

Yu-Shiba-Rusinov states in color-superconducting quark matter

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The high-density region of the QCD phase diagram displays an intricate competition between color superconductivity and the QCD Kondo effect due to color exchange in quark matter containing a single heavy quark impurity. We explore the characteristic impurity-induced superconducting subgap states arising in such systems by generalizing the surrogate model solver, recently considered in the context of condensed matter physics [Phys. Rev. B **108**, L220506 (2023)]. The method consists of approximating the full superconducting bulk by only a small number of effective levels whose parameters are chosen so as to best reproduce the Matsubara frequency dependence of the impurity-bulk hybridization function. We numerically solve a surrogate QCD Kondo model describing a quantum impurity color exchange coupled to a two-color two-flavor superconducting bulk. The results directly indicate the presence of multiple phase transitions as the coupling of the impurity to the bulk is increased, due to the interplay between various overscreened states. The methods introduced here are straightforward enough to be extended to more realistic QCD scenarios.

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I. INTRODUCTION

Heavy quarks are essential probes for the study of hot and dense matter arising in heavy-ion collisions and compact neutron or quark stars [1–4]. In this context, the heavy charm or bottom quarks may be regarded as impurities immersed in a medium composed of light up, down, and strange quarks, with the complex non-Abelian interactions specific to quantum chromodynamics (QCD) then leading to a subtle interplay of various many-body effects. In particular, heavy quarks or hadrons may experience the Kondo effect through interactions (in the color or isospin channels) with the nuclear or quark medium [5–16]. At low baryon densities, this QCD Kondo effect competes with the formation of the chiral condensate and at high baryon densities with the formation of a diquark condensate [17,18]. In the latter case, the Fermi surface is subject to the Cooper instability due to the attractive interactions between light quarks, which favor a color-superconducting ground state at low temperatures [19,20].

The interplay of Kondo screening and superconductivity has been investigated for decades in condensed matter physics [21–30], with the complex properties of the impurity models describing Kondo systems having been mapped out using a complementary set of techniques, such as the scaling approach [31], Wilson’s numerical renormalization group [32], the Bethe ansatz solution [33–36], and various large- N expansion schemes [37,38]. The interest in such problems has been revived in recent years following the discovery of the Kondo effect in quantum dots (QDs) [39–44], which essentially behave as tunable magnetic impurities. When tunnel coupled to superconducting leads, they induce so-called Andreev bound states inside the superconducting gap. These subgap states range from Yu-Shiba-Rusinov (YSR) states [21–23], induced by odd occupied Coulomb blockaded QDs with a local magnetic moment, to a localized quasiparticle excitation above an induced gap on proximitized QDs with smaller charging energy [45–48].

As the finite BCS gap of the superconducting quasiparticle excitations essentially cuts off all logarithmic singularities and hinders an actual Kondo problem, the calculational effort required to obtain the low-lying subgap bound states turns out to be rather modest. This is due to the smooth and well-behaved dependence of the BCS Green’s functions on the Matsubara frequency, which admits a rapidly converging few-level approximation. The latter allowed us to match the

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numerically exact single-impurity numerical renormalization group results with a low-dimensional surrogate BCS model and to then explore the subgap physics of more complicated multi-impurity and/or multiterminal problems [49–51].

Impurity problems of similar complexity naturally arise when considering the extra color and flavor degrees of freedom inherent to QCD. The main purpose of this work is, thus, to show that the above surrogate model solver (SMS) is an efficient method for studying the low-lying states induced by quantum impurities in high-density color-superconducting matter. This constitutes an unbiased approach that goes beyond the limiting classical spin approximation [18] and is, thus, able to account for the various phase transitions expected for an impurity with multiple quantum numbers, as shown below.

Furthermore, our approach provides a complementary perspective to mean-field-based studies of the QCD Kondo effect (see, e.g., the recent Ref. [16] and references therein) and to recent exact solutions [52,53]. Let us note here that the QCD Kondo effect is expected to affect the transport properties of quark matter at low temperatures, e.g., to enhance the electrical resistivity and to suppress the shear viscosity [6,9]. The anisotropic behavior of these quantities could then be linked to the inhomogeneous distribution of heavy impurities in low-energy heavy-ion collisions, e.g., in experiments at RHIC, GSI-FAIR, and J-PARC which will involve high-density quark matter with charm quark impurities [6]. As the existence of the Kondo phase has direct consequences on the equation of state for light quark matter with charm impurities, it is expected to influence the possible masses and sizes of quark stars [54]. In the context of the neutrino emissivity as a cooling mechanism of neutron stars, the Kondo effect may prove to be a suppression mechanism for the emissivity associated with the two-flavor color-superconducting phase, leading to cooling rates in closer agreement to the data [11].

The paper is organized as follows. We define in Sec. II the single-impurity QCD Kondo model, to which we apply the SMS approximation in Sec. III. In Sec. IV, we outline and comment on the numerical results regarding the phase diagrams of the surrogate Kondo models and their corresponding correlation functions. The final Sec. V is devoted to the conclusions and perspectives.

