

Vacuum moduli space of the minimal supersymmetric standard model

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A starting point in the study of the minimal supersymmetric Standard Model (MSSM) is the vacuum moduli space, which is a highly complicated algebraic variety; it is the image of an affine variety $X \subset \mathbb{C}^{49}$ under a symplectic quotient map ϕ to $\mathbb{C}^{973}$. Previous work [Y.-H. He *et al.*, The geometry of generations, *Commun. Math. Phys.* **339**, 149 (2015)] computed the vacuum moduli space of the electroweak sector; geometrically this corresponds to studying a restriction of $\phi: \mathbb{C}^{13\phi^{\text{res}}} \rightarrow \mathbb{C}^{22}$. We analyze the geometry of the full vacuum moduli space for superpotentials W_{minimal} (without neutrinos) and W_{MSSM} (with neutrinos) in $\mathbb{C}^{973}$. In both cases, we prove that X consists of three irreducible components X_1 , X_2 , and X_3 , and determine the images M_i of the X_i under ϕ . For W_{minimal} we show they have, respectively, dimensions 1, 15, and 29, and prove that each of the M_i is a rational variety, while for W_{MSSM} we show that M_3 is the only component. Restricting the M_i to the electroweak sector, we recover the results in He *et al.* [The geometry of generations, *Commun. Math. Phys.* **339**, 149 (2015)]. We describe the components of the vacuum moduli space geometrically in terms of incidence varieties to a product of Segre varieties.

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I. INTRODUCTION

The Standard Model of particle physics [1–4] is the most precisely tested theory in the history of science [5]. It explains the electromagnetic, strong, and weak interactions as well as the Higgs mechanism and classifies the known elementary particles [6–8]. Though recent collider experiments hint at intriguing mismatches between theory and observation, to date there is no 5σ discovery of particle physics beyond the Standard Model [9]. We know, however, that new physics is compulsory. For example, the Standard Model does not explain what dark matter is, the mechanism by which neutrinos acquire mass and what these masses are, or the requisite CP violation necessary to

accommodate a matter/antimatter asymmetry in the early Universe [10,11]. We must also ultimately reconcile quantum field theory with gravitation [12].

A. Supersymmetry

We wish to understand not only how particle physics is organized but also whence this structure originates. The best understood top down constructions of Standard Model-like physics arise from Calabi-Yau compactifications of ten-dimensional superstring theory down to a four-dimensional effective theory in $\mathbb{R}^{1,3}$ [13,14]. By virtue of having a Ricci-flat metric on the Calabi-Yau space, such heterotic compactifications enjoy $\mathcal{N} = 1$ supersymmetry in four dimensions and are source free, leading-order solutions to the Einstein equation. Supersymmetry posits a mass degeneracy between fermionic and bosonic degrees of freedom. (See Refs. [15,16] for reviews.) This is not seen empirically in nature; there is not, for instance, a scalar particle with the same mass as the electron. At best, supersymmetry is realized at low energies as a softly broken symmetry. Nevertheless, it is a theoretically appealing idea; it explicates the hierarchy between the electroweak symmetry breaking scale and the Planck mass, it allows for the unification of gauge couplings, it supplies

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candidates for weakly interacting dark matter, and it can enhance the amount of CP violation within the Standard Model. The expectation of detecting superpartners of Standard Model particles at the Large Hadron Collider is so far sadly dashed by data. There is no experimental corroboration of supersymmetry at scales below 1 TeV [9,17]. Optimistically, perhaps this only means that parameters in a supersymmetric Standard Model are somewhat finely tuned and the superpartners are accessible at a higher-energy regime to be probed by the next generation of particle accelerators. We are agnostic about this possibility.

Because it is the simplest theory consistent with observation and compatible with the modern framework for quantum gravity, we adopt the view that the minimal supersymmetric Standard Model (MSSM) [18] remains important to study in its own right. Since cosmological measurements indicate that our Universe today most closely resembles de Sitter space [19,20], which is incompatible with supersymmetry [21,22], we of course view this analysis as an initial step. The MSSM is a modestly unreal world, but its physics is not excluded by data.

