

## Path-integral predictions for preasymptotic quantum tunneling

Joshua Lin<sup>1,2,\*</sup> Bruno Scheihing-Hitschfeld<sup>3,†</sup> and Thomas Steingasser<sup>1,4,5,‡</sup>

<sup>1</sup>*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

<sup>2</sup>*Physics Division, Argonne National Laboratory, Lemont, Illinois 60439, USA*

<sup>3</sup>*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA*

<sup>4</sup>*Black Hole Initiative at Harvard University, 20 Garden Street, Cambridge, Massachusetts 02138, USA*

<sup>5</sup>*Departamento de Física Teórica, Universidad Autónoma de Madrid, and IFT-UAM/CSIC, Cantoblanco, 28049, Madrid, Spain*



(Received 21 December 2025; accepted 30 April 2026; published 6 July 2026)

When tunneling occurs out of generic initial states, a significant fraction of probability is lost at early times, during which the dynamics is governed by excited resonance states. However, first-principles analyses based on path-integrals have only captured the leading asymptotic behavior, during which the tunneling rate is dominated by the false vacuum contribution. In this work, we discuss the behavior in the preasymptotic regime from a first-principles path-integral perspective. We demonstrate how the relevant expressions can be evaluated systematically through semiclassical methods in the recently developed steadyon picture. This approach allows one to trace the role of the relevant physical scales, making transparent the underlying assumptions and approximations, and offering a clear path to establishing a systematically improvable framework to evaluate tunneling rates nonperturbatively.

DOI: [10.1103/PhysRevD.114.L011903](https://doi.org/10.1103/PhysRevD.114.L011903)

**Introduction.** Quantum tunneling is one of the most universal features of quantum systems, governing phenomena that may take place at every length or energy scale, from nuclear decay rates to out-of-equilibrium phenomena in the early Universe. A common feature of tunneling phenomena is that they rarely occur out of exact, idealized false vacuum states. And yet, modern first-principles path-integral based investigations of tunneling have only analyzed tunneling out of the latter state. While this is sufficient to describe the long-term asymptotics of the relevant probabilities, it fails to capture the dominant mechanism of probability loss at intermediate times. In this regime, the tunneling rate is dominated by contributions from excited states.

In this Letter, we show how this behavior can be captured through a combination of the steadyon picture for the semiclassical evaluation of real-time path-integrals [1,2] with the modern, first-principles based “direct approach” [3,4], extending the advantages of this approach to the analysis of the early-time transient dynamics. While the specific details of the transient dynamics are highly dependent

on the initial state, numerical investigations suggest that a significant fraction of the overall probability flux is lost during this transient period. Moreover, many applications rely on tunneling to occur during these early times.

On the one hand, a popular approach to tunneling rate calculations is the instanton picture, which preserves all of the familiarity and power of Euclidean path-integrals, at the cost of being restrictive on the class of states it can be used. On the other hand, when the initial state can be expanded in terms of resonant states, the tunneling rate can alternatively be calculated using the WKB approximation. Obtaining these states can, however, prove challenging, especially in multiple dimensions or field theory. Moreover, obtaining a transparent connection between the tunneling rate and the WKB method has been found to give to a surprising number of subtleties already for a point particle in one spatial dimension [3]. We will demonstrate how our approach can be applied to *generic* wave functions, before relating it to the WKB regime and resonant states, while at the same time showing how it extends the advantages of the instanton-based approach.

For simplicity, we focus in this work on the simple example of a point particle in one spatial dimension. However, due to our analysis’ connection to the direct approach, it inherits its straightforward generalization to higher-dimensional systems, on which we comment in the final section. We denote the basin surrounding the false vacuum  $x_{\text{FV}}$  as  $\mathcal{F}$ , and its complement (the real vacuum region) by  $\mathcal{R}$  (see Fig. 1). We can, without loss of generality, assume that  $x_f < x_r$  for all  $x_f \in \mathcal{F}$  and  $x_r \in \mathcal{R}$ . We restrict

\*Contact author: [joshua.lin@anl.gov](mailto:joshua.lin@anl.gov)

†Contact author: [bscheihi@kitp.ucsb.edu](mailto:bscheihi@kitp.ucsb.edu)

‡Contact author: [thomas.steingasser@uam.es](mailto:thomas.steingasser@uam.es)

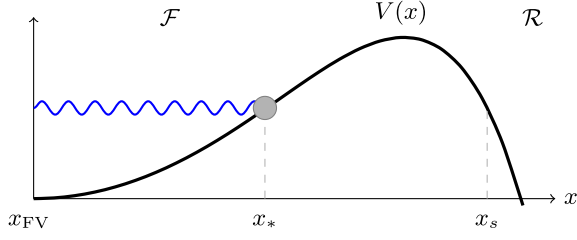


FIG. 1. At  $t = 0$ , the particle is fully localized within the well  $\mathcal{F}$ .  $x_*$  denotes the point beyond which its wave function decays exponentially.  $x_s$  denotes the energetically degenerate point on the other side of the potential barrier.

ourselves to a brief but comprehensive discussion and provide extensive, detailed derivations in our companion article [5].

