

Matching between collinear twist-3 and TMD fragmentation function contributions to polarized hyperon production in SIDIS

Riku Ikarashi,¹ Yuji Koike^{1,2}, and Shinsuke Yoshida^{3,4}

¹*Graduate School of Science and Technology, Niigata University, Ikarashi 2-no-cho, Niigata 950-2181, Japan*

²*Department of Physics, Niigata University, Ikarashi 2-no-cho, Niigata 950-2181, Japan*

³*State Key Laboratory of Nuclear Physics and Technology, Institute of Quantum Matter, South China Normal University, Guangzhou 510006, China*

⁴*Guangdong Basic Research Center of Excellence for Structure and Fundamental Interactions of Matter, Guangdong Provincial Key Laboratory of Nuclear Science, Guangzhou 510006, China*



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We investigate the consistency between the collinear twist-3 factorization and the transverse-momentum-dependent (TMD) factorization for the transversely polarized hyperon production in semi-inclusive deep inelastic scattering, $ep \rightarrow e\Lambda^\uparrow X$. In particular, we focus on the contributions from the twist-3 fragmentation functions (FFs) and the TMD polarizing FF in the region of the hyperon's intermediate transverse momentum P_T , $\Lambda_{\text{QCD}} \ll P_T \ll Q$, where both frameworks are valid. In this region the polarizing FF can be expressed in terms of the twist-3 FFs including the purely gluonic ones, and the resulting TMD factorization formula in the Ji-Ma-Yuan scheme for $ep \rightarrow e\Lambda^\uparrow X$ agrees with the small- P_T limit of the corresponding twist-3 cross section. This matching of the two calculations indicates that the two frameworks describe the same effect in quantum chromodynamics (QCD) and provide complementary frameworks for the process in different kinematic regions.

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I. INTRODUCTION

Understanding the mechanism for the transverse polarization of hyperons produced in unpolarized collisions,¹ such as $pp \rightarrow \Lambda^\uparrow X$, $ep \rightarrow (e)\Lambda^\uparrow X$ and $e^+e^- \rightarrow \Lambda^\uparrow X$ etc., has been one of the central issues in high-energy QCD. This hyperon polarization, as well as $p^\uparrow p \rightarrow hX$, $ep^\uparrow \rightarrow (e)hX$ ($h = \pi, \gamma$, jet etc.), are the typical examples of the transverse single spin asymmetry (SSA), and have been studied at length in the literature. Collinear twist-2 factorization in perturbative QCD, which has been successful in describing many unpolarized collision processes, cannot accommodate large SSAs [1]. It has been understood by now that effects of particular multiparton correlations and/or intrinsic transverse momentum of partons either in the initial hadrons or in the fragmenting process can cause large SSAs. Systematic methods of handling those effects have been formulated in the framework of the collinear twist-3 factorization [2–14] and the transverse

momentum dependent (TMD) factorization [15,16]. Collinear twist-3 factorization is designed to describe SSAs in the region of the large transverse momentum of the final hadron $\Lambda_{\text{QCD}} \ll P_T \lesssim Q$ where Q is a typical hard scale in the process, while the TMD factorization describes those in the small- P_T region $\Lambda_{\text{QCD}} \lesssim P_T \ll Q$. Relation between these two frameworks has also been studied for some processes, and it has been shown that they match in the region of the intermediate transverse momentum ($\Lambda_{\text{QCD}} \ll P_T \ll Q$) for $ep^\uparrow \rightarrow e\pi X$ and Drell-Yan processes etc. [17–22]. Although the twist-3 calculation so far has been mostly limited to the leading order (LO) with respect to the QCD coupling due to the technical complexities, there have been some studies to include next-to-leading order (NLO) corrections for $ep^\uparrow \rightarrow e\pi X$ [23–27], Drell-Yan process [28,29], and $e^+e^- \rightarrow \Lambda^\uparrow X$ [30] etc. More recently, a complete NLO twist-3 distribution function (DF) contribution to the fully differential cross section for $ep^\uparrow \rightarrow hX$ ($h = \pi$, jet) has been derived and its factorization property has been proven [31,32].

In this paper we investigate the consistency between the collinear twist-3 factorization and the TMD factorization at LO in the intermediate transverse momentum region for the hyperon polarization in semi-inclusive deep inelastic scattering (SIDIS) $ep \rightarrow e\Lambda^\uparrow X$. In the collinear twist-3

¹In this paper we collectively denote spin-1/2 hyperon as Λ .

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factorization, two effects participate in this process: One is (i) the twist-3 unpolarized DF in the initial proton convoluted with the twist-2 transversity fragmentation function (FF) for the final hyperon and the corresponding partonic hard cross section, and the other is (ii) the twist-3 FFs for the final hyperon convoluted with the unpolarized DF in the proton and the partonic hard cross section. In (ii), two contributions exist; (a) the twist-3 quark FFs and (b) the twist-3 gluon FFs. The complete LO twist-3 cross section formula for $ep \rightarrow e\Lambda^\dagger X$ has been derived for (i) in [21,33], for (ii)(a) in [33] and for (ii)(b) in [34]. In the TMD factorization, corresponding to the above (i) and (ii), there are two contributions: (I) the Boer-Mulders function in the initial proton combined with the transversity FF for the hyperon [35] and (II) the polarizing FF combined with the unpolarized DF in the proton contribute to the hyperon polarization [36]. So far, it has been shown that (i) and (I) match at intermediate transverse momentum $\Lambda_{\text{QCD}} \ll P_T \ll Q$ [21], while no such relation has been established between (ii) and (II).

The purpose of this paper is to investigate the relation between (ii) and (II) for $ep \rightarrow e\Lambda^\dagger X$. So far such study on the matching between the two frameworks has been mostly for the distribution side, and the study on the FF contribution to SSA has been performed only for the process $ep^\dagger \rightarrow e\pi X$ in [22], which shows the twist-3 FF contribution matches with that from the Collins function [37]. This paper extends the study to $ep \rightarrow e\Lambda^\dagger X$. For (ii) and (II), extra complication occurs compared to $ep^\dagger \rightarrow e\pi X$: Owing to the chiral-even nature of the relevant twist-3 FFs and the polarizing FF, a purely gluonic channel [above (ii) (b)] enters the calculation, for which the complete twist-3 cross section was derived only recently [34]. Here we will show that the TMD polarizing FF contribution can be expressed in terms of the collinear twist-3 quark and gluon FFs in the region of large intrinsic transverse momentum, and the resulting TMD factorization formula agrees with that obtained from the small- P_T limit of the collinear twist-3 cross section.

This paper is organized as follows: In Sec. II, we summarize the kinematics for $ep \rightarrow e\Lambda^\dagger X$. In Sec. III, we summarize the collinear twist-3 FFs and the TMD polarizing FF relevant to the present study. In Sec. IV, we recall the structure of the twist-3 FF contribution for $ep \rightarrow e\Lambda^\dagger X$ from [33,34] and derive its small- P_T limit. In Sec. V, we present a perturbative calculation of the TMD polarizing FF in the region of the large intrinsic transverse momentum, and show that the two frameworks indeed match at $\Lambda_{\text{QCD}} \ll P_T \ll Q$. Section VI is devoted to a brief summary.

II. KINEMATICS FOR $ep \rightarrow e\Lambda^\dagger X$

Here we briefly summarize the kinematics for the process following [33,34],

$$e(\ell) + p(p) \rightarrow e(\ell') + \Lambda^\dagger(P_h, S_\perp) + X, \quad (1)$$

where ℓ, ℓ', p and P_h are the momenta of each particle and $S_\perp$ is the transverse spin vector for Λ . With the virtual photon's momentum $q = \ell - \ell'$, we introduce the five Lorentz invariants as

$$S_{ep} \equiv (p + \ell)^2 \simeq 2p \cdot \ell, \quad Q^2 \equiv -q^2, \\ x_{bj} \equiv \frac{Q^2}{2p \cdot q}, \quad z_f \equiv \frac{p \cdot P_h}{p \cdot q}, \quad q_T \equiv \sqrt{-q_t^2}, \quad (2)$$

where

$$q_t^\mu \equiv q^\mu - \frac{P_h \cdot q}{p \cdot P_h} p^\mu - \frac{p \cdot q}{p \cdot P_h} P_h^\mu \quad (3)$$

is a spacelike momentum satisfying $q_t \cdot p = q_t \cdot P_h = 0$. As in [33,34], we work in the hadron frame [38] (See Fig. 1), where p^μ and q^μ are collinear and take the following form:

$$p^\mu = \frac{Q}{2x_{bj}}(1, 0, 0, 1), \quad (4)$$

$$q^\mu = (0, 0, 0, -Q). \quad (5)$$

Defining the azimuthal angles for the hadron plane and the lepton plane as χ and ϕ , respectively, as shown in Fig. 1, P_h^μ and ℓ^μ can be written as

$$P_h^\mu = \frac{z_f Q}{2} \left(1 + \frac{q_T^2}{Q^2}, \frac{2q_T}{Q} \cos \chi, \frac{2q_T}{Q} \sin \chi, -1 + \frac{q_T^2}{Q^2} \right), \quad (6)$$

$$\ell^\mu = \frac{Q}{2} (\cosh \psi, \sinh \psi \cos \phi, \sinh \psi \sin \phi, -1), \quad (7)$$

where $\cosh \psi \equiv \frac{2x_{bj}S_{ep}}{Q^2} - 1$. With this parametrization, the magnitude of the transverse momentum of the hyperon $P_{hT} \equiv P_T$ is given by

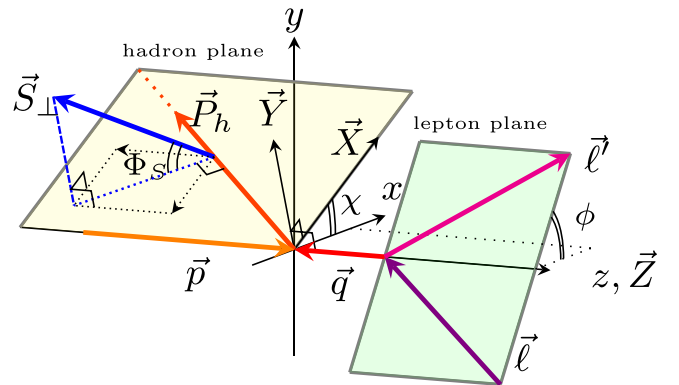


FIG. 1. Hadron frame.

$$P_T = z_f q_T. \quad (8)$$

For the calculation of the cross section, we introduce four axes by

$$\begin{aligned} T^\mu &\equiv \frac{1}{Q}(q^\mu + 2x_{bj}p^\mu) = (1, 0, 0, 0), \\ Z^\mu &\equiv -\frac{q^\mu}{Q} = (0, 0, 0, 1), \\ X^\mu &\equiv \frac{1}{q_T} \left[\frac{P_h^\mu}{z_f} - q^\mu - \left(1 + \frac{q_T^2}{Q^2}\right) x_{bj} p^\mu \right] = (0, \cos\chi, \sin\chi, 0), \\ Y^\mu &\equiv \epsilon^{\mu\nu\rho\sigma} Z_\nu T_\rho X_\sigma = (0, -\sin\chi, \cos\chi, 0), \end{aligned} \quad (9)$$

where the actual form in the hadron frame is given after the last equality in each equation. The final hyperon resides in the XZ -plane and the transverse spin vector of the hyperon can be written as

$$S_\perp^\mu = \cos\theta \cos\Phi_S X^\mu + \sin\Phi_S Y^\mu - \sin\theta \cos\Phi_S Z^\mu, \quad (10)$$

where θ is the polar angle of $\vec{P}_h$ as measured from the Z -axis and Φ_S is the azimuthal angle of $\vec{S}_\perp$ around $\vec{P}_h$ as measured from the XZ -plane. From (6), the polar angle θ is written as

$$\cos\theta = \frac{P_{hz}}{|\vec{P}_h|} = \frac{q_T^2 - Q^2}{q_T^2 + Q^2}, \quad (11)$$

$$\sin\theta = \frac{P_T}{|\vec{P}_h|} = \frac{2q_T Q}{q_T^2 + Q^2}. \quad (12)$$

