Analytic approach to boundary integrability with application to mixed-flux $AdS_3 \times S^3$

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Boundary integrability provides rare analytic control over field theories with interfaces, from quantum impurity problems to open string dynamics. We propose a new approach for integrable boundaries in two-dimensional sigma models that uses the divisor structure of the Lax connection to identify admissible reflection maps, including when standard parity-based constructions fail. Applied to open strings on $AdS_3 \times S^3$ with mixed Neveu-Schwarz–Neveu-Schwarz–Ramond-Ramond flux, where the Neveu-Schwarz–Neveu-Schwarz flux breaks world sheet parity, we find two interesting branches of integrable boundaries: one restricted to pure Ramond-Ramond flux and another admitting D-branes wrapping twisted conjugacy classes for generic flux. At the Wess-Zumino-Witten point, the latter reduces to the known conformal D-branes, opening a path to comparison with conformal perturbation theory. More broadly, our framework suggests generalizations of standard lattice constructions that may enlarge existing classifications of integrable boundaries.

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I. INTRODUCTION

Field theories with interfaces play an important role in both high-energy and condensed-matter physics. In open string theory, world sheet boundary conditions describe D-branes, while in condensed-matter and lattice systems they model boundary critical phenomena and impurities such as the Kondo effect. When the bulk theory is integrable, the situation becomes especially interesting, as an infinite set of conserved charges provides a rare source of analytic control for interacting dynamics. The presence of an interface probes this structure in a nontrivial way. Geometric symmetries are typically broken, and it is far from obvious that an infinite tower of conserved charges can survive. When they do, notions of boundary integrability have e.g. opened access to exact reflection data [1] and methods for computing defect observables in AdS/CFT [2]. Identifying which physical interfaces are compatible with integrable dynamics is therefore a basic structural question. In this work, we address it for two-dimensional sigma models using simple analytical arguments from a novel perspective.

The starting point is an integrable sigma model on $\mathbb{R}_\tau \times S^1_\sigma$ with an on-shell flat Lax connection $\mathcal{L}(\mathsf{x})$,[1] i.e.

$$\mathrm{d}\mathcal{L}(\mathsf{x}) + \mathcal{L}(\mathsf{x}) \wedge \mathcal{L}(\mathsf{x}) = 0, \quad \forall\ \mathsf{x} \in \mathbb{CP}^1, \tag{1}$$

which represents the sigma-model equations of motion for every value of a spectral parameter x. In general $\mathcal{L}(\mathsf{x})$ is a matrix-valued one-form and meromorphic in $\mathsf{x} \in \mathbb{CP}^1$. Then, under periodic boundary conditions $\mathcal{L}(\sigma + 2\pi; \mathsf{x}) = \mathcal{L}(\sigma; \mathsf{x})$, the monodromy matrix $T_{\mathcal{L}}(2\pi, 0; \mathsf{x})$, with

$$T_{\mathcal{L}}(b, a; \mathsf{x}) \coloneqq \overleftarrow{\mathcal{P}} \exp\left(-\int_a^b d\sigma \mathcal{L}_\sigma(\mathsf{x})\right), \quad a, b \in \mathbb{R}, \tag{2}$$

is a generator of conserved charges, as its eigenvalues are independent of the choice of spatial slice for every value of x (see e.g. the lecture notes [3]).

On an interval $\sigma \in [0, \pi]$, boundary integrability requires a modification of this generator. The prototype is obtained by following Sklyanin's construction [4,5], where one forms a *double-row* monodromy matrix $T_{\mathrm{b}}(\mathsf{x})$ by gluing $T_{\mathcal{L}}(\pi, 0; \mathsf{x})$ over the interval $\sigma \in [0, \pi]$ to a reflected copy. On the lattice, Sklyanin's approach applies to integrable models with crossing-symmetric R-matrices of difference form [5]. The reflection compatible with integrability then has two components: a spatial flip reversing the order of sites and a spectral reflection $\mathsf{x} \mapsto \rho(\mathsf{x}) = -\mathsf{x}$ implementing

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[1]For brevity, and where possible, we will suppress explicit time ($\tau \in \mathbb{R}_\tau \coloneqq \mathbb{R}$) or space ($\sigma \in S^1_\sigma \coloneqq S^1$ with radius 1) dependence.

crossing symmetry. Additional boundary degrees of freedom are encoded in reflection matrices $K_{\sigma_\star}(\mathsf{x})$ at the end points $\sigma_\star = 0, \pi$, which must satisfy the boundary Yang-Baxter equation to preserve integrability. In the sigma model, Sklyanin's equivalent definition of the full reflected copy can be formulated as applying a world sheet parity map $p\colon(\tau,\sigma) \mapsto (\tau, 2\pi - \sigma)$ combined with a spectral reflection $\rho\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ defined such that the Lax connection transforms covariantly, i.e.[2]

$$p^*\mathcal{L}(\mathsf{x}) = \mathcal{L}(\rho(\mathsf{x})), \tag{3}$$

over the full world sheet Σ domain, e.g. [4,6–8]. Integrable boundary conditions are then obtained by evaluating (3) at $\sigma_\star = 0, \pi$, up to a residual gauge freedom by boundary reflection matrices $K_{\sigma_\star}(\mathsf{x}) \coloneqq K_{\sigma_\star}(\tau;\mathsf{x})$. The corresponding generator of conserved charges is $T_{\mathrm{b}}(\mathsf{x}) = K_0(\mathsf{x}) \times T_{\mathcal{L}}(2\pi,\pi;\mathsf{x})K_\pi^{-1}(\mathsf{x})T_{\mathcal{L}}(\pi,0;\mathsf{x}) = K_0(\mathsf{x})T_{\mathcal{L}}^{-1}(\pi,0;\rho(\mathsf{x})) \times K_\pi^{-1}(\mathsf{x})T_{\mathcal{L}}(\pi,0;\mathsf{x})$, using (3).[3]

There are however limitations to taking this prescription directly as a definition of boundary integrability for sigma models. In fact, there need not exist a $\rho(\mathsf{x})$ that implements (3) and thus makes $\mathcal{L}(\mathsf{x})$ covariant after a parity flip. This is typically the case when the action contains a parity-odd term, such as a Wess-Zumino (WZ) term or a B field, as in many integrable deformations. In those cases one may consider the lattice equivalent, where $\rho(\mathsf{x})$ is dictated by the crossing properties of the bulk R-matrix. However such input is often unavailable (or indirect through the sigma-model S-matrix), and the defining integrability object is only the Lax connection $\mathcal{L}(\mathsf{x})$. There is then no given principle to select an appropriate generator of conserved charges.

In this work, the main idea is to address this problem by allowing a simple generalization of the reflected copy defined by (3).[4] Not only at the boundary points $\sigma_\star$, but also in the definition of the full reflected region can one allow for different gauge representatives. That is, $\rho(\mathsf{x})$ can be derived as the spectral reflection that consistently solves

$$p^*\mathcal{L}(\mathsf{x}) = \mathcal{L}^U(\rho(\mathsf{x})) = U\mathcal{L}(\rho(\mathsf{x}))U^{-1} - \mathrm{d}UU^{-1}, \tag{4}$$

for all $\tau \in \mathbb{R}$ and $\sigma \in [0,\pi]$ and for some matrix $U \coloneqq U(\tau,\sigma;\mathsf{x})$ that is defined over the full world sheet. While *a priori* a modest change compared to (3), we will see that this extension can lead to very distinct classes of integrable boundary conditions which go beyond those obtained from parity covariance alone.

In Sec. II, we give our approach to identify the new admissible gluing data $(U,\rho(\mathsf{x}))$ by means of the analytic structure of the bulk Lax connection. In particular, we will explain how matching the zero and pole divisors of the two Lax representatives constrains these possibilities. For each admissible bulk gluing class $(U,\rho(\mathsf{x}))$, the residual boundary reflection matrices $K_{\sigma_\star}(\mathsf{x})$ then select the compatible integrable interfaces through (9).

