

Nontopological electromagnetic hedgehogs

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We study classical localized configurations—solitons—in a theory of self-interacting complex Proca field with the global $U(1)$ symmetry. We focus on spherically symmetric solitons near the nonrelativistic limit, which are supported by the quartic interactions of the neutral Proca field. Such solitons can source the radial electric (magnetic) field if one introduces a parity-even (parity-odd) coupling of the Proca field to the electromagnetic field tensor. We discuss the conditions of existence of such nontopological “electromagnetic hedgehogs” and their properties.

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I. INTRODUCTION

Nontopological solitons [1] are stationary nonlinear particlelike solutions of field equations in theories with global or local symmetries. They exist due to self-interaction of the field, gravitational attraction, or some background potential. The best-known example of nontopological solitons is Q balls [2,3] arising in theories of complex scalar field with the global or local [4,5] $U(1)$ symmetry. The symmetry leads to the conserved charge that can prevent the Q ball, whose stability is not guaranteed by topology, from dissolving into free particles. Q balls found numerous applications in particle physics and cosmology and provide valuable insights into behavior of other solutions of classical equations of motion [6].

Analogues of Q balls exist in theories of self-interacting complex massive vector (Proca) fields [7]. The vector Q balls, self-gravitating Proca stars [8]—analogues of scalar Boson stars [9,10]—and their gauged counterparts [11] have been extensively studied in recent years [12–16]. The motivation comes partially from cosmology, where massive vector particles can be a dark matter candidate [17], and spatially localized structures made of these particles are exotic compact objects [18].

Theories of self-interacting vector fields are effective field theories. In the exceptional case of non-Abelian gauge theory, the well-known weakly coupled UV completion

involves the Higgs mechanism, but this approach is not applicable in the general case.¹ Moreover, as was shown in [19–21], pathologies in models of self-interacting Proca fields may appear already at the classical level. This does not preclude viewing relativistic massive vector theories as low-energy effective theories. Solitons and other classical configurations can be studied in this framework [22,23], which we adopt in this paper.

In this paper, we study nongravitating solitons in the theory of massive complex vector field V^μ with quartic self-interactions. The theory possesses the global $U(1)$ symmetry which ensures the existence of solitons. Interestingly, one can introduce a coupling of V^μ to the electromagnetic field tensor $F_{\mu\nu}$. Such coupling may arise as the low-energy limit of the gauge- and global $U(1)$ -invariant coupling between V^μ and the gauge field of the $U(1)_Y$ group of the Standard Model. Depending on the parity of the coupling term, the interaction will endow V^μ particles with the electric or magnetic dipole moment, and the vector soliton will act as a source of electric or magnetic field confined to the bulk of the soliton. We will be interested in spherically symmetric configurations giving rise to radially polarized fields. These objects are somewhat similar to nontopological magnetic monopoles studied in [22], although in our case the fields decay exponentially fast beyond the bulk of the soliton.

The paper is organized as follows. In Sec. II, we introduce the model and derive the field equations and conserved currents. In Sec. III, we introduce the field ansatz for solitons and discuss their general properties. We then find the solitons numerically, paying special attention to the nonrelativistic limit and the regime where the thin-wall approximation is applicable. We also touch upon the

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¹For example, spin-1 atoms and ρ mesons are composite particles in the complete strongly coupled theories.

validity of the effective field theory description for our solutions. In Sec. IV, we add the electromagnetic field to the picture of the solitons, and in Sec. V we conclude. Throughout the paper, we put $\hbar = c = 1$ and use the metric signature $(+, -, -, -)$.

II. SETUP

Consider a $(3 + 1)$ -dimensional theory of the complex massive neutral field V^μ coupled to the electromagnetic field tensor $F_{\mu\nu}$. We require the global $U(1)$ symmetry of this theory and the gauge-invariant coupling of V^μ to $F_{\mu\nu}$. The general parity- (P) even Lagrangian reads as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}V_{\mu\nu}^*V^{\mu\nu} + \frac{i\gamma}{2}F_{\mu\nu}W^{\mu\nu} - U(V^\nu, V^{*\mu}), \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, and $W_{\mu\nu} = V_\mu^* V_\nu - V_\nu^* V_\mu$. Furthermore, γ is a dimensionless constant, and $U(V^\nu, V^{*\mu})$ is a $U(1)$ -invariant potential of self-interaction.

The real interaction term $\frac{i\gamma}{2}F_{\mu\nu}W^{\mu\nu}$ was first introduced in [24]. In particle physics, this interaction describes three-vector-boson coupling which was considered in electroweak phenomenology [25] and studied in a low-energy effective theory of strongly interacting spin-1 particles [26]. In the nonrelativistic limit, the term $\frac{i\gamma}{2}F_{\mu\nu}W^{\mu\nu}$ generates the magnetic dipole moment $(\gamma/2M)$ [27,28] and the electric quadrupole moment $(-\gamma/M^2)$ [25,26] of a vector particle.