II. COLOR-SUPERCONDUCTING KONDO MODEL

Following Ref. [18], we consider a toy model of two-color two-flavor Dirac fermions, invariant under the $SU(2) \times SU(2) \times SU(2)$ that acts on the spin(σ)-flavor (τ)-color(t) indices:

$$\hat{\mathcal{H}}_0 = \int d^3x \left\{ \psi^\dagger (i v \boldsymbol{\sigma} \cdot \nabla - \mu) \psi - \frac{\Delta}{2} [\psi^T \sigma_2 \tau_2 t_2 \psi + \text{H.c.}] \right\}, \quad (1)$$

suitable to describe the two-flavor color-superconducting phase that may be realized in the interior of compact stars [55,56]. In this phase, quarks with one out of three colors remain gapless and neutral under the unbroken gauge group $SU(2) \subset SU(3)$, their interactions with the gapped quarks and impurities being suppressed at low energies. Thus, we are led to the above model involving two-color two-flavor Dirac fermions with a Majorana mass Δ .

We will consider for simplicity a single quark chirality as in Ref. [18], where the subgap states associated with a classical colorful impurity were computed by employing the T-matrix method. In this context, “classical” refers to the fact that the impurity is treated as a (pseudo-)spin-dependent external potential, with an interaction term of the form $\hat{\mathcal{H}}_{\text{cl}} = \int d^3x \psi^\dagger U(\mathbf{x}) \psi$. For example, the potential corresponding to a localized colorful impurity would be given by $U(\mathbf{x}) = u \delta(\mathbf{x}) t_3$, with u parametrizing its strength.

In this work, by using the surrogate methodology outlined in the next section, we are able to go beyond the limiting case of a frozen (classical) color pseudospin configuration to a fully dynamical quantum variable. Thus, we consider here a quantum impurity color exchange coupled to the bulk medium:

$$\hat{\mathcal{H}}_{\text{exc}} = J_0 \mathbf{T}_{\text{imp}} \cdot \psi^\dagger(\mathbf{0}) \mathbf{t} \psi(\mathbf{0}), \quad (2)$$

leading to the total QCD Kondo model Hamiltonian

$$\hat{\mathcal{H}}_{\text{Kondo}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\text{exc}}. \quad (3)$$

Above, the light-quark field ψ carries spin ($\uparrow, \downarrow$), color ($\uparrow, \downarrow$), and flavor (u, d) quantum numbers, and $\mathbf{T}_{\text{imp}}$ represents the impurity’s color pseudospin with magnitude $T = 1/2$. The above Eq. (3) may be seen as a reduced version of the model found in Eqs. (46) and (47) in Ref. [18], involving a four-fermion term which effectively accounts for the gluon-mediated color-current interaction. By focusing only on color exchange, we do not take into account any potential scattering, spin-exchange, and spin-color exchange interactions. This is justified as heavy quarks do not experience spin-dependent interactions in quark matter [1,3,57].

Equation (1) may be rewritten in the Nambu basis $\Psi_{\mathbf{k}} = (\psi_{\mathbf{k}} \psi_{-\mathbf{k}})^T$ as

$$\hat{\mathcal{H}}_0 = \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Psi_{\mathbf{k}}^\dagger \mathcal{H}_{\mathbf{k}} \Psi_{\mathbf{k}}, \quad (4)$$

with

$$\mathcal{H}_{\mathbf{k}} = \begin{pmatrix} (-v\mathbf{k} \cdot \boldsymbol{\sigma} - \mu) \otimes \mathbb{1}_4 & -\Delta \sigma_2 \tau_2 t_2 \\ -\Delta \sigma_2 \tau_2 t_2 & (-v\mathbf{k} \cdot \boldsymbol{\sigma}^T + \mu) \otimes \mathbb{1}_4 \end{pmatrix}, \quad (5)$$

where $\mathbb{1}_4 = \mathbb{1}_2^{\text{color}} \otimes \mathbb{1}_2^{\text{flavor}}$. In the following, we will need the momentum integrated Green's function for $G_0(\mathbf{k}, i\omega) = 1/(i\omega - \mathcal{H}_{\mathbf{k}})$ which will be reinterpreted below Eq. (11) as the tunneling self-energy Σ_T :

$$\Sigma_T \equiv J_0 \int \frac{d^3\mathbf{k}}{(2\pi)^3} G_0(\mathbf{k}, i\omega) = J_0 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\mathcal{X}^2 - 4\mu^2 v^2 k^2} \times \begin{pmatrix} [(i\omega - \mu)\mathcal{X} - 2\mu v^2 k^2] \mathbb{1}_8 & -\Delta \sigma_2 \tau_2 t_2 \mathcal{X} \\ -\Delta \sigma_2 \tau_2 t_2 \mathcal{X} & [(i\omega + \mu)\mathcal{X} + 2\mu v^2 k^2] \mathbb{1}_8 \end{pmatrix}, \quad (6)$$

where $\mathcal{X} = -\omega^2 - \mu^2 - \Delta^2 - v^2 k^2$. For more computational details, see Sec. IV B in Ref. [18]. For a weak superconductor $\Delta \ll \mu$, this simplifies to $\mathcal{X} \approx -\mu^2 - v^2 k^2$. Furthermore, if we restrict the domain of integration to a thin shell of half-bandwidth D around the Fermi surface as in the conventional BCS theory, we get $\mathcal{X} \approx -2\mu^2$ and

$$\Sigma_T = \mathcal{J} \begin{pmatrix} -i\omega \mathbb{1}_8 & \Delta \sigma_2 \tau_2 t_2 \\ \Delta \sigma_2 \tau_2 t_2 & -i\omega \mathbb{1}_8 \end{pmatrix} g(\omega), \quad (7)$$

