this approach is that inverting the matrix $q_{-S}(S)$ can become quite a challenge around complicated backgrounds. This motivated the authors of [63,64] to develop a second approach in [65].

2. Redefined fluctuation variables approach

In bigravity, where there are just two metrics, $g_{\mu\nu}^{(1)} \equiv g_{\mu\nu}$ and $g_{\mu\nu}^{(2)} \equiv f_{\mu\nu}$, and a single interaction $S_{g \rightarrow f} \equiv S$ (so $S_{f \rightarrow g} = S^{-1}$), the necessity to solve the Sylvester matrix equation can be circumvented by redefining the metric perturbations in a clever way. It was shown in [65] that by absorbing S into the definition of the fluctuation variables as,

$$\delta g_{\mu\nu} = 2\delta_{(\mu}^{\lambda} S_{\nu)}^{\sigma} \delta g'_{\sigma\lambda}, \quad (\text{A16})$$

$$\delta f_{\mu\nu} = 2\delta_{(\mu}^{\lambda} [S^{-1}]_{\nu)}^{\sigma} \delta f'_{\sigma\lambda}, \quad (\text{A17})$$

then so long as S and $-S$ do not share any common eigenvalues (i.e., no zeroes), δS and δS^{-1} can be uniquely determined in terms of $\delta g'$ and $\delta f'$ as,

$$\delta S = -g^{-1} \delta g' S^2 + S^{-1} g^{-1} \delta f' S^{-1}, \quad (\text{A18})$$

$$\delta S^{-1} = -f^{-1} \delta f' S^{-2} + S f^{-1} \delta g' S, \quad (\text{A19})$$

symmetric under $g \leftrightarrow f$ exchange. These expressions are significantly simpler than Eq. (A11), and indeed the authors used them to demonstrate the existence of the ghost-killing constraints at the level of the linearized field equations in a covariant manner.

However, there are two problems with this approach, as alluded to in the introduction. The first is that by making the field redefinitions (A16) and (A17), the kinetic part of the field equations now contains derivatives acting on S and S^{-1} , which is undesirable, but ultimately not catastrophic. The real problem is that this procedure does not work as soon as the theory contains additional interacting metrics, i.e., once one begins to consider true multigravity theories. For example, suppose $g_{\mu\nu}$ interacts with both $f_{\mu\nu}$ and an additional metric $h_{\mu\nu}$; by redefining the perturbations of $g_{\mu\nu}$ and $f_{\mu\nu}$ via Eqs. (A16) and (A17) to determine $\delta S_{g \rightarrow f}$, one loses the ability to do the same thing for $S_{g \rightarrow h}$ because the perturbations of $g_{\mu\nu}$ have already been redefined—it is only ever possible to absorb *one* interaction into the field redefinitions. Thus, in multigravity theories, up until now one was forced to succumb to solving the Sylvester matrix equation for each $\delta S_{i \rightarrow j}$, which was not ideal. The improved procedure we have developed in this paper, however, works for any number of metrics, does not require any such field redefinitions to be made, and trades the complicated matrix equations for the algebraic Lorentz constraints.

3. Mazuet and Volkov's vielbein approach

The approach from [66,67], like the one we developed in Sec. III, is able to circumvent the difficulties associated with the square root matrices by working instead with the vielbein perturbations. However, their work was applied specifically to four-dimensional dRGT gravity, with fixed reference metric (where equivalence between metric and vielbein formulations is automatic as there is only one interaction), rather than a generic multigravity theory; we will see that many of their calculations become challenging practically once more dynamical metrics or extra dimensions become involved.

In the notation of [66,67], there are again two metrics: the physical metric $g_{\mu\nu}^{(1)} \equiv g_{\mu\nu}$, with associated vielbein $e_{\mu}^{(1)a} \equiv e_{\mu}^a$, and the fixed reference metric $g_{\mu\nu}^{(2)} \equiv f_{\mu\nu}$, with associated vielbein $e_{\mu}^{(2)a} \equiv \phi_{\mu}^a$. They write the background field equations for the physical metric explicitly in terms of traces and powers of a matrix they call γ , defined by,

$$\gamma^{\mu}_{\nu} = e^{\mu}_a \phi_{\nu}^a. \quad (\text{A20})$$

Of course, the matrix γ is nothing more than $S_{g \rightarrow f} \equiv S$ [cf. Eq. (33)], so we will denote it S going forward, in keeping with our own notation throughout the main paper, but one should keep in mind the notational difference if one wishes to compare directly with [66,67].