As a physical example, the classical field V_μ describes a gas of spin-1 atoms in a Bose-Einstein condensate [29]. In this case, the global $U(1)$ symmetry of the theory (1) corresponds to the $U(1)_B$ symmetry for the baryon number conservation. This motivates us to pay special attention to the solitons in the nonrelativistic (low-energy) regime of the theory. At the quantum level, the model (1) is an effective theory that is valid below the cutoff scale $\sim M/\sqrt{\gamma}$ or a scale associated with the vector self-interaction. Note that the model cannot be considered as a low-energy limit of some renormalizable non-Abelian gauge theory. One possibility for a UV completion is to match the model to some strongly coupled theory at low energies [26]. We will follow the approach of [22,23,30] and study classical solutions in the model assuming that the coupling constants in (1) are independent. Note also that, if the massive vector field is a dark matter candidate, there is a strong bound on γ [31], and the cutoff of the theory is determined by the vector field self-interaction [15].

The Lagrangian (1) leads to the Maxwell equations

$$\partial_\mu F^{\mu\nu} = i\gamma\partial_\mu W^{\mu\nu}, \quad (2)$$

which, for configurations with trivial boundary conditions at spatial infinity, can be integrated to yield

$$F^{\mu\nu} = i\gamma W^{\mu\nu}. \quad (3)$$

Next, we choose the most general P-even fourth-order potential of self-interaction,

$$U = -M^2 V_\mu^* V^\mu - \frac{\alpha}{2}(V_\mu^* V^\mu)^2 - \frac{\beta}{2}(V_\mu^* V^{*\mu})(V_\nu V^\nu), \quad (4)$$

with α, β dimensionless constants. Substituting Eqs. (2) and (4) to (1), we arrive at the Lagrangian containing the vector field only,

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}V_{\mu\nu}^*V^{\mu\nu} - \tilde{U}(V^\nu, V^{*\mu}), \\ \tilde{U} &= -M^2 V_\mu^* V^\mu - \frac{\tilde{\alpha}}{2}(V_\mu^* V^\mu)^2 - \frac{\tilde{\beta}}{2}(V_\mu^* V^{*\mu})(V_\nu V^\nu), \end{aligned} \quad (5)$$

where $\tilde{\alpha} = \alpha + \gamma^2$ and $\tilde{\beta} = \beta - \gamma^2$.

Let us discuss the possibility of enhancing the Lagrangian (1) with the P-odd term $\frac{i\tilde{\gamma}}{2}\tilde{F}_{\mu\nu}W^{\mu\nu}$, where $\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\rho\lambda}F^{\rho\lambda}$ and $\tilde{\gamma}$ is a dimensionless coupling constant. In the nonrelativistic limit, this interaction provides the vector boson with the electric dipole and magnetic quadrupole moment. Considering this coupling instead of $\frac{i\gamma}{2}F_{\mu\nu}W^{\mu\nu}$ leads again to the effective theory (5) with $\tilde{\alpha} = \alpha - 4\tilde{\gamma}^2$ and $\tilde{\beta} = \beta + 4\tilde{\gamma}^2$. Note that both P-even and P-odd couplings provide the P-even contributions to the effective Lagrangian. This is because $W_{\mu\nu}$ (or $\tilde{W}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}W^{\rho\sigma}$) gets squared when moving from (1) to (5). If both γ and $\tilde{\gamma}$ are nonzero, the P-odd interaction term $\frac{i\tilde{\gamma}}{2}W^{\mu\nu}\tilde{W}_{\mu\nu}$ should also be added to the vector field Lagrangian. In this paper, we will be interested in configurations supporting the electric (or magnetic) field in the center-of-mass frame, hence the latter term is identically zero for our solitons.

The theory (5) possesses a conserved current corresponding to the global $U(1)$ symmetry,

$$j^\nu = i(V^{*\mu\nu}V_\mu - V^{\mu\nu}V_\mu^*). \quad (6)$$

Additionally, there is a “topological” current

$$j_T^\nu = i\partial_\mu W^{\mu\nu}, \quad (7)$$

which is manifestly conserved due to the antisymmetry of $W_{\mu\nu}$. The $U(1)$ current defines the global charge Q of solitonic solutions; the classically stable (unstable) soliton provides a local minimum (maximum) of the Hamiltonian of the theory at a fixed value of Q [3]. As for the current j_T^ν , the associated charge will vanish identically on the solutions studied below.