*Probability flux and tunneling rate.* The tunneling rate can be defined through the probability to find the particle within  $\mathcal{F}$  at a given time  $t$  through

$$P_{\mathcal{F}}(t) = P_{\mathcal{F}}(0) \cdot e^{-\int_0^t \Gamma(t') dt'}, \quad (1)$$

$$\Rightarrow \Gamma(t) = -\frac{\dot{P}_{\mathcal{F}}(t)}{P_{\mathcal{F}}(t)} = \frac{\dot{P}_{\mathcal{R}}(t)}{P_{\mathcal{F}}(t)}, \quad (2)$$

where in the last step, we have used that the probability flow out of  $\mathcal{F}$  is the same as that into  $\mathcal{R}$ . These probabilities can be expressed in terms of the theory's Feynman propagator  $D_F$  as

$$\begin{aligned} P_{\Omega}(t) &= \int_{\Omega} dx \langle x | e^{-iHt} | \Psi \rangle \langle \Psi | e^{iHt} | x \rangle \\ &= \int_{\Omega} dx \int dx_1 \int dx_2 \psi(x_1) \psi^*(x_2) \langle x | e^{-iHt} | x_1 \rangle \langle x_2 | e^{iHt} | x \rangle \\ &= \int_{\Omega} dx \int dx_1 \int dx_2 \psi(x_1) \psi^*(x_2) D_F(x, t | x_1) D_F^*(x, t | x_2), \end{aligned} \quad (3)$$

whose structure we will use to calculate  $\dot{P}_{\mathcal{R}}(t)$  and  $P_{\mathcal{F}}(t)$ . Here, and in the remainder of this article, the absence of a time label is equivalent to  $t = 0$ . First, in order to calculate  $\dot{P}_{\mathcal{R}}(t)$ , we follow Ref. [4] and introduce the auxiliary propagator  $\bar{D}_F$  defined by

$$\bar{D}_F(x_s, t_s | x_i) \equiv \int_{x(0)=x_i}^{x(t_s)=x_s} \mathcal{D}x e^{iS[x]} \delta(F_{x_s}[x] - t_s), \quad (4)$$

which satisfies

$$D_F(x_f, t | x_i) = \int_0^t dt_s D_F(x_f, t | x_s, t_s) \bar{D}_F(x_s, t_s | x_i). \quad (5)$$

We denote by  $x_*$  the largest  $x \in \mathcal{F}$  for which the wave function has non-negligible support, and by  $x_s$ , the closest

energetically degenerate point in  $\mathcal{R}$ ; see Fig. 1. For the example of a resonance state (which we discuss at length later)  $|\tilde{E}\rangle$  with  $\tilde{E} = E - i\Gamma/2$ ,  $x_*$  is the classical turning point of a classical particle with energy  $E$ , beyond which the state's wave function decays exponentially. The functional  $F_{x_s}[x(t)]$  then maps the path  $x(t)$  onto the time when it first crosses the point  $x_s$ —i.e., when the analog classical particle completely crosses the barrier.

Next, we make use of the first implicit assumption underlying Eq. (1), namely the absence of backtunneling. In terms of the propagator, this assumption translates to asserting that the propagation of particles from  $x_s$  back into  $\mathcal{F}$  is negligible, and thus, the integration over  $\mathcal{R}$  in  $\dot{P}_{\mathcal{R}}$  may be extended to all of space,

$$\int_{\mathcal{R}} dx_f D_F(x_f, t | x_s, t_s) D_F^*(x_f, t | x_s, t'_s) = D_F(x_s, t'_s | x_s, t_s). \quad (6)$$

It is then straightforward to show that

$$\begin{aligned} \dot{P}_{\mathcal{R}}(t) &= \int dx_1 \int dx_2 \psi(x_1) \psi^*(x_2) \\ &\quad \times D_F(x_s, t | x_1) \bar{D}_F^*(x_s, t | x_2) + \text{c.c.} \end{aligned} \quad (7)$$