The background field equations (in vacuum) are expressed as (Eq. (3.30) in [67]),

$$\begin{aligned} G_{\mu\nu} + \beta_0 g_{\mu\nu} &+ \beta_1 [e_1(S) g_{\mu\nu} - S_{\mu\nu}] \\ &+ \beta_2 [S_{\mu\nu}^2 - e_1(S) S_{\mu\nu} + e_2(S) g_{\mu\nu}] \\ &+ \beta_3 [-S_{\mu\nu}^3 + e_1(S) S_{\mu\nu}^2 - e_2(S) S_{\mu\nu} + e_3(S) g_{\mu\nu}] \\ &= 0, \end{aligned} \quad (\text{A21})$$

where, e.g., $S_{\mu\nu}^2$ is shorthand for $S_{\mu\lambda} S^{\lambda}_{\nu}$. The W -tensor in Eq. (A21) is in the form of Eq. (A1), just with the S matrix expressed in terms of the vierbeins rather than the metrics.

Equation (A21) is linearized by perturbing the vielbein of the physical metric¹³ as $e_{\mu}^a \rightarrow \bar{e}_{\mu}^a + \delta e_{\mu}^a$, then computing the variation of S in terms of the vielbein perturbations, which in principle contain all 16 components—both physical and nonphysical. The authors of [66,67] do *not* explicitly separate the vierbein perturbations into metric and local Lorentz degrees of freedom as we do in our Eq. (38). Instead, they elect to simply project the vierbein perturbations onto the background vielbein, resulting in a description where all 16 degrees of freedom contained within δe_{μ}^a are carried by the following nonsymmetric tensor:

$$X^{\mu}_{\nu} \equiv \bar{e}^{\mu}_a \delta e_{\nu}^a. \quad (\text{A22})$$

A benefit of our approach is that, thanks to Eq. (38), one is able to explicitly keep track of which degrees of freedom within $X_{\mu\nu}$ are unphysical—one may identify [cf. Eqs. (41)–(43)],

$$X^{(1)\mu}_{\nu} = \frac{\kappa}{2} (\delta g - 4\omega)_{\nu}^{\mu}, \quad (\text{A23})$$

$$X^{(2)\mu}_{\nu} = -\frac{\kappa^2}{8} (\delta g^2 - 8\omega\delta g + 16\omega^2)_{\nu}^{\mu}, \quad (\text{A24})$$

$$\begin{aligned} X^{(3)\mu}_{\nu} &= \frac{\kappa^3}{16} (\delta g^3 - 4\delta g^2\omega + 16\omega^2\delta g + 32\omega^3)_{\nu}^{\mu}, \\ &\vdots \end{aligned} \quad (\text{A25})$$

from which one can clearly see that, e.g., $\kappa\delta g_{\mu\nu} = 2X_{(\mu\nu)}^{(1)}$, as in Eq. (3.22) of [67].

To linear order, directly from Eq. (A20), one finds that, in terms of the tensor $X_{\mu\nu}$, the variation of S^{μ}_{ν} is¹⁴:

¹³The reference vielbein, ϕ_{μ}^a , is not perturbed, as it is completely fixed in dRGT massive gravity.

¹⁴If the reference metric were dynamical, Eq. (A26) would instead read $\delta S^{\mu}_{\nu} = -\bar{S}^{\lambda}_{\nu} X^{(g)\mu}_{\lambda} + \bar{S}^{\mu}_{\lambda} X^{(f)\lambda}_{\nu}$, which is why the difference expressions $[X]_{i,j}$ [c.f. Eq. (55)] show up in our effective mass terms from Sec. III.