To investigate a supersymmetric gauge theory, we must examine its matter content and interactions. The superpotential $W(\Phi)$ is a holomorphic function of the superfields that encodes this information. The vacuum is the simultaneous solution to the F-flatness conditions obtained from taking derivatives of the superpotential with respect to the scalar components of the chiral superfields and the D-flatness conditions associated to the generators of the gauge group, which for the MSSM is $G = SU(3)_C \times SU(2)_L \times U(1)_Y$.¹ This describes the set of zero energy states modulo gauge transformations. In general, the field configurations defined by scalar vacuum expectation values (VEVs) for which the potential determined by the F-terms and D-terms vanishes does not specify a unique point in field space. Rather, there is a *vacuum moduli space* of solutions [24].² This is a manifold $\mathcal{M}$, conveniently described in algebraic geometry. Moreover, since the superpotential does not acquire quantum corrections in perturbation theory, the vacuum moduli space $\mathcal{M}$ is a robust and defining feature of the theory.³ The gauge invariant operators (GIOs) of a supersymmetric field theory provide coordinates on $\mathcal{M}$. In this work, we complete the

program initiated in [27–30] and explicitly solve for the full classical vacuum moduli space of the MSSM with a renormalizable superpotential.

B. Dramatis personae

Throughout this paper, we will use the standard notation that the matter content and the superpotential (taken to be the minimal renormalizable one) of the MSSM are

$Q_{a,\alpha}^i$	Doublet quarks	$W_{\text{MSSM}} = C^0 H_a \bar{H}_\beta \epsilon^{\alpha\beta} + C_{ij}^1 Q_{a,\alpha}^i u_a^j H_\beta \epsilon^{\alpha\beta} + C_{ij}^2 Q_{a,\alpha}^i d_a^j \bar{H}_\beta \epsilon^{\alpha\beta} + C_{ij}^3 e^i L_a^j \bar{H}_\beta \epsilon^{\alpha\beta} + C_{ij}^4 \nu^i \nu^j + C_{ij}^5 \nu^i L_a^j H_\beta \epsilon^{\alpha\beta},$
u_a^i	Singlet up-type quarks	
d_a^i	Singlet down-type quarks	
L_a^i	Doublet leptons	
e^i	Singlet leptons	$W_{\text{minimal}} = W_{\text{MSSM}}$ with ν^i set to 0,
ν^i	Neutrinos	
H_a	Up-type Higgs	
$\bar{H}_a$	Down-type Higgs	

where $i = 1, 2, 3$ (corresponding to generation or family), $a = 1, 2, 3$ (corresponding to color), and $\alpha, \beta = 1, 2$ (corresponding to the weak interaction). In the superpotential, the coupling constants C^A are taken to be generic (complex) numbers.

C. Outline of paper and main results

Much of the introduction should be familiar to high-energy physicists. The context for why solving for the vacuum moduli space of the MSSM is an interesting problem to study is largely directed to algebraic geometers. The organization of the remainder of the paper is as follows.

In Sec. II, we describe an algorithm for finding the vacuum moduli space of a four-dimensional $\mathcal{N} = 1$ supersymmetric field theory in terms of the fields and GIOs. In order to apply this algorithm and to set notation, we present the MSSM matter content and GIOs. We close Sec. II with a discussion of previous work on the problem. In parallel to the introduction addressed to algebraic geometers, this section couches the necessary mathematics in the language of $\mathcal{N} = 1$ quantum field theory in a manner that we hope is accessible to particle theorists. In Sec. III, we analyze the master space of the MSSM, which is a complex affine algebraic variety $X \subset \mathbb{C}^{49}$. Algebraically it corresponds to the Jacobian ideal of the superpotential W ; geometrically it defines the singular locus of the hypersurface defined by the vanishing of W . We prove X consists of three irreducible components X_1, X_2 , and X_3 , and give explicit defining equations for each component. In Sec. IV, we begin our analysis of the vacuum moduli space (henceforth abbreviated VMS). We denote by M_i the image of X_i under the symplectic quotient map. We show that both X_1 and X_2 map to relatively small subspaces of $\mathbb{C}^{973}$ (to $\mathbb{C}^3$ and $\mathbb{C}^{72}$, respectively). We show that M_1 is a line, and that M_2 is birational to a variety in $\mathbb{C}^{39}$ which is an incidence correspondence between Segre varieties. Restriction to the electroweak sector corresponds algebraically to setting