With the kinematical variables defined above, the differential cross section takes the following form:

$$\frac{d^6\sigma}{dx_{bj}dQ^2dz_f dq_T^2 d\phi d\chi} = \frac{\alpha_{em}^2}{128\pi^4 x_{bj}^2 S_{ep}^2 Q^2} \times z_f L^{\rho\sigma}(\ell, \ell') W_{\rho\sigma}(p, q, P_h), \quad (13)$$

where $\alpha_{em} = e^2/(4\pi)$ is the QED coupling constant, $L^{\rho\sigma} = 2(\ell^\rho \ell'^\sigma + \ell^\sigma \ell'^\rho) - Q^2 g^{\rho\sigma}$ is the leptonic tensor and $W_{\rho\sigma}$ is the hadronic tensor which can be expanded in terms of the six tensors [38] $\mathcal{V}_k^{\rho\sigma}$ ($k = 1, \dots, 4, 8, 9$) defined by

$$\begin{aligned} \mathcal{V}_1^{\mu\nu} &= X^\mu X^\nu + Y^\mu Y^\nu, & \mathcal{V}_2^{\mu\nu} &= g^{\mu\nu} + Z^\mu Z^\nu, \\ \mathcal{V}_3^{\mu\nu} &= T^\mu X^\nu + X^\mu T^\nu, \\ \mathcal{V}_4^{\mu\nu} &= X^\mu X^\nu - Y^\mu Y^\nu, & \mathcal{V}_8^{\mu\nu} &= T^\mu Y^\nu + Y^\mu T^\nu, \\ \mathcal{V}_9^{\mu\nu} &= X^\mu Y^\nu + Y^\mu X^\nu. \end{aligned} \quad (14)$$

Although there are two azimuthal angles, ϕ and χ , the cross section depends on the relative angle $\varphi \equiv \phi - \chi$ only, and

the φ -dependence of $L^{\rho\sigma} W_{\rho\sigma}$ in (13) is carried by the coefficients $\mathcal{A}_k(\varphi) \equiv L_{\mu\nu} \mathcal{V}_k^{\mu\nu} / Q^2$ which are given by

$$\begin{aligned} \mathcal{A}_1(\varphi) &= 1 + \cosh^2\psi, & \mathcal{A}_2(\varphi) &= -2, \\ \mathcal{A}_3(\varphi) &= -\cos\varphi \sinh 2\psi, \\ \mathcal{A}_4(\varphi) &= \cos 2\varphi \sinh^2\psi, & \mathcal{A}_8(\varphi) &= -\sin\varphi \sinh 2\psi, \\ \mathcal{A}_9(\varphi) &= \sin 2\varphi \sinh^2\psi. \end{aligned} \quad (15)$$

The cross section (13) can be expressed in terms of S_{ep} , Q^2 , x_{bj} , z_f , q_T^2 , φ and Φ_S .

III. COLLINEAR TWIST-3 FFS AND TMD FF FOR $ep \rightarrow e\Lambda^\dagger X$

Here we summarize the FFs contributing to $ep \rightarrow e\Lambda^\dagger X$ in the collinear twist-3 factorization and the TMD factorization.

A. Collinear twist-3 FFs

1. Twist-3 quark FFs

Three kinds of twist-3 quark FFs [39] contribute to $ep \rightarrow e\Lambda^\dagger X$.

(1) Intrinsic FFs

They are defined from the light cone correlation function of the quark fields as

$$\begin{aligned} \Delta(z) &= \frac{1}{N} \sum_X \int \frac{d\lambda}{2\pi} e^{-i\lambda} \langle 0 | \psi(0) | \Lambda^\dagger X \rangle \langle \Lambda^\dagger X | \bar{\psi}(\lambda w) | 0 \rangle \\ &= \gamma_5 \mathcal{S}_\perp P_h \frac{H_1(z)}{z} + M_h \epsilon^{\alpha S_\perp w P_h} \gamma_\alpha \frac{D_T(z)}{z} \\ &\quad + M_h \gamma_5 \mathcal{S}_\perp \frac{G_T(z)}{z} + \dots, \end{aligned} \quad (16)$$

where z is the longitudinal momentum fraction transferred to the hyperon, and N is the number of color SU(N), and $|\Lambda^\dagger\rangle$ is the state vector for the hyperon with mass M_h , momentum P_h ($P_h^2 = M_h^2$), and the transverse spin vector $S_\perp$ ($S_\perp^2 = -1$). In the twist-3 accuracy, P_h can be treated as lightlike and w is the another lightlike vector satisfying $P_h \cdot w = 1$, i.e., for $P_h = (|\vec{P}_h|, \vec{P}_h)$, $w = \frac{1}{2|\vec{P}_h|^2} (|\vec{P}_h|, -\vec{P}_h)$. Here and below, we use the notation $\epsilon^{\alpha S_\perp w P_h} \equiv \epsilon^{\alpha\mu\nu\lambda} S_{\perp\mu} w_\nu P_{h\lambda}$ and the gauge-link operators which restore the color-gauge invariance of FFs are implicit. $H_1(z)$ is the twist-2 transversity fragmentation contributing to $ep \rightarrow e\Lambda^\dagger X$ and $pp \rightarrow \Lambda^\dagger X$ combined with the twist-3 unpolarized distribution in the nucleon $E_F(x_1, x_2)$ [33,40–42], which we shall not discuss here. $D_T(z)$ and $G_T(z)$ are the *intrinsic* twist-3 quark fragmentation function, in particular, $D_T(z)$ is T-odd and contributes to

$ep \rightarrow e\Lambda^\uparrow X$. In (16) and below, the flavor indices for the quark FFs are suppressed for simplicity.

(2) Kinematical FFs

Twist-3 kinematical quark fragmentation functions are defined from the correlation functions containing the transverse derivative of the fields

$$\begin{aligned}\Delta_\delta^\alpha(z) &= \frac{1}{N} \sum_X \int \frac{d\lambda}{2\pi} e^{-i\frac{\lambda}{z}} \langle 0 | \psi(0) | \Lambda^\uparrow X \rangle \\ &\quad \times \langle \Lambda^\uparrow X | \bar{\psi}(\lambda w) | 0 \rangle \overleftrightarrow{\partial}^\alpha \\ &= -iM_h e^{\alpha S_\perp w P_h} P_h \frac{D_{1T}^{\perp(1)}(z)}{z} \\ &\quad + iM_h S_\perp^\alpha \gamma_5 P_h \frac{G_{1T}^{\perp(1)}(z)}{z} + \dots, \quad (17)\end{aligned}$$

where $\bar{\psi}(\lambda w) \overleftrightarrow{\partial}^\alpha \equiv \lim_{\xi \rightarrow 0} (d/d\xi_\alpha) \bar{\psi}(\lambda w + \xi)$. $D_{1T}^{\perp(1)}(z)$ and $G_{1T}^{\perp(1)}(z)$ are called *kinematical* fragmentation functions, with its superscript (1) denoting the k_T^2/M_h^2 -moment of the TMD fragmentation functions [36]. (See Eq. (32) below.) $G_{1T}^{\perp(1)}(z)$ is T-even, while $D_{1T}^{\perp(1)}(z)$ is T-odd and contributes to $ep \rightarrow e\Lambda^\uparrow X$.

(3) Dynamical FFs

They are defined as

$$\begin{aligned}\Delta_F^\alpha(z, z_1) &= \frac{1}{N} \sum_X \iint \frac{d\lambda d\mu}{(2\pi)^2} e^{-i\frac{\lambda}{z_1}} e^{-i\mu(\frac{1}{z_2} - \frac{1}{z_1})} \\ &\quad \times \langle 0 | \psi(0) | \Lambda^\uparrow X \rangle \langle \Lambda^\uparrow X | \bar{\psi}(\lambda w) g F^{\alpha w}(\mu w) | 0 \rangle \\ &= M_h e^{\alpha S_\perp w P_h} P_h \frac{\hat{D}_{FT}^*(z, z_1)}{z} \\ &\quad - iM_h S_\perp^\alpha \gamma_5 P_h \frac{\hat{G}_{FT}^*(z, z_1)}{z} + \dots, \quad (18)\end{aligned}$$

$\hat{D}_{FT}(z, z_1)$ and $\hat{G}_{FT}(z, z_1)$ are complex functions with the asterisk * denoting the complex conjugate. Their real parts are T-even, while their imaginary parts are T-odd and contribute to $ep \rightarrow e\Lambda^\uparrow X$.

2. Twist-3 gluon FFs

Twist-3 gluon fragmentation functions can be also classified into three types [30,43,44].

(1) Intrinsic FFs

Intrinsic fragmentation functions are defined from the light cone correlation functions of the field strength $F_a^{\mu\nu}$

$$\begin{aligned}\hat{\Gamma}^{\alpha\beta}(z) &= \frac{1}{N^2 - 1} \sum_X \int \frac{d\lambda}{2\pi} e^{-i\frac{\lambda}{z}} \langle 0 | F_a^{w\beta}(0) | \Lambda^\uparrow X \rangle \langle \Lambda^\uparrow X | F_a^{w\alpha}(\lambda w) | 0 \rangle \\ &= iM_h e^{\alpha\beta w S_\perp} \Delta \hat{G}_{3T}(z) + M_h e^{P_h w S_\perp \{\alpha\beta\}} \Delta \hat{G}_{3\bar{T}}(z) + \dots, \quad (19)\end{aligned}$$

where $g_\perp^{\alpha\beta} = g^{\alpha\beta} - P_h^\alpha w^\beta - P_h^\beta w^\alpha$ and $\{\alpha\beta\}$ indicates the symmetrization of the Lorentz indices. $\Delta \hat{G}_{3T}(z)$ and $\Delta \hat{G}_{3\bar{T}}(z)$ are the intrinsic twist-3 fragmentation functions. $\Delta \hat{G}_{3\bar{T}}(z)$ is T-odd and contributes to the hyperon polarization.

(2) Kinematical FFs

Kinematical fragmentation functions are defined from the correlation functions containing the derivative of the field strengths

$$\begin{aligned}\hat{\Gamma}_\partial^{\alpha\beta\gamma}(z) &= \frac{1}{N^2 - 1} \sum_X \int \frac{d\lambda}{2\pi} e^{-i\frac{\lambda}{z}} \langle 0 | F_a^{w\beta}(0) | \Lambda^\uparrow X \rangle \langle \Lambda^\uparrow X | F_a^{w\alpha}(\lambda w) | 0 \rangle \overleftrightarrow{\partial}^\gamma \\ &= -i\frac{M_h}{2} g_\perp^{\alpha\beta} e^{P_h w S_\perp \gamma} \hat{G}_T^{(1)}(z) + \frac{M_h}{2} e^{P_h w \alpha\beta} S_\perp^\gamma \Delta \hat{G}_T^{(1)}(z) - i\frac{M_h}{8} (e^{P_h w S_\perp \{\alpha\beta\} \gamma} + e^{P_h w \gamma \{\alpha\beta\}}) \Delta \hat{H}_T^{(1)}(z) + \dots, \quad (20)\end{aligned}$$

where $\Delta \hat{G}_T^{(1)}(z)$ is T-even, while $\hat{G}_T^{(1)}(z)$ and $\Delta \hat{H}_T^{(1)}(z)$ are T-odd contributing to the hyperon polarization, and can be written as the k_T^2/M_h^2 -moment of the TMD gluon fragmentation functions [44].