The Einstein energy-momentum tensor of the theory (5) takes the form

$$\begin{aligned}
 T_{\mu\nu} = & -V_{\lambda\mu}^* V_{\nu}^{\lambda} - V_{\lambda\mu} V_{\nu}^{*\lambda} + g_{\mu\nu} \frac{1}{2} V_{\lambda\rho}^* V^{\lambda\rho} \\
 & - \frac{1}{2} \left(\frac{\partial \tilde{U}}{\partial V^{\mu}} V_{\nu} + \frac{\partial \tilde{U}}{\partial V^{\nu}} V_{\mu} \right) \\
 & - \frac{1}{2} \left(\frac{\partial \tilde{U}}{\partial V^{*\mu}} V_{\nu}^* + \frac{\partial \tilde{U}}{\partial V^{*\nu}} V_{\mu}^* \right) + \tilde{U}, \quad (8)
 \end{aligned}$$

and the energy density is

$$T_{00} = V_{0i}^* V_{0i} + \frac{1}{2} V_{ij}^* V_{ij} - \frac{\partial \tilde{U}}{\partial V_0} V_0 - \frac{\partial \tilde{U}}{\partial V_0^*} V_0^* + \tilde{U}. \quad (9)$$

The latter determines the energy E of the solitonic solutions. Note that T_{00} coincides with the energy density of the original theory (1) provided that Eq. (3) is fulfilled, that is, if there are no extra sources of the electromagnetic field.

III. CLASSICAL SOLUTIONS IN VECTOR THEORY

In this section, we study nontopological vector solitons in the theory (5). By Eq. (3), any spatially localized solution with nonzero $W^{\mu\nu}$ can host the electric (magnetic) field that is confined to the bulk of the solution and vanishes exponentially fast outside it. As explained above, we are interested in the solitons that admit the nonrelativistic limit. Additionally, we look for solutions that are kinematically stable against dissolution into a gas of free particles. This is ensured by requiring that $E < MQ$.

To facilitate the study, we switch to the dimensionless units as follows:

$$V^{\nu} = \frac{M \tilde{V}^{\nu}}{\sqrt{|\tilde{\alpha}|}}, \quad \kappa = \frac{\tilde{\beta}}{|\tilde{\alpha}|}, \quad x_i x^i = \frac{r^2}{M^2}, \quad t = \frac{\tau}{M}. \quad (10)$$

The action S of the theory becomes $S = |\tilde{\alpha}|^{-1} \tilde{S}$ where $\tilde{S}$ depends only on κ . As we will see, the physically interesting region of parameters is $|\kappa| \sim 1$. The validity of the semiclassical approach requires $|\tilde{\alpha}|/4\pi \ll 1$.

In units (10), the vector field equations become

$$\partial_{\mu} \tilde{V}^{\mu\nu} + \tilde{V}^{\nu} \pm (\tilde{V}_{\mu}^* \tilde{V}^{\mu}) \tilde{V}^{\nu} + \kappa (\tilde{V}_{\mu} \tilde{V}^{\mu}) \tilde{V}^{*\nu} = 0, \quad (11)$$

where the sign of the third term coincides with the sign of $\tilde{\alpha}$.

A. Condensate

Before studying solitons, it is useful to consider spatially homogeneous solutions of Eq. (11)—vector condensates. There are two types of relevant condensates. The first one is described by the following solution:

$$\tilde{V}_0 = 0, \quad \tilde{V}^1 = \sqrt{\frac{1-w^2}{\pm 1 + \kappa}} e^{-i w \tau}, \quad \tilde{V}^i = 0, \quad i = 2, 3. \quad (12)$$

Here w is the angular frequency in units of the field mass, and the sign in the denominator follows the sign of $\tilde{\alpha}$. In the nonrelativistic limit, $w \rightarrow 1$, this condensate approaches the classical ground state $\tilde{V}^{\mu} = 0$. At $w < 1$, the solution (12) describes weakly interacting vector particles in a spatially homogeneous bound state. The existence of such bound state is important for the nonrelativistic solitons that are physically formed out of the gas of weakly interacting particles. Thus, we require $\pm 1 + \kappa > 0$ for our solitons. Another way to obtain this constraint is to study directly the nonrelativistic limit of solitons with a particular ansatz; see, e.g., [7].

The second condensate that we consider has nonzero W^{01} and F^{01} , and it allows us to further narrow the range of physically interesting parameters. The ansatz is qualitatively similar to the one used in [7],

$$\tilde{V}_0 = i u e^{-i w \tau}, \quad \tilde{V}^1 = v e^{-i w \tau}, \quad \tilde{V}^i = 0, \quad i = 2, 3, \quad (13)$$

where u and v are real constants depending on w . Using equations of motion (11), we obtain that the bound-state solution of the form (13) exists at $-1 < \kappa < 0$. The existence of this bound state is important for the kinematic stability of the solitons. Indeed, the solitons with $E < MQ$ are often associated with the thin-wall limit [3]. Near this limit, the bulk of the soliton, which makes the dominant contribution to its charge and energy, is close to the respective condensate solution. If the latter is kinematically stable, so is the soliton. As we will see, the ansatz (13) describes the bulk of our solitons in the thin-wall regime. Thus, we require $-1 < \kappa < 0$, which, together with the bound $\pm 1 + \kappa > 0$, also implies that $\tilde{\alpha} > 0$.