¹In this work, we do not differentiate between G and G/N , where N is a subgroup of $\mathbb{Z}_6$. This subtlety distinguishes different grand unified theories [23].

²It should be emphasized that throughout we only consider the so-called Higgs branch of the vacuum moduli space. The Coulomb branch [25], on which there has been much recent activity, will be left to a separate investigation.

³The superpotential does receive nonperturbative corrections, e.g., from instantons. This means that the quantum moduli space can be different from the classical moduli space. Supersymmetric quantum chromodynamics (SQCD) with $N_c = N_f$ is an example of a theory that exhibits this phenomenon; the origin in the classical moduli space is excised in the true, quantum moduli space [26].

$Q = u = d = 0$, and we show the restriction of M_2 yields exactly the result in [27]. In Sec. V, we tackle the largest component of the master space, X_3 . This component is of dimension 41 in $\mathbb{C}^{49}$, and maps to a 783-dimensional subspace of the target space $\mathbb{C}^{973}$; it is a complete intersection defined by four linear forms and four quadrics. We prove that the image M_3 is a rational variety of dimension 29. To do this, we factor the map given by the 13 types of nonvanishing GIOs on X_3 into individual steps, and construct a tower of projections terminating in a 29-dimensional base variety in $\mathbb{C}^{63}$, for which we give explicit defining equations. In Sec. VI, we determine the effect of adding neutrinos to the superpotential, and the concomitant changes on the master space and vacuum moduli space.

Theorem 1.1. The master space X and the vacuum moduli space $\mathcal{M}$ of W_{minimal} are complex affine algebraic varieties. They both have three irreducible components, which intersect as below. Numbers indicate (affine) dimensions of components.

We determine explicit defining equations for birational models of each of these varieties, and describe the corresponding birational models geometrically:

- (i) M_1 is a line.
- (ii) M_2 is a rational variety of dimension 15.
- (iii) M_3 is a rational variety of dimension 29. The intersection structure and dimensions are summarized Fig. 1.

In Sec. VI, we prove for W_{MSSM} that the vacuum moduli space consists of one irreducible component; the vanishing ideal is the ideal of M_3 along with the neutrino variables.

For both W_{MSSM} and W_{minimal} , we show that the restriction to $Q = u = d = 0$ recovers previous results of [27] on the electroweak sector of the vacuum moduli space.

II. PRELIMINARIES

A. Defining the vacuum moduli space

To set notation, we briefly review textbook material [15,16,31–33]. The action of an $\mathcal{N} = 1$ supersymmetric quantum field theory in four dimensions is

$$S = \int d^4x \left[\int d^4\theta K(\Phi_i^\dagger, \Phi_i) + \left(\frac{1}{4g^2} \left(\int d^2\theta \text{tr} \mathcal{W}_\alpha \mathcal{W}^\alpha + \int d^2\bar{\theta} \text{tr} \bar{\mathcal{W}}_\alpha \bar{\mathcal{W}}^\alpha \right) + \int d^2\theta W(\Phi_i) + \int d^2\bar{\theta} \bar{W}(\Phi_i^\dagger) \right) \right]. \quad (2.1)$$