(3) Dynamical FFs

Dynamical gluon FFs are defined from the correlation functions of three field-strengths. Two types of FFs exist, depending on the contraction of the color indices by two structure constants of color SU(N) $-if^{abc}$, d^{abc}

$$\begin{aligned}\hat{\Gamma}_{FA}^{\alpha\beta\gamma}\left(\frac{1}{z_1}, \frac{1}{z_2}\right) &= \frac{-if^{abc}}{N^2 - 1} \sum_X \iint \frac{d\lambda d\mu}{(2\pi)^2} e^{-i\frac{\lambda}{z_1}} e^{-i\mu(\frac{1}{z_2} - \frac{1}{z_1})} \langle 0 | F_b^{w\beta}(0) | \Lambda^\uparrow X \rangle \langle \Lambda^\uparrow X | F_a^{w\alpha}(\lambda w) g F_c^{w\gamma}(\mu w) | 0 \rangle \\ &= -M_h \left(\hat{N}_1 \left(\frac{1}{z_1}, \frac{1}{z_2} \right) g_\perp^{\alpha\gamma} e^{P_h w S_\perp \beta} + \hat{N}_2 \left(\frac{1}{z_1}, \frac{1}{z_2} \right) g_\perp^{\beta\gamma} e^{P_h w S_\perp \alpha} - \hat{N}_2 \left(\frac{1}{z_2} - \frac{1}{z_1}, \frac{1}{z_2} \right) g_\perp^{\alpha\beta} e^{P_h w S_\perp \gamma} \right), \quad (21)\end{aligned}$$

and

$$\begin{aligned} \hat{\Gamma}_{FS}^{\alpha\beta\gamma}\left(\frac{1}{z_1}, \frac{1}{z_2}\right) &= \frac{d_{abc}}{N^2 - 1} \sum_X \iint \frac{d\lambda d\mu}{(2\pi)^2} e^{-i\frac{\lambda}{z_1}} e^{-i\mu(\frac{1}{z_2} - \frac{1}{z_1})} \langle 0 | F_b^{w\beta}(0) | \Lambda^\uparrow X \rangle \langle \Lambda^\uparrow X | F_a^{w\alpha}(\lambda w) F_c^{w\gamma}(\mu w) | 0 \rangle \\ &= -M_h \left(\hat{O}_1\left(\frac{1}{z_1}, \frac{1}{z_2}\right) g_\perp^{\alpha\gamma} e^{P_h w S_\perp \beta} + \hat{O}_2\left(\frac{1}{z_1}, \frac{1}{z_2}\right) g_\perp^{\beta\gamma} e^{P_h w S_\perp \alpha} + \hat{O}_2\left(\frac{1}{z_2} - \frac{1}{z_1}, \frac{1}{z_2}\right) g_\perp^{\alpha\beta} e^{P_h w S_\perp \gamma} \right), \end{aligned} \quad (22)$$

where the support of each function is $1/z_2 > 1$ and $1/z_2 > 1/z_1 > 0$. Four functions, $\hat{N}_{1,2}(\frac{1}{z_1}, \frac{1}{z_2})$ and $\hat{O}_{1,2}(\frac{1}{z_1}, \frac{1}{z_2})$, are complex functions and their imaginary parts are T-odd, contributing to SSAs, and satisfies the following symmetry properties:

$$\hat{N}_1\left(\frac{1}{z_1}, \frac{1}{z_2}\right) = -\hat{N}_1\left(\frac{1}{z_2} - \frac{1}{z_1}, \frac{1}{z_2}\right), \quad \hat{O}_1\left(\frac{1}{z_1}, \frac{1}{z_2}\right) = \hat{O}_1\left(\frac{1}{z_2} - \frac{1}{z_1}, \frac{1}{z_2}\right). \quad (23)$$

For the complete description of the hyperon polarization, another type of dynamical FFs constructed from the gluon's field strength and the quark and antiquark correlation functions are necessary

$$\begin{aligned} \tilde{\Delta}^\alpha\left(\frac{1}{z_1}, \frac{1}{z_2}\right) &= \frac{1}{N} \sum_X \iint \frac{d\lambda d\mu}{(2\pi)^2} e^{-i\frac{\lambda}{z_1}} e^{-i\mu(\frac{1}{z_2} - \frac{1}{z_1})} \langle 0 | F_a^{w\alpha}(\mu w) | \Lambda^\uparrow X \rangle \langle \Lambda^\uparrow X | g\bar{\psi}(\lambda w) t^a \psi(0) | 0 \rangle \\ &= M_h \left(e^{\alpha P_h w S_\perp} P_h \tilde{D}_{FT}\left(\frac{1}{z_1}, \frac{1}{z_2}\right) + i S_\perp^\alpha \gamma_5 P_h \tilde{G}_{FT}\left(\frac{1}{z_1}, \frac{1}{z_2}\right) \right). \end{aligned} \quad (24)$$

The support of these functions is $1/z_1 > 0$, $1/z_2 < 0$ and $1/z_1 - 1/z_2 > 1$. They are complex functions, imaginary parts of which are T-odd, contributing to SSAs. We remark that, although the above correlation function $\tilde{\Delta}^\alpha$ contains quark and antiquark fields like Δ_F^α in (18), it is more appropriate to classify $\tilde{D}_{FT}$ and $\tilde{G}_{FT}$ as the twist-3 dynamical *gluon* FFs [34,43].

3. Relations among the twist-3 FFs

The twist-3 FFs introduced in the previous subsections obey the constraint relations among themselves, which are known as the QCD equation-of-motion (EOM) relations and the Lorentz invariance relations (LIRs). (See [39,43] for the derivation of the complete sets of those relations for the twist-3 quark and gluon FFs.) They play crucial roles to guarantee the gauge and frame-independence of various twist-3 cross sections including $pp \rightarrow \Lambda^\uparrow X$ [13,14,39]. Here we quote those relations relevant to the present study from [39,43], since they are also indispensable to derive the consistency between the TMD factorization and the collinear twist-3 factorization for $ep \rightarrow e\Lambda^\uparrow X$.

For the T-odd twist-3 quark FFs, the EOM relation reads (Eq. (22) of [39])

$$\int_z^\infty \frac{dz_1}{z_1^2} \frac{1}{1/z - 1/z_1} (\Im \hat{D}_{FT}(z, z_1) - \Im \hat{G}_{FT}(z, z_1)) = \frac{D_T(z)}{z} + D_{1T}^{\perp(1)}(z), \quad (25)$$

and the LIR reads (Eq. (42) of [39])

$$-\frac{2}{z} \int_z^\infty \frac{dz_1}{z_1^2} \frac{\Im \hat{D}_{FT}(z, z_1)}{(1/z_1 - 1/z)^2} = \frac{D_T(z)}{z} + \frac{\partial}{\partial(1/z)} \frac{D_{1T}^{\perp(1)}(z)}{z}. \quad (26)$$

For the T-odd twist-3 gluon FFs, EOM relation reads (Eq. (50) of [43])

$$\begin{aligned} &\frac{1}{z} \Delta \hat{G}_{3T}(z) + \Im \tilde{D}_{FT}(z) - \frac{1}{2} (\hat{G}_T^{(1)}(z) + \Delta \hat{H}_T^{(1)}(z)) \\ &= \int d\left(\frac{1}{z'}\right) \frac{1}{1/z - 1/z'} \Im \left(2\hat{N}_1\left(\frac{1}{z'}, \frac{1}{z}\right) + \hat{N}_2\left(\frac{1}{z'}, \frac{1}{z}\right) - \hat{N}_2\left(\frac{1}{z} - \frac{1}{z'}, \frac{1}{z}\right) \right), \end{aligned} \quad (27)$$

where $\tilde{D}_{FT}(z)$ is defined as

$$\tilde{D}_{FT}(z) \equiv \frac{2}{C_F} \int_0^{1/z} d\left(\frac{1}{z'}\right) \tilde{D}_{FT}\left(\frac{1}{z'}, \frac{1}{z'} - \frac{1}{z}\right), \quad C_F = \frac{N^2 - 1}{2N}. \quad (28)$$

The LIRs combined with another EOM relation (as obtained from Eqs. (53), (63), and (68) of [43]) read

$$\begin{aligned} \frac{1}{z} \frac{\partial \hat{G}_T^{(1)}(z)}{\partial(1/z)} + 2(\Im \tilde{D}_{FT}(z) - \hat{G}_T^{(1)}(z)) &= 4 \int d\left(\frac{1}{z'}\right) \frac{1}{1/z - 1/z'} \Im \left(\hat{N}_1\left(\frac{1}{z'}, \frac{1}{z}\right) - \hat{N}_2\left(\frac{1}{z} - \frac{1}{z'}, \frac{1}{z}\right) \right) \\ &+ 2 \int d\left(\frac{1}{z'}\right) \frac{1/z}{(1/z - 1/z')^2} \Im \left(\hat{N}_1\left(\frac{1}{z'}, \frac{1}{z}\right) + \hat{N}_2\left(\frac{1}{z'}, \frac{1}{z}\right) - 2\hat{N}_2\left(\frac{1}{z} - \frac{1}{z'}, \frac{1}{z}\right) \right), \end{aligned} \quad (29)$$

and

$$\begin{aligned} \frac{1}{z} \frac{\partial \Delta \hat{H}_T^{(1)}(z)}{\partial(1/z)} + 4(\Im \tilde{D}_{FT}(z) - \Delta \hat{H}_T^{(1)}(z)) &= 8 \int d\left(\frac{1}{z'}\right) \frac{1}{1/z - 1/z'} \Im \left(\hat{N}_1\left(\frac{1}{z'}, \frac{1}{z}\right) + \hat{N}_2\left(\frac{1}{z'}, \frac{1}{z}\right) \right) \\ &+ 4 \int d\left(\frac{1}{z'}\right) \frac{1/z}{(1/z - 1/z')^2} \Im \left(\hat{N}_1\left(\frac{1}{z'}, \frac{1}{z}\right) + \hat{N}_2\left(\frac{1}{z'}, \frac{1}{z}\right) \right). \end{aligned} \quad (30)$$

These relations allow one to rewrite the derivatives of the kinematical FFs in terms of other FFs.

B. TMD polarizing FF

The hyperon polarization at small- P_T can be described in the framework of the TMD factorization [15,16]. Relevant TMD FF is the polarizing FF defined by [36]

$$\begin{aligned} \Delta(z, \vec{k}_\perp) &= \sum_X \int \frac{d\xi^+ d^2 \vec{\xi}_\perp}{2z(2\pi)^3} e^{ik \cdot \xi} \langle 0 | \psi(\xi) | \Lambda^\uparrow X \rangle \langle \Lambda^\uparrow X | \bar{\psi}(0) | 0 \rangle \big|_{\xi^- = 0} \\ &= \frac{1}{2M_h P_h^-} \epsilon^{\alpha P_h k_\perp S_\perp} \gamma_\alpha D_{1T}^\perp(z, z^2 \vec{k}_\perp^2) + \dots, \end{aligned} \quad (31)$$

where $z = P_h^-/k^-$ is the longitudinal momentum fraction (for the hyperon moving in the $-z$ direction). The hyperon's transverse momentum $\vec{P}_T$ with respect to the quark's momentum is related to the above $\vec{k}_\perp$ (which is defined in the frame with $P_{hT} = 0$) as $\vec{P}_T = -z \vec{k}_\perp$. $D_{1T}^\perp(z_f, z_f^2 \vec{k}_\perp^2)$ is the T-odd polarizing FF. The kinematical twist-3 quark FF $D_{1T}^{\perp(1)}$ in (17) is the $\vec{k}_\perp^2/M_h^2$ -moment of $D_{1T}^\perp$

$$D_{1T}^{\perp(1)}(z) = z^2 \int d^2 \vec{k}_\perp \left(\frac{\vec{k}_\perp^2}{2M_h^2} \right) D_{1T}^\perp(z, z^2 \vec{k}_\perp^2). \quad (32)$$

$D_{1T}^\perp$ convoluted with the TMD unpolarized distribution in the nucleon contributes to the hyperon polarization in the TMD factorization.