B. Solitons

Consider the following spherically symmetric radial ansatz for the vector field (see [7] and references therein):

$$\tilde{V}_0 = i u(r) e^{-i w \tau}, \quad \tilde{V}^i = \frac{\tilde{x}^i}{r} v(r) e^{-i w \tau}, \quad (14)$$

where $\tilde{x}^i = M x^i$, $u(r)$, $v(r)$ are profile functions of the vector field, and w is the angular frequency in units of the field mass. Substituting to Eq. (11), we obtain

$$\begin{aligned}
 u'' + \frac{2}{r} u' - w \left(v' + \frac{2}{r} v \right) - u - (u^2 - v^2) u \\
 - \kappa (u^2 + v^2) u = 0, \\
 w u' + (1 - w^2) v + (u^2 - v^2) v - \kappa (u^2 + v^2) v = 0, \quad (15)
 \end{aligned}$$

where prime means the derivative with respect to r , and the sign of the cubic term is fixed by the requirement $\tilde{\alpha} > 0$. We solve these equations numerically with the appropriate boundary conditions for u and v . The latter are set by the

regularity at the origin ($u' \rightarrow 0$, $v \rightarrow 0$, $v' \rightarrow \text{const}$, $r \rightarrow 0$) and approaching the classical ground state at infinity ($u, v \rightarrow 0$, $r \rightarrow \infty$).

With the ansatz (14), the global $U(1)$ charge and energy of the soliton are given by

$$Q = \frac{8\pi}{|\tilde{\alpha}|} \int_0^\infty v(wv - u')r^2 dr, \quad (16)$$

$$E = \frac{4\pi M}{|\tilde{\alpha}|} \int_0^\infty ((wv - u')^2 +$$

$$+ (u^2 + v^2) + \frac{1}{2}(u^2 - v^2)(3u^2 + v^2) + \frac{\kappa}{2}(u^2 + v^2)(3u^2 - v^2))r^2 dr. \quad (17)$$

Taking the derivatives with respect to $\omega \equiv Mw$ in these expressions and using the equations of motion, it is straightforward to show that

$$\frac{\partial E}{\partial \omega} = \omega \frac{\partial Q}{\partial \omega}, \quad (18)$$

which is the well-known general relation for nontopological solitons. We use it to check the consistency of our numerical routines.

The rest of the paper is dedicated to solving numerically Eq. (15) with the solitonic boundary conditions and discussing the properties of the obtained solutions. Our procedure is as follows. We fix the value of κ from the range $-1 < \kappa < 0$, in which one expects solutions both in the nonrelativistic and the thin-wall (kinematically stable) limits. Using the shooting method, we first find solutions close to the nonrelativistic limit, $w \rightarrow 1$. In this limit, the field amplitudes tend to zero, and the validity of the effective theory is ensured.

Then we look for solutions with smaller w by an iterative procedure. We find the frequency w_s at which $E = MQ$; at $w < w_s$ the solitons are kinematically stable. Figure 1 shows an exemplary solution with $E < MQ$ living close to w_s , at $\kappa = -0.9$. Depending on the value of κ , we may enter the thin-wall regime as we further decrease w ; see Fig. 2 for the illustration. The thin-wall limit corresponds to $w \rightarrow w_{\min} > 0$.

Let us discuss the limit $w \rightarrow w_{\min}$ in more detail. There are two types of solutions near this limit; both of them are kinematically stable. Solutions of the first type—the proper thin-wall solitons—are found in the range $-1 < \kappa \lesssim 0.6$. For such solitons, the region of the central depression of the fields is followed by a broad bulk where the magnitudes of u, v approach the values of the condensate solution (13) with the same frequency. Outside the bulk, the fields rapidly vanish. This structure is illustrated in Fig. 2. The size of the soliton, its total charge, and energy grow indefinitely as $w \rightarrow w_{\min}$.

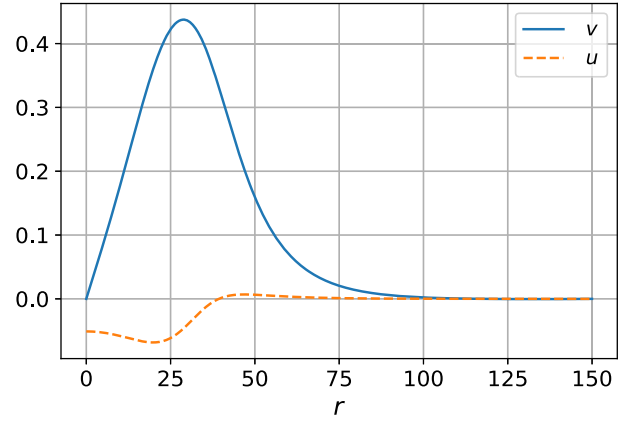


FIG. 1. Vector soliton (14) at $\kappa = -0.9$ and $w = 0.998$. This solution is kinematically stable, $E < MQ$.