To further develop this expression, we introduce two more auxiliary propagators,

$$\underline{D}_F(x_s, t_s | x_*, t_*) \equiv \int_{x(t_*)=x_*}^{x(t_s)=x_s} \mathcal{D}x e^{iS[x]} \delta(G_{x_*}[x] - t_*), \quad (8)$$

$$\begin{aligned} \bar{\underline{D}}_F(x_s, t_s | x_*, t_*) &\equiv \int_{x(t_*)=x_*}^{x(t_s)=x_s} \mathcal{D}x e^{iS[x]} \delta(G_{x_*}[x] - t_*) \delta(F_{x_s}[x] - t_s). \end{aligned} \quad (9)$$

The functional  $G_{x_*}[x]$  now maps the path  $x(t)$  onto the time the path  $x(t)$  crosses  $x_*$  for the *last* time. These new functions can be related to  $D_F$  and  $\bar{D}_F$  through the integral relations,

$$D_F(x_s, t | x_i) = \int_0^t dt_* \underline{D}_F(x_s, t | x_*, t_*) D_F(x_*, t_* | x_i), \quad (10)$$

$$\bar{D}_F(x_s, t | x_i) = \int_0^t dt_* \bar{\underline{D}}_F(x_s, t | x_*, t_*) D_F(x_*, t_* | x_i). \quad (11)$$

These decompositions require that  $x_i < x_* < x_s$ , which can strictly only be achieved if these expressions were to be integrated against wave functions that are fully localized within  $\mathcal{F}$ . This makes transparent the implicit assumption that most of the probability is contained within  $\mathcal{F}$  before reaching  $x_*$ .

When applicable, these decompositions allow one to recombine the wave function with one of the emerging propagators through

$$\int dx_1 D_F(x_*, t_* | x_1) \psi(x_1) = \psi(x_*, t_*). \quad (12)$$

This further simplifies Eq. (7) to

$$\begin{aligned} \dot{P}_{\mathcal{R}} = & \int_0^t d\Delta t \underline{D}_F(x_s, \Delta t | x_*) \psi(x_*, t - \Delta t) \\ & \times \int_0^t d\Delta t' \bar{\underline{D}}_F^*(x_s, \Delta t' | x_*) \psi^*(x_*, t - \Delta t') + \text{c.c.} \end{aligned} \quad (13)$$

This relation applies to *any* wave function subject to our previously made assumptions—the localization within  $\mathcal{F}$  beyond some point  $x_* \in \mathcal{F}$  and the absence of backtunneling. Moreover, it is *exact* beyond these assumptions, not relying on any further expansion.

*Semiclassical evaluation.* The propagators in Eq. (13) can be evaluated semiclassically in the so-called *steadyon picture* developed in Refs. [1,2]. These works suggest regularizing the theory by a deformation of the Hamiltonian by introducing an infinitesimal imaginary part,  $H \rightarrow (1 - i\epsilon)H$ . In this deformed theory, the path-integrals representing the propagators can be evaluated through a stationary phase approximation around a complex solution  $\bar{x}(t)$ ,

$$\underline{D}_F(x_s, \Delta t | x_*) \sim \bar{\underline{D}}_F^*(x_s, \Delta t | x_*) \sim e^{iS[\bar{x}]}. \quad (14)$$

For the probability flux, this implies that

$$\dot{P}_{\mathcal{R}} \sim e^{-2\text{Im}(S[\bar{x}])}. \quad (15)$$

The complex solution  $\bar{x}(t) = \bar{x}_{\text{Re}}(t) + i\bar{x}_{\text{Im}}(t)$ , called *steadyon*, emerges as a solution to the complexified equation of motion,

$$m\ddot{\bar{x}}(t') = (1 - 2i\epsilon)V'(\bar{x}(t')). \quad (16)$$

This equation alone determines  $\bar{x}(t)$  only up to its boundary conditions, which depend on the time interval  $\Delta t$ . Thus, the integral over  $\Delta t$  in Eq. (13) amounts to an integral over contributions from different steadyons, which we can carry out through another (stationary phase) approximation for the combination of propagator with wave function. In addition to the direct dependence of the action on  $\Delta t$ , this induces an indirect dependence of the action on  $\Delta t$  through the form of the steadyon. Taking into account both of these dependencies, the condition for a saddle point in  $\Delta t$  determines the initial velocity of the steadyon. Neglecting terms suppressed by the regularization parameter  $\epsilon$  and using conservation of energy in the deformed theory, we find

$$\begin{aligned} i \frac{d}{d\Delta t} \log \psi(x_*, t - \Delta t) &= \frac{d}{d\Delta t} S(\Delta t) \\ &= (1 + i\epsilon) \frac{m}{2} \dot{\bar{x}}(0)^2 - (1 - i\epsilon)V(\bar{x}_*). \end{aligned} \quad (17)$$