IV. TWIST-3 FF CONTRIBUTION TO $ep \rightarrow e\Lambda^\uparrow X$ AT $\Lambda_{\text{QCD}} \ll P_T \ll Q$

A. Structure of the twist-3 cross section

Using the fragmentation functions defined in the previous section, the twist-3 quark and gluon fragmentation function contribution to $ep \rightarrow e\Lambda^\uparrow X$ was derived, respectively, in [33,34]. Here we summarize their basic structures.

(1) Twist-3 quark FF contribution

$$\begin{aligned}
& \frac{d^6 \Delta \sigma^{\text{tw3-quark}}}{dx_{bj} dQ^2 dz_f dq_T^2 d\phi d\chi} \\
&= \frac{\alpha_{em}^2 \alpha_s (-M_h)}{16\pi^2 x_{bj}^2 S_{ep}^2 Q^2} \sum_{k=1,2,3,4,8,9} \mathcal{A}_k(\phi - \chi) \mathcal{S}_k \int_{x_{\min}}^1 \frac{dx}{x} \sum_{f_1=q,G} e_q^2 f_1(x) \int_{z_{\min}}^1 \frac{dz}{z} \delta\left(\frac{q_T^2}{Q^2} - \left(1 - \frac{1}{\hat{x}}\right)\left(1 - \frac{1}{\hat{z}}\right)\right) \\
&\times \left[\frac{D_T(z)}{z} \hat{\sigma}_T^k - \left\{ \frac{d}{d(1/z)} \frac{D_{1T}^{(1)}(z)}{z} \right\} \hat{\sigma}_{\perp D}^k - D_{1T}^{(1)}(z) \hat{\sigma}_{\perp}^k \right. \\
&+ \int \frac{dz'}{z'^2} \mathcal{P}\left(\frac{1}{1/z - 1/z'}\right) \left\{ \Im \hat{D}_{FT}(z, z') \left[\frac{z'}{z} \hat{\sigma}_{DF1}^k + \frac{1}{z} \left(\frac{1}{z'} - \frac{1}{(1 - q_T^2/Q^2)z_f} \right)^{-1} \hat{\sigma}_{DF2}^k \right] \right. \\
&\left. \left. + \Im \hat{G}_{FT}(z, z') \left[\frac{z'}{z} \hat{\sigma}_{GF1}^k + \frac{1}{z} \left(\frac{1}{z'} - \frac{1}{(1 - q_T^2/Q^2)z_f} \right)^{-1} \hat{\sigma}_{GF2}^k \right] \right\} \right], \tag{33}
\end{aligned}$$

where $\sum_{f_1=q,G}$ indicates the sum over the twist-2 unpolarized quark distributions $q(x)$'s for quark flavor $q = u, d, \bar{u}, \dots$, and the twist-2 unpolarized gluon distribution $G(x)$, with e_q representing the fractional electric charge for the quark or antiquark. Note that the flavor indices for the twist-3 quark FFs are implicit in the above equation. In (33), the lower limits of x and z integrations are, respectively, given by $x_{\min} = x_{bj}(1 + \frac{z_f}{1-z_f} \frac{q_T^2}{Q^2})$ and $z_{\min} = z_f(1 + \frac{x_{bj}}{1-x_{bj}} \frac{q_T^2}{Q^2})$ and $\Im \hat{D}_{FT}$ etc. indicates the imaginary part of $\hat{D}_{FT}$. $\mathcal{A}_k(\phi - \chi)$'s are given in (15), and $\mathcal{S}_k = \sin \Phi_S$ ($k = 1, \dots, 4$) and $\mathcal{S}_{8,9} = \cos \Phi_S$. The partonic hard cross sections $\hat{\sigma}_T^k, \hat{\sigma}_{\perp D}^k, \hat{\sigma}_{\perp}^k, \hat{\sigma}_{DF1}^k, \hat{\sigma}_{DF2}^k, \hat{\sigma}_{GF1}^k$ and $\hat{\sigma}_{GF2}^k$ which, of course, depend on whether $f_1 = q$ or G are the functions of $\hat{x} \equiv x_{bj}/x, \hat{z} \equiv z_f/z, Q$ and q_T and can be found in [33].

(2) Twist-3 gluon FF contribution

$$\begin{aligned}
& \frac{d^6 \Delta \sigma^{\text{tw3-gluon}}}{dx_{bj} dQ^2 dz_f dq_T^2 d\phi d\chi} \\
&= \frac{\alpha_{em}^2 \alpha_s M_h}{16\pi^2 x_{bj}^2 S_{ep}^2 Q^2} \sum_k \mathcal{A}_k(\phi - \chi) \mathcal{S}_k \int_{x_{\min}}^1 \frac{dx}{x} \int_{z_{\min}}^1 \frac{dz}{z} z^2 \sum_q e_q^2 q(x) \delta\left(\frac{q_T^2}{Q^2} - \left(1 - \frac{1}{\hat{x}}\right)\left(1 - \frac{1}{\hat{z}}\right)\right) \\
&\times \left\{ \hat{G}_T^{(1)}(z) \hat{\sigma}_G^k + \Delta \hat{H}_T^{(1)}(z) \hat{\sigma}_H^k \right. \\
&+ \int d\left(\frac{1}{z'}\right) \left[\frac{1}{1/z - 1/z'} \Im \left(\hat{N}_1 \left(\frac{1}{z'}, \frac{1}{z} \right) \hat{\sigma}_{N1}^k + \hat{N}_2 \left(\frac{1}{z'}, \frac{1}{z} \right) \hat{\sigma}_{N2}^k + \hat{N}_2 \left(\frac{1}{z} - \frac{1}{z'}, \frac{1}{z} \right) \hat{\sigma}_{N3}^k \right) \right. \\
&+ \frac{1}{z} \left(\frac{1}{1/z - 1/z'} \right)^2 \Im \left(\hat{N}_1 \left(\frac{1}{z'}, \frac{1}{z} \right) \hat{\sigma}_{DN1}^k + \hat{N}_2 \left(\frac{1}{z'}, \frac{1}{z} \right) \hat{\sigma}_{DN2}^k + \hat{N}_2 \left(\frac{1}{z} - \frac{1}{z'}, \frac{1}{z} \right) \hat{\sigma}_{DN3}^k \right) \\
&+ \frac{1}{1/z - 1/z'} \Im \left(\hat{O}_1 \left(\frac{1}{z'}, \frac{1}{z} \right) \hat{\sigma}_{O1}^k + \hat{O}_2 \left(\frac{1}{z'}, \frac{1}{z} \right) \hat{\sigma}_{O2}^k + \hat{O}_2 \left(\frac{1}{z} - \frac{1}{z'}, \frac{1}{z} \right) \hat{\sigma}_{O3}^k \right) \\
&+ \frac{1}{z} \left(\frac{1}{1/z - 1/z'} \right)^2 \Im \left(\hat{O}_1 \left(\frac{1}{z'}, \frac{1}{z} \right) \hat{\sigma}_{DO1}^k + \hat{O}_2 \left(\frac{1}{z'}, \frac{1}{z} \right) \hat{\sigma}_{DO2}^k + \hat{O}_2 \left(\frac{1}{z} - \frac{1}{z'}, \frac{1}{z} \right) \hat{\sigma}_{DO3}^k \right) \\
&+ \int d\left(\frac{1}{z'}\right) \frac{2}{C_F} \left[\Im \tilde{D}_{FT} \left(\frac{1}{z'}, \frac{1}{z} - \frac{1}{z} \right) \left(\hat{\sigma}_{DF1}^k + \frac{1}{z} \frac{1}{1/z - 1/z'} \hat{\sigma}_{DF2}^k + \frac{z'}{z} \hat{\sigma}_{DF3}^k \right. \right. \\
&+ \frac{1}{1 - (1 - q_T^2/Q^2)z_f/z'} \hat{\sigma}_{DF4}^k + \frac{1}{1 - (1 - q_T^2/Q^2)z_f(1/z - 1/z')} \hat{\sigma}_{DF5}^k \Big) \\
&+ \Im \tilde{G}_{FT} \left(\frac{1}{z'}, \frac{1}{z} - \frac{1}{z} \right) \left(\frac{1}{z} \frac{1}{1/z - 1/z'} \hat{\sigma}_{GF2}^k + \frac{z'}{z} \hat{\sigma}_{GF3}^k \right. \\
&\left. \left. + \frac{1}{1 - (1 - q_T^2/Q^2)z_f/z'} \hat{\sigma}_{GF4}^k + \frac{1}{1 - (1 - q_T^2/Q^2)z_f(1/z - 1/z')} \hat{\sigma}_{GF5}^k \right) \right] \Big\}, \tag{34}
\end{aligned}$$

where the flavor indices for $\tilde{D}_{FT}$ and $\tilde{G}_{FT}$ are again implicit and each partonic hard cross section is given in [34]. The total twist-3 fragmentation function contribution to $ep \rightarrow e\Lambda^\uparrow X$ is the sum of (33) and (34), and is decomposed into the five structure functions based on the dependence on the azimuthal angles as

$$\begin{aligned} \frac{d^6 \Delta \sigma^{\text{tw}3}}{dx_{bj} dQ^2 dz_f dq_T^2 d\phi d\chi} &= \frac{d^6 \Delta \sigma^{\text{tw}3\text{-quark}}}{dx_{bj} dQ^2 dz_f dq_T^2 d\phi d\chi} + \frac{d^6 \Delta \sigma^{\text{tw}3\text{-gluon}}}{dx_{bj} dQ^2 dz_f dq_T^2 d\phi d\chi} \\ &= \mathcal{F}_1 \sin \Phi_S + \mathcal{F}_2 \sin \Phi_S \cos(\phi - \chi) + \mathcal{F}_3 \sin \Phi_S \cos 2(\phi - \chi) \\ &\quad + \mathcal{F}_4 \cos \Phi_S \sin(\phi - \chi) + \mathcal{F}_5 \cos \Phi_S \sin 2(\phi - \chi). \end{aligned} \quad (35)$$