The Φ_i are chiral superfields, which transform in representations R_i of the gauge group G . We define the gauge field strength from the chiral spinor superfield,

$$\mathcal{W}_\alpha = i\bar{D}^2 e^{-V} D_\alpha e^V, \quad (2.2)$$

with V a vector superfield taking values in the Lie algebra $\mathfrak{g}$. The Kähler potential includes the kinetic terms:

$$K(\Phi_i^\dagger, \Phi_i) = \Phi_i^\dagger e^V \Phi_i + \dots, \quad (2.3)$$

and the superpotential is a holomorphic function of the chiral superfields that describes interactions. The integrals are over spacetime and superspace coordinates. The action (2.1) is invariant under the transformations,

$$\Phi \mapsto h\Phi, \quad e^V \mapsto (h^{-1})^\dagger e^V h^{-1}, \quad (2.4)$$

where $h = e^{i\Lambda}$ and Λ is a chiral superfield.

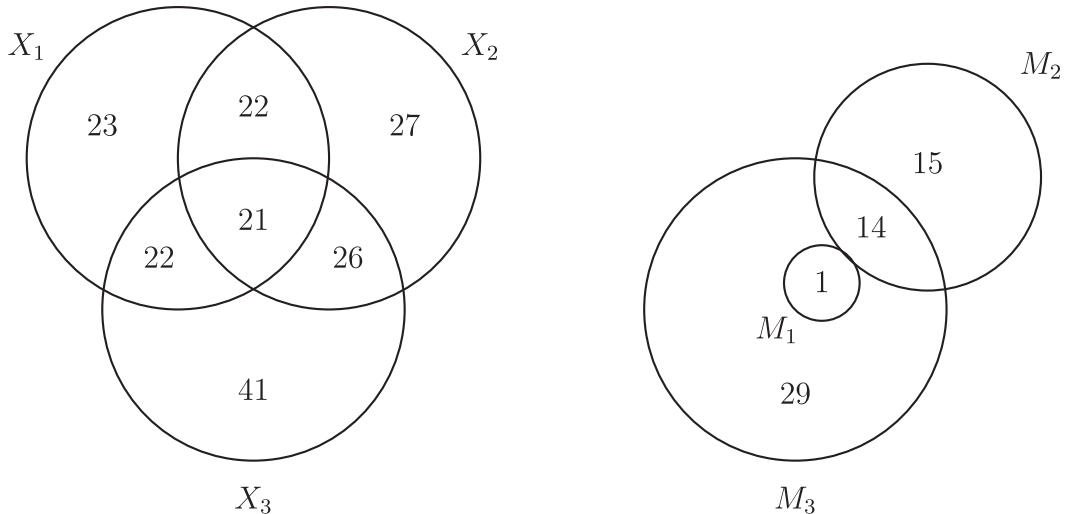


FIG. 1. The components X_i and M_i and dimensions of intersections.

The potential of the theory is written in terms of the lowest components of the chiral superfields:

$$V(\phi, \phi^*) = \sum_i |F^i|^2 + \frac{1}{4} \sum_a (D^a)^2. \quad (2.5)$$

The F-terms and D-terms are, respectively,

$$F^i = \frac{\partial W}{\partial \phi_i}, \quad (2.6)$$

$$D^a = g \sum_i \phi_i^* T^a \phi_i, \quad (2.7)$$

where the T^a , $a = 1, \dots, \dim G$, are the generators of $\mathfrak{g}$.⁴ There is an F-term for every chiral superfield. Supersymmetry imposes the condition that the eigenvalues of the Hamiltonian operator are non-negative. Thus, the minimum energy states—viz., the *vacua*—have zero energy, which means the potential must be zero. Since each term in (2.5) must individually vanish, vacua are defined by the field configurations for which $F^i = 0$ and $D^a = 0$, for all i and for all a .

The vacuum is generically a manifold rather than an isolated point.⁵ From a physical perspective, at a particular point in the vacuum geometry, the scalar components of some of the chiral superfields may have nonzero VEVs. The gauge groups under which such a field is charged are (at least partially) broken at that point.