This equation can be understood as determining the steadyon's (complex) initial velocity, which together with the boundary

conditions  $\bar{x}(0) = x_*$  and  $\bar{x}(\Delta t) = x_s$  unambiguously determines both the steadyon and  $\Delta t$ , all up to terms of order  $\epsilon$ .

*Resonant states.* A particularly relevant class of states in this context are resonant states  $|\tilde{E}_i\rangle$  [3,6–10]. These states are constructed by imposing outgoing boundary conditions at the barrier, and because of that, they are only approximate solutions to Schrödinger's equation with complex eigenvalues  $\tilde{E}_i = E_i - i\Gamma_i/2$ . Exact solutions will feature some backtunneling [11,12]. Ignoring the effects of backtunneling (equivalently, requiring that  $\Gamma_i > 0$  for all  $i$ ), the resonant states are ordered such that  $i < j$  implies  $\Gamma_i < \Gamma_j$ .

For each such state, we find (with the  $\epsilon$  regularization)

$$\psi(x_*, t - \Delta t) \simeq e^{-iE_i t - \epsilon E_i t} \cdot e^{iE_i \Delta t + \epsilon E_i \Delta t} \cdot \psi(x_*), \quad (18)$$

which is nothing more than the usual time evolution for a state with energy  $E_i$  in the  $\epsilon$ -deformed theory, neglecting the (small) effect of the (yet-to-be-calculated) tunneling rate  $\Gamma_i$ .<sup>1</sup> To evaluate Eq. (17), it is now sufficient to observe that  $V(x_*) = E_i$ , which implies immediately that  $\dot{\bar{x}}(0) \sim \epsilon \rightarrow 0$ . This leads to a class of steadyons introduced in Ref. [1] as *periodic steadyons*. These solutions are associated with a physical time  $\Delta t_{\text{per}}$ , which depends on  $\epsilon$  through  $\Delta t_{\text{per}} = N\epsilon^{-1}t_{\text{sys}}$ , where  $N$  is an  $\mathcal{O}(1)$  coefficient and  $t_{\text{sys}}$  the system's typical time scale, in agreement with our previous interpretation of the integral over  $\Delta t$  as an integral over different saddle points parametrized by  $\Delta t$ .

We can now observe that the deformation of the Hamiltonian is equivalent to an infinitesimal Wick rotation for systems without an explicit time dependence. In this picture, the steadyon emerges as a solution of its respective equation on a rotated contour in the complex-time plane, and its action as a line integral over this contour. As both  $\bar{x}(t)$  and its action are analytic functions, its action is invariant under a deformation of this contour into one segment parallel to the real-time axis and one parallel to the Euclidean-time axis, as shown in Fig. 2.

Using the periodic steadyon's characteristic boundary condition  $\dot{\bar{x}}(0) = 0$  together with  $\epsilon = 0$  (corresponding to the contour in real-time direction), it is easy to see that Eq. (16) leads to a classical motion of the point particle within the false vacuum with strictly real action.

Along the segment parallel to the Euclidean-time axis, Eq. (16) reduces to the Euclidean equation of motion, and the action calculated along it to the Euclidean action up to a factor of  $i$  from the line element. The condition  $\dot{\bar{x}}(0) = \mathcal{O}(\epsilon)$  meanwhile again implies a motion out of rest. Taken together, these simple observations imply that the projection of the steadyon onto the Euclidean-time segment reproduces the periodic Euclidean-time instanton. Its contribution to the overall action is therefore the Euclidean

<sup>1</sup>See Ref. [5] for a detailed discussion.

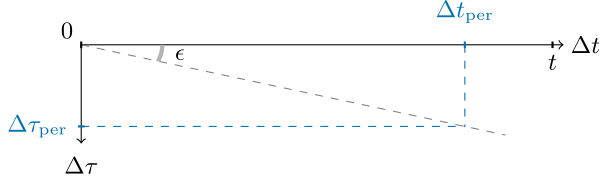


FIG. 2. Assigning the factor  $1 - i\epsilon$  to the time variable rather than the Hamiltonian allows to interpret the steadyon as a solution along a diagonal contour in the complex time plane. Its projections on the real- and imaginary-time axis yield the familiar classical and Euclidean-time solutions, respectively.

action  $S_E$  of this solution times a factor of  $i$ . This, in turn, implies that  $S_{\text{Im}} = S_E$ .