B. Twist-3 cross section at $\Lambda_{\text{QCD}} \ll P_T \ll Q$

In order to investigate the consistency between the twist-3 cross section (35) and the corresponding TMD factorization formula, one needs to examine the behavior of (35) in the region of the small transverse momentum $P_T = z_f q_T \ll Q$. To this end, we note

$$\delta\left(\frac{q_T^2}{Q^2} - \left(1 - \frac{1}{\hat{x}}\right)\left(1 - \frac{1}{\hat{z}}\right)\right)\Big|_{q_T \ll Q} = \hat{x} \hat{z} \left(\frac{\delta(\hat{x} - 1)}{(1 - \hat{z})_+} + \frac{\delta(\hat{z} - 1)}{(1 - \hat{x})_+} + \delta(\hat{x} - 1)\delta(\hat{z} - 1) \ln \frac{Q^2}{q_T^2} \right), \quad (36)$$

where $+$ -function is defined as

$$\int_z^1 dx T(x) \frac{1}{(1-x)_+} \equiv \int_z^1 dx \frac{T(x) - T(1)}{1-x} + T(1) \ln(1-z) \quad (37)$$

for an arbitrary function $T(x)$. Using the actual form of the partonic hard cross sections, one finds that only $\mathcal{F}_1$ in (35) receives the leading contribution with respect to q_T/Q

$$\frac{d^6 \Delta \sigma^{\text{tw}3}}{dx_{bj} dQ^2 dz_f dq_T^2 d\phi d\chi}\Big|_{q_T \ll Q} = \mathcal{F}_1|_{q_T \ll Q} \sin \Phi_S, \quad (38)$$

with

$$\begin{aligned} \mathcal{F}_1|_{q_T \ll Q} &= \frac{-4\alpha_s M_h \sigma_0}{(2\pi)^2 q_T^3} \sum_q e_q^2 \left[q(x_{bj}) \int \frac{dz}{z} \left(A - \frac{1}{4} B \right) + C_F D_{1T}^{\perp(1)}(z_f) \int \frac{dx}{x} q(x) \frac{1 + \hat{x}^2}{(1 - \hat{x})_+} \right. \\ &\quad \left. + \frac{1}{2} D_{1T}^{\perp(1)}(z_f) \int \frac{dx}{x} G(x) (1 - 2\hat{x} + 2\hat{x}^2) + 2C_F q(x_{bj}) D_{1T}^{\perp(1)}(z_f) \ln \frac{Q^2}{q_T^2} \right], \end{aligned} \quad (39)$$

where $\sigma_0 = \alpha_{\text{em}}^2 (1 - y + y^2/2)/Q^4$ with $y \equiv p \cdot q / p \cdot \ell = Q^2 / (x_{bj} S_{ep})$. A and B in (39) are, respectively, the twist-3 quark and gluon FF contributions and are given by²

$$\begin{aligned} A &= \frac{D_T(z)}{z} \left(-C_F (1 + 2\hat{z}) - \frac{1}{2N} \frac{1 + \hat{z}^2}{\hat{z}} \right) \\ &\quad + \left(\frac{\partial}{\partial(1/z)} \frac{D_{1T}^{\perp(1)}(z)}{z} \right) \left(-\frac{1}{2N} \frac{1 + \hat{z}^2}{\hat{z}} \right) + D_{1T}^{\perp(1)}(z) \left(C_F \frac{\hat{z}(1 + \hat{z})}{(1 - \hat{z})_+} \right) \\ &\quad + \int d\left(\frac{1}{z'}\right) \frac{1/z}{1/z - 1/z'} \Im \hat{D}_{FT}(z, z') \left\{ \frac{1}{1/z' - 1/z_f} \frac{1}{2N} \frac{2 - \hat{z}}{\hat{z}} + \frac{1}{1/z' - 1/z_f} \left(C_F + \frac{1}{2N} \right) (1 + \hat{z}) \right\} \\ &\quad + \int d\left(\frac{1}{z'}\right) \frac{1/z}{1/z - 1/z'} \Im \hat{G}_{FT}(z, z') \left\{ \frac{1}{1/z' - 1/z_f} \frac{1}{2N} - \frac{1}{1/z' - 1/z_f} \left(C_F + \frac{1}{2N} \right) (1 - \hat{z}) \right\}, \end{aligned} \quad (40)$$

²Note that A and B also carry the flavor indices from twist-3 FFs, which are suppressed here.

and

$$\begin{aligned}
B = & 2C_F z^2 \left[\frac{(1-\hat{z})(-2+\hat{z}^2)}{\hat{z}^2} \hat{G}_T^{(1)}(z) - 2 \frac{(1-\hat{z})}{\hat{z}} \Delta \hat{H}_T^{(1)}(z) + \int d\left(\frac{1}{z'}\right) \frac{1}{1/z-1/z'} \frac{(1-\hat{z})}{\hat{z}^2} \right. \\
& \times \Im \left\{ 4(2-3\hat{z}+\hat{z}^2) \hat{N}_1\left(\frac{1}{z'}, \frac{1}{z}\right) + 2(2-3\hat{z}) \hat{N}_2\left(\frac{1}{z'}, \frac{1}{z}\right) - 2(2-3\hat{z}+2\hat{z}^2) \hat{N}_2\left(\frac{1}{z}-\frac{1}{z'}, \frac{1}{z}\right) \right\} \\
& + \int d\left(\frac{1}{z'}\right) \frac{1}{z} \left(\frac{1}{1/z-1/z'} \right)^2 \frac{(1-\hat{z})}{\hat{z}^2} \\
& \times \Im \left\{ (4-4\hat{z}+\hat{z}^2) \hat{N}_1\left(\frac{1}{z'}, \frac{1}{z}\right) + (4-4\hat{z}+\hat{z}^2) \hat{N}_2\left(\frac{1}{z'}, \frac{1}{z}\right) - 2(2-2\hat{z}+\hat{z}^2) \hat{N}_2\left(\frac{1}{z}-\frac{1}{z'}, \frac{1}{z}\right) \right\} + \int d\left(\frac{1}{z'}\right) \frac{1}{1/z-1/z'} \frac{(1-\hat{z})}{\hat{z}^2} \\
& \times \Im \left\{ 4(2-\hat{z}) \hat{O}_1\left(\frac{1}{z'}, \frac{1}{z}\right) + 2(4-3\hat{z}+\hat{z}^2) \hat{O}_2\left(\frac{1}{z'}, \frac{1}{z}\right) + 2(4-3\hat{z}+\hat{z}^2) \hat{O}_2\left(\frac{1}{z}-\frac{1}{z'}, \frac{1}{z}\right) \right\} + \int d\left(\frac{1}{z'}\right) \frac{1}{z} \left(\frac{1}{1/z-1/z'} \right)^2 \frac{(1-\hat{z})}{\hat{z}^2} \\
& \times \Im \left\{ (4-4\hat{z}+\hat{z}^2) \hat{O}_1\left(\frac{1}{z'}, \frac{1}{z}\right) + (4-4\hat{z}+\hat{z}^2) \hat{O}_2\left(\frac{1}{z'}, \frac{1}{z}\right) + 2(2-2\hat{z}+\hat{z}^2) \hat{O}_2\left(\frac{1}{z}-\frac{1}{z'}, \frac{1}{z}\right) \right\} \\
& + \frac{1}{C_F} \int d\left(\frac{1}{z'}\right) \Im \tilde{D}_{FT}\left(\frac{1}{z'}, \frac{1}{z'} - \frac{1}{z}\right) \\
& \times \left\{ -4 \frac{(1-\hat{z})}{\hat{z}^2} (2-3\hat{z}+\hat{z}^2) + \frac{1}{N} \frac{1}{z} \frac{1}{1/z-1/z'} \frac{-1}{\hat{z}} (2-\hat{z}) + \frac{1}{N} \frac{1}{1-z_f/z'} (-\hat{z})(-2+\hat{z}) \right\} \\
& \left. + \frac{1}{C_F} \int d\left(\frac{1}{z'}\right) \Im \tilde{G}_{FT}\left(\frac{1}{z'}, \frac{1}{z'} - \frac{1}{z}\right) \left\{ \frac{1}{N} \frac{1/z}{1/z-1/z'} + \frac{1}{N} \frac{1}{1-z_f/z'} (-\hat{z}^2) \right\} \right]. \tag{41}
\end{aligned}$$

We note that, at $P_T/Q \rightarrow 0$, $\vec{P}_h$ and $\vec{q}$ become parallel and the azimuthal angle Φ_S can be regarded as the angle around $\vec{q}$ as is seen from Fig. 1.

V. TMD FACTORIZATION FOR $ep \rightarrow e\Lambda^\dagger X$ AT $P_T \gg \Lambda_{\text{QCD}}$

A. Framework

When the transverse momentum of the hyperon P_T is much smaller than the hard scale, i.e., $\Lambda_{\text{QCD}} \lesssim P_T = z_f q_T \ll Q$, the TMD factorization can be applied to $ep \rightarrow e\Lambda^\dagger X$. In this framework, the $\sin \Phi_S$ asymmetry can be described in terms of the polarizing FF $D_{1T}^\perp$ [36], and the corresponding factorization formula in the Ji-Ma-Yuan scheme [15,16] can be written as

$$\begin{aligned}
\frac{d^6 \sigma}{dx_{bj} dQ^2 dz_f dq_T^2 d\phi d\chi} = & -z_f^2 \sigma_0 \sin \Phi_S \int d^2 \vec{k}_\perp d^2 \vec{p}_\perp d^2 \vec{\lambda}_\perp \delta^2(\vec{k}_\perp - \vec{p}_\perp + \vec{\lambda}_\perp - \vec{P}_T/z_f) \\
& \times \frac{\vec{p}_\perp^2}{q_T M_h} \sum_q e_q^2 q(x_{bj}, k_\perp^2) D_{1T}^\perp(z_f, z_f^2 p_\perp^2) S^{-1}(\lambda_\perp^2) H(Q^2), \tag{42}
\end{aligned}$$

where $q(x_{bj}, k_\perp^2)$ is the TMD unpolarized quark DF, $H(Q^2)$ is the hard factor which is calculable in perturbative QCD and takes the form of $H(Q^2) = 1 + \mathcal{O}(\alpha_s)$, and the soft factor $S(\lambda_\perp^2)$ is the vacuum expectation value of the Wilson line representing the soft gluon radiation with S^{-1} canceling the effect of the soft gluon radiation in the TMD functions. In order to see the consistency with the collinear twist-3 factorization, we see the behavior of (42) in large- q_T region, $q_T \gg \Lambda_{\text{QCD}}$. The dominant contribution of (42) in this region is obtained by setting one of $\vec{k}_\perp$, $-\vec{p}_\perp$ and $\vec{\lambda}_\perp$ in the δ -function equal to $\vec{P}_T/z_f$ with other two transverse momenta set to zero. Then (42) becomes

$$\begin{aligned}
\frac{d^6 \sigma}{dx_{bj} dQ^2 dz_f dq_T^2 d\phi d\chi} = & -2\sigma_0 \sin \Phi_S \frac{M_h}{q_T} \sum_q e_q^2 (q(x_{bj}, q_T^2) D_{1T}^{\perp(1)}(z_f) \\
& + \frac{P_T^2}{2M_h^2} q(x_{bj}) D_{1T}^\perp(z_f, P_T^2) + q(x_{bj}) D_{1T}^{\perp(1)}(z_f) S^{-1}(q_T^2)), \tag{43}
\end{aligned}$$

where we have used the following relation,

$$\int d^2\vec{k}_\perp q(x_{bj}, \vec{k}_\perp^2) = q(x_{bj}), \quad (44)$$

and $D_{1T}^{\perp(1)}(z_f)$ is the twist-3 quark FF obtained as the $p_\perp^2/M_h^2$ -moment of the TMD polarizing FF³

$$D_{1T}^{\perp(1)}(z_f) = z_f^2 \int d^2 p_\perp \frac{\vec{p}_\perp^2}{2M_h^2} D_{1T}^\perp(z_f, z_f^2 p_\perp^2). \quad (45)$$