with ω being the Matsubara frequency and

$$g(\omega) \equiv \frac{1}{\pi} \int_{-D}^D d\xi \frac{1}{\xi^2 + \Delta^2 + \omega^2} = \frac{2}{\pi} \frac{\arctan\left(\frac{D}{\sqrt{\Delta^2 + \omega^2}}\right)}{\sqrt{\Delta^2 + \omega^2}}, \quad (8)$$

where we denoted the dimensionless color-exchange coupling by $\mathcal{J} = \pi\rho(\mu)J_0/2$, which is given in terms of the density of states at the Fermi energy in the normal state, $\rho(\mu) = \mu^2/(2\pi^2 v^3)$. Let us note that the thin shell approximation introduces only mild quantitative effects for the resulting phase diagram [18]. We thus leave the rigorous treatment of the above Eq. (6) for future works devoted to more realistic applications.

III. SURROGATE KONDO MODELS

We now proceed to formulate the SMS prescription for generic superconducting Kondo models. The presence of fermionic bilinears in the exchange term of Eq. (2) prevents us from directly applying the SMS methodology as in the case of Anderson impurities linearly (tunnel) coupled to the superconducting bulk [49]. We may, nevertheless, consider the partition function corresponding to Eq. (3) for the bulk fermions with a frozen impurity configuration, expressed as the coherent state path integral

$$Z[T_{\text{imp}}] = \int \mathcal{D}\psi e^{J_0 T_{\text{imp}} \cdot (\sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger) t (\sum_{\mathbf{k}'} \psi_{\mathbf{k}'}) + \sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger G_0^{-1} \psi_{\mathbf{k}}}, \quad (9)$$

where the summation over Matsubara frequencies is implied. The exchange coupling term may be eliminated by a Hubbard-Stratonovich transformation introducing a

local auxiliary η field, leading to

$$Z[T_{\text{imp}}] = \int \mathcal{D}\psi \mathcal{D}\eta e^{\eta^\dagger (T_{\text{imp}} t) \eta + [\sqrt{J_0} \sum_{\mathbf{k}} \eta^\dagger \psi_{\mathbf{k}} + \text{H.c.}] + \sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger G_0^{-1} \psi_{\mathbf{k}}}, \quad (10)$$

and, since the ψ fermions are now linearly (tunnel) coupled to the local auxiliary η field, they may be integrated out to give

$$Z[T_{\text{imp}}] = \int \mathcal{D}\eta e^{\eta^\dagger (T_{\text{imp}} t) \eta + \eta^\dagger \Sigma_T \eta}, \quad (11)$$

written in terms of the tunneling self-energy defined in Eq. (6) above.

At this point, the g function of Eq. (8), and thus the tunneling self-energy itself, of the full continuous BCS model may be approximated within the SMS approach [49] by that corresponding to a small number $\tilde{L}$ of effective levels:

$$\tilde{g}_{\text{even}}(\omega) \equiv 2 \sum_{\ell=1}^K \frac{\gamma_\ell}{\xi_\ell^2 + \Delta^2 + \omega^2}, \quad \tilde{L} = 2K, \\ \tilde{g}_{\text{odd}}(\omega) \equiv \frac{\gamma_0}{\Delta^2 + \omega^2} + \tilde{g}_{\text{even}}(\omega), \quad \tilde{L} = 2K + 1. \quad (12)$$

Note that each SC level with energy ξ contributes with a factor of $(\xi^2 + \Delta^2 + \omega^2)^{-1}$ to the g function of Eq. (8). A $\tilde{g}$ function as in Eq. (12) may, thus, be obtained by integrating out an effective superconductor with the same gap, Δ , as the original one and whose $\tilde{L}$ discrete levels with energies $\pm|\tilde{\xi}_\ell|$ are coupled to the η field via tunneling matrix elements $\tilde{t}_\ell \sim \sqrt{\gamma_\ell}$. The discrete $\tilde{g}$ -function approximation, thus, retains all the properties of the full impurity-bath hybridization, e.g., causality for the surrogate hybridization function $\tilde{\Sigma}_T$ defined as in Eq. (7) in terms of the $\tilde{g}$ function. The parameters $\{\gamma_\ell, \tilde{\xi}_\ell\}$ define the surrogate model Hamiltonian as a discretized version of Eq. (4). They are determined in Ref. [49] by minimizing the cost function $\chi^2 = \sum_j |g(\omega_j) - \tilde{g}(\omega_j)|^2$, evaluated on a logarithmically spaced grid of frequencies ω_j in order to ensure a good fit at subgap frequencies. The numerical values used in this work are the same as in Ref. [49] and correspond to $D = 10\Delta$, namely, $\gamma_0 = 0.9618$ for $\tilde{L} = 1$, $\gamma_1 = 1.2642$ and $\tilde{\xi}_1 = 1.3099$ for $\tilde{L} = 2$, and $\gamma_0 = 0.508$, $\gamma_1 = 1.8454$, and $\tilde{\xi}_1 = 2.7546$ for $\tilde{L} = 3$ (all in units of Δ). The results presented below are insensitive to the precise value of the shell width D ; see also Appendix B.