$$\delta S_\nu^\mu = -\bar{S}_\nu^\lambda X^\mu{}_\lambda. \quad (\text{A26})$$

The effective mass term, Eq. (3.26) in [67], is then computed by the direct substitution of Eq. (A26) into Eq. (A21), with the result,

$$\begin{aligned} \mathcal{M}_{\mu\nu} = & \beta_1 [\bar{S}^\lambda{}_\mu X_{\lambda\nu} - g_{\mu\nu} \bar{S}_{\alpha\beta} X^{\alpha\beta}] \\ & + \beta_2 [-\bar{S}^\alpha{}_\mu \bar{S}^\beta{}_\nu X_{\alpha\beta} - (\bar{S}^2)^\lambda{}_\mu X_{\lambda\nu} + \bar{S}_{\mu\nu} \bar{S}_{\alpha\beta} X^{\alpha\beta} \\ & + \bar{S} \bar{S}^\lambda{}_\mu X_{\lambda\nu} + g_{\mu\nu} ((\bar{S}^2)_{\alpha\beta} X^{\alpha\beta} - \bar{S} \bar{S}_{\alpha\beta} X^{\alpha\beta})] \\ & + \beta_3 \det \bar{S} [X_{\mu\lambda} (\bar{S}^{-1})^\lambda{}_\nu - X (\bar{S}^{-1})_{\mu\nu}]. \end{aligned} \quad (\text{A27})$$

One can check that this equation agrees precisely with the $\mathcal{B}$ -tensor terms from our Eq. (53), upon expanding out the generalized deltas (the β_3 term also requires use of the Cayley-Hamilton relation—Eq. (3.10) in [67]—to replace the dependence on $\bar{S}^{-1}$ with a dependence on $\bar{S}$ itself). However, the structure of each of the terms is arguably clearer in our expressions, holds for multiple dynamical metrics, and easily extends to general spacetime dimension as we showed. Using their approach, these generalizations would likely prove a small challenge.

For completeness, we note that the antisymmetric part of their linearized field equations leads to $\mathcal{M}_{[\mu\nu]} = 0$ as we expect; taking Eq. (A27) for the effective mass term, this implies the following six constraint equations:

$$\bar{S}^\lambda{}_\mu X_{\lambda\nu} = \bar{S}^\lambda{}_\nu X_{\lambda\mu}, \quad (\text{A28})$$

which are of course equivalent to our linearized Lorentz constraints (57). It is shown explicitly in [67] that five more constraints arise from the linearized field equations: four from taking their divergence, and one extra from taking an additional divergence then subtracting off the trace—exactly as occurs in standard FP theory around an Einstein background [1]. In total, there are 11 constraints; consequently, the tensor $X_{\mu\nu}$ propagates five physical degrees of freedom, as befitting a massive spin-2 field in four dimensions. However, its direct relationship to the metric perturbations is not immediately clear without our Eq. (38).

Finally, we would like to comment on one additional interesting thing the authors of [66,67] do, which works well in dRGT gravity but is much more difficult (as well as less useful) in multigravity. After computing the effective mass term (A27), the authors decide to attempt to remove all dependence of the linearized field equations on the matrix $\bar{S}_{\mu\nu}$, instead expressing them only in terms of the tensor $X_{\mu\nu}$ and powers of the background curvature of the physical metric. To do so, they treat the background field equations (A21) as algebraic, polynomial equations for $\bar{S}_{\mu\nu}$ that may be solved in terms of $\bar{g}_{\mu\nu}$ and $\bar{R}_{\mu\nu}$. Since they work in $D = 4$, these equations are cubic, so the solutions have the structure,

$$\bar{S}_{\mu\nu} = y_0 \bar{g}_{\mu\nu} + y_1 \bar{R}_{\mu\nu} + y_2 \bar{R}_{\mu\nu}^2 + y_3 \bar{R}_{\mu\nu}^3, \quad (\text{A29})$$

for some set of coefficients y_m , which can (in principle) be solved for by injecting the above ansatz into Eq. (A21). Upon solving for the y_m , one can also (in principle) determine the effective mass term as a function of $\bar{g}_{\mu\nu}$ and $\bar{R}_{\mu\nu}$ by substituting the solution for $\bar{S}_{\mu\nu}$ into Eq. (A27); one finds a similar structure,

$$\mathcal{M}_{\mu\nu} = B_0 \bar{g}_{\mu\nu} + B_1 \bar{R}_{\mu\nu} + B_2 \bar{R}_{\mu\nu}^2 + B_3 \bar{R}_{\mu\nu}^3, \quad (\text{A30})$$

for some new coefficients B_m .