In Sec. III, we will illustrate our approach on the distinguished example of the string sigma model on $\mathrm{AdS}_3 \times S^3 \times M_4$ with mixed Neveu-Schwarz–Neveu-Schwarz (NSNS) and Ramond-Ramond (RR) flux; for reviews see [11]. Its classical Lax connection has been known for all flux values since the work of Cagnazzo and Zarembo [12] and illustrates the above limitation: the NSNS term spoils parity invariance of the action and consequently the parity-based criterion (3) alone does not admit a $\rho(\mathsf{x})$ at nontrivial NSNS flux. Moreover, while world sheet methods have led to S-matrices [13], a dual CFT and lattice formulation remains absent for generic fluxes. This makes the mixed-flux model an ideal setting to demonstrate our approach to boundary integrability.

Boundary integrability of the mixed-flux $\mathrm{AdS}_3 \times S^3$ background is also inherently motivated by exact world sheet studies. At the pure NSNS point, the bosonic theory is described by an exactly solvable Wess-Zumino-Witten (WZW) model on $SU(1,1) \times SU(2)$ [14], in which CFT-preserving D-branes wrap twisted conjugacy classes that admit a boundary CFT (BCFT) description [15,16]. Turning on RR flux corresponds to a marginal deformation and, for pointlike branes, its effect on the open-string spectrum has been analyzed through boundary conformal perturbation theory, leading to all-orders resummed results, while in the closed-string sector the RR perturbation is singular [17]. The boundary sector thus provides one of the few controlled windows on mixed-flux physics. It was in fact suggested already in [12] that identifying integrable boundary conditions could provide further structural information at finite flux and offer a bridge to the BCFT picture. The present work realizes this at the classical bosonic level. Imposing our analytic criteria on the mixed-flux Lax, we find two branches of integrable boundary conditions: one compatible with the sigma-model action only in the pure RR case, while the other persists for generic flux and includes all D-branes wrapping twisted conjugacy classes. Interestingly, the former arises by identifying $\rho(\mathsf{x})$ from the covariance of $\mathcal{L}(\mathsf{x})$ under parity alone, whereas the latter requires combining parity with a gauge transformation of the Lax connection. It is in the second branch that we have a rigid class of integrable branes whose embedding coincides with the conformal WZW branes. We show that their

[2]Here p^* denotes the standard pullback of the map p, which maps the original strip to the reflected copy. That is, $p^*\mathcal{L} = (\mathcal{L}_\tau \circ p)\mathrm{d}\tau - (\mathcal{L}_\sigma \circ p)\mathrm{d}\sigma$. Similarly, we can rewrite (3) as $p^*\mathcal{L} = \rho^*\mathcal{L}$, as we will sometimes do below. The purpose of this gluing is to solve fields on $p(\Sigma)$ in terms of fields on Σ for consistent $\rho(\mathsf{x})$.

[3]The original and reflected rows $T_{\mathcal{L}}$ together form the noncontractible spatial circle of the doubled construction $[0,\pi] \cup [\pi,2\pi] \sim S^1$, with K_0 and K_π acting as transition matrices at the gluing points.

[4]For other alternative approaches via parity, yet with additional assumptions, see [9]. The approach we take here moreover admits a natural origin in four-dimensional Chern-Simons theory, as will be reported elsewhere [10].

worldvolume two-form can remain flux independent, with the mixed-flux contribution encoded entirely in dynamical reflection matrices $K_{\sigma_\star}(\mathsf{x})$.[5]

Outlook. For mixed-flux $\mathrm{AdS}_3 \times S^3$, our results thus provide a natural setup to compare integrability-based data with conformal perturbation theory around the solvable WZW point, where the same twisted conjugacy classes underlie the BCFT description. Including the fermionic sector and analysing the open string spectrum would enable comparison with the all-order spectra of [17] and open the way to extending the analysis beyond the pointlike brane sector. We leave this for future work.

More broadly, our approach may extend to other sigma models with parity-odd terms, such as integrable deformations, and also suggests a possible generalization for lattice models. Traditionally, the boundary Yang-Baxter equation assumes that both rows in the double-row monodromy share the same R-matrix. By allowing different representatives in each row, mirroring different gauge choices of $\mathcal{L}(\mathsf{x})$, a natural question is whether classifications can be enlarged to include new integrable interfaces that yield distinct states after reflection. Given the scarcity of integrable models with interfaces—for defect observables in holography, benchmarking impurity physics, engineering quantum circuits, etc.—this could provide valuable new ground for applications of exact methods.

II. ANALYTIC CRITERIA

To analyze integrable two-dimensional sigma models with boundaries, let us first collect some necessary background on the bulk integrable sigma model. Crucially, there is a Lax connection $\mathcal{L}(\mathsf{x})$, generally an object depending locally on the fields and in an $n \times n$ matrix representation of the loop algebra $\mathbb{C}(\mathsf{x}) \otimes \mathfrak{g}$, and with the defining property that it is flat (1) on shell for all $\mathsf{x} \in \mathbb{CP}^1$. The flatness condition (1) can be contrasted with an ordinary current conservation law, $\mathrm{d}\star J = 0$ for a matrix-valued current $J \in \mathfrak{g}$, whose conserved charge is local and obtained by integrating $\star J$ over a spatial slice. At generic x, the quadratic term $\mathcal{L}(\mathsf{x}) \wedge \mathcal{L}(\mathsf{x})$ for non-Abelian $\mathcal{L}(\mathsf{x})$ then introduces a nonlocality in the corresponding conserved object. In particular, under periodic conditions on S^1_σ, the path-ordered exponential $T_{\mathcal{L}}(2\pi, 0; \mathsf{x})$, defined in (2), is a generating function of generically nonlocal conserved charges. However, at special points x_0 in $\mathbb{CP}^1$, namely where $\mathcal{L}(\mathsf{x}_0) = 0$, the flatness condition at first order around $\mathsf{x} = \mathsf{x}_0$ implies that $\star\mathcal{L}'(\mathsf{x}_0)$ is an ordinary conserved current. The first nontrivial term of $T_{\mathcal{L}}(2\pi, 0; \mathsf{x})$ around x_0 then indeed is a local conserved charge. Hence zeros of the Lax connection in $\mathbb{CP}^1$ hold a special role for the sigma-model physics.

The above statements refer, however, to a particular Lax representative. In fact, $\mathcal{L}(\mathsf{x})$ is not unique: it is defined up to local flatness-preserving gauge transformations

$$\mathcal{L}^U(\mathsf{x}) \coloneqq U\mathcal{L}(\mathsf{x})U^{-1} - \mathrm{d}UU^{-1}, \tag{5}$$

with $U \coloneqq U(\tau, \sigma; \mathsf{x})$ an invertible matrix that is such that $\mathcal{L}^U(\mathsf{x})$ remains in the relevant algebraic realization. Under such gauge transformations one has

$$T_{\mathcal{L}^U}(2\pi, 0; \mathsf{x}) = U(\tau, 2\pi; \mathsf{x})T_{\mathcal{L}}(2\pi, 0; \mathsf{x})U(\tau, 0; \mathsf{x})^{-1}. \tag{6}$$

Thus, for an admissible U that is periodic along the spatial circle, the two monodromies are spectrally equivalent. Their conserved charges are therefore independent of the chosen representative. Nevertheless, the spectral points at which their locality is manifest directly from the Lax connection or the monodromy matrix do depend on that representative. Indeed, U may vary along the spatial slice, and the inhomogeneous term $\partial_\sigma UU^{-1}$ can change or remove the spectral points at which $\mathcal{L}^U_\sigma(\mathsf{x})$ vanishes. This dependence, which does not alter the conserved spectral data of the periodic model, becomes relevant when different representatives are assigned to the two rows of the boundary construction, as we will see.

In what follows we will thus denote the gauge orbit of the Lax connection by $[\mathcal{L}]$ and associate with it the collection

$$\mathcal{Z}([\mathcal{L}]) \coloneqq \{(\mathsf{x}_0, U) | \mathcal{L}^U(\mathsf{x}_0) = 0\}. \tag{7}$$

For a particular representative $\mathcal{L}^U(\mathsf{x})$, we can denote its zero divisor by $\mathrm{D}_{\mathrm{z}}[\mathcal{L}^U] \coloneqq \{\mathsf{x}_0 \in \mathbb{CP}^1 | \mathcal{L}^U(\mathsf{x}_0) = 0\}$. Similarly we will write $\mathrm{D}_{\mathrm{p}}[\mathcal{L}^U]$ for its pole divisor. Note that the condition $\mathcal{L}^U(\mathsf{x}_0) = 0$ is equivalent to $\mathcal{L}(\mathsf{x}_0) = U^{-1}\mathrm{d}U$, so it requires $\mathcal{L}(\mathsf{x}_0)$ to be locally pure gauge, or equivalently off-shell flat, which is a strong requirement.