To solve for the vacuum, we shall work with G^c the complexification of the gauge group. The complexification of a real compact Lie group G is a complex Lie group G^c containing G as a real subgroup such that there is a Lie algebra isomorphism between the two. In particular, the complexification of the group $SU(N)$ is $SL(N, \mathbb{C})$, and the complexification of $U(1)$ is $\mathbb{C}^*$. A theorem rediscovered several times [38–41] establishes that the (classical) *vacuum moduli space* is the symplectic quotient of the master space, which is the manifold of scalar field VEVs that satisfy the F-term equations:

$$\mathcal{M} = \mathcal{F} // G = \mathcal{F} / G^c. \quad (2.8)$$

To explicate this statement, consider a polynomial ring defined by the chiral superfields:

$$A = \mathbb{C}[\Phi_1, \dots, \Phi_m]. \quad (2.9)$$

We define the *master space* [42,43] as the space obtained by quotienting this ring by the ideal of the F-term constraints:

$$\mathcal{F} = A/X, \quad X = \left\langle \partial_i W := \frac{\partial W}{\partial \phi_i} = 0 \right\rangle, \quad i = 1, \dots, m. \quad (2.10)$$

Luty and Taylor [41] show that for every solution to the F-terms, there is a solution to the D-terms in the completion of the orbit of the complexified gauge group G^c . The GIOs supply a basis for the D-orbits; if the basis has n elements, this allows us to define a map from the master space to $\mathbb{C}^n$. To determine the image of the map (a geometric object) we introduce a second polynomial ring,

$$B = \mathbb{C}[O_1, \dots, O_n], \quad (2.11)$$

where the $\{O_j\}$ are a minimal basis of G -invariants, which are composite operators constructed from the elementary fields Φ_i . Then the vacuum moduli space $\mathcal{M}$ is the vanishing locus of the polynomials in the kernel of the ring map:

$$\pi: B \rightarrow \mathcal{F}. \quad (2.12)$$

B. An algorithm

A mathematical implementation of this prescription is straightforward [44,45]:

- (i) The input is the action S , from which we read off the chiral superfields Φ_i , the gauge group G , and the superpotential interactions in W .
- (ii) Knowing the representations R_i under which the Φ_i transform, we construct a minimal list of GIOs, $\{O_j\}$. Here, $i = 1, \dots, m$ and $j = 1, \dots, n$.
- (iii) Define the polynomial ring

$$R = \mathbb{C}[\phi_1, \dots, \phi_m, y_1, \dots, y_n], \quad (2.13)$$

where the y_j are auxiliary variables.

- (iv) Construct the ideal

$$J = \langle \partial_i W, y_j - O_j \rangle. \quad (2.14)$$

- (v) Using Gröbner bases, eliminate the variables ϕ_i from $J \subset R$. This leaves the ideal $I_{\mathcal{M}} \subset \mathbb{C}[y_1, \dots, y_n]$. This is the ideal corresponding to the vacuum moduli space, $\mathcal{M}$.

An example illustrates the central idea. Consider the conifold theory [46] whose quiver is shown in Fig. 2 and whose superpotential is

⁴We have neglected the Fayet-Iliopoulos D-terms [34] associated to $U(1)$ factors, but these can be included as well.

⁵This is true even within the Standard Model itself. In the Abelian Higgs mechanism, the renormalizable potential in four dimensions is quartic; $V(\phi) = \lambda(|\phi|^2 - \phi_0^2)^2$ and has the shape of a Mexican hat or the bottom of a wine bottle. Its minimum is achieved at $\phi = e^{i\theta}\phi_0$. The vacuum geometry is S^1 , corresponding to the phase described by θ . Spontaneous symmetry breaking selects a particular phase and breaks the $SU(2)_L \times U(1)_Y$ electroweak symmetry to the $U(1)_{\text{EM}}$ of electromagnetism [35–37]. This trivializes the vacuum to a point.