Having obtained a suitable surrogate approximation for the tunneling self-energy, we may now retrace our steps through the path integrations. In doing so, we use the representation

$$e^{\eta^\dagger \tilde{\Sigma}_T \eta} = \int \mathcal{D}\tilde{c} e^{\sum_\ell \tilde{c}_\ell^\dagger \tilde{G}_0^{-1} \tilde{c}_\ell + (\sum_\ell \eta^\dagger \sqrt{\mathcal{J}} \gamma_\ell \tilde{c}_\ell + \text{H.c.})}, \quad (13)$$

which leads, upon integrating out the η field, to the surrogate partition function

$$\tilde{Z}[\mathbf{T}_{\text{imp}}] = \int \mathcal{D}\tilde{c} e^{\mathcal{J} \mathbf{T}_{\text{imp}} \cdot (\sum_\ell \sqrt{\gamma_\ell} \tilde{c}_\ell^\dagger) t (\sum_{\ell'} \sqrt{\gamma_{\ell'}} \tilde{c}_{\ell'}) + \sum_\ell \tilde{c}_\ell^\dagger \tilde{G}_0^{-1} \tilde{c}_\ell}. \quad (14)$$

This corresponds to the surrogate Kondo Hamiltonian

$$\tilde{\mathcal{H}}_{\text{Kondo}} = \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_{\text{exc}}, \quad (15)$$

where

$$\tilde{\mathcal{H}}_0 = \sum_{\ell=1}^{\tilde{L}} \tilde{\xi}_\ell \tilde{c}_\ell^\dagger \tilde{c}_\ell - \frac{\Delta}{2} (\tilde{c}_\ell^\dagger \sigma_2 \tau_2 t_2 \tilde{c}_\ell + \text{H.c.}) \quad (16)$$

and

$$\tilde{\mathcal{H}}_{\text{exc}} = \mathcal{J} \mathbf{T}_{\text{imp}} \cdot \sum_{\ell, \ell'=1}^{\tilde{L}} \sqrt{\gamma_\ell \gamma_{\ell'}} \tilde{c}_\ell^\dagger \tilde{c}_{\ell'}. \quad (17)$$

Here, we suppressed the spin-color-flavor indices for the $\tilde{c}_\ell$ operators. Explicitly, $\tilde{c}_{\ell, \sigma \tau f}$ would create a fermion of spin $\sigma = \uparrow, \downarrow$, color $\tau = \uparrow, \downarrow$, and flavor $f = u, d$ on the level ℓ . Explicitly, for the simplest $\tilde{L} = 1$ or zero-bandwidth (ZBW) case, the pairing term reads

$$\begin{aligned} \tilde{\mathcal{H}}_0^{(\tilde{L}=1)} = & -i\Delta u_{\uparrow\uparrow}^\dagger d_{\downarrow\downarrow}^\dagger + i\Delta u_{\uparrow\uparrow}^\dagger d_{\downarrow\downarrow}^\dagger + i\Delta u_{\downarrow\downarrow}^\dagger d_{\uparrow\uparrow}^\dagger - i\Delta u_{\downarrow\downarrow}^\dagger d_{\uparrow\uparrow}^\dagger \\ & + \text{H.c.} \end{aligned} \quad (18)$$

Before discussing the results obtained by solving the above surrogate Kondo models, let us note that, by increasing the number of surrogate levels $\tilde{L}$, the errors in the g -function fit are rapidly and systematically reduced by several orders of magnitude across (and beyond) the fitting range. This ensures a fast convergence of the SMS subgap spectra and related observables (assuming the ratio of the SC gap to the Kondo temperature is reasonably small). Finally, we refer the reader to Appendix A for a short discussion on the simpler surrogate Kondo models for standard BCS problems.

IV. NUMERICAL RESULTS

We solved for the lowest-lying states in each fermion-parity sector of the surrogate Kondo models with $\tilde{L} = 1$ by exact diagonalization and for $\tilde{L} = 1, 2, 3$ by the density matrix renormalization group (DMRG) in the matrix-product-state formulation, which is straightforward to implement with the ITensor library [58–61]. For the

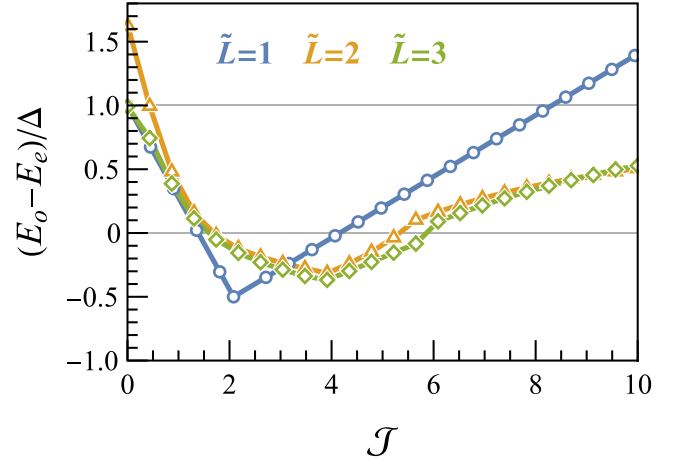


FIG. 1. Energy difference $E_o - E_e$ (in units of Δ) between the lowest even- and odd-fermion-parity states of the surrogate Kondo models for $\tilde{L} = 1, 2, 3$ vs the dimensionless color-exchange coupling $\mathcal{J}$.