Since we calculate $q(x_{bj}, q_T^2)$, $D_{1T}^\perp(z_f, P_T^2)$, $S^{-1}(q_T^2)$ in the LO with respect to the QCD coupling constant, the hard factor is set equal to one. In order to investigate the behavior of (43) at $\Lambda_{\text{QCD}} \ll q_T \ll Q$, we utilize the fact that the q_T -dependence of each factor in (43) can be calculated by perturbative QCD [17–20]. A perturbative calculation of the TMD DFs and FFs at large intrinsic transverse momentum involves the rapidity divergence, if one uses the light cone vectors n^μ and w^μ ($n^2 = w^2 = 0$). To regularize this divergence, we introduce nonlightlike vectors v^μ and $\tilde{v}^\mu$

$$v = (v^+, v^-, \vec{0}_\perp), \quad \tilde{v} = (\tilde{v}^+, \tilde{v}^-, \vec{0}_\perp), \quad (46)$$

and define

$$\rho^2 = \frac{(2v \cdot \tilde{v})^2}{v^2 \tilde{v}^2}. \quad (47)$$

As $v^+ \rightarrow 0$, $\tilde{v}^- \rightarrow 0$, v and $\tilde{v}$ become lightlike ($v^2 = \tilde{v}^2 = 0$) and $\rho^2 \rightarrow \infty$. The soft factor has been obtained as [18,19]

$$S^{-1}(q_T^2) = -\frac{\alpha_s C_F}{2\pi^2 q_T^2} (\ln \rho^2 - 2). \quad (48)$$

The transverse momentum dependence of the TMD functions can be calculated by perturbative QCD at $q_T \gg \Lambda_{\text{QCD}}$, and they can be expressed in terms of the collinear PDFs and FFs. The result for the unpolarized DF is well known [16,19]

³The relations between the TMD and the collinear functions as given in (44) and (45) do not hold in a rigorous sense in the level of the renormalized functions because the renormalization scale structure differs between a TMD function and the corresponding collinear function. However, recent works [45–47] show that these relations effectively hold under a specific choice of the renormalization scales and by introducing an appropriate UV cutoff in the momentum integral.

$$\begin{aligned} q(x_{bj}, q_T^2) &= \frac{\alpha_s C_F}{2\pi^2 q_T^2} \int \frac{dx}{x} q(x) \left[\frac{1 + \hat{x}^2}{(1 - \hat{x})_+} \right. \\ &\quad \left. + \delta(1 - \hat{x}) \left(\ln \frac{x_{bj}^2 \zeta^2}{q_T^2} - 1 \right) \right] \\ &\quad + \frac{\alpha_s}{2\pi^2 q_T^2} \frac{1}{2} \int \frac{dx}{x} G(x) (1 - 2\hat{x} + 2\hat{x}^2), \end{aligned} \quad (49)$$

where $\zeta^2 \equiv (2v \cdot p)^2/v^2$ ($\zeta \rightarrow \infty$ as $v^+ \rightarrow 0$) is the regulator for the rapidity divergence.

B. Perturbative calculation of the polarizing FF

$D_{1T}^\perp(z, z^2 k_\perp^2)$ at $k_\perp \gg \Lambda_{\text{QCD}}$

1. Factorization of the TMD FF into the twist-3 FFs and the hard part

Next we consider the perturbative calculation of the polarizing FF $D_{1T}^\perp$ at $q_T \gg \Lambda_{\text{QCD}}$. Following the method for the Siverson function and the Collins function [18,19,22], one can factorize the correlation function for the TMD FF (31) into the collinear twist-3 FFs and the hard part. To this end, we decompose (31) into the quark and gluon pieces

$$\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} = \left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]^q + \left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]^g, \quad (50)$$

where the first and the second terms, respectively, correspond to the quark- and gluon-fragmentation channels, and they are given as

$$\begin{aligned} \left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]^q &\equiv \text{Tr} \int \frac{d^4 \ell}{(2\pi)^4} \Delta(\ell) S(\ell) \\ &\quad + \text{Tr} \iint \frac{d^4 \ell d^4 \ell'}{(2\pi)^8} \Delta_{La}^\alpha(\ell, \ell') S_\alpha^{La}(\ell, \ell') \\ &\quad + [L \rightarrow R]. \end{aligned} \quad (51)$$

$$\begin{aligned} \left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]^g &\equiv \text{Tr} \int \frac{d^4 \ell}{(2\pi)^4} \hat{\Gamma}_{ab}^{\mu\nu}(\ell) S_{\mu\nu}^{ab}(\ell) \\ &\quad + \frac{1}{2} \text{Tr} \iint \frac{d^4 \ell d^4 \ell'}{(2\pi)^8} \hat{\Gamma}_{Labc}^{\mu\nu\lambda}(\ell, \ell') S_{\mu\nu\lambda}^{Labc}(\ell, \ell') \\ &\quad + \text{Tr} \iint \frac{d^4 \ell d^4 \ell'}{(2\pi)^8} \tilde{\Delta}_{La}^\alpha(\ell, \ell') \tilde{S}_\alpha^{La}(\ell, \ell') + [L \rightarrow R], \end{aligned} \quad (52)$$

where Tr indicates sum over the spinor or Lorentz indices depending on channels. $D_{1T}^\perp$ is identified as the coefficient of $\epsilon^{\alpha P_h k_\perp S_\perp} \gamma_\alpha$ in $\Delta(z_f, \vec{k}_\perp)$

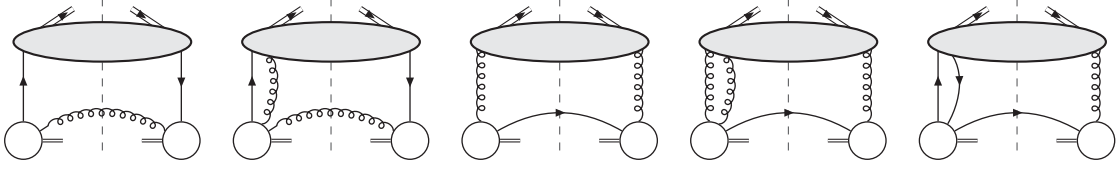


FIG. 2. Feynman diagrams contributing to $D_{1T}^\perp$ at $q_T \gg \Lambda_{\text{QCD}}$. Mirror diagrams also contribute. From left to right, each diagram corresponds to $\Delta(\ell)S(\ell)$, $\Delta_{La}^a(\ell, \ell')S_{\alpha}^{La}(\ell, \ell')$, $\hat{\Gamma}_{ab}^{\mu\nu}(\ell)S_{\mu\nu}^{ab}(\ell)$, $\hat{\Gamma}_{Labc}^{\mu\nu\lambda}(\ell, \ell')S_{\mu\nu\lambda}^{Labc}(\ell, \ell')$, $\tilde{\Delta}_{La}^a(\ell, \ell')\tilde{S}_{\alpha}^{La}(\ell, \ell')$. Upper portion in each diagram represents the fragmentation matrix elements (for the twist-3 FFs), while the lower white circles correspond to the partonic hard part. In the second, fourth, and fifth diagram, momenta of the parton lines in the left of the cut are ℓ and $\ell' - \ell$ flowing upward.

$$\text{Tr}\Delta(z_f, \vec{k}_\perp) \frac{\mathcal{M}}{2} = \frac{\epsilon^{wP_h k_\perp S_\perp}}{M_h P_h} D_{1T}^\perp(z_f, z_f^2 k_\perp^2) + \dots, \quad (53)$$

where we have dropped the terms corresponding to the unpolarized FF. Correspondingly, we pick up terms proportional to $\epsilon^{wP_h k_\perp S_\perp}$ in the right-hand side (rhs) of (50). $\Delta(\ell)$ and $\Delta_{La}^a(\ell, \ell')$ in (51) are the fragmentation matrix elements corresponding, respectively, to $\mathcal{FT} \sum_X \langle 0 | \psi | \Lambda^\dagger X \rangle \times \langle \Lambda^\dagger X | \bar{\psi} | 0 \rangle$ and $\mathcal{FT} \sum_X \langle 0 | \psi | \Lambda^\dagger X \rangle \langle \Lambda^\dagger X | \bar{\psi} g A_a^\alpha | 0 \rangle$ as defined in Eqs. (18) and (19) of [12]. $\hat{\Gamma}_{ab}^{\mu\nu}(\ell)$, $\hat{\Gamma}_{Labc}^{\mu\nu\lambda}(\ell, \ell')$ and $\tilde{\Delta}_{La}^a(\ell, \ell')$ in (52) are the gluon fragmenting matrix elements corresponding, respectively, to $\mathcal{FT} \sum_X \langle 0 | A_b^\mu | \Lambda^\dagger X \rangle \langle \Lambda^\dagger X | A_a^\mu | 0 \rangle$, $\mathcal{FT} \sum_X \langle 0 | A_b^\mu | \Lambda^\dagger X \rangle \times \langle \Lambda^\dagger X | A_a^\mu g A_c^\lambda | 0 \rangle$, and $\mathcal{FT} \sum_X \langle 0 | g A_a^\alpha | \Lambda^\dagger X \rangle \langle \Lambda^\dagger X | \psi \bar{\psi} | 0 \rangle$ as defined in Eqs. (32), (33) and (35) of [34]. $S(\ell)$, $S_{\alpha}^{La}(\ell, \ell')$, $S_{\mu\nu}^{ab}(\ell)$, $S_{\mu\nu\lambda}^{Labc}(\ell, \ell')$ and $\tilde{S}_{\alpha}^{La}(\ell, \ell')$ are the partonic hard part before collinear expansion in each channel. The symbol $L, (R)$ represents that two fragmenting parton lines reside in the left (right) of the cut, and $[L \rightarrow R]$ indicates the contribution from the mirror diagram. Figure 2 shows the generic diagrams for the twist-3 FF contribution to $D_{1T}^\perp$. These diagrams show the TMD FF in which a quark with momentum $(z_f, \vec{k}_\perp)$ splits into partons with momenta represented by ℓ (and/or ℓ') and eventually fragments into the polarized hyperon through the twist-3 FF. The mixing with purely gluonic channels also occurs. Each term in (51) and (52) is not gauge invariant, but the reorganization of the collinear expansion combined with the Ward-Takahashi identities lead to a gauge-invariant twist-3 FF contribution to the process.

Although the hard parts $S(\ell)$ and $S_{\alpha}^{La}(\ell, \ell') \dots$ contains the Wilson lines, they satisfy the following Ward-Takahashi identities which are formally the same as in the case of the calculation of the twist-3 cross section [12,34]. They read

$$(\ell' - \ell)^\alpha S_{\alpha}^{La}(\ell, \ell') = T^a S(\ell') + G^a(\ell, \ell'), \quad (54)$$

for the quark fragmentation channel, and

$$\ell^\mu S_{\mu\nu}^{ab}(\ell) = \ell^\nu S_{\mu\nu}^{ab}(\ell) = 0, \quad (55)$$

$$\ell^\mu S_{\mu\nu\lambda}^{Labc}(\ell, \ell') = \frac{if^{abc}}{N^2 - 1} S_{\lambda\nu}(\ell'), \quad (56)$$

$$(\ell' - \ell)^\lambda S_{\mu\nu\lambda}^{Labc}(\ell, \ell') = \frac{-if^{abc}}{N^2 - 1} S_{\mu\nu}(\ell'), \quad (57)$$

$$\ell'^\nu S_{\mu\nu\lambda}^{Labc}(\ell, \ell') = 0, \quad (58)$$

$$(\ell - \ell')^\alpha \tilde{S}_{\alpha}^{La}(\ell, \ell') = 0, \quad (59)$$

for the gluon fragmentation channel with $S_{\lambda\nu}(\ell') \equiv \delta^{ab} S_{\lambda\nu}^{ab}(\ell')$. Although the ghostlike terms $G^a(\ell, \ell')$ appear in the quark fragmentation channel, they eventually drop in the calculation [12]. In the gluon fragmentation channel, such ghostlike terms do not appear as in the calculation of the twist-3 cross section [34]. Accordingly, (51) and (52) can be analyzed by using the same method used in the derivation of the twist-3 FF contribution to $ep \rightarrow e\Lambda^\dagger X$ in [33,34].