Let us now consider open boundaries. As described in the introduction, we approach this by employing a mirror trick, where one glues the original interval $\sigma \in [0, \pi]$ to a reflected, artificial, region $p(\sigma) \in [\pi, 2\pi]$. We then relate the degrees of freedom in the reflected region to the original ones via the generalized gluing map (4). The corresponding double-row monodromy matrix then takes the form

$$T_{\mathrm{b}}(\mathsf{x}) \coloneqq K_0(\mathsf{x})T^{-1}_{\mathcal{L}^U}(\pi, 0; \rho(\mathsf{x}))K_\pi(\mathsf{x})^{-1}T_{\mathcal{L}}(\pi, 0; \mathsf{x}). \tag{8}$$

Evaluated along a spatial slice as in (2), $T_{\mathrm{b}}(\mathsf{x})$ then generates an infinite tower of conserved charges under the following *integrable boundary conditions*:

$$\mathcal{L}_\tau(\mathsf{x}) \overset{\mathrm{b}}{=} K_{\sigma_\star}(\mathsf{x})\mathcal{L}^U_\tau(\rho(\mathsf{x}))K^{-1}_{\sigma_\star}(\mathsf{x}) - \partial_\tau K_{\sigma_\star}(\mathsf{x})K^{-1}_{\sigma_\star}(\mathsf{x}), \tag{9}$$

[5]This is compatible with the λ-deformed WZW case, where integrable D-branes also remain twisted conjugacy classes but for which the gauge-invariant worldvolume two-form acquires a nontrivial λ dependence given the λ-deformed metric [18].

with "$\overset{\mathrm{b}}{=}$" denoting evaluation at the boundary $\sigma_\star = 0, \pi$ and $K_{\sigma_\star} \coloneqq K(\tau; \mathsf{x})$ an invertible *reflection matrix* defined along $\partial\Sigma = (\tau, \sigma_\star = 0, \pi)$ and whose action is such that the right-hand side remains in the relevant algebraic realization of the Lax connection. These matrices encode additional possible boundary degrees of freedom.[6] One may also explicitly verify that $\partial_\tau \mathrm{Tr} T_{\mathrm{b}}(\mathsf{x})^m = 0$ for any integer m, using flatness of the Lax and the boundary conditions (9), so that expanding the eigenvalues of $T_{\mathrm{b}}(\mathsf{x})$ in x indeed produces conserved objects. Throughout, we use boundary integrability in this Lax sense of a conserved double-row monodromy and its spectral invariants.

Let us emphasize that, although only the restriction of U to the boundary enters (9), its role is fundamentally distinct from that of $K_{\sigma_\star}$. The matrix U is defined throughout the bulk and identifies the transformation of the Lax under parity, thereby selecting the Lax representative used in the reflected region. Its relative action between the two rows is therefore part of the bulk gluing data. In the next subsection, we will fix the gluing data $(U, \rho(\mathsf{x}))$ from the bulk analytic structure, prior to solving for the boundary data $K_{\sigma_\star}$. This viewpoint generalizes the prescription of [4,6–8,18–20] beyond parity-covariant gluing, as described in the introduction.

Remark. The gauge transformation matrices do not need to belong to the Lie group G of $\mathfrak{g}$, yet their conjugation should preserve $\mathfrak{g}$. Constant matrices, e.g. $K_{\sigma_\star} = w$, can thus implement (outer) automorphisms, $W(y) = wyw^{-1} \in \mathfrak{g}$ for $y \in \mathfrak{g}$, as in [7]. In suitable realizations, such matrices also accommodate the common alternative convention for $T_{\mathcal{L}}$ in the reflected region, where the inverse in (8) is replaced by transposition.

A. Fixing U and $\rho(\mathsf{x})$ from bulk analyticity

Our main point is that U and $\rho(\mathsf{x})$ can be fixed from the analytic structure of $\mathcal{L}(\mathsf{x})$, mirroring the lattice logic where $\rho(\mathsf{x})$ is derived from bulk properties of the R-matrix (rather than boundary K-matrices).

A first useful criterion is to choose "physical" representatives for the unreflected and reflected rows, such that $T_{\mathrm{b}}(\mathsf{x})$ can generate a set of local charges. Concretely, we thus select two elements of the gauge orbit that realize nontrivial elements in $\mathcal{Z}([\mathcal{L}])$. Without loss of generality, we can assume that the given representative for $\mathcal{L}$ is already calibrated to do so, and thus we take $(\mathsf{x}_0, \mathbb{1}), (\mathsf{x}_0', U) \in \mathcal{Z}([\mathcal{L}])$, i.e. with

$$\mathcal{L}(\mathsf{x}_0) = 0, \qquad \mathcal{L}^U(\mathsf{x}_0') = 0. \tag{10}$$

We then let $\mathcal{L}$ be the reference representative in the unreflected row on $\sigma \in [0, \pi]$ and $\mathcal{L}^U$ the representative for the reflected row.[7]

In practice, the set $\mathcal{Z}([\mathcal{L}])$, and thus all relevant gauges to construct $T_{\mathrm{b}}(\mathsf{x})$ under our requirements, is found by scanning for those points $\mathsf{x} \in \mathbb{CP}^1$ where $\mathcal{L}(\mathsf{x})$ becomes locally pure gauge. Relaxing these criteria for $T_{\mathrm{b}}(\mathsf{x})$ would formally enlarge the possibilities of finding integrable boundary conditions but could obscure the existence of residual local symmetries which are usually desired for spectral analyses.

Given relevant representatives $\mathcal{L}$ and $\mathcal{L}^U$, the spectral reflection $\rho(\mathsf{x})$ is then constrained directly by the bulk gluing map (4). Since this map must return to the original row when applied twice, we require that $\rho\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is a nontrivial involutive Möbius transformation

$$\rho(\mathsf{x}) = \frac{a\mathsf{x} + b}{c\mathsf{x} - a}, \qquad a^2 + bc \neq 0, \tag{11}$$

with $\rho^2(\mathsf{x}) = \mathsf{x}$ and $a, b, c \in \mathbb{C}$ to be determined by consistency. In fact, because (4) is an equality of meromorphic Lax one-forms, their complete zero and pole divisors must be mapped into one another, i.e.

$$\mathrm{D}_{\mathrm{p}}[\mathcal{L}] = \mathrm{D}_{\mathrm{p}}[\rho^* \mathcal{L}^U], \qquad \mathrm{D}_{\mathrm{z}}[\mathcal{L}] = \mathrm{D}_{\mathrm{z}}[\rho^* \mathcal{L}^U]. \tag{12}$$

The zeros selected above then obey $\rho(\mathsf{x}_0) = \mathsf{x}_0'$. This allows local charges to be extracted from the leading terms of the expansion of $T_{\mathrm{b}}(\mathsf{x})$ around x_0.[8]

Once the relevant Lax representatives have been classified modulo constant gauge transformations, an involutive Möbius transformation can only fix marked points or exchange them in pairs while preserving their analytic type. For a finite set of such representatives, as in the example discussed below, this leaves only finitely many candidates (U, ρ) modulo constant gauge transformations. Each choice of (U, ρ), together with all subsequent $K_{\sigma_\star}$ solutions of (9), can lead to different integrable boundary conditions and hence preserve different combinations of the bulk local charges.

Let us end by clarifying the scope of this prescription. Within the class of admissible local representatives for which both rows possess a zero, every solution of the bulk gluing map (4) must satisfy the analytic matching conditions (12) above. These conditions are necessary but not sufficient, so the resulting candidates (U, ρ) must still be checked against the full gluing map. In fact, consistency under two applications additionally requires the resulting composite gauge

[6]Note furthermore that (9) corresponds precisely to evaluating the time component of (4) at $\sigma_\star$ up to allowing residual gauge transformations by $K_{\sigma_\star}$.