Having identified the dominant contribution to the  $\Delta t$  integrals, it is straightforward to obtain  $\dot{P}_{\mathcal{R}}$  to leading order as

$$\dot{P}_{\mathcal{R}} \sim e^{-2\epsilon E_i t} \exp(-S_E[x_{b,\text{per}}] + 2E_i \Delta\tau_{\text{per}}), \quad (19)$$

where  $x_{b,\text{per}}$  denotes the *periodic bounce solution* describing the motion from  $x_*$  to  $x_s$  and back.

To obtain the tunneling rate through Eq. (7), we calculate the probability  $P_{\mathcal{F}}$  taking into account the deformation of the Hamiltonian. Using Eq. (19), we find

$$\begin{aligned} P_{\mathcal{F}}(t) &= \int_{\mathcal{F}} |\psi_i(t, x)|^2 dx \\ &= \int_{\mathcal{F}} e^{-2\epsilon E_i t} |\psi_i(x)|^2 \simeq e^{-2\epsilon E_i t}. \end{aligned} \quad (20)$$

In the last step, we have made use of the approximation that a majority of the wave function remains in the false vacuum basin  $\mathcal{F}$ . Thus, we find that this approach is to leading order indeed capable of recovering the familiar result obtained by relying on the WKB approximation, [5]

$$\Gamma = \frac{\dot{P}_{\mathcal{R}}(t)}{P_{\mathcal{F}}(t)} \simeq |\psi_i(x_*)|^2 \exp(-S_E[x_{b,\text{per}}] + 2E_i \Delta\tau_{\text{per}}). \quad (21)$$

*Resonant expansion.* The resonant expansion assumes that states  $|\Psi\rangle$  localised within  $\mathcal{F}$  can be expanded as a sum over normalized resonant states  $|\tilde{E}_i\rangle$  [3,6–10],

$$|\Psi\rangle = \sum_{i=1}^{\infty} \psi_i |\tilde{E}_i\rangle. \quad (22)$$

In the long-time limit, the wave function is dominated by the lowest resonance  $|\tilde{E}_1\rangle$ , and the probability  $P_{\mathcal{F}}(t)$  decays exponentially in time with  $P_{\mathcal{F}} \sim e^{-\Gamma_1 t}$ . However, at pre-asymptotic times, there are corrections due to the higher resonant states, which in general are nonorthogonal because of the non-Hermiticity of the Hamiltonian restricted to the false-vacuum sector with outgoing boundary conditions.

Including the effects of  $n$  of the lowest resonant states,  $P_{\mathcal{F}}(t)$  is given by the resonant expansion,

$$P_{\mathcal{F}}(t) \approx \sum_{i,j=1}^n \text{Re}\{\psi_j^* \langle \tilde{E}_j | \tilde{E}_i \rangle \psi_i e^{-i(E_i - E_j)t}\} e^{-(\Gamma_i + \Gamma_j)t/2}, \quad (23)$$

where due to the nonorthogonality of the resonant states, the nondiagonal overlap matrix  $\langle \tilde{E}_j | \tilde{E}_i \rangle$  introduces *oscillations* into the decay of  $P_{\mathcal{F}}(t)$ . This behavior can be clearly seen in Fig. 3(b), where we show the probability contained within  $\mathcal{F}$  as a function of time for the initial condition and potential shown in Fig. 3(a): following a short period in which there is contamination from higher-energy resonant states, the early-time decline of  $P_{\mathcal{F}}$  is dominated by the contribution from the second resonance  $|\tilde{E}_2\rangle$ . As  $\Gamma_2 > \Gamma_1$ , this contribution becomes subdominant at larger times, as the second resonance has fully decayed. Between these two regimes we observe a transient dominated by the term mixing  $\tilde{E}_1$  and  $\tilde{E}_2$ , manifesting in oscillatory behavior.<sup>2</sup>

In principle, one could treat every variable in the resonant expansion Eq. (23) as a quantity to be predicted. However, the fact that the expansion takes this form is already predictive, as it implies a precise form of the time dependence that can be tested against numerical simulations, and also serves as a test of independent calculations of the resonant energies  $E_i$  and tunneling rates  $\Gamma_i$ —e.g., those of the WKB approximation. The matrix elements  $\langle \tilde{E}_i | \tilde{E}_j \rangle$  are also parameters characterizing the false vacuum, and crucially account for the presence of the oscillatory behavior in the probability. On the other hand,  $\psi_{i,j}$  characterize the initial state. In practice, numerically fitting the resonant expansion as shown in Fig. 3 works well, and the extracted values of  $E_i$ ,  $\Gamma_i$ , and  $\langle \tilde{E}_i | \tilde{E}_j \rangle$  are stable with respect to changing the initial state, as explored in our companion article [5]. This demonstrates that they are intrinsic properties of the system.