This way, we obtain from (51) the twist-3 quark FF contribution to the polarizing FF in a gauge-invariant form as

$$\begin{aligned} & \left[\text{Tr}\Delta(z_f, \vec{k}_\perp) \frac{\mathcal{M}}{2} \right]_{\text{q-tw3}} \\ &= \int \frac{dz}{z^2} \left[\text{Tr}\Delta(z) S(P_h/z) - i \text{Tr}\Omega_\beta^\alpha \Delta_\partial^\beta(z) \frac{\partial S(k)}{\partial k^\alpha} \right]_{\text{c.l.}} \\ &+ 2\Re i \int \frac{dz'}{z'^2} \frac{1}{1/z - 1/z'} \text{Tr}\Omega_\beta^\alpha \Delta_F^\beta(z', z) S_{La}(1/z', 1/z) \Big], \end{aligned} \quad (60)$$

where $\Omega_\beta^\alpha \equiv g_\beta^\alpha - P_h^\alpha w_\beta$, and Δ_∂^β and Δ_F^β are, respectively, the correlation functions for the intrinsic, kinematical and dynamical twist-3 quark FFs given in (16), (17) and (18). Likewise from (52), one obtains the gauge-invariant twist-3 gluon FF contribution to the polarizing FF as

$$\begin{aligned}
\left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]^{g\text{-tw3}} &= \Omega_\mu^\alpha \Omega_\nu^\beta \int d\left(\frac{1}{z}\right) z^2 \hat{\Gamma}^{\mu\nu}(z) S_{\alpha\beta}(1/z) - i \Omega_\mu^\alpha \Omega_\nu^\beta \Omega_\lambda^\gamma \int d\left(\frac{1}{z}\right) z^2 \hat{\Gamma}_\partial^{\mu\nu\lambda}(z) \frac{\partial S_{\alpha\beta}(k)}{\partial k^\gamma} \Big|_{\text{c.l.}} \\
&+ \Re \left[i \Omega_\mu^\alpha \Omega_\nu^\beta \Omega_\lambda^\gamma \iint d\left(\frac{1}{z}\right) d\left(\frac{1}{z'}\right) \frac{zz'}{1/z - 1/z'} \right. \\
&\times \left\{ -\frac{if^{abc}}{N} \hat{\Gamma}_{FA}^{\mu\nu\lambda}\left(\frac{1}{z'}, \frac{1}{z}\right) + \frac{Nd^{abc}}{N^2 - 4} \hat{\Gamma}_{FS}^{\mu\nu\lambda}\left(\frac{1}{z'}, \frac{1}{z}\right) \right\} S_{\alpha\beta\gamma}^{Labc}\left(\frac{1}{z'}, \frac{1}{z}\right) \Big] \\
&+ \Re \left[i \Omega_\mu^\alpha \iint d\left(\frac{1}{z}\right) d\left(\frac{1}{z'}\right) z \text{Tr} \tilde{\Delta}^\mu\left(\frac{1}{z'}, \frac{1}{z'} - \frac{1}{z}\right) \tilde{S}_\alpha^L\left(\frac{1}{z'}, \frac{1}{z'} - \frac{1}{z}\right) \right], \tag{61}
\end{aligned}$$

where $\hat{\Gamma}^{\mu\nu}$, $\hat{\Gamma}_\partial^{\mu\nu\lambda}$, $\hat{\Gamma}_{FA}^{\mu\nu\lambda}$, $\hat{\Gamma}_{FS}^{\mu\nu\lambda}$ and $\tilde{\Delta}^\mu$ are given in (19), (20), (21), (22), and (24).

2. Quark fragmentation channel

We first calculate the twist-3 quark FF contribution to the TMD polarizing FF (60). Figure 3 shows the relevant diagrams. Some diagrams contain Wilson lines, which are absent from the hard parts for the twist-3 quark FF contribution to the cross section.

The contribution from $\Delta(z)$ [the first term in the bracket $[\cdots]$ of (60)] becomes

$$\left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]_\Delta^{\text{q-tw3}} = \frac{4\alpha_s M_h C_F \epsilon^{k_\perp P_h S_\perp w}}{(2\pi)^2 P_h^- \vec{k}_\perp^4 z_f^2} \int \frac{dz}{z} \frac{D_T(z)}{z} \frac{1}{\hat{z}}. \tag{62}$$

The contribution from $\Delta_\partial(z)$ [the second term in the bracket $[\cdots]$ of (60)] is obtained as

$$\begin{aligned}
\left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]_{\Delta_\partial}^{\text{q-tw3}} &= \frac{4\alpha_s M_h C_F \epsilon^{k_\perp P_h S_\perp w}}{(2\pi)^2 P_h^- \vec{k}_\perp^4 z_f^2} \int \frac{dz}{z} \left[D_{1T}^{\perp(1)}(z) \left\{ \frac{1 + \hat{z}}{(1 - \hat{z})_+} + \delta(1 - \hat{z}) \left(\ln \frac{\hat{\zeta}^2}{\vec{k}_\perp^2 z_f^2} - 1 \right) \right\} \right. \\
&\left. + \left(\frac{\partial}{\partial(1/z)} \frac{D_{1T}^{\perp(1)}(z)}{z} \right) \frac{1 + \hat{z}^2}{\hat{z}} \right], \tag{63}
\end{aligned}$$

where $\hat{\zeta}^2 \equiv (2P_h \cdot \tilde{v})^2 / \tilde{v}^2$ ($\hat{\zeta} \rightarrow \infty$ as $\tilde{v}^- \rightarrow 0$) regularizes the rapidity divergence for the TMD FF.

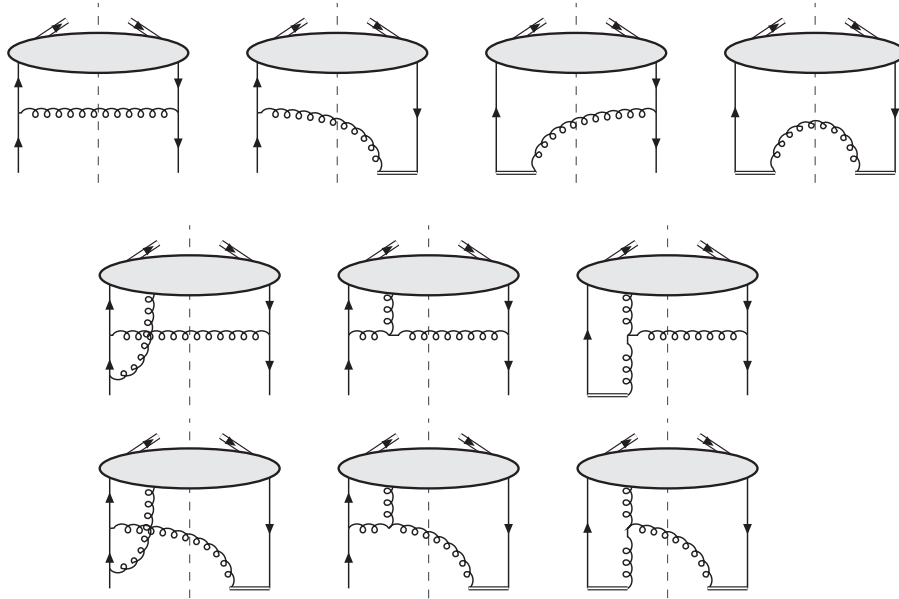


FIG. 3. Twist-3 quark FF contribution to the TMD polarizing FF at large $k_\perp$. Diagrams in which a gluon line directly connects the twist-3 FF and the Wilson line become zero, so that they are not shown.

The contribution from $\Delta_F(z', z)$ [the third term in the bracket $[\dots]$ of (60)] is obtained to be

$$\begin{aligned} \left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]_{\Delta_F}^{\text{q-tw3}} &= \frac{4\alpha_s M_h \epsilon^{k_\perp P_h S_\perp w}}{(2\pi)^2 P_h^- \vec{k}_\perp^4 z_f^2} \left[\int \frac{dz}{z} d\left(\frac{1}{z'}\right) \frac{1/z}{1/z - 1/z'} \right. \\ &\quad \times \left\{ \Im \hat{D}_{FT}(z, z') \left(\frac{1}{1/z'} \frac{1}{2N} \frac{2 - \hat{z}}{\hat{z}} + \frac{1}{1/z' - 1/z_f} \frac{N}{2} (1 + \hat{z}) \right) \right. \\ &\quad \left. + \Im \hat{G}_{FT}(z, z') \left(\frac{1}{1/z'} \frac{1}{2N} - \frac{1}{1/z' - 1/z_f} \frac{N}{2} (1 - \hat{z}) \right) \right\} \\ &\quad - \int \frac{dz}{z} d\left(\frac{1}{z'}\right) \frac{C_F}{1/z - 1/z'} (\Im \hat{D}_{FT}(z, z') - \Im \hat{G}_{FT}(z, z')) (1 + \hat{z}) \\ &\quad \left. - \int \frac{dz}{z} \frac{-2}{z} d\left(\frac{1}{z'}\right) \frac{1}{(1/z - 1/z')^2} \Im \hat{D}_{FT}(z, z') \frac{N}{2} \frac{1 + \hat{z}^2}{\hat{z}} \right]. \end{aligned} \quad (64)$$

Taking the sum of (62), (63), and (64) and using the EOM relation (25) and the LIR (26), one obtains the twist-3 quark FF contribution to the polarizing FF $D_{1T}^\perp$ as

$$D_{1T}^\perp(z_f, P_T^2)|^{\text{quark}} = \frac{\alpha_s}{2\pi^2} \frac{2M_h^2 z_f^2}{P_T^4} \left[\int \frac{dz}{z} A + C_F D_{1T}^{\perp(1)}(z_f) \left(\ln \frac{\hat{s}^2}{P_T^2} - 1 \right) \right], \quad (65)$$

where A is defined in (40), which appeared in the small q_T -limit of the twist-3 quark FF contribution to the cross section for $ep \rightarrow e\Lambda^\uparrow X$.

3. Gluon fragmentation channel

Next we proceed to calculate the twist-3 gluon FF contribution to the TMD polarizing FF (61). Relevant Feynman diagrams are shown in Fig. 4.