[7]Note that the double-row generator $T_{\mathrm{b}}(\mathsf{x})$ built from representatives $\mathcal{L}^{U_+}$ and $\mathcal{L}^{U_-}$ is spectrally equivalent to the one built from $\mathcal{L}$ and $\mathcal{L}^{U=U_+^{-1}U_-}$ upon the boundary transformation $K_{\sigma_\star} \mapsto U_+^{-1}(\sigma_\star) K_{\sigma_\star} U_+(\sigma_\star)$. Thus only the relative choice of Lax representative is meaningful, as a common gauge transformation leaves the spectrum of $T_{\mathrm{b}}(\mathsf{x})$ unchanged.

[8]Note that $K_{\sigma_\star}$ is σ independent and thus does not alter locality (assuming $K_{\sigma_\star}$ is regular at x_0 and is itself local).

transformation to act trivially on the Lax connection, as it does in the examples below. In addition, the integrable boundary conditions (9) must be consistent with the sigma-model variational problem (cf. Appendix A). Furthermore, the requirement of a zero in the row representatives can be relaxed, which may enlarge the space of solutions. The corresponding double-row monodromies may then still generate conserved spectral invariants, but an expansion producing local conserved charges would no longer apply. Our prescription is thus intended to capture the class relevant for local analyses rather than all possible reflected Lax pairs.

III. MIXED-FLUX $AdS_3 \times S^3$

We now apply our approach to boundary integrability in the mixed-flux $AdS_3 \times S^3$ coset sigma model [12] and analyze the simplest resulting integrable boundary conditions. The relevant background ingredients for this analysis are reviewed in Appendixes A and B, to which we refer the reader for conventions and notation.

For simplicity, we restrict to the bosonic sector ($g_F = \tilde{g}_F = \mathbb{1}$) of the mixed-flux $AdS_3 \times S^3$ coset sigma-model action (B6) and fix the coset gauge $\tilde{g}_B = \mathbb{1}$, so that (B11) applies. The Lax connection (B8) then simplifies to

$$\mathcal{L}(\mathsf{x}) = f_1(\mathsf{x})\mathcal{J}^B + f_2(\mathsf{x}) \star \mathcal{J}^B, \tag{13}$$

with

$$f_1(\mathsf{x}) = \frac{q_R}{\mathsf{x}^2-1} + \frac{q_R+1}{2}, \quad f_2(\mathsf{x}) = \frac{\mathsf{x} q_R}{1-\mathsf{x}^2} + \frac{q_{NS}}{2}, \tag{14}$$

where $q_R, q_{NS} \in \mathbb{R}$ denote the amount of RR and NSNS flux of the full theory, respectively, which are related as $q_R^2 + q_{NS}^2 = 1$ [12].

A. Marked points, spectral reflections and boundary conditions

There are two interesting Lax representatives in the gauge orbit of $\mathcal{L}(\mathsf{x})$ that realize nontrivial elements in $\mathcal{Z}([\mathcal{L}])$. In fact, for $\mathcal{L}(\mathsf{x})$ to be pure gauge, a necessary requirement is $f_2(\mathsf{x}) = 0$, which has the following two solutions:

$$\mathsf{x}_0 = \frac{q_R - 1}{q_{NS}}, \qquad \mathsf{x}_0' = \frac{q_R + 1}{q_{NS}}. \tag{15}$$

Upon $q_R^2 + q_{NS}^2 = 1$ one then finds[9]

$$\mathcal{L}(\mathsf{x}_0) = 0, \qquad \mathcal{L}(\mathsf{x}_0') = g_B^{-1} \mathrm{d} g_B. \tag{16}$$

TABLE I. The inequivalent Möbius involutions $\rho(\mathsf{x})$.

i	Zeros	Poles	$\rho_i(\mathsf{x})$	Gluing maps
1	$\rho_1(\mathsf{x}_0) = \mathsf{x}_0$	$\rho_1(\mathsf{x}_\pm) = \mathsf{x}_\mp$	$\frac{-\mathsf{x}-q_{NS}}{q_{NS}\mathsf{x}+1}$, $q_{NS} \neq 1$	$p^*\mathcal{L} = \rho^*\mathcal{L}$
2	$\rho_2(\mathsf{x}_0) = \mathsf{x}_0'$	$\rho_2(\mathsf{x}_\pm) = \mathsf{x}_\mp$	$-\mathsf{x}^{-1}$	$p^*\mathcal{L} = \rho^*\mathcal{L}^{g_B}$
3	$\rho_3(\mathsf{x}_0) = \mathsf{x}_0$	$\rho_3(\mathsf{x}_\pm) = \mathsf{x}_\pm$	x^{-1}, $q_R = 0$	$p^*\mathcal{L} = \rho^*\mathcal{L}$
4	$\rho_4(\mathsf{x}_0) = \mathsf{x}_0'$	$\rho_4(\mathsf{x}_\pm) = \mathsf{x}_\pm$	x^{-1}, $q_{NS} = 0$	$p^*\mathcal{L} = \rho^*\mathcal{L}^{g_B}$

Hence, modulo constant gauge transformations, the two relevant elements of $\mathcal{Z}([\mathcal{L}])$ are represented by

$$(\mathsf{x}_0, \mathbb{1}), \qquad (\mathsf{x}_0', g_B). \tag{17}$$

Both $\mathcal{L}(\mathsf{x})$ and $\mathcal{L}^{g_B}(\mathsf{x})$ are thus admissible representatives for each row in $T_b(\mathsf{x})$.

Both $\mathcal{L}(\mathsf{x})$ and $\mathcal{L}^{g_B}(\mathsf{x})$ have the same pole divisor: two simple poles at $\mathsf{x}_\pm = \pm 1$. Given (12), there are then four possible reflections $\rho_i(\mathsf{x})$, depending on whether the poles and zeros are fixed or permuted. The behavior of the zeros further determines the necessary gauges for the Lax connections: when $\rho(\mathsf{x}_0) = \mathsf{x}_0$ we must take $U = \mathbb{1}$, while for $\rho(\mathsf{x}_0) = \mathsf{x}_0'$ we must take $U = g_B$.[10] In Table I, we summarize the possibilities, the resulting solutions of (11) and the resulting gluing maps (4). The latter are equivalent at the boundary to the integrable boundary conditions (9) for trivial $K_{\sigma_\star}$ matrices.

Let us first consider cases 3 and 4, in which the poles are fixed. Case 3, defined additionally by $\rho_3(\mathsf{x}_0) = \mathsf{x}_0$, is formally solved by $\rho_3(\mathsf{x}) = \mathsf{x}^{-1}$ only at $q_R = 0$. At this point, however, the apparent zero $\mathsf{x}_0 = -1$ collides with a pole, while the pole residues vanish and the bosonic Lax connection becomes independent of x. We therefore discard this degenerate case.[11] Case 4, defined by $\rho_4(\mathsf{x}_0) = \mathsf{x}_0'$ and thus with $U = g_B$, is solved by $\rho_4(\mathsf{x}) = \mathsf{x}^{-1}$ only at $q_{NS} = 0$. In this pure RR limit, we have $\mathsf{x}_0 = 0$ and $\mathsf{x}_0' = \infty$, and the spectral reflection is in principle admissible. Suppose then that $K_{\sigma_\star}(\mathsf{x})$ is regular and nondegenerate at $\mathsf{x}_\pm$. Defining $W_\pm(y) \coloneqq K_{\sigma_\star}(\mathsf{x}_\pm) y K_{\sigma_\star}^{-1}(\mathsf{x}_\pm)$, the residue conditions following from (9) at $\mathsf{x}_\pm$ consequently are $(1 + W_\pm \mathrm{Ad}_{g_B})\mathcal{J}^B_\mp \overset{b}{=} 0$. Whenever these two operators are invertible—as is generally the case at generic target-space points—the conditions imply, after translating to the $AdS_3 \times S^3$ fields as in (A3) and using coordinates $\sigma^\pm = \tau \pm \sigma$, that

$$\partial_+ X^\mu \overset{b}{=} 0, \qquad \partial_- X^\mu \overset{b}{=} 0, \quad \forall\ \mu = 1, \ldots, 6. \tag{18}$$

[9] When keeping the coset gauge for $\tilde{g}_B$ general, $\mathcal{L}(\mathsf{x})$ would be pure gauge only at $\mathsf{x} = \mathsf{x}_0'$, with $U = g$. In the pure RR case this point reduces to $\mathsf{x} = \infty$. This corresponds to the standard setup for coset models where one expands $\mathcal{L}^g(\mathsf{x})$ to obtain local charges. The possibility of having an additional pure gauge location when using $\tilde{g}_B = \mathbb{1}$ (i.e. at $\mathsf{x} = \mathsf{x}_0$ with $U = \mathbb{1}$) is however special to $AdS_3 \times S^3$ (permutation) cosets.