DMRG calculations, we employed a maximum bond dimension of 5000 and an energy convergence threshold of $10^{-4}\Delta$. For the $\tilde{L} = 1$ case, we found a perfect numerical agreement between the exact diagonalization and DMRG results.

The results regarding the evolution of the lowest-lying subgap states in each fermion-parity sectors are shown in Fig. 1. First, all surrogate models agree on the crucial qualitative features of the phase diagram, directly indicating the existence of two distinct phase transitions with increasing coupling strength. From a more quantitative perspective, we notice some minor shortcomings in both the $\tilde{L} = 1$ and $\tilde{L} = 2$ surrogates. On the one hand, the overestimation of the strong-coupling excitation energy in the $\tilde{L} = 1$ model is encountered also in the standard BCS case [49] and is related by the inability of the simple ZBW approximation (and, to a lesser extent, of the other odd- $\tilde{L}$ models) to properly screen the impurity. On the other hand, the $\tilde{L} = 2$ model overestimates the excitation energy at weak couplings; this is caused by the absence of the $\tilde{\xi}_0 = 0$ level which would normally accommodate the lowest-lying screening quasiparticle at $E_{\text{qp}} = \Delta$, as is the case in the odd- $\tilde{L}$ models. Nevertheless, the $\tilde{L} = 2, 3$ models agree to a very good extent over the entire range of moderate to strong couplings shown in Fig. 1, thus validating the SMS convergence for the subgap spectrum. Even if naively one could argue that smaller values of $\tilde{L}$ than in the standard BCS case are needed to achieve a good convergence (see Ref. [49] and also Appendix A), the complexity of the surrogate model space also grows faster in the present case due to the extra color-flavor quantum numbers.

To gain more insight into the impurity screening processes across the phase diagram, we plot in Fig. 2(a) the evolution of the lowest-lying states in each sector of definite SC

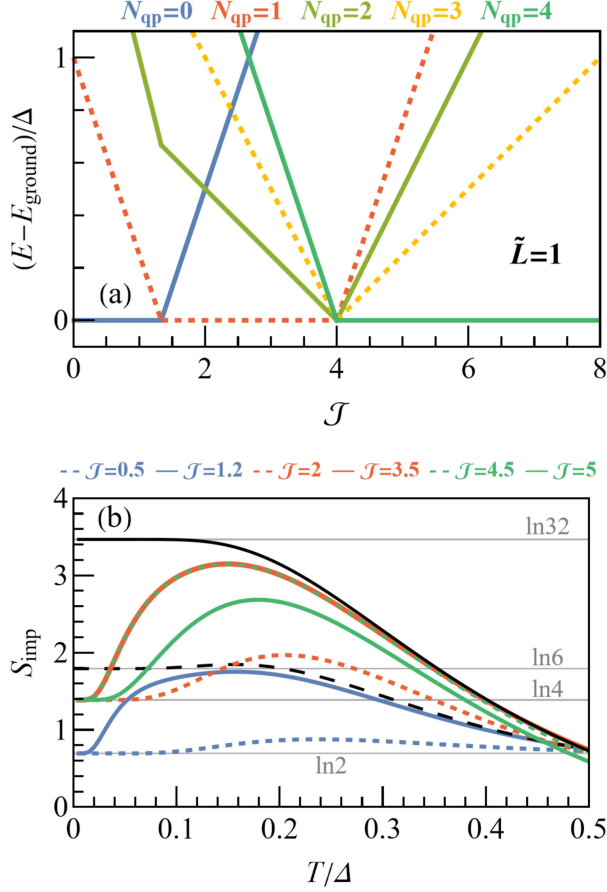


FIG. 2. (a) Energies (relative to the ground state) of the lowest-lying states in each sector of definite SC quasiparticle number $N_{\text{qp}} = 0, \dots, 4$ vs the dimensionless color-exchange coupling J . Continuous (dashed) lines denote states with even (odd) fermion parity. (b) Impurity entropy vs temperature for various color-exchange couplings J across the various phases (two values are shown for each phase). The black curves denote the exact phase transition values, $J = 4/3$ (dashed curve) and $J = 4$ (continuous curve). The results in both panels are obtained by a $\tilde{L} = 1$ (ZBW) model with $\gamma/\Delta = 1$.

quasiparticle number $N_{\text{qp}} = 0, \dots, 4$ for the ZBW ($\tilde{L} = 1$) model. In this simple case, their energies $E_{N_{\text{qp}}}$ may be computed analytically: $E(N_{\text{qp}})/\Delta = N_{\text{qp}}(1 - J/4) - J/2$ at $\gamma/\Delta = 1$ for $N_{\text{qp}} \geq 1$, all relative to the unscreened state's constant reference value $E(N_{\text{qp}} = 0) = 0$.

As expected, the first phase transition is due to a single quasiparticle excitation becoming favored, with increasing coupling, over the SC ground state with no quasiparticles. Note that the former is a fourfold degenerate state due to the screening of the impurity's color pseudospin, say, $\uparrow\uparrow$, by each of the four spin-flavor combinations $u_{\uparrow\uparrow}, u_{\downarrow\uparrow}, d_{\uparrow\uparrow}$, and $d_{\downarrow\uparrow}$. At larger couplings, the states with extra screening quasiparticles become increasingly favored. Incidentally, the present symmetric coupling situation features the special point $J = 4$ where all quasiparticle excitations become degenerate.