The contribution from $\hat{\Gamma}(z)$ to (61) (the first term) can be obtained as

$$\left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]_{\hat{\Gamma}}^{\text{g-tw3}} = \frac{2\alpha_s M_h C_F \epsilon^{k_\perp P_h S_\perp w}}{(2\pi)^2 P_h^- \vec{k}_\perp^4 z_f^2} \int \frac{dz}{z} \frac{1}{z} \Delta \hat{G}_{3T}(z) \frac{1 - \hat{z}}{\hat{z}^2} 2(2 - \hat{z}) \hat{z}^2. \quad (66)$$

The contribution from $\hat{\Gamma}_\partial(z)$ to (61) (the second term) reads

$$\begin{aligned} \left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]_{\hat{\Gamma}_\partial}^{\text{g-tw3}} &= \frac{2\alpha_s M_h C_F \epsilon^{k_\perp P_h S_\perp w}}{(2\pi)^2 P_h^- \vec{k}_\perp^4 z_f^2} \int \frac{dz}{z} z^2 \frac{1 - \hat{z}}{\hat{z}^2} \left[(4 - 3\hat{z} + \hat{z}^2) \hat{G}_T^{(1)}(z) + (2 - \hat{z}) \Delta \hat{H}_T^{(1)}(z) \right. \\ &\quad \left. - \left(\frac{1 - \hat{z}}{z} \frac{\partial \Delta \hat{H}_T^{(1)}(z)}{\partial(1/z)} + \frac{2 - 2\hat{z} + \hat{z}^2}{z} \frac{\partial \hat{G}_T^{(1)}(z)}{\partial(1/z)} \right) \right]. \end{aligned} \quad (67)$$

The contribution from $\hat{\Gamma}_{FS}$ to (61) reads

$$\begin{aligned} \left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]_{\hat{\Gamma}_{FS}}^{\text{g-tw3}} &= -\frac{4\alpha_s M_h C_F \epsilon^{k_\perp P_h S_\perp w}}{4(2\pi)^2 \vec{k}_\perp^4 z_f^2 P_h^-} \int \frac{dz}{z} d\left(\frac{1}{z'}\right) \frac{z'}{(1/z - 1/z')^2} \frac{(1 - \hat{z})}{\hat{z}^2 \hat{z}'} \\ &\quad \times \Im \left[(-2 + \hat{z})(\hat{z}^2 + 2\hat{z}'^2 - 2\hat{z}(1 + \hat{z}')) \hat{O}_1\left(\frac{1}{z'}, \frac{1}{z}\right) \right. \\ &\quad \left. + (2\hat{z}^3 - 2\hat{z}'^2 - 2\hat{z}^2(2 + \hat{z}') + \hat{z}(4 + 2\hat{z}' + \hat{z}'^2)) \hat{O}_2\left(\frac{1}{z'}, \frac{1}{z}\right) \right. \\ &\quad \left. + (-4\hat{z}^2 + \hat{z}^3 - 2\hat{z}'^2 + \hat{z}(4 + 2\hat{z}' + \hat{z}'^2)) \hat{O}_2\left(\frac{1}{z} - \frac{1}{z'}, \frac{1}{z}\right) \right]. \end{aligned} \quad (68)$$

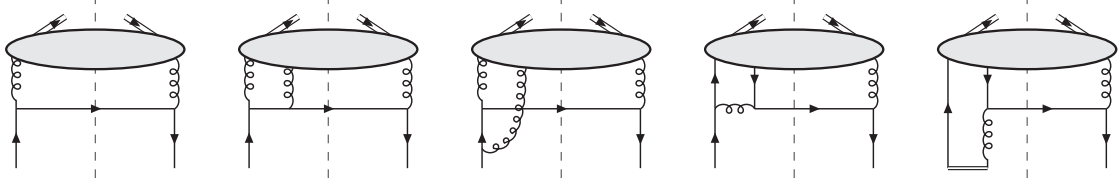


FIG. 4. Feynman diagrams for the twist-3 gluon FF contribution to the polarizing FF at large P_T . Diagrams in which a gluon line coming out of the upper blob directly couples to the Wilson line become either zero or suppressed at large $\hat{\zeta}^2$, so that they are not shown here.

Using the second relation of (23) and the fact that the change of the integration variables $1/z' \leftrightarrow 1/z - 1/z'$ leaves $\int_0^{1/z} d(\frac{1}{z'})$ invariant, together with the following relations

$$\frac{z'}{1/z - 1/z'} = z'z + \frac{z}{1/z - 1/z'}, \quad \int d\left(\frac{1}{z'}\right) \frac{z'}{1/z - 1/z'} \hat{O}_1 = \int d\left(\frac{1}{z'}\right) \frac{2z}{1/z - 1/z'} \hat{O}_1, \quad (69)$$

(68) can be rewritten as

$$\left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]_{\hat{\Gamma}_{FS}}^{\text{g-tw3}} = -\frac{\alpha_s M_h \epsilon^{k_\perp P_h S_\perp w}}{(2\pi)^2 \vec{k}_\perp^4 P_h^- z_f^2} \int \frac{dz}{z} B_{\hat{O}}, \quad (70)$$

where $B_{\hat{O}}$ represents the sum of the terms in (41) containing $\hat{\mathcal{O}}_{1,2}$, i.e., the same as the $\hat{O}_{1,2}$ contribution to the twist-3 cross section $\mathcal{F}_1|_{q_T \ll Q}$ in (39).

Likewise the contribution from $\hat{\Gamma}_{FA}$ to (61) can be obtained as

$$\begin{aligned} \left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]_{\hat{\Gamma}_{FA}}^{\text{g-tw3}} &= -\frac{4\alpha_s M_h C_F \epsilon^{k_\perp P_h S_\perp w}}{4(2\pi)^2 \vec{k}_\perp^4 z_f^2 P_h^-} \int \frac{dz}{z} d\left(\frac{1}{z'}\right) \frac{1 - \hat{z}}{\hat{z}^2} 2z^2 \times \Im \left[\frac{1/z}{(1/z - 1/z')^2} (-(-2 + \hat{z})^2) \hat{N}_1 \left(\frac{1}{z'}, \frac{1}{z} \right) \right. \\ &\quad \left. + \frac{1/z}{(1/z - 1/z')^2} (-\hat{z}^2 + 4\hat{z} - 4) \hat{N}_2 \left(\frac{1}{z'}, \frac{1}{z} \right) + \frac{1/z}{(1/z - 1/z')^2} (2\hat{z}^2 - 4\hat{z} + 4) \hat{N}_2 \left(\frac{1}{z} - \frac{1}{z'}, \frac{1}{z} \right) \right], \end{aligned} \quad (71)$$

where we have used the first relation of (23).

The contribution from $\tilde{\Delta}$ to (61) (the last term) can be obtained as

$$\begin{aligned} \left[\text{Tr} \Delta(z_f, \vec{k}_\perp) \frac{\mathcal{W}}{2} \right]_{\tilde{\Delta}}^{\text{g-tw3}} &= -\frac{2\alpha_s M_h \epsilon^{k_\perp P_h S_\perp w}}{(2\pi)^2 \vec{k}_\perp^4 z_f^2 P_h^- N} \int \frac{dz}{z} d\left(\frac{1}{z'}\right) \times \left\{ \Im \tilde{G}_{FT} \left(\frac{1}{z'}, \frac{1}{z'} - \frac{1}{z} \right) \left[\frac{z}{1/z - 1/z'} + \frac{z_f^2}{-1 + \hat{z}} \right] \right. \\ &\quad \left. + \Im \tilde{D}_{FT} \left(\frac{1}{z'}, \frac{1}{z'} - \frac{1}{z} \right) z^2 \left[\frac{1}{z} \frac{2 - \hat{z}}{1/z - 1/z'} \frac{-1}{\hat{z}} + \frac{-\hat{z}(-2 + \hat{z})}{1 - \hat{z}'} \right] \right\}. \end{aligned} \quad (72)$$

The total contribution from the gluon fragmentation channel to $D_{1T}^\perp$ can be obtained from the sum of (66), (67), (70), (71), and (72). In taking the sum, one should note that the intrinsic, kinematical, and dynamical twist-3 gluon FFs are related among themselves by the EOM relation (27) and the LIRs, (29) and (30), except that $\hat{\mathcal{O}}_{1,2}$ and $\tilde{\mathcal{G}}_{FT}$ are independent from others. Here we employ the following procedure: We express the intrinsic FF $\Delta \hat{G}_{3\bar{T}}$ in terms of the others by using (27), and also eliminate the derivatives of the kinematical FFs by using (29) and (30). This gives the final form for the twist-3 gluon FF contribution to $D_{1T}^\perp$ as

$$D_{1T}^\perp(z_f, P_T^2)|^{\text{gluon}} = \frac{\alpha_s}{2\pi^2} \frac{2M_h^2 z_f^2}{P_T^4} \int \frac{dz}{z} \left(-\frac{1}{4} B \right), \quad (73)$$

where B is defined in (41) which appeared in the $q_T \ll Q$ -limit of the twist-3 cross section for $ep \rightarrow e\Lambda^+ X$.

C. TMD factorization formula for $ep \rightarrow e\Lambda^\uparrow X$ at $\Lambda_{\text{QCD}} \ll P_T \ll Q$

Taking the sum of the twist-3 quark and gluon FF contributions, (65) and (73), the polarizing FF $D_{1T}^\perp(z_f, P_T^2)$ at $P_T \gg \Lambda_{\text{QCD}}$ can be written in terms of the twist-3 FFs as

$$D_{1T}^\perp(z_f, P_T^2) = \frac{\alpha_s}{2\pi^2} \frac{2M_h^2 z_f^2}{P_T^4} \left[\int \frac{dz}{z} \left(A - \frac{1}{4} B \right) + C_F D_{1T}^{\perp(1)}(z_f) \left(\ln \frac{\hat{\xi}^2}{P_T^2} - 1 \right) \right], \quad (74)$$

where A and B are identical to those in (40) and (41). Inserting (48), (49), and (74) into the TMD cross section formula (43), one finally obtains

$$\begin{aligned} \frac{d^6\sigma}{dx_{bj}dQ^2dz_fdq_T^2d\phi d\chi} &= \frac{-4\alpha_s M_h \sigma_0}{(2\pi)^2 q_T^3} \sin\Phi_S \sum_q e_q^2 \left[q(x_{bj}) \int \frac{dz}{z} \left(A - \frac{1}{4} B \right) \right. \\ &\quad + C_F D_{1T}^{\perp(1)}(z_f) \int \frac{dx}{x} q(x) \frac{1 + \hat{x}^2}{(1 - \hat{x})_+} + \frac{1}{2} D_{1T}^{\perp(1)}(z_f) \int \frac{dx}{x} G(x) (1 - 2\hat{x} + 2\hat{x}^2) \\ &\quad \left. + C_F q(x_{bj}) D_{1T}^{\perp(1)}(z_f) \left\{ \left(\ln \frac{x_{bj}^2 \hat{\xi}^2}{q_T^2} - 1 \right) + \left(\ln \frac{\hat{\xi}^2}{z_f^2 q_T^2} - 1 \right) - (\ln \rho^2 - 2) \right\} \right]. \quad (75) \end{aligned}$$

where three terms in $\{\}$ come from $q(x_{bj}, q_T^2)$, $D_{1T}^\perp(z_f, z_f^2 q_T^2)$ and $S^{-1}(q_T^2)$. If one sets $x_{bj}^2 \hat{\xi}^2 = \hat{\xi}^2 / z_f^2 = \rho Q^2$ as a regularization scheme for the rapidity divergence [16,18], the terms in $\{\}$ of (75) become $2 \ln \frac{Q^2}{q_T^2}$, and (75) completely agrees with (38). This way we have confirmed that the twist-3 FF contribution and the TMD polarizing FF contribution to $ep \rightarrow e\Lambda^\uparrow X$ are consistent at $\Lambda_{\text{QCD}} \ll P_T \ll Q$.

VI. CONCLUSION

In this paper we have investigated the consistency between the collinear twist-3 FF contribution and the TMD polarizing FF contribution at $\Lambda_{\text{QCD}} \ll P_T \ll Q$ for $ep \rightarrow e\Lambda^\uparrow X$ in LO QCD. Although the collinear twist-3 FFs give rise to all five azimuthal asymmetries for this process, only $\sin\Phi_S$ -asymmetry survives in the leading inverse power of P_T/Q . In the TMD factorization, the $\sin\Phi_S$ -asymmetry is caused by the TMD polarizing FF. Calculating the dependence of the polarizing FF on the intrinsic transverse momentum $k_\perp$ in perturbative QCD at

$k_\perp \gg \Lambda_{\text{QCD}}$, the FF can be expressed in terms of the collinear twist-3 FFs including purely gluonic ones. Inserting this expression into the TMD factorization formula, we found the result completely agrees with that obtained as the small P_T -limit of the collinear twist-3 FF contribution to the cross section. This indicates that the two frameworks describe the polarization consistently in the whole kinematics region.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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