[10] Note that $\rho(\mathsf{x}_0') = \mathsf{x}_0'$ with *both* Lax representatives in the gauge $U = g_B$ is equivalent to $\rho(\mathsf{x}_0) = \mathsf{x}_0$ with $U = \mathbb{1}$ in both rows up to redefining $K_{\sigma_\star}$, as explained in footnote 7.

[11] Moreover, for trivial $K_{\sigma_\star}$ one has $T_b(\mathsf{x}) = \mathbb{1}$, while (9) becomes an identity and therefore supplies no boundary conditions to be matched with the sigma-model variational conditions (A2).

They constrain both the temporal and spatial derivatives of every field at the boundary and are therefore overdetermined and incompatible with the sigma-model variational conditions (A2).[12]

More interesting cases arise when poles are permuted.

Branch 1. Firstly, we have

$$\rho_1(\mathsf{x}) = -\frac{\mathsf{x}+q_{\rm NS}}{q_{\rm NS}\mathsf{x}+1}, \qquad q_{\rm NS}\neq 1, \tag{19}$$

which fixes x_0 and thus takes $U=\mathbb{1}$. For $q_{\rm NS}=0$, this reduces to the standard reflection $\rho_1(\mathsf{x})=-\mathsf{x}$ obtained from parity-covariant gluing (3) in pure RR cosets [7,21]. For general $q_{\rm NS}$, mapping the residues of the integrable boundary conditions (9) for trivial $K_{\sigma_\star}$, i.e. $\mathcal{L}_\tau(\mathsf{x}) \overset{\rm b}{=} \mathcal{L}_\tau(\rho_1(\mathsf{x}))$, now gives

$$\mathcal{J}^{\rm B}_+ \overset{\rm b}{=} \frac{1-q_{\rm NS}}{1+q_{\rm NS}}\mathcal{J}^{\rm B}_- \quad \Rightarrow \quad \partial_\sigma X^\mu \overset{\rm b}{=} -q_{\rm NS}\partial_\tau X^\mu, \tag{20}$$

where we again used coordinates $\sigma^\pm=\tau\pm\sigma$ and translated to the $\mathrm{AdS}_3\times S^3$ fields X^μ as in (A3). In this diagonal form, the above conditions can only be compatible with the sigma-model boundary conditions (A2) at $q_{\rm NS}=0$, i.e. *only in the pure RR case*. Indeed, $q_{\rm NS}\neq 0$ would require $\mathcal{F}_{\mu\mu}\neq 0$, which is impossible because $\mathcal{F}_{\mu\nu}$ must be antisymmetric. Consequently, for $q_{\rm NS}=0$ the relation (20) implies that the integrable D-brane is space filling ($p=5$) with a vanishing worldvolume flux $\mathcal{F}_{mn}=0$. Allowing more general matrices $K_{\sigma_\star}$ can however change the dimension p and the flux $\mathcal{F}$. For example, for a constant $K_{\sigma_\star}=w$ implementing an automorphism W, Eq. (9) additionally implies $W^2=\mathbb{1}$, such that W must have eigenvalues $\lambda_\pm=\pm 1$ of multiplicity $m_\pm$, respectively, with $m_++m_-=\dim(\mathrm{AdS}_3\times S^3)=6$. Going to a diagonal frame $X^\mu\to\tilde{X}^\mu(X)$ and demanding compatibility with (A2) gives again valid conditions only at $q_{\rm NS}=0$, given by

$$\partial_\sigma \tilde{X}^{\mu_+} \overset{\rm b}{=} 0, \qquad \partial_\tau \tilde{X}^{\mu_-} \overset{\rm b}{=} 0, \tag{21}$$

where $\mu_\pm$ has cardinality $m_\pm$. For the automorphism given in (B15), one finds e.g. $m_+=4$ and then this corresponds to a D-brane with $p=3$ and $\mathcal{F}=0$. Nontrivial worldvolume fluxes are typically introduced by dynamical and x-dependent $K_{\sigma_\star}$-matrices, as happens for several interesting D-branes in $\mathrm{AdS}_5\times S^5$ and $\mathrm{AdS}_4\times\mathbb{CP}^3$ [21,22]. In the future, it would be interesting to compare with the x-dependent quantum $K_{\sigma_\star}$-matrices of [23] known for pure RR flux. For nontrivial mixed flux, an interesting question is also whether compatibility between (19) and (A2) requires dynamical reflection data $K_{\sigma_\star}$.

[12]This conclusion may be weakened on exceptional loci where either operator develops a kernel or for reflection matrices that are degenerate or singular at $\mathsf{x}_\pm$ and can modify the pole structure. We do not analyze these possibilities here.

Branch 2. Both poles and zeros are permuted by

$$\rho_2(\mathsf{x}) = -\mathsf{x}^{-1}, \tag{22}$$

with no restriction on $q_{\rm NS}$ and $q_{\rm R}$. In this branch we have $U=g_{\rm B}$. Including automorphisms W as above and mapping residues, the integrable boundary conditions (9) for trivial $K_{\sigma_\star}$, i.e. $\mathcal{L}_\tau(\mathsf{x}) \overset{\rm b}{=} W[\mathcal{L}^{g_{\rm B}}_\tau(\rho_2(\mathsf{x}))]$, now simplify for $q_{\rm R}\neq 0$ to

$$\mathcal{J}^{\rm B}_\pm \overset{\rm b}{=} -W[\mathcal{R}^{\rm B}_\mp] = -(W\mathrm{Ad}_{g_{\rm B}})[\mathcal{J}^{\rm B}_\mp], \tag{23}$$

implying the constraint $(W\mathrm{Ad}_{g_{\rm B}})^2 \overset{\rm b}{=} \mathbb{1}$. Interestingly, for $q_{\rm R}=0$, where $\mathcal{L}(\mathsf{x})$ and $\mathcal{L}^{g_{\rm B}}(\mathsf{x})$ become x independent, the boundary conditions enhance to $\mathcal{J}^{\rm B}_+ \overset{\rm b}{=} -W[\mathcal{R}^{\rm B}_-]$ without further constraints. These are precisely the conformal boundary conditions of WZW models which correspond to D-branes embedded as twisted conjugacy classes in $SU(1,1)\times SU(2)$.[13] For $W=\mathbb{1}$ this yields *unstable* $dS_2\times S^2$ branes [16], which we will exclude in the following, while including the nontrivial outer automorphism (B16) with $K_{\sigma_\star}=w_{\rm o}$ yields *stable* $\mathrm{AdS}_2\times S^2$ branes [16]. In terms of the fields $X^\mu=\{\psi,\omega,t;\alpha,\beta,\gamma\}$ given in (B12), the latter boundary conditions read

$$\begin{aligned} 3\partial_\tau\psi &\overset{\rm b}{=} 0, & \partial_\sigma\omega &\overset{\rm b}{=} \tanh\psi\cosh\omega\,\partial_\tau t,\\ & & \partial_\sigma t &\overset{\rm b}{=} \tanh\psi\,\mathrm{sech}\omega\,\partial_\tau\omega,\\ \partial_\tau\alpha &\overset{\rm b}{=} 0, & \partial_\sigma\beta &\overset{\rm b}{=} -\cot\alpha\sin\beta\,\partial_\tau\gamma,\\ & & \partial_\sigma\gamma &\overset{\rm b}{=} \cot\alpha\csc\beta\,\partial_\tau\beta. \end{aligned} \tag{24}$$

The conjugacy classes are thus fixed at $\{\psi;\alpha\}$ and extend along $X^m=\{t,\omega;\beta,\gamma\}$. With (A2) and (B13) the worldvolume flux can then be extracted from (24) as

$$\mathcal{F} = \frac{\sinh 2\psi\cosh\omega}{2}\mathrm{d}\omega\wedge\mathrm{d}t + \frac{\sin 2\alpha\sin\beta}{2}\mathrm{d}\gamma\wedge\mathrm{d}\beta. \tag{25}$$

For mixed flux, however, the constraint $(W\mathrm{Ad}_{g_{\rm B}})^2 \overset{\rm b}{=} \mathbb{1}$ fixes the D-brane conjugacy classes on specific loci, i.e. at

$$\psi=0, \qquad \alpha=\left\{0,\frac{\pi}{2},\pi\right\}, \tag{26}$$

on which the worldvolume flux becomes trivial, $\mathcal{F}=0$. We next show that this restriction can be lifted by allowing nontrivial reflection data (i.e. beyond $K_{\sigma_\star}=w_{\rm o}$).