Beyond this value, the ground state will involve all four spin-flavor combinations screening the impurity's color.

Overall, the ZBW ground state displays an $N_{\text{qp}} = 0 \rightarrow 1 \rightarrow 4$ evolution with increasing coupling, lacking the phases with two or three screening quarks which appear only as parity-flipping excitations in the subgap spectrum.

The possibility of computing the full subgap spectrum with the ZBW approach also allows for a glimpse into the low-temperature thermodynamics of our heavy-quark impurity model. It was recently pointed out in Ref. [62] that the YSR regime displays a pronounced nonmonotonic entropy behavior, due to the thermal activation of subgap states (this contrasts with a conventional Kondo problem, where the impurity entropy is monotonic). We illustrate in Fig. 2(b) the corresponding temperature dependence obtained across the phase diagram for the impurity entropy $S_{\text{imp}} = -\partial_T F_{\text{imp}}(T)$, where the free energy $F_{\text{imp}}(T) = -T \ln Z/Z_0$ is obtained from the ratio of partition functions with and without impurity.

At very low temperatures, S_{imp} is in direct correspondence with the nature of the ground state impurity screening process. At small couplings, the impurity is essentially unscreened, with S_{imp} taking the expected value $\ln 2$. For intermediate couplings, the impurity may be screened by each one of four distinct spin-flavor combinations, the resulting entropy thus being $\ln 4$. Finally, at large couplings, the impurity is simultaneously overscreened by four quasiparticles leading to a $T = 3/2$ ground state multiplet with an entropy of $\ln 4$. Note that the impurity entropy is naturally largest at the phase transition points where multiple subgap states become degenerate [see the black curves in Fig. 2(b)]. Away from these points, the flow of the entropy interpolates between the $T = 0$ value and the black curve's tails, thus leading to the intermediate- T nonmonotonic hump also discussed in Ref. [62]. The entropy behavior is, thus, qualitatively similar in the screened and overscreened phases.

For the more complex $\tilde{L} \geq 2$ models with multiple sets of distinct quasiparticle excitations, it becomes advantageous to describe the impurity screening in terms of the color pseudospin quantum numbers and associated correlation functions; see Figs. 3 and 4. In Figs. 3(a) and 3(b), the only qualitative feature that is specific to the $\tilde{L} \geq 2$ surrogates is the gradual screening of the impurity's color with increasing coupling, as opposed to the ZBW case, where the lowest-lying state remains unscreened for weak to moderate couplings. Beyond this first regime, all surrogates agree on the existence of a finite region with $T_{\text{tot}} = 1/2$ and $\langle \vec{T}_{\text{imp}} \cdot \vec{T}_{\text{SC}} \rangle = -1$, corresponding to an overscreened $T_{\text{SC}} = 1$ state. Here, we define

$$\vec{T}_{\text{tot}} = \vec{T}_{\text{imp}} + \vec{T}_{\text{SC}}. \quad (19)$$

Finally, at strong coupling, all models indicate $T_{\text{tot}} = 3/2$ and $\langle \vec{T}_{\text{imp}} \cdot \vec{T}_{\text{SC}} \rangle = -1.5$, corresponding to a $T_{\text{SC}} = 2$ state.