In practice, this process can be debilitatingly difficult for a massive gravity theory with generic β_m . Indeed, it was only done explicitly in [67] for two simplified models: one containing only nonvanishing β_1 , and the other containing only nonvanishing β_3 . As before, the procedure only gets more difficult in higher dimensions, as the order of the polynomial equation the one must solve for $\bar{S}_{\mu\nu}$ (and hence the highest power of $\bar{R}_{\mu\nu}$ in the above expressions) increases. Furthermore, it only really makes sense to do this at all for dRGT gravity, where the reference metric is a fixed input to the theory: by expressing $\bar{S}_{\mu\nu}$ in the form suggested by Eq. (A29), one effectively has to tune the reference metric to ensure that an arbitrary $\bar{g}_{\mu\nu}$ will always satisfy its background field equations; when the reference metric has its own dynamics, there is no guarantee that this tuning will satisfy both sets of field equations simultaneously. In this case, if one wished to proceed, one would have to solve both sets of background field equations simultaneously, leading to an expression for $\bar{S}_{\mu\nu}$ given in terms of the background curvatures of *both* metrics. This represents a significant increase in complexity, which of course only gets worse when there are additional dynamical metrics and hence more simultaneous equations to solve. Instead, when one is working with a genuine multigravity theory, we argue that one should always first solve the background field equations to determine the form of the various background metrics. Once this is done, all of the background $\bar{S}_{i \rightarrow j}$ matrices will be known (provided that one is able to compute the matrix square roots, or equally, the associated vielbeins and their inverses); it then becomes a relatively simple task to substitute these matrices into Eq. (53) to find the structure of the effective mass terms and Eq. (57) to determine the linearized Lorentz constraints.

APPENDIX B: RECOVERY OF NONPROPORTIONAL SCHWARZSCHILD PERTURBATION STRUCTURE FROM OUR GENERAL BLACK HOLE PERTURBATION EXPRESSIONS

In this Appendix we demonstrate how to recover the perturbation structure around nonproportional, nonrotating black holes in four-dimensional multigravity, given in [56,59,118],

from our general black hole perturbation expressions (102) and (103).

To start, we need to expand a little bit on how multimetric black hole solutions are actually constructed. Let us assume that we are in vacuum, so the black holes are uncharged. With the ansatz (100) for the metrics, the Einstein tensors are simply,

$$G^{(i)\mu}{}_{\nu} = -\frac{\Lambda}{a_i^2} \delta_{\nu}^{\mu}, \quad (\text{B1})$$

irrespective of the scalar functions ϕ_i , while the W tensors have components,

$$\begin{aligned} W^{(i)\mu}{}_{\nu} = & \delta_{\nu}^{\mu} \left[\sum_j \sum_{m=0}^D \beta_m^{(i,j)} \binom{D-1}{m} \left(\frac{a_j}{a_i} \right)^m \right. \\ & + \sum_k \sum_{m=0}^D \beta_{D-m}^{(k,i)} \binom{D-1}{m} \left(\frac{a_k}{a_i} \right)^m \Big] \\ & + l^{\mu} l_{\nu} \left[\sum_j \frac{a_j}{a_i} (\phi_i - \phi_j) \sigma_{i,j}^{(+)} \right. \\ & \left. + \sum_k \frac{a_k}{a_i} (\phi_i - \phi_k) \sigma_{i,k}^{(-)} \right]. \end{aligned} \quad (\text{B2})$$

Again we see the ubiquitous $\sigma_{i,j}^{(\pm)}$ parameters [cf. Eq. (67) and (68)] rearing their heads in the off-diagonal terms, while the diagonal terms take the same form as they do around proportional solutions [cf. Eq. (60)].

To solve the field equations, two things must happen: first, the diagonal (δ_{ν}^{μ}) part of the W tensors must specify the value of the effective cosmological constant via Eq. (61); second, the off-diagonal ($l^{\mu} l_{\nu}$) components of the W tensors must vanish. The latter is accomplished by setting either $\phi_i = \phi_j$ or $\sigma_{i,j}^{(+)} = 0$ along each interaction link. This can be achieved in three distinct ways:

- (i) Set *all* $\phi_i = \phi_j$. All metrics are proportional, and one is free to use the results of Sec. IV A to determine the perturbation structure.
- (ii) Set *all* $\sigma_{i,j}^{(+)} = 0$ (remember, the σ 's are proportional to one another so this also means all $\sigma_{j,i}^{(-)} = 0$). None of the metrics are proportional.
- (iii) Set $\phi_i = \phi_j$ along some interaction links but $\sigma_{i,j}^{(+)} = 0$ along others. Some, but not all, of the metrics are proportional.