Before extending branch 2, let us emphasize the distinction between branches 1 and 2. While branch 2 is admissible at generic mixed flux, including pure RR flux, branch 1 gives rise to additional integrable boundary conditions only in the pure RR limit. This has a simple origin in the $\mathbb{Z}_2$ symmetries of the bulk sigma-model action

[13]In the WZW limit, integrability is instead encoded by the Kac-Moody algebra.

(B6) already visible in the bosonic sector. At pure RR flux, $q_{\rm NS}=0$, the bosonic action is invariant under world sheet parity, $\sigma\to 2\pi-\sigma$. Adding the WZ term however breaks this symmetry. Yet for generic mixed flux the action is invariant under a combination of world sheet parity and group element inversion:

$$\sigma\to 2\pi-\sigma,\qquad g_{\rm B}\to g_{\rm B}^{-1}. \tag{27}$$

These two symmetries underlie the spectral reflections (4) associated with branches 1 and 2, respectively, and explain why the pure RR case enjoys more types of integrable boundary conditions. It would be interesting to relate this to crossing properties of [13].

B. Extension to all twisted conjugacy classes

To restore the full family of stable $\mathrm{AdS}_2\times S^2$ twisted conjugacy classes with two-form (25) at generic (ψ,α), we allow for appropriate matrices $K_{\sigma_\star}(\mathsf{x})$ while keeping the same reflection $\rho_2(\mathsf{x})=-\mathsf{x}^{-1}$. For this purpose we reverse the logic and use the WZW boundary conditions (24) as input to (9) to determine the required $K_{\sigma_\star}(\mathsf{x})$.

We find that it is sufficient to consider the ansatz

$$K_{\sigma_\star}(\tau;\mathsf{x})=\left(\mathbb{1}+\sum_{k=1}^{6}u_k(X^\mu;\mathsf{x})T_k\right)w_{\rm o}, \tag{28}$$

with T_k the $\mathfrak{su}(1,1)\oplus\mathfrak{su}(2)$ generators given in (B3) and u_k depending on τ only through the target-space fields $X^\mu=\{\psi,\omega,t;\alpha,\beta,\gamma\}$. Projecting the resulting system (9) onto the generators yields six coupled first-order partial differential equations in the world sheet variables (τ,σ). Using (24), we rewrite them in terms of $\partial_\sigma\{\psi,\alpha\}$ and $\partial_\tau\{t,\omega,\beta,\gamma\}$ only. Requiring that u_k does not depend on $\partial_\sigma\{\psi,\alpha\}$ (i.e. u_k depends on the fields but not their derivatives) gives a set of algebraic conditions solved by

$$\frac{u_2(\tau;\mathsf{x})}{u_1(\tau;\mathsf{x})}=\cot t,\qquad \frac{u_3(\tau;\mathsf{x})}{u_1(\tau;\mathsf{x})}=\tanh\omega\csc t,$$
$$\frac{u_5(\tau;\mathsf{x})}{u_4(\tau;\mathsf{x})}=-\cot\gamma,\qquad \frac{u_6(\tau;\mathsf{x})}{u_4(\tau;\mathsf{x})}=-\cot\beta\csc\gamma. \tag{29}$$

While this reduces the unfixed functions to u_1 and u_4 only, we remarkably find that the resulting six partial differential equations can still be solved consistently. We simply obtain

$$u_1=-\sqrt{2}\frac{\sinh 2\psi\cosh\omega\sin t}{z(\mathsf{x})+\cosh 2\psi},$$
$$u_4=\frac{\sqrt{2}}{i}\frac{\sin 2\alpha\sin\beta\sin\gamma}{z(\mathsf{x})-\cos 2\alpha}, \tag{30}$$

with the rational function $z(\mathsf{x})$ defined as

$$z(\mathsf{x})=\frac{(\mathsf{x}-1)(q_{\rm NS}(\mathsf{x}+1)+q_{\rm R}(\mathsf{x}-1))}{(\mathsf{x}+1)((1-q_{\rm NS})(\mathsf{x}-1)+q_{\rm R}(\mathsf{x}+1))}. \tag{31}$$

The resulting solution $K_{\sigma_\star}$ is regular and nondegenerate at $\mathsf{x}_\pm$. On the loci (26) we obtain $u_k=0$ and hence $K_{\sigma_\star}=w_{\rm o}$, as well as whenever the denominator of $z(\mathsf{x})$ vanishes, which occurs at $\mathsf{x}=\mathsf{x}_0$ and at $q_{\rm R}=0$ (the WZW point). This is therefore fully consistent with the previous subsection. Away from these special points, we find that branes wrapping twisted conjugacy classes, as in boundary WZW models, remain compatible with integrability, with the mixed-flux deformation carried entirely by the nontrivial reflection data $K_{\sigma_\star}(\mathsf{x})$ rather than the brane geometry.

We will report on the extension to the fermionic sector, including the analysis of residual symmetries from $T_{\rm b}(\mathsf{x})$ and the open-string spectrum, elsewhere.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: SIGMA-MODEL BOUNDARY CONDITIONS

We formulate integrable boundary conditions through a relation on the Lax connection (9), which guarantees conservation of charges generated by $T_{\rm b}(\mathsf{x})$. At the level of the sigma model, this must translate into boundary conditions on the fields X^μ, $\mu=0,\ldots,D-1$, that are compatible with the variational principle.

The bosonic action of a standard two-derivative sigma model on $\Sigma=\mathbb{R}_\tau\times[0,\pi]_\sigma$ can be written as[14]

[14]Here we use the flat sigma-model metric $\eta_{\alpha\beta}=\mathrm{diag}(1,-1)$, with $\sigma^\alpha=(\tau,\sigma)$, and $\epsilon_{\tau\sigma}=1$. In general the target-space fields (metric and B-field) are curved, $G_{\mu\nu}\coloneqq G_{\mu\nu}(X)$ and $B_{\mu\nu}\coloneqq B_{\mu\nu}(X)$ and T is the sigma-model coupling (i.e. string tension). As an action of open strings (A1) is written in conformal gauge.

$$S=\frac{T}{2}\int_{\Sigma}\mathrm{d}^2\sigma\partial_\alpha X^\mu(\eta^{\alpha\beta}G_{\mu\nu}+\epsilon^{\alpha\beta}B_{\mu\nu})\partial_\beta X^\nu+\int_{\partial\Sigma}\mathrm{d}\tau A_m\partial_\tau X^m, \tag{A1}$$

where $m,n=0,\ldots,p\leq D-1$ are the directions along which $X^\mu|_{\partial\Sigma}$ is allowed to fluctuate, i.e. (for open strings) those along a Dp-brane with a $U(1)$ gauge field $A_m\coloneqq A_m(X)$ and field strength $F_{mn}=\partial_m A_n-\partial_n A_m$. The boundary equations of motion are then solved by

$$\begin{aligned}&\text{Dirichlet}\colon\ \partial_\tau X^{\hat{m}}\overset{\text{b}}{=}0,\\&\text{gen. Neumann}\colon\ G_{mn}\partial_\sigma X^n\overset{\text{b}}{=}\mathcal{F}_{mn}\partial_\tau X^n,\end{aligned} \tag{A2}$$

with $\hat{m}=p+1,\ldots,D-1$ and $\mathcal{F}_{mn}(X)=B_{mn}(X)+T^{-1}F_{mn}(X)$ the gauge-invariant worldvolume flux. Here we assumed adapted coordinates in which the target-space fields split orthogonally in the Neumann (m) and Dirichlet directions ($\hat{m}$). Note that the conditions (A2) are rather restrictive, given the symmetry properties of each term in the generalized Neumann conditions.