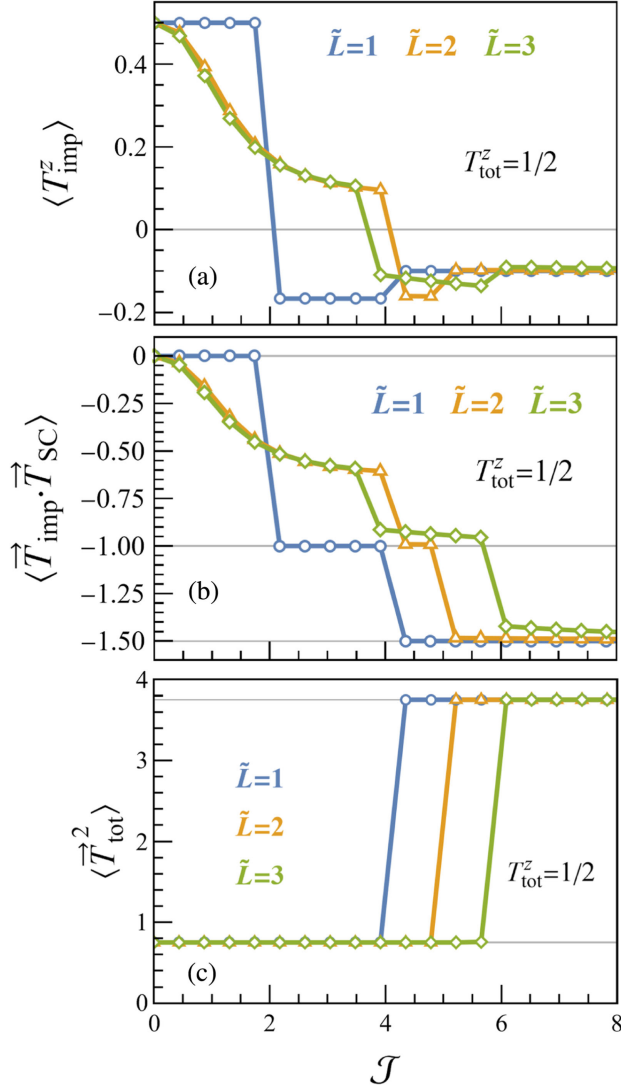


FIG. 3. Color pseudospin properties of the lowest $T_{\text{tot}}^z = 1/2$, even fermion-parity states vs the dimensionless color-exchange coupling $\mathcal{J}$, for the surrogate Kondo models $\tilde{L} = 1, 2, 3$. (a) Average impurity color pseudospin $\langle T_{\text{imp}}^z \rangle$. (b) Impurity-bulk color pseudospin correlation function $\langle \vec{T}_{\text{imp}} \cdot \vec{T}_{\text{SC}} \rangle$, with grid lines at 0, -1, and -1.5. (c) Total color pseudospin squared $\langle \vec{T}_{\text{tot}}^2 \rangle$, with grid lines at 0.75 and 3.75.

The situation in the odd-fermion parity sector is also qualitatively similar for all surrogates, with the lowest-lying state switching with increasing coupling from a color singlet $T_{\text{tot}} = 0$, $T_{\text{SC}} = 1/2$ to an $T_{\text{tot}} = 2$, $T_{\text{SC}} = 3/2$ overscreened state, as seen from Fig. 4. We note that, in this sector for $T_{\text{tot}}^z = 0$, the impurity's average color pseudospin projection also vanishes, $\langle T_{\text{imp}}^z \rangle = 0$.

Overall, the screening picture given by the ZBW model is qualitatively preserved by the $\tilde{L} = 2, 3$ models, with the notable exception of the weak coupling regime where finite-bandwidth effects are necessary to reproduce the gradual (under)screening of the impurity's color. The $\tilde{L} = 2, 3$

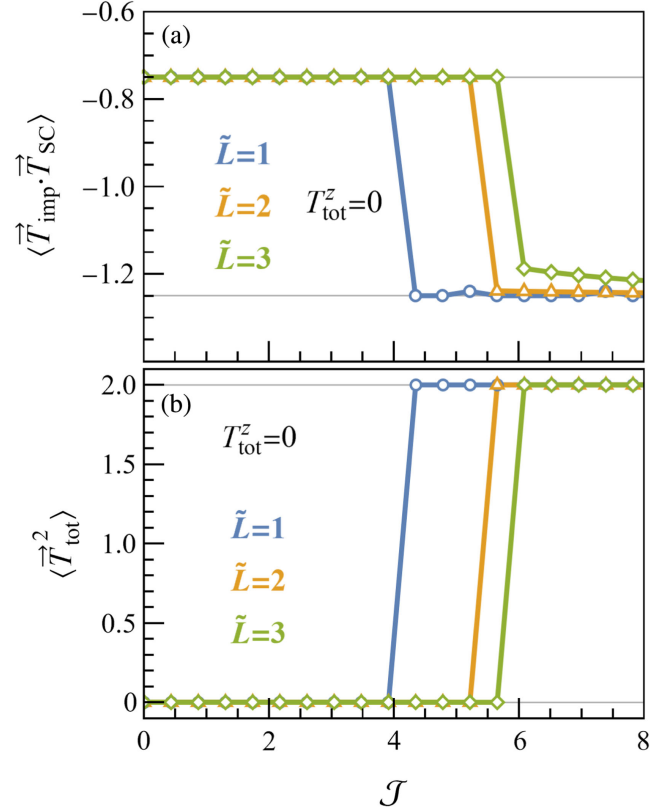


FIG. 4. Color pseudospin properties of the lowest $T_{\text{tot}}^z = 0$, odd fermion-parity states vs the dimensionless color-exchange coupling $\mathcal{J}$, for the surrogate Kondo models $\tilde{L} = 1, 2, 3$. (a) Impurity-bulk color pseudospin correlation function $\langle \vec{T}_{\text{imp}} \cdot \vec{T}_{\text{SC}} \rangle$, with grid lines at -0.75 and -1.25. (b) Total color pseudospin squared $\langle \vec{T}_{\text{tot}}^2 \rangle$, with grid lines at 0 and 2.

surrogates are generally in a good semiquantitative agreement, as seen from Figs. 1, 3, and 4.

V. CONCLUSIONS

We have presented a study of the subgap states induced by a quantum impurity in high-density quark matter, where color superconductivity competes with the QCD Kondo effect. We have obtained the subgap spectrum of a two-color two-flavor QCD Kondo model with minimal numerical effort by a generalization of the few-level surrogate model solver [49] to Kondo systems. The main qualitative features of the phase diagram are already present in the crudest one-level (zero-bandwidth) approximation of the superconducting bulk, and good quantitative convergence is achieved with the inclusion of one or two suitably chosen extra levels.

Concretely, we have found that with increasing color-exchange coupling the impurity experiences two phase transitions. At a moderate coupling, we switch from an unscreened or underscreened impurity to the phase where its color is screened by one of the four spin-flavor light-quark

species. This gets directly superseded at a larger coupling by a phase where all four species are involved in the screening process. The alternative screened phases involving two or three quarks coupled to the impurity (previously hypothesized in Ref. [18]) appear only as excitations in our subgap spectrum.

Because of its simplicity, our approach may prove instrumental in exploring more involved QCD scenarios, e.g., displaying the full SU(3) color symmetry and/or the dynamical behavior of the coupling as a function of density. The method developed here could also be used as a nonperturbative alternative to the mean-field approximation of Ref. [9] in estimating the effective interaction of the heavy quark with the light-quark medium, a preliminary step necessary in the actual computation of various transport coefficients.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [63].