*Generalization.* The direct approach offers a direct path towards generalizing these results [3,4,13]. For multidimensional resonant states, the decompositions (5), (10), and (11) need to be supplemented with integrals over the submanifolds of energetically degenerate points,  $\Sigma_*$  and  $\Sigma_s$ , respectively,

$$\int dt_* \rightarrow \int dt_* \int_{\Sigma_*} dx_*, \quad \int dt_s \rightarrow \int dt_s \int_{\Sigma_s} dx_s. \quad (24)$$

On the level of the steadyon, this amounts to receiving one contribution from every suitable pair of initial and final points within  $\Sigma_*$  and  $\Sigma_s$ , respectively. The generalization of Eqs. (16) and (17) meanwhile is straightforward, as it amounts to replacing the one-dimensional steady solution by its multidimensional counterpart. Depending on the

<sup>2</sup>Note that we had assumed the absence of backtunneling for the calculation of the tunneling rate for each *individual* resonant state, which remains valid. Our results imply that capturing this oscillatory behavior in a direct evaluation  $\Gamma(t)$  for a superposition of resonance states would require the inclusion of leading-order backtunneling effects.

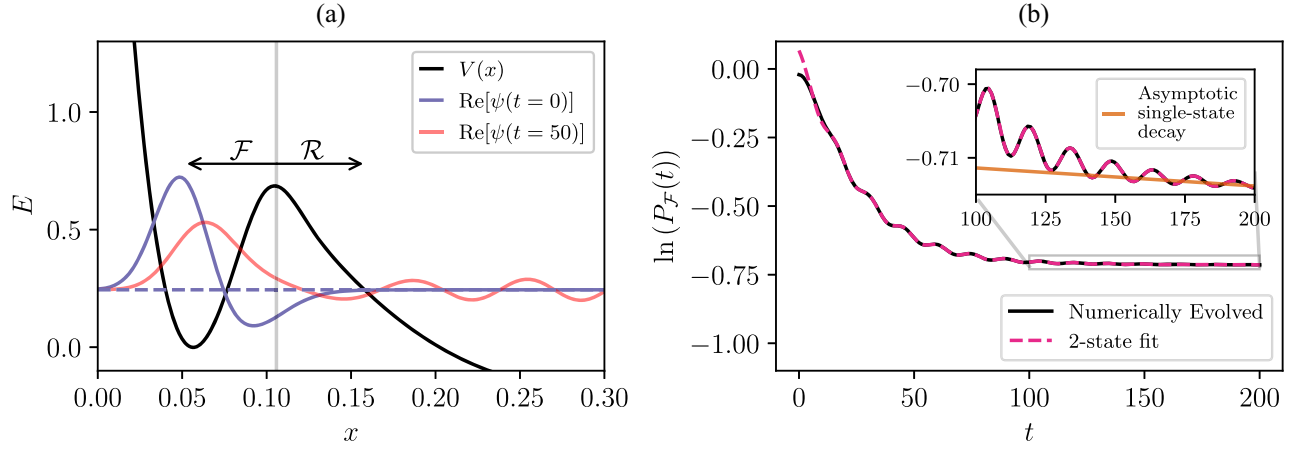


FIG. 3. (a): A one-dimensional potential  $V(x)$  (inspired by the potential used in Refs. [3,4]) with a barrier separating a false vacuum  $\mathcal{F}$  from a true vacuum  $\mathcal{R}$ . For an example initial wave function  $\psi(t=0)$ , time evolution causes it to tunnel through the barrier, where the mass is fixed to  $m = 6 \times 10^3$  in natural units ( $\hbar = 1$ ). (b): The time evolution of the probability  $P_{\mathcal{F}}(t)$  of finding the particle in  $\mathcal{F}$  is well-described by a two-state resonant expansion, up to corrections at early times due to contributions from higher energy resonant states. At large times, the decay of  $P_{\mathcal{F}}(t)$  approaches an exponential decay described by a single resonant state, shown in the inset figure.

properties of the initial state, the integral over the initial position  $\vec{x}_*$  can impose additional constraints on  $\vec{x}(0)$ .