The problem addressed in this work can be restated as determining those Dirichlet and Neumann conditions which are guaranteed to preserve classical integrability. For the sigma models considered here, the Lax connection $\mathcal{L}(\mathsf{x})\in\mathbb{C}(\mathsf{x})\otimes\mathfrak{g}$ is a linear combination of (projections of) $\mathfrak{g}$-valued one-form currents $\mathcal{J}$ and $\star\mathcal{J}$ with x-dependent coefficients. The currents themselves are linear in $\mathrm{d}X^\mu$ so that one can introduce a frame field E as

$$\mathcal{J}_\alpha=E_\mu{}^A(X)T_A\partial_\alpha X^\mu,\qquad(\star\mathcal{J})_\alpha=\mathcal{J}_\beta\epsilon^\beta{}_\alpha, \tag{A3}$$

with T_A the generators of $\mathfrak{g}$. The conditions (9) for chosen $(\rho,U,K_{\sigma_\star})$ therefore yield linear gluing relations between $\partial_\sigma X^\mu$ and $\partial_\tau X^\mu$ at $\partial\Sigma$ which must be matched to the expressions (A2) for consistency with the variational principle and hence determine the dimension p and the flux $\mathcal{F}_{mn}$ (equivalently F_{mn} for given B_{mn}). Likewise, one may start from local D-brane data $(p,\mathcal{F})$ which determine the relations (A2) and solve Eqs. (4) and (9) for $(\rho,U,K_{\sigma_\star})$.

APPENDIX B: MIXED-FLUX $\mathrm{AdS}_3\times S^3$ BULK SIGMA MODEL

In this appendix, we collect the background ingredients for the boundary integrability analysis of the mixed NSNS-RR flux string on $\mathrm{AdS}_3\times S^3\times M_4$, with M_4 a four-dimensional compact manifold required to complete the model to a type IIB superstring (see [11] for reviews). Since the integrable structure relevant for our purposes is encoded in the $\mathrm{AdS}_3\times S^3$ sector, we will restrict attention to its Green-Schwarz formulation as a $\mathbb{Z}_4$-graded supercoset sigma model [12] (see also [24]) while keeping M_4 implicit throughout.

1. Supercoset

Concretely, the world sheet degrees of freedom in this sector are described by $\hat{F}/F^{(0)}$ with

$$\hat{F}=G\times G,\qquad G=PSU(1,1|2), \tag{B1}$$

and $F^{(0)}$ the bosonic diagonal subgroup of $\hat{F}$, i.e. $(h,h)\in F^{(0)}$ for $h\in SU(1,1)\times SU(2)$. The Lie superalgebra $\hat{\mathfrak{f}}=\mathfrak{g}\oplus\mathfrak{g}$, with $\mathfrak{g}=\mathfrak{psu}(1,1|2)$, of $\hat{F}$ admits a $\mathbb{Z}_4$ grading

$$\hat{\mathfrak{f}}=\mathfrak{f}^{(0)}\oplus\mathfrak{f}^{(1)}\oplus\mathfrak{f}^{(2)}\oplus\mathfrak{f}^{(3)}, \tag{B2}$$

with $\mathfrak{f}^{(0)}$ the Lie algebra corresponding to $F^{(0)}$. The bosonic subalgebra $\mathfrak{su}(1,1)\oplus\mathfrak{su}(2)$ is generated by

$$\begin{aligned}&T_{1,2}=\frac{1}{\sqrt{2}}(\sigma_{1,3}\oplus 0),\qquad T_3=\frac{i}{\sqrt{2}}(\sigma_2\oplus 0),\\&T_{4,5,6}=\frac{1}{\sqrt{2}}(0\oplus\sigma_{1,2,3}),\end{aligned} \tag{B3}$$

with $\sigma_{1,2,3}$ the Pauli matrices. For a complete superalgebra realization of $\mathfrak{g}$ and $\hat{\mathfrak{f}}$, see e.g. Appendix C of [25].

Let $f(\tau,\sigma)\in\hat{F}=G\times G$ be a supercoset representative with $\mathfrak{J}=f^{-1}\mathrm{d}f\in\hat{\mathfrak{f}}$ the left-invariant current. We will work in a block-diagonal matrix representation, where $f=\mathrm{diag}(g,\tilde{g})$ and $\mathfrak{J}=\mathrm{diag}(\mathcal{J},\tilde{\mathcal{J}})$ with $g,\tilde{g}\in G$, and implement the $\mathbb{Z}_4$ grading as

$$\begin{aligned}&\mathfrak{J}^{(0)}=\begin{pmatrix}\mathcal{J}^{(0)}&\\&\mathcal{J}^{(0)}\end{pmatrix},\qquad\mathfrak{J}^{(1)}=\begin{pmatrix}\mathcal{J}^{(1)}&\\&-i\mathcal{J}^{(1)}\end{pmatrix},\\&\mathfrak{J}^{(2)}=\begin{pmatrix}\mathcal{J}^{(2)}&\\&-\mathcal{J}^{(2)}\end{pmatrix},\qquad\mathfrak{J}^{(3)}=\begin{pmatrix}\mathcal{J}^{(3)}&\\&i\mathcal{J}^{(3)}\end{pmatrix},\end{aligned} \tag{B4}$$

where $\mathfrak{J}^{(i)}\in\mathfrak{f}^{(i)}$ and $\mathcal{J}^{(0)}=\frac{1}{2}P_{\mathrm{e}}(\mathcal{J}+\tilde{\mathcal{J}})$, $\mathcal{J}^{(2)}=\frac{1}{2}P_{\mathrm{e}}(\mathcal{J}-\tilde{\mathcal{J}})$, $\mathcal{J}^{(1)}=\frac{1}{2}P_{\mathrm{o}}(\mathcal{J}+i\tilde{\mathcal{J}})$, $\mathcal{J}^{(3)}=\frac{1}{2}P_{\mathrm{o}}(\mathcal{J}-i\tilde{\mathcal{J}})$, with P_{e} (P_{o}) projectors on the Grassmann-even (odd) parts of $\mathfrak{g}$. We further define two inequivalent bilinear forms on $\hat{\mathfrak{f}}$ as

$$\mathrm{STr}(\mathfrak{J})=\mathrm{Str}(\mathcal{J})+\mathrm{Str}(\tilde{\mathcal{J}}),\qquad\widetilde{\mathrm{STr}}(\mathfrak{J})=\mathrm{STr}(\mathfrak{W}\mathfrak{J}), \tag{B5}$$

with Str the supertrace of $\mathfrak{g}$ and $\mathfrak{W}=\mathrm{diag}(\mathbb{1},-\mathbb{1})$.