Solutions of the second type are found in the range $-0.6 \lesssim \kappa \lesssim 0$. They do not approach the thin-wall limit even though the condensate (13) still exists. The obstruction to the limit lies in the structure of the differential equations (15). To see it, we express v' in terms of $u, \chi \equiv u'$ and v [which brings Eq. (15) to the standard form for the Runge-Kutta method],

$$v' = \frac{w}{1 + (u^2 - v^2) - \kappa(u^2 + v^2) - 2v^2(1 + \kappa)} \times \left(-\frac{2}{w}(1 - \kappa)uv\chi + \frac{2(\chi - wv)}{r} - u - u(u^2 - v^2) - \kappa u(u^2 + v^2) \right). \quad (19)$$

Notice that the denominator can vanish at some $u, v > 0$. When this happens, the smoothness of the solution is lost. We see indeed that $v(r)$ develops a cusp at some finite r when the stiffness point is approached; see Fig. 3 for

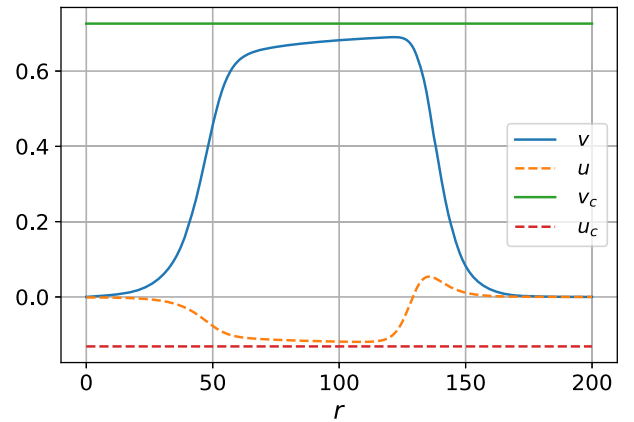


FIG. 2. Kinematically stable vector soliton (14) at $\kappa = -0.9$ and $w = 0.99$. Also shown is the condensate solution (13) (u_c, v_c) for the same parameters.

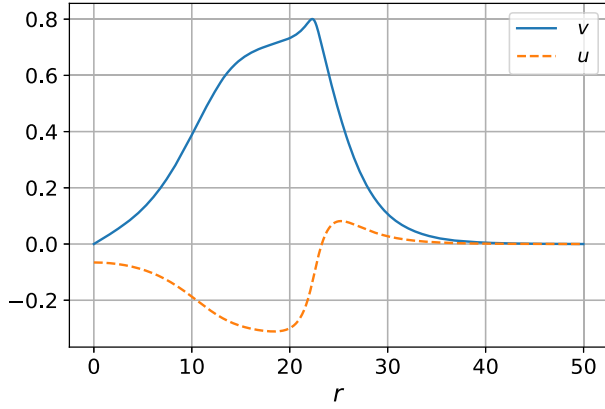


FIG. 3. Kinematically stable vector soliton (14) at $\kappa = -0.55$ and $w = 0.96$. We see that the cusp is approached in the transverse components of the field.

illustration. Because of the cusp, we cannot reach the thin-wall regime. In fact, the stiffness problem is common for classical vector field equations of the type studied here [19–21]. It indicates the breakdown of the effective theory (5) for solitons near the stiffness point.

Having obtained the family of solitons parametrized by w , we can study relations between their charge Q and energy E . For example, Fig. 4 shows the parametric dependence of $E/M - Q$ on Q , at $\kappa = -0.55$. We see the lower branch heading toward the thin-wall limit and the upper branch connecting to the nonrelativistic limit. The branches are joined at a point where $\partial E/\partial w = \partial Q/\partial w = 0$. The qualitatively same dependencies are obtained for other allowed values of κ (for $\kappa \gtrsim -0.4$, however, the soliton develops the cusp at $E/M - Q > 0$, and the kinematically stable region is inaccessible). The behavior of E and Q is quite similar to the one in the Friedberg-Lee-Sirlin model of Q balls made of two scalar fields [32].