APPENDIX A: SURROGATE KONDO MODELS FOR STANDARD BCS SUPERCONDUCTORS

We include here further numerical results supporting the use of the prescription of Eq. (15) as a surrogate Kondo model. In particular, we consider a spin-1/2 impurity coupled to a standard BCS bulk stripped of its extra QCD-specific degrees of freedom:

$$\tilde{\mathcal{H}}_{\text{Kondo}} = \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_{\text{exc}}, \quad (\text{A1})$$

where

$$\tilde{\mathcal{H}}_0 = \sum_{\ell=1}^{\tilde{L}} \sum_{\sigma=\uparrow\downarrow} \xi_{\ell\sigma} \tilde{c}_{\ell\sigma}^\dagger \tilde{c}_\ell - \sum_{\ell=1}^{\tilde{L}} \Delta (\tilde{c}_{\ell\downarrow} \tilde{c}_{\ell\uparrow} + \text{H.c.}) \quad (\text{A2})$$

and

$$\tilde{\mathcal{H}}_{\text{exc}} = \mathcal{J} \mathcal{S}_{\text{imp}} \cdot \sum_{\ell, \ell'=1}^{\tilde{L}} \sqrt{\gamma_\ell \gamma_{\ell'}} \tilde{c}_\ell^\dagger \frac{\boldsymbol{\sigma}}{2} \tilde{c}_{\ell'}. \quad (\text{A3})$$

This Kondo model may be obtained as a limit of the Anderson model considered in Ref. [49] for a large Coulomb interaction energy U , which suppresses the impurity's charge fluctuations and effectively reduces it to a spin degree of freedom. In fact, the subgap spectrum

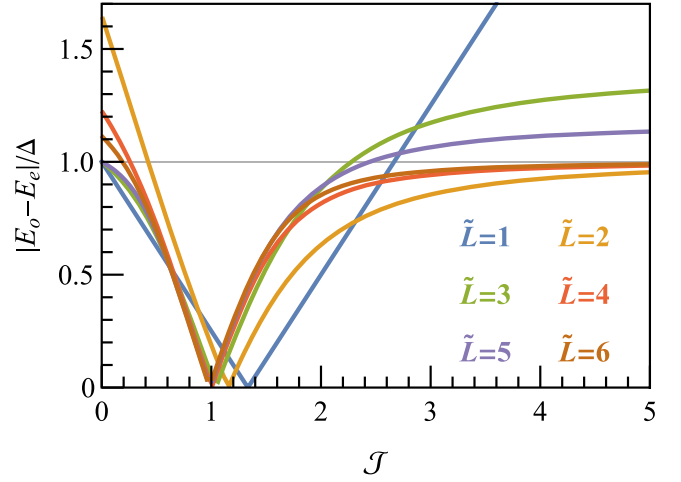


FIG. 5. Energy difference $E_o - E_e$ (in units of Δ) between the lowest even- and odd-fermion-parity states of the surrogate Kondo models of Eq. (A1) for $\tilde{L} = 1, \dots, 6$ vs the dimensionless exchange coupling $\mathcal{J}$.

shown in Fig. 5 for this simple case is in excellent agreement with the full Anderson model results with $U = 15\Delta$; cf. Fig. 3 in Ref. [49]. In particular, the $\tilde{L} = 1$ (ZBW) subgap state energy shown in Fig. 5 as a blue line reads $(E_o - E_e)/\Delta = 1 - 3\mathcal{J}/4$.

APPENDIX B: ADDITIONAL RESULTS

We provide here additional details regarding the influence of the precise value of the chosen shell width on the subgap spectrum. We represent in Fig. 6 the $\tilde{L} = 2$ results for various bandwidths $D/\Delta = 6, 10, 14$, showing minimal quantitative differences overall. Across this range of shell widths, the local variations in the excitation energies are at most of the order of 0.1Δ .

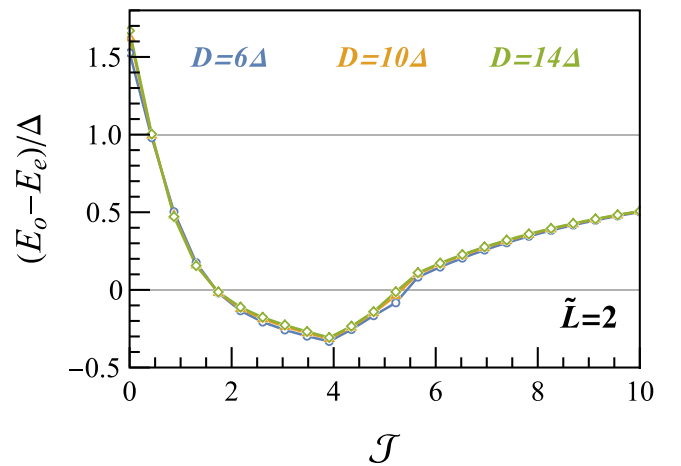


FIG. 6. Bandwidth dependence of the energy difference $E_o - E_e$ (in units of Δ) between the lowest even- and odd-fermion-parity states of the surrogate Kondo model of Eq. (15) for $\tilde{L} = 2$ vs the dimensionless exchange coupling $\mathcal{J}$.

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