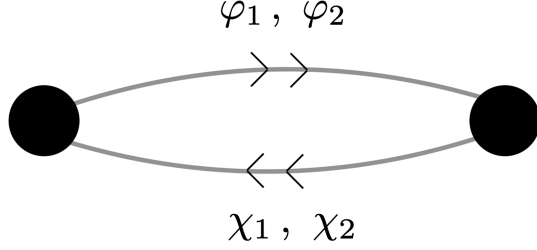


FIG. 2. The quiver of the conifold.

$$W = \text{tr}(\varphi_1 \chi_1 \varphi_2 \chi_2 - \varphi_1 \chi_2 \varphi_2 \chi_1). \quad (2.15)$$

The gauge group is $SU(N) \times SU(N)$. The matter content of the theory consists of two chiral superfields φ_1 and φ_2 that transform as $(N, \bar{N})$ and two chiral superfields χ_1 and χ_2 that transform as $(\bar{N}, N)$. The tensor product of the fundamental and antifundamental representations of $SU(N)$ gives the singlet plus the adjoint. A GIO is constructed by contracting fundamental indices (raised) with antifundamental indices (lowered):

$$\varphi_i \chi_j := \text{tr} \varphi_i \chi_j = \sum_{a=1}^N \sum_{b=1}^N (\varphi_i)_b^a (\chi_j)_a^b. \quad (2.16)$$

Since we take traces over the color indices, the resulting operator is a singlet—i.e., it transforms in the $(\mathbf{1}, \mathbf{1})$ representation of $SU(N) \times SU(N)$ —and is thus gauge invariant. To streamline notation we will leave the traces implicit when writing a GIO. The full list of GIOs for the conifold theory is

$$\{O_1, O_2, O_3, O_4\} = \{\varphi_1 \chi_1, \varphi_1 \chi_2, \varphi_2 \chi_1, \varphi_2 \chi_2\}. \quad (2.17)$$

For simplicity, we take $N = 1$ so that the fields transform in $U(1)$. As the fields are functions that commute, the superpotential vanishes, and there are no F-terms. There is one relation among the GIOs:

$$\mathcal{M} = O_1 O_4 - O_2 O_3 = 0. \quad (2.18)$$

This is the vacuum moduli space of the theory. It is as well the defining equation of the conifold itself. The gauge theory on the worldvolume of a D3-brane at the conifold singularity is reproduced by the quiver in Fig. 2. This is a crucial observation; the vacuum moduli space of the $\mathcal{N} = 1$ theory probes the transverse space to the brane [47]. This applies to a host of theories, e.g., to branes at ADE singularities on ALE spaces. Since N parallel D3-branes at the singularity are mutually BPS, the vacuum moduli space of the $SU(N) \times SU(N)$ theory is given by $\text{Sym}^N(\mathcal{M})$, the N -fold symmetric product.

C. Matter content and GIOs of the MSSM

As we aim to apply the algorithm from Sec. II B to determine the vacuum moduli space of the MSSM, let us review the field content and the minimal basis of GIOs. The matter fields transform in fundamental, antifundamental, and singlet representations of the product group $G = SU(3)_C \times SU(2)_L \times U(1)_Y$. The complexification is therefore $G^c = SL(3, \mathbb{C}) \times SL(2, \mathbb{C}) \times \mathbb{C}^*$.

To set notation, we take Latin indices from the middle of the alphabet $i, j, \dots$ to be flavor (generation) indices that run over 1, 2, 3. Whereas the gauge indices of G get contracted, multiplicities for the GIOs occur because the flavor indices can remain. Latin indices from the beginning of the alphabet $a, b, \dots$ also run over 1, 2, 3; these label the color $SU(3)_C$. Greek indices $\alpha, \beta, \dots$ run over 1, 2 and label $SU(2)_L$. The $U(1)_Y$ hypercharge assignments are given treating all the fields as being left handed. The chiral superfields in the MSSM contain the fermions found in the Standard Model. As there are fields $Q_{a,\alpha}^i$ charged under all the factors of the gauge group, unlike the example of the conifold, the MSSM is not a quiver gauge theory.⁶ A pair of Higgs doublets are necessary for two reasons; the superpotential must be holomorphic and we must ensure that gauge anomalies associated to triangle diagrams vanish.⁷ There are 52 component fields in total that we list in Table I. The minimal list of 976 GIOs constructed from these is quoted in Table VI in the Appendix [27,29,50].