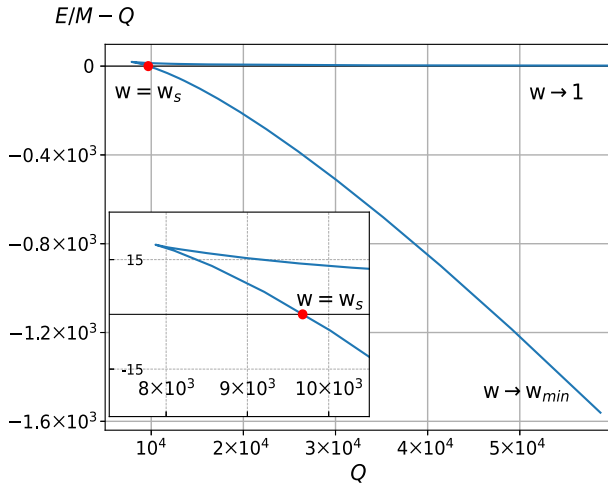


FIG. 4. $E/M - Q$ as a function of Q for vector solitons (14) in the theory (5) with $\tilde{\alpha} = 1$ and $\kappa = \tilde{\beta}/|\tilde{\alpha}| = -0.55$.

Importantly, when $\kappa \rightarrow -1$, the frequency w_s , at which the solitons become kinematically stable, tends to 1. This means that we can find solutions that are both nonrelativistic and kinematically stable. The stiffness problem never occurs to them. Note that, even though w changes very little for such solutions, the possible values of Q and E are not bounded from above, thanks to the thin-wall limit. All these properties are favorable for considering such solitons as “hosts” of the electric (or magnetic) field.

C. Applicability of classical theory

Let us discuss the validity of the solutions obtained above. We work in the regime $\gamma \ll \alpha \lesssim 1$. One of these inequalities is required by the semiclassical approximation: as discussed above, the classical action of the theory with the potential (4) scales as $|\alpha|^{-1}$, hence quantum corrections are suppressed by the powers of small coupling α and additional loop factor $1/(4\pi)^2$. The strong coupling scale of the theory is estimated as $\Lambda_\alpha \sim M\sqrt{4\pi}/\alpha^{1/4}$ [33]. It should be larger than the interacting energy of particles in the soliton, which is satisfied automatically for our solutions at small coupling constants. Note that, even though the magnitude of the field in the soliton may exceed the cutoff scale, $V^\mu \sim M/\sqrt{\alpha} > \Lambda_\alpha$, this does not mean the breakdown of the semiclassical approximation, which follows from the loop expansion controlled by the powers of α .² The second inequality is motivated by considering V_μ particles as a dark matter candidate. For example, the direct constraint on the magnetic dipole moment of electromagnetically interacting vector particles with masses of order 100 MeV is $\gamma < 10^{-8}$ [31].

The charge (16) and energy (17) of our solutions satisfy $Q \gg 1$, $E/M \gg 1$, meaning that the solitons are composed of many quanta of the field V^μ . Furthermore, the size of the soliton is much larger than its Compton wavelength $\sim E^{-1}$. Finally, from Fig. 4 we see that $|E - MQ| \ll MQ$, meaning that the binding energy of V^μ particles inside the solitons is small compared to their rest energy, and the particles are nonrelativistic.

Note, finally, that the solitons with the cusp, such as the one presented in Fig. 3, are beyond the applicability limit of the effective theory, since they have large quantum corrections [19].

IV. ELECTROMAGNETIC HEDGEHOGS

According to Eq. (3), the vector solitons can confine the electromagnetic field. As discussed above, we choose $\gamma \ll \alpha \lesssim 1$. At $-1 < \kappa < 0$, we have found solitons that are nonrelativistic, kinematically stable, and have a wide range of physical parameters Q and E . These spherically

²A similar procedure is used to validate the semiclassical analysis of the Higgs mechanism for parametrically large vacuum expectation values [34].

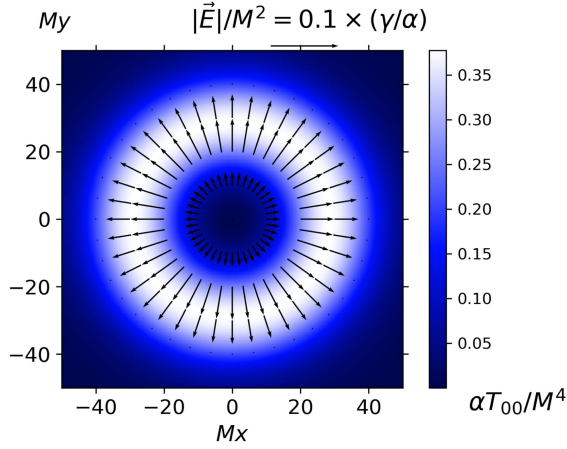


FIG. 5. The equatorial cross section of the electric hedgehog—radial spherically symmetric nontopological vector soliton—in the theory (1). We take $\kappa = \beta/\alpha = -0.9$, $w = 0.998$, and $\gamma \ll \alpha$. The arrows indicate the value $|\vec{E}|/M^2$ and the direction of the electric field.

symmetric solutions support radial fields—“hedgehogs” (see Fig. 5 for illustration). For the P-even interaction between $F_{\mu\nu}$ and $W^{\mu\nu}$, it is the radial electric field trapped in the bulk of the soliton, while for the dual theory ($\gamma = 0$, $\tilde{\gamma} \neq 0$), it is the radial magnetic field. Note that these nontopological solitons are not magnetic monopoles, since the magnetic field decreases exponentially fast at large distances. This localization of the field is the feature of the theory (1).