The latter two branches require one to fine-tune the $\beta_m^{(i,j)}$ interaction coefficients to get a valid solution, and we argued in [58,59] that they should actually be pathological anyway, as setting $\sigma_{i,j}^{(+)} = 0$ leads to one of the heavy spin-2 fields becoming massless asymptotically far from the black hole, in violation of Boulanger *et al.*'s no-go theorem [60]. Nevertheless, until this pathology is explicitly confirmed,

let us determine the structure of the perturbations for non-proportional solutions (option 2). Setting all the σ 's equal to 0 in Eqs. (102) and (103) means that the contribution of the mass terms to the linearized field equations (53) take the form,

$$\begin{aligned} [\mathcal{B}_{i,j}^{(+)}]_{\nu}^{\mu}{}^{\alpha}{}_{\beta} (\bar{S}_{i \rightarrow j})^{\beta}{}_{\lambda} \chi^{\lambda}{}_{\alpha} \\ = \mathcal{C}_{i,j}^{(+)} (\phi_i - \phi_j) [2\chi^{\mu}{}_{\alpha} l^{\alpha} l_{\nu} + \delta_{\nu}^{\mu} \chi^{\alpha}{}_{\beta} l^{\beta} l_{\alpha} - \chi^{\alpha}{}_{\nu} l^{\mu} l_{\alpha}], \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} [\mathcal{B}_{i,k}^{(-)}]_{\nu}^{\mu}{}^{\alpha}{}_{\beta} (\bar{S}_{i \rightarrow k})^{\beta}{}_{\lambda} \chi^{\lambda}{}_{\alpha} \\ = \mathcal{C}_{i,k}^{(-)} (\phi_i - \phi_k) [2\chi^{\mu}{}_{\alpha} l^{\alpha} l_{\nu} + \delta_{\nu}^{\mu} \chi^{\alpha}{}_{\beta} l^{\beta} l_{\alpha} - \chi^{\alpha}{}_{\nu} l^{\mu} l_{\alpha}], \end{aligned} \quad (\text{B4})$$

where we have defined the parameters,

$$\mathcal{C}_{i,j}^{(+)} = \frac{a_j^2}{a_i^2} \beta_2^{(i,j)} + \frac{a_j^3}{a_i^3} \beta_3^{(i,j)}, \quad (\text{B5})$$

$$\mathcal{C}_{i,k}^{(-)} = \frac{a_k^2}{a_i^2} \beta_2^{(k,i)} + \frac{a_k}{a_i} \beta_3^{(k,i)}, \quad (\text{B6})$$

related by,

$$\mathcal{C}_{i,j}^{(+)} = \left(\frac{a_j}{a_i} \right)^4 \mathcal{C}_{j,i}^{(-)}. \quad (\text{B7})$$

Note that β_1 is not present in these expressions, as we have used $\sigma_{i,j}^{(\pm)} = 0$ to express it in terms of β_2 and β_3 .

Now we can specify our background further to the one used in [56,59,118], namely, $D = 4$ multi-Schwarzschild-(A)dS in (advanced) Eddington-Finkelstein coordinates, where the various functions from Eq. (100) are explicitly,

$$g_{\mu\nu}^{(\Lambda)} dx^{\mu} dx^{\nu} = - \left(1 - \frac{\Lambda}{3} r^2 \right) dv^2 + 2dvdr + r^2 d\Omega_2^2, \quad (\text{B8})$$

$$g^{(\Lambda)\mu\nu} \partial_{\mu} \partial_{\nu} = 2\partial_v \partial_r + \left(1 - \frac{\Lambda}{3} r^2 \right) \partial_r^2 + \frac{\partial_{\theta}^2 + \csc^2 \theta \partial_{\phi}^2}{r^2}, \quad (\text{B9})$$

$$l_{\mu} dx^{\mu} = dv, \quad (\text{B10})$$

$$l^{\mu} \partial_{\mu} = \partial_r, \quad (\text{B11})$$

$$\phi_i = \frac{r_{s,i}}{2r}, \quad (\text{B12})$$

with $r_{s,i}$ the Schwarzschild radius associated to each metric. With these choices, the linearized Lorentz constraints (104) become,

$$[\omega_{mn}]_{i,j} = 0, \quad (\text{B13})$$

$$4[\omega_{0m}]_{i,j} = -\frac{1}{4r} (r_{s,i} - r_{s,j}) [\delta g_{1m}]_{i,j}. \quad (\text{B14})$$