For a scalar field, the tunneling process is controlled by the initial wave functional  $\Psi[\phi]$  and its energy functional,

$$U[\phi] = \int d^3x \frac{1}{2} |\nabla \phi|^2 + V(\phi). \quad (25)$$

For arbitrary initial states, the combined saddle point condition for the path and  $\Delta t$  integral can be straightforwardly obtained as

$$\ddot{\phi} = (1 - 2i\epsilon)[\Delta \bar{\phi} + V'(\bar{\phi})] \quad (26)$$

$$i \frac{d}{dt} \log \Psi[\phi_*](t - \Delta t) = \frac{1}{2} \int \dot{\phi}(0)^2 d^3x - U[\phi_*]. \quad (27)$$

Using Eq. (25), the field space analogue of the hypersurfaces  $\Sigma_*$  and  $\Sigma_s$  for a resonant state can be constructed through the condition  $U[\phi_*] = E = U[\phi_s]$  for all  $\phi_* \in \Sigma_*$  and  $\phi_s \in \Sigma_s$ . Equation (27) thus implies that also for a field, the dominant contribution to the tunneling arises from the periodic steadyon and its associated periodic instanton.

*Discussion.* We have shown that tunneling at preasymptotic times can be described in terms of the tunneling rates out of linear combinations of individual resonance states. We have provided a first-principles derivation of these rates, combining the direct approach of [3,4] with the more recently developed steadyon picture [1,2].

Our analysis inherits and expands the advantages of the direct approach analysis of the asymptotic false vacuum rate.

Unlike the analysis based on the WKB approximation, it makes evident the role of physical time for the tunneling. On a conceptual level, it establishes a clear connection between real and imaginary time, generalizing the Wick-rotation

based argument for the false vacuum analysis. Moreover, it allows for a clear interpretation of the relevant time scales. The ability of the steadyons to serve as approximate saddle points of the real-time path-integrals that they are used to evaluate requires  $t \gg t_{\text{sys}}$ . On the other hand, the existence of an upper bound on the time  $t$  where this approximation is valid follows from two factors. First, we demanded no backtunneling, which was already responsible for this bound for the false vacuum analysis. However, our analysis illustrates a second factor: The periodic instanton emerges only after specifying that the initial state is an approximate eigenstate of the Hamiltonian. As soon as the resonance “disintegrates” the factor  $\psi(x_*, \Delta t)$  changes, and the periodic instanton is no longer the optimal solution. Having this clear form, however, in principle also allows one to interpolate beyond the disintegration of the resonance state, given knowledge of how  $\psi(x_*, \Delta t)$  evolves past this point.

More generally, our approach is easily generalized to more general cases for which the WKB approximation might not work, enabling a direct calculation without first decomposing the state of interest in terms of resonances.

Even though we restricted ourselves to the calculation of the leading-order exponent, our expression for the probability flux in Eq. (13) is true nonperturbatively. It thus allows for a systematically improvable perturbative treatment. The representation of our results in terms of path-integrals also simplifies investigations concerning gauge invariance [14–16], as well as calculations in situations in which resummations are necessary to address large back-reactions [17], e.g., for finite-temperature processes [2].

Our analysis also inherits the direct approach’s simple generalization to higher-dimensional systems by integrating the points  $x_*$  and  $x_s$  over the submanifolds of energetically degenerate points, compared to the WKB method in multiple dimensions [18]. The resulting expression can be understood

as the overall tunneling rate being described by a collection of instantons from all classical turning points to any of the energetically degenerate points beyond the barrier. Our analysis moreover makes explicit the definition of the regions  $\mathcal{F}$  and  $\mathcal{R}$  through the localization of the wave function.