Figure 5 shows the solution with $\kappa = \beta/\alpha = -0.9$, $w = 0.998$. The solution confines the electric field in its central ringlike region. The maximum of the field strength in International System of Units (V/m) is

$$|\vec{E}| \sim \frac{\gamma}{\alpha} \times 10^5 \frac{M^2}{\text{eV}^2 \text{m}}. \quad (20)$$

Since $w \approx 1$, the solution can be described in the nonrelativistic language where it has a clear physical interpretation. In the nonrelativistic regime, one can rewrite the equation of motion for V^i in the form of the Gross-Pitaevskii equation, with additional terms describing magnetic dipole and electric quadrupole interactions, and use the transversality condition $\partial_\mu V^\mu = 0$ to exclude V_0 [27]. Note that the magnetic dipole moment vanishes on the ansatz (14) due to antisymmetry of ϵ_{ijk} . This means that the leading electromagnetic interaction is between the electric field and the electric quadrupole moment of the vector field. It is this interaction that provides the soliton with the structure of the electric hedgehog shown in Fig. 5.

V. CONCLUSION

In this work, we studied solitons in a theory of complex vector field V^μ coupled to the electromagnetic field as in Eq. (1). We obtained kinematically stable solutions close to the nonrelativistic limit that can confine radial electric or magnetic field in their interior. Importantly, we worked in the limit in which the coupling of V^μ to the electromagnetic field is much smaller than other couplings in the theory. In cosmology, this is motivated by the fact that V^μ can be a dark matter candidate. On the other hand, the theory (1) can also be used as a mean field description of a Bose-Einstein condensate of atoms or molecules with the large dipole moment $\sim \gamma/M$ [35]. The classical solutions found here are readily adapted to larger values of γ .

Let us outline possible directions for future work. One interesting generalization of the present study would be to consider the coupling of the global vector field to a non-Abelian gauge field [36]. The gauge field lies in the adjoint representation, and one can consider the dipole interaction for the neutral vector field V in the fundamental representation. The interaction term

$$i\text{Tr}(V_\mu^\dagger F_{\mu\nu} V_\nu) + \text{H.c.}$$

is invariant under $U_g(1)$ and the gauge group.

Another natural generalization is to use gravity as a binding force of the vector soliton, rather than the field self-interaction. This would allow one to disentangle the parameters of the soliton from its coupling to the electromagnetic field. Next, one can add a Higgs-like field in the theory (5) to obtain configurations with lower values of w . Solitons in a resulting theory—“Proca-Higgs balls” and their gravitating counterparts—are a subject of recent research [23,37]. Finally, one can study small perturbations of the solutions presented here. In particular, it is important to analyze linear classical stability of the vector solitons and of the electromagnetic field trapped by the soliton.