As in the cosmological case, the spatial components are solved trivially but the mixed space-time components are not.

Regarding the metric equations, substituting into Eqs. (B3) and (B4), then into Eq. (53), one finds that the linearized field equations around this background are,

$$\bar{\mathcal{E}}^{(i)\mu}{}_{\nu}{}^{\alpha}{}_{\beta} \delta g^{(i)\beta}{}_{\alpha} - \Lambda \left(\delta g^{(i)\mu}{}_{\nu} - \frac{1}{2} \delta^{\mu}_{\nu} \delta g^{(i)} \right) + \frac{\kappa_i a_i^2}{2} \left[\sum_j [\chi_{i,j}^{(+)}]_{\nu}^{\mu} + \sum_k [\chi_{i,k}^{(-)}]_{\nu}^{\mu} \right] = 0, \quad (\text{B15})$$

defining the tensors,

$$[\chi_{i,j}^{(+)}]_{\nu}^{\mu} = - \left(\frac{a_j}{a_i} \right)^4 [\chi_{j,i}^{(-)}]_{\nu}^{\mu} = \mathcal{C}_{i,j}^{(+)}(\phi_i - \phi_j) [\Delta_{i,j}]_{\nu}^{\mu}, \quad (\text{B16})$$

where the matrix $\Delta_{i,j}$ has components,

$$\Delta_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ [\delta g^2_2]_{i,j} + [\delta g^3_3]_{i,j} & 0 & -[\delta g^0_2]_{i,j} & -[\delta g^0_3]_{i,j} \\ -[\delta g^2_1]_{i,j} & 0 & [\delta g^0_1]_{i,j} & 0 \\ -[\delta g^3_1]_{i,j} & 0 & 0 & [\delta g^0_1]_{i,j} \end{bmatrix}. \quad (\text{B17})$$

One should note that *none* of the nontrivial Lorentz parameters enter Eq. (B17) owing to the form of the inverse metric (B9)—one has,

$$\begin{aligned} [\delta g^0_m]_{i,j} - 4[\omega^0_m]_{i,j} &= [\delta g_{1m}]_{i,j} - 4[\omega_{1m}]_{i,j} \\ &= [\delta g_{1m}]_{i,j} \\ &= [\delta g^0_m]_{i,j}. \end{aligned} \quad (\text{B18})$$

Therefore, on this particular background, simply knowing the form of the background $\bar{S}_{i \rightarrow j}$ matrices is enough to fully specify the form of the linearized field equations.

Since the background is nonrotating, it is spherically symmetric. This means that the metric perturbations can be decomposed into a complete basis of tensor spherical harmonics. In Fourier space, the decomposition reads,

$$\delta g_{\mu\nu}^{(i)}(v, r, \theta, \phi) = \sum_{l,m} \frac{a_i^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega v} \delta \tilde{g}_{\mu\nu}^{(i)lm}(\omega, r, \theta, \phi), \quad (\text{B19})$$

where,

$$\delta \tilde{g}_{\mu\nu}^{(i)lm} = \delta \tilde{g}_{\mu\nu}^{(i)\text{ax},lm} + \delta \tilde{g}_{\mu\nu}^{(i)\text{pol},lm}. \quad (\text{B20})$$