2. Action and integrability

The bulk sigma-model action with mixed-flux contributions is [12,24]

$$S_{\rm cl} = \frac{1}{2}\int_\Sigma \mathrm{STr}(\mathfrak{F}^{(2)} \wedge \star\mathfrak{F}^{(2)} + q_{\rm R}\mathfrak{F}^{(1)} \wedge \mathfrak{F}^{(3)}) + q_{\rm NS}\int_{\mathcal{B}} \widetilde{\mathrm{STr}}\Big(\frac{2}{3}\mathfrak{F}^{(2)} \wedge \mathfrak{F}^{(2)} \wedge \mathfrak{F}^{(2)} + \mathfrak{F}^{(1)} \wedge \mathfrak{F}^{(3)} \wedge \mathfrak{F}^{(2)} + \mathfrak{F}^{(3)} \wedge \mathfrak{F}^{(1)} \wedge \mathfrak{F}^{(2)}\Big), \tag{B6}$$

where $q_{\rm R}$ and $q_{\rm NS}$, respectively, quantify the RR and NSNS flux couplings and $\mathcal{B}$ is a three-dimensional manifold such that $\Sigma = \partial\mathcal{B}$. Throughout we will consider the branch $0 \le q_{\rm NS} \le 1$. The full bosonic $\mathrm{AdS}_3 \times S^3 \times T^4$ world sheet theory is obtained by reinstating the string tension $T = (2\pi\alpha')^{-1}$ and supplementing the above action with four flat bosonic fields, which we keep implicit. The pure NSNS background $q_{\rm R} = 0$, $q_{\rm NS} = 1$ corresponds to the WZW model, i.e. in the bosonic subsector the $SL(2,R) \times SU(2)$ WZW description of strings on $\mathrm{AdS}_3 \times S^3$ [14]. The usual WZ level k is related to the above parameters by $2\pi T q_{\rm NS} = k$ with $k \in \mathbb{Z}$ due to the compact $SU(2)$ isometry subgroup.

Classical integrability, κ-symmetry and conformal invariance to one loop in α' require the following relation between the flux parameters [12]:

$$q_{\rm R}^2 + q_{\rm NS}^2 = 1. \tag{B7}$$

The flat Lax connection then takes the form

$$\mathcal{L}(\mathsf{x}) = \mathcal{J}^{(0)} + q_{\rm R}\frac{\mathsf{x}^2+1}{\mathsf{x}^2-1}\mathcal{J}^{(2)} + \Big(q_{\rm NS} - \frac{2q_{\rm R}\mathsf{x}}{\mathsf{x}^2-1}\Big)\star\mathcal{J}^{(2)} + \Big(\mathsf{x} + \frac{q_{\rm R}}{1-q_{\rm NS}}\Big)\sqrt{\frac{q_{\rm R}(1-q_{\rm NS})}{\mathsf{x}^2-1}}\mathcal{J}^{(1)} + \Big(\mathsf{x} - \frac{q_{\rm R}}{1+q_{\rm NS}}\Big)\sqrt{\frac{q_{\rm R}(1+q_{\rm NS})}{\mathsf{x}^2-1}}\mathcal{J}^{(3)}. \tag{B8}$$

At the pure RR point $q_{\rm R} = 1$, $q_{\rm NS} = 0$ this Lax corresponds to [26], while at the pure NSNS or WZW point it degenerates and does not evidently capture the full dynamics. In that case one can instead work with holomorphic current algebras rather than Lax connections.

3. Parametrization and background

The sigma model (A1) is invariant under local, right-acting, $F^{(0)}$ gauge transformations. We can therefore take

$$g = g_{\rm F}g_{\rm B}, \qquad \tilde{g} = \tilde{g}_{\rm F}\tilde{g}_{\rm B} = \tilde{g}_{\rm F}, \tag{B9}$$

in $f = \mathrm{diag}(g, \tilde{g})$, where $g_{\rm B} \in SU(1,1) \times SU(2)$, $\tilde{g}_{\rm B} = \mathbb{1}$ and $g_{\rm F}, \tilde{g}_{\rm F} \in \exp(P_{\rm o}\mathfrak{g})$. In this case, we have

$$\mathcal{J} = \mathcal{J}^{\rm B} + g_{\rm B}^{-1}\mathcal{J}^{\rm F}g_{\rm B}, \qquad \tilde{\mathcal{J}} = \tilde{\mathcal{J}}^{\rm F}, \tag{B10}$$

with $\mathcal{J}^{\rm B} = g_{\rm B}^{-1}\mathrm{d}g_{\rm B} \in \mathfrak{su}(1,1) \oplus \mathfrak{su}(2)$, $\mathcal{J}^{\rm F} = g_{\rm F}^{-1}\mathrm{d}g_{\rm F} \in \mathfrak{g}$ and $\tilde{\mathcal{J}}^{\rm F} = \tilde{g}_{\rm F}^{-1}\mathrm{d}\tilde{g}_{\rm F} \in \mathfrak{g}$. In the main text, we furthermore focus on the bosonic sector of $\mathfrak{psu}(1,1|2)$ and thus take $g_{\rm F}, \tilde{g}_{\rm F} = \mathbb{1}$. Altogether, this implies the simplifications

$$\mathcal{J}^{(0)}|_{g_{\rm B}} = \mathcal{J}^{(2)}|_{g_{\rm B}} = \frac{1}{2}\mathcal{J}^{\rm B}, \qquad \mathcal{J}^{(1,3)}|_{g_{\rm B}} = 0. \tag{B11}$$

We will then parametrize the bosonic element $g_B = \mathrm{diag}(g_{SU(1,1)}|g_{SU(2)})$ as

$$g_{SU(1,1)} = \sinh\psi\sigma_3 + \cosh\psi e^{\frac{it}{2}\sigma_2}e^{\omega\sigma_1}e^{\frac{it}{2}\sigma_2},$$
$$g_{SU(2)} = e^{-\frac{i\gamma}{2}\sigma_3}e^{\frac{i\beta}{2}\sigma_1}e^{i\alpha\sigma_3}e^{-\frac{i\beta}{2}\sigma_1}e^{\frac{i\gamma}{2}\sigma_3}, \tag{B12}$$

where $\psi, \omega, t \in \mathbb{R}^3$ are global coordinates on AdS_3, with $\{\omega, t\}$ parametrizing AdS_2 fibered over ψ, while the coordinates $\beta \in [0,\pi)$ and $\gamma \in [0, 2\pi)$ parametrize an S^2 fibered over the interval $\alpha \in [0,\pi)$, forming the three-sphere S^3. The metric and NSNS three-form $H = \mathrm{d}B$ can then be expressed as

$$\mathrm{d}s^2 = \mathrm{d}\psi^2 + \cosh^2\psi(-\cosh^2\omega\mathrm{d}t^2 + \mathrm{d}\omega^2) + \mathrm{d}\alpha^2 + \sin^2\alpha(\mathrm{d}\beta^2 + \sin^2\beta\mathrm{d}\gamma^2),$$
$$H = 2q_{\rm NS}\cosh^2\psi\cosh\omega\mathrm{d}\psi \wedge \mathrm{d}t \wedge \mathrm{d}\omega + 2q_{\rm NS}\sin^2\alpha\sin\beta\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\gamma. \tag{B13}$$

When including the fermionic sector, the RR three-form flux is $F_3 = e^{-\Phi_0}q_{\rm R}(H/q_{\rm NS})$ for constant dilaton $\Phi(X) = \Phi_0$.

4. Outer automorphisms of $SU(1,1) \times SU(2)$

Automorphisms W of $\mathfrak{su}(1,1) \oplus \mathfrak{su}(2)$ which preserve the ad-invariant metric form the group $SO(2,1) \times SO(3)$. Explicit matrix realizations can be defined as $W(y) = wyw^{-1} = y^A W_A{}^B T_B$ for $y = y^A T_A \in \mathfrak{g}$. The elements relevant to the boundary integrability discussion of the main text are those W which are involutive, $W^2 = \mathbb{1}$. Up to conjugation (and combinations), the ones that do not mix $\mathfrak{su}(1,1)$ and $\mathfrak{su}(2)$ and that (i) are inner are

$$W_{inn} = \begin{cases} \mathbb{1}_3 \oplus \mathbb{1}_3 \\ \mathrm{diag}(-1,-1,1) \oplus \mathbb{1}_3 \\ \mathbb{1}_3 \oplus \mathrm{diag}(-1,-1,1) \end{cases} \in SO^+(2,1) \times SO(3), \tag{B14}$$

or (ii) are outer are

$$W_{\rm out.} = \mathrm{diag}(-1,1,-1) \oplus \mathbb{1}_3 \in SO^-(2,1) \times SO(3). \tag{B15}$$

The latter is realized as $W_{\rm out.}(X) = w_{\rm o}Xw_{\rm o}^{-1}$ with

$$w_{\rm o} = \mathrm{diag}(1,-1) \oplus \mathbb{1}_2 \notin SU(1,1) \times SU(2). \tag{B16}$$

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