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- [1] T. D. Lee and Y. Pang, Nontopological solitons, *Phys. Rep.* **221**, 251 (1992).
- [2] G. Rosen, Particlelike solutions to nonlinear complex scalar field theories with positive-definite energy densities, *J. Math. Phys. (N.Y.)* **9**, 996 (1968).
- [3] S. R. Coleman, Q-balls, *Nucl. Phys.* **B262**, 263 (1985); **B269**, 744(A) (1986).
- [4] G. Rosen, Charged particlelike solutions to nonlinear complex scalar field theories, *J. Math. Phys. (N.Y.)* **9**, 999 (1968).
- [5] K.-M. Lee, J. A. Stein-Schabes, R. Watkins, and L. M. Widrow, Gauged q balls, *Phys. Rev. D* **39**, 1665 (1989).
- [6] E. Y. Nugaev and A. V. Shkerin, Review of nontopological solitons in theories with $U(1)$ -symmetry, *J. Exp. Theor. Phys.* **130**, 301 (2020).
- [7] A. Y. Loginov, Nontopological solitons in the model of the self-interacting complex vector field, *Phys. Rev. D* **91**, 105028 (2015).
- [8] R. Brito, V. Cardoso, C. A. R. Herdeiro, and E. Radu, Proca stars: Gravitating Bose–Einstein condensates of massive spin 1 particles, *Phys. Lett. B* **752**, 291 (2016).
- [9] S. L. Liebling and C. Palenzuela, Dynamical boson stars, *Living Rev. Relativity* **26**, 1 (2023).
- [10] L. Visinelli, Boson stars and oscillatons: A review, *Int. J. Mod. Phys. D* **30**, 2130006 (2021).
- [11] I. Salazar Landea and F. García, Charged Proca stars, *Phys. Rev. D* **94**, 104006 (2016).
- [12] Y. Brihaye, T. Delplace, and Y. Verbin, Proca Q balls and their coupling to gravity, *Phys. Rev. D* **96**, 024057 (2017).
- [13] M. Minamitsuji, Vector boson star solutions with a quartic order self-interaction, *Phys. Rev. D* **97**, 104023 (2018).
- [14] J. Heeck, A. Rajaraman, R. Riley, and C. B. Verhaaren, Proca Q-balls and Q-shells, *J. High Energy Phys.* **10** (2021) 103.
- [15] K. Aoki and M. Minamitsuji, Highly compact Proca stars with quartic self-interactions, *Phys. Rev. D* **107**, 044045 (2023).
- [16] Z. Wang, T. Helfer, and M. A. Amin, General relativistic polarized Proca stars, *Phys. Rev. D* **109**, 024019 (2024).
- [17] D. Antypas *et al.*, New horizons: Scalar and vector ultralight dark matter, [arXiv:2203.14915](https://arxiv.org/abs/2203.14915).
- [18] V. Cardoso and P. Pani, Testing the nature of dark compact objects: A status report, *Living Rev. Relativity* **22**, 4 (2019).
- [19] Z.-G. Mou and H.-Y. Zhang, Singularity problem for interacting massive vectors, *Phys. Rev. Lett.* **129**, 151101 (2022).
- [20] A. Coates and F. M. Ramazanoğlu, Intrinsic pathology of self-interacting vector fields, *Phys. Rev. Lett.* **129**, 151103 (2022).
- [21] A. Coates and F. M. Ramazanoğlu, Treatments and placebos for the pathologies of effective field theories, *Phys. Rev. D* **108**, L101501 (2023).
- [22] K.-M. Lee and E. J. Weinberg, Nontopological magnetic monopoles and new magnetically charged black holes, *Phys. Rev. Lett.* **73**, 1203 (1994).
- [23] C. Herdeiro, E. Radu, and E. dos Santos Costa Filho, Proca-Higgs balls and stars in a UV completion for Proca self-interactions, *J. Cosmol. Astropart. Phys.* **05** (2023) 022.
- [24] H. C. Corben and J. Schwinger, The electromagnetic properties of mesotrons, *Phys. Rev.* **58**, 953 (1940).
- [25] K. Hagiwara, R. D. Peccei, D. Zeppenfeld, and K. Hikasa, Probing the weak boson sector in $e^+e^- \rightarrow W^+W^-$, *Nucl. Phys.* **B282**, 253 (1987).
- [26] Z. Davoudi and W. Detmold, Composite vector particles in external electromagnetic fields, *Phys. Rev. D* **93**, 014509 (2016).
- [27] W. Pauli, Relativistic field theories of elementary particles, *Rev. Mod. Phys.* **13**, 203 (1941).
- [28] T. D. Lee and C.-N. Yang, Theory of charged vector mesons interacting with the electromagnetic field, *Phys. Rev.* **128**, 885 (1962).
- [29] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of Bose-Einstein condensation in trapped gases, *Rev. Mod. Phys.* **71**, 463 (1999).
- [30] A. M. Pombo, J. a. M. S. Oliveira, and N. M. Santos, Coupled scalar-Proca soliton stars, *Phys. Rev. D* **108**, 044044 (2023).
- [31] X. Chu, J. Hisano, A. Ibarra, J.-L. Kuo, and J. Pradler, Multipole vector dark matter below the GeV scale, *Phys. Rev. D* **108**, 015029 (2023).
- [32] R. Friedberg, T. D. Lee, and A. Sirlin, A class of scalar-field soliton solutions in three space dimensions, *Phys. Rev. D* **13**, 2739 (1976).
- [33] M. Porrati and R. Rahman, Intrinsic cutoff and acausality for massive spin 2 fields coupled to electromagnetism, *Nucl. Phys.* **B801**, 174 (2008).
- [34] S. R. Coleman and E. J. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking, *Phys. Rev. D* **7**, 1888 (1973).
- [35] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, The physics of dipolar bosonic quantum gases, *Rep. Prog. Phys.* **72**, 126401 (2009).
- [36] S. L. Glashow, The renormalizability of vector meson interactions, *Nucl. Phys.* **10**, 107 (1959).
- [37] M. Brito, C. Herdeiro, N. Sanchis-Gual, E. dos Santos Costa Filho, and M. Zilhão, Self-interactions can (also) destabilize bosonic stars, *Classical Quantum Gravity* **41**, 195005 (2024).