The superscript “ax” stands for axial (i.e., parity-odd) perturbations, and the superscript “pol” stands for polar (i.e., parity-even) perturbations. The spherical symmetry ensures that the field equations for the axial and polar perturbations decouple, as do those for different harmonic indices l . The explicit expressions for the axial and polar perturbation matrices are [suppressing (i) indices, and indicating symmetric components with asterisks] [57,118],

$$\delta \tilde{g}_{\mu\nu}^{\text{ax},lm} = \begin{bmatrix} 0 & 0 & h_0^{lm}(\omega, r) \csc \theta \partial_{\phi} Y_{lm} & -h_0^{lm}(\omega, r) \sin \theta \partial_{\theta} Y_{lm} \\ * & 0 & h_1^{lm}(\omega, r) \csc \theta \partial_{\phi} Y_{lm} & -h_1^{lm}(\omega, r) \sin \theta \partial_{\theta} Y_{lm} \\ * & * & -h_2^{lm}(\omega, r) \csc \theta X_{lm} & h_2^{lm}(\omega, r) \sin \theta Z_{lm} \\ * & * & * & h_2^{lm}(\omega, r) \sin \theta X_{lm} \end{bmatrix}, \quad (\text{B21})$$

$$\delta \tilde{g}_{\mu\nu}^{\text{pol},lm} = \begin{bmatrix} H_0^{lm}(\omega, r) Y_{lm} & H_1^{lm}(\omega, r) Y_{lm} & \eta_0^{lm}(\omega, r) \partial_{\theta} Y_{lm} & \eta_0^{lm}(\omega, r) \partial_{\phi} Y_{lm} \\ * & H_2^{lm}(\omega, r) Y_{lm} & \eta_1^{lm}(\omega, r) \partial_{\theta} Y_{lm} & \eta_1^{lm}(\omega, r) \partial_{\phi} Y_{lm} \\ * & * & r^2 \left[\begin{matrix} K^{lm}(\omega, r) Y_{lm} \\ + G^{lm}(\omega, r) Z_{lm} \end{matrix} \right] & r^2 G^{lm}(\omega, r) X_{lm} \\ * & * & * & r^2 \sin^2 \theta \left[\begin{matrix} K^{lm}(\omega, r) Y_{lm} \\ - G^{lm}(\omega, r) Z_{lm} \end{matrix} \right] \end{bmatrix}. \quad (\text{B22})$$

Here, $Y_{lm}(\theta, \phi)$ are the ordinary (scalar) spherical harmonics, the functions $X_{lm}(\theta, \phi)$ and $Z_{lm}(\theta, \phi)$ are given by,

$$X_{lm}(\theta, \phi) = 2\partial_{\phi}(\partial_{\theta} Y_{lm} - \cot \theta Y_{lm}), \quad (\text{B23})$$

$$Z_{lm}(\theta, \phi) = \partial_{\theta}^2 Y_{lm} - \cot \theta \partial_{\theta} Y_{lm} - \csc^2 \theta \partial_{\phi}^2 Y_{lm}, \quad (\text{B24})$$

and all of the remaining functions of (ω, r) are free.

Raising an index with the background metric (100), written in the Eddington-Finkelstein coordinates of Eqs. (B8)–(B12), then substituting into Eq. (B17), one finds that the interaction matrix $\Delta_{i,j}$, in Fourier space, explicitly reads as follows:

$$\Delta_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2[K^{lm}]_{i,j}Y_{lm} & 0 & -\left(\csc\theta\partial_\phi Y_{lm}[h_1^{lm}]_{i,j} + \partial_\theta Y_{lm}[\eta_1^{lm}]_{i,j}\right) & \left(\sin\theta\partial_\theta Y_{lm}[h_1^{lm}]_{i,j} - \partial_\phi Y_{lm}[\eta_1^{lm}]_{i,j}\right) \\ -\left(\csc\theta\partial_\phi Y_{lm}[h_1^{lm}]_{i,j} + \partial_\theta Y_{lm}[\eta_1^{lm}]_{i,j}\right) & 0 & [H_2^{lm}]_{i,j}Y_{lm} & 0 \\ \frac{1}{r^2\sin^2\theta}\left(\partial_\theta Y_{lm}[h_1^{lm}]_{i,j} - \csc\theta\partial_\phi Y_{lm}[\eta_1^{lm}]_{i,j}\right) & 0 & 0 & [H_2^{lm}]_{i,j}Y_{lm} \end{bmatrix}. \quad (\text{B25})$$

This is precisely the perturbation matrix around nonproportional multi-Schwarzschild black holes that was derived in [56,59,118] [up to the canonical normalization of the metric perturbations implicit in Eq. (55), which we have included here but which was not included in those works]. Everything works as intended.

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