*Acknowledgments.* We acknowledge the hospitality of the Munger residence and the Kavli Institute for Theoretical Physics at the University of California, Santa Barbara, where parts of this research were conducted. We also thank Matt D. Charles for contributions to an inspiring work environment and Sebastian Schenk for helpful discussions. Portions of this work were conducted in MIT's *Center for Theoretical Physics—a Leinweber institute* and partially supported by the U.S. Department of Energy under Contract No. DE-SC0012567. While at MIT, JL's contributions were supported by the U.S. Department of Energy under Contract No. DE-SC0011090, by the SciDAC5 Award No. DE-SC0023116, and additionally, by the National Science Foundation under Cooperative Agreement Grant No. PHY-2019786 (The NSF AI Institute for Artificial Intelligence and Fundamental Interactions, <http://iaifi.org/>). While at Argonne National Laboratory, JL's contributions were supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics through Contract No. DE-AC02-06CH11357, and by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics, Early Career Award through

Contract No. DE-SCL0000017. This research was supported in part by Grant No. NSF PHY-2309135 to the Kavli Institute for Theoretical Physics (KITP). BSH's contributions were also supported by Grant No. 994312 from the Simons Foundation. TS's contributions to this work were made possible by the Walter Benjamin Programme of the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)—512630918. This project was also supported in part by the Black Hole Initiative at Harvard University, with support from the Gordon and Betty Moore Foundation and the John Templeton Foundation. He also acknowledges partial financial support from the Spanish Research Agency (Agencia Estatal de Investigación) through the Grants IFT Centro de Excelencia Severo Ochoa No. CEX2020-001007-S and No. PID2022-137127NB-I00 funded by MCIN/AEI/10.13039/501100011033/ FEDER, UE. This project has received funding/support from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Staff Exchange Grant Agreement No. 101086085 -ASYMMETRY.

The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of these Foundations.

*Data availability.* There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

- 
- [1] T. Steingasser and D. I. Kaiser, Toward quantum tunneling from excited states: Recovering imaginary-time instantons from a real-time analysis, *Phys. Rev. D* **111**, 096009 (2025).
  - [2] T. Steingasser, M. König, and D. I. Kaiser, Finite-temperature instantons from first principles, *Phys. Rev. D* **110**, L111902 (2024).
  - [3] A. Andreassen, D. Farhi, W. Frost, and M. D. Schwartz, Precision decay rate calculations in quantum field theory, *Phys. Rev. D* **95**, 085011 (2017).
  - [4] A. Andreassen, D. Farhi, W. Frost, and M. D. Schwartz, Direct approach to quantum tunneling, *Phys. Rev. Lett.* **117**, 231601 (2016).
  - [5] J. Lin, B. Scheihing-Hitschfeld, and T. Steingasser, Quantum tunneling from excited states in the steadyon picture, [arXiv:2511.08670](https://arxiv.org/abs/2511.08670).
  - [6] I. Y. Kobzarev, L. B. Okun, and M. B. Voloshin, Bubbles in metastable vacuum, *Yad. Fiz.* **20**, 1229 (1974).
  - [7] E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, Perturbation theory at large order. 1. The  $\phi^{2N}$  interaction, *Phys. Rev. D* **15**, 1544 (1977).
  - [8] C. G. Callan, Jr. and S. R. Coleman, The fate of the false vacuum. 2. First quantum corrections, *Phys. Rev. D* **16**, 1762 (1977).
  - [9] J. Zinn-Justin, Quantum field theory and critical phenomena, *Int. Ser. Monogr. Phys.* **113**, 1 (2002).
  - [10] M. Razavy, *Quantum Theory of Tunneling* (World Scientific Publishing Company Pte Limited, Singapore, 2013), 10.1007/978-3-642-01850-3.
  - [11] G. Gamow, Zur Quantentheorie des Atomkernes, *Z. Phys.* **51**, 204 (1928).
  - [12] A. J. F. Siegert, On the derivation of the dispersion formula for nuclear reactions, *Phys. Rev.* **56**, 750 (1939).
  - [13] T. Steingasser, New perspectives on solitons and instantons in the standard model and beyond, Ph.D. thesis, Munich University, 2022, 10.5282/edoc.30495.
  - [14] D. Metaxas and E. J. Weinberg, Gauge independence of the bubble nucleation rate in theories with radiative symmetry breaking, *Phys. Rev. D* **53**, 836 (1996).
  - [15] A. Andreassen, W. Frost, and M. D. Schwartz, Consistent use of the standard model effective potential, *Phys. Rev. Lett.* **113**, 241801 (2014).
  - [16] A. Andreassen, W. Frost, and M. D. Schwartz, Consistent use of effective potentials, *Phys. Rev. D* **91**, 016009 (2015).
  - [17] S. R. Coleman and E. J. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking, *Phys. Rev. D* **7**, 1888 (1973).
  - [18] T. Banks, C. M. Bender, and T. T. Wu, Coupled anharmonic oscillators. 1. Equal mass case, *Phys. Rev. D* **8**, 3346 (1973).