We first consider the MSSM with the superpotential,

$$\begin{aligned} W_{\text{minimal}} = & C^0 \sum_{\alpha,\beta} H_\alpha \bar{H}_\beta \epsilon^{\alpha\beta} + \sum_{i,j} C_{ij}^1 \sum_{\alpha,\beta,a} Q_{a,\alpha}^i u_a^j H_\beta \epsilon^{\alpha\beta} \\ & + \sum_{i,j} C_{ij}^2 \sum_{\alpha,\beta,a} Q_{a,\alpha}^i d_a^j \bar{H}_\beta \epsilon^{\alpha\beta} \\ & + \sum_{i,j} C_{ij}^3 e^i \sum_{\alpha,\beta} L_\alpha^j \bar{H}_\beta \epsilon^{\alpha\beta}. \end{aligned} \quad (2.19)$$

This neglects neutrinos, which we add in Sec. VI; we have been explicit about sums on gauge and flavor indices. The C^A are coefficient matrices whose elements are generic. We do not introduce hierarchies by hand. In principle, the coefficients in the superpotential are real numbers, but for purposes of the calculation, they could just as well be complex. A computation shows that if the C^A are invertible, we may take them to be identity matrices. The superpotential includes the μ -term for the Higgs doublets and Yukawa couplings for the quarks and leptons leading to Dirac masses for particles in the Standard Model Lagrangian upon electroweak symmetry breaking. All of the terms in (2.19) are renormalizable. On phenomenological

⁶It embeds into one, however [48,49].

⁷If, for instance, the Standard Model group embeds in an $SO(10)$ or $SU(5)$ grand unified theory, we may include additional pairs of Higgs doublets. In this work, we do not consider this possibility.

TABLE I. Indices and field content conventions for the MSSM. The enumeration counts each component superfield individually.

Field	Representation	Number	Description
$Q_{a,\alpha}^i$	$(\mathbf{3}, \mathbf{2}, \frac{1}{6})$	18	Doublet quarks
u_a^i	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})$	9	Singlet up-type quarks
d_a^i	$(\bar{\mathbf{3}}, \mathbf{1}, \frac{1}{3})$	9	Singlet down-type quarks
L_α^i	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$	6	Doublet leptons
e^i	$(\mathbf{1}, \mathbf{1}, 1)$	3	Singlet leptons
ν^i	$(\mathbf{1}, \mathbf{1}, 0)$	3	Neutrinos
H_α	$(\mathbf{1}, \mathbf{2}, \frac{1}{2})$	2	Up-type Higgs
$\bar{H}_\alpha$	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$	2	Down-type Higgs

grounds, we exclude the renormalizable GIOs LH , $LL\bar{e}$, and QdL that violate R-parity from the superpotential (2.19).

Because they are GIOs in their own right, we can write different terms involving neutrinos. The MSSM superpotential we consider becomes

$$W_{\text{MSSM}} = W_{\text{minimal}} + W_{\text{neutrinos}}, \quad (2.20)$$

$$W_{\text{neutrinos}} = \sum_{i,j} C_{ij}^4 \nu^i \nu^j + \sum_{i,j} C_{ij}^5 \nu^i \sum_{\alpha,\beta} L_\alpha^j H_\beta \epsilon^{\alpha\beta}. \quad (2.21)$$

The two terms in (2.21) give rise to Majorana and Dirac masses for the neutrino. We do not include linear terms (tadpoles), which can be absorbed via field redefinition, or R-parity violating ν^3 terms. Adopting a conservative view, we assume that there are exactly three species of active neutrinos. That is to say, consistent with experiment, we do not include any sterile neutrinos.

D. History and previous work on the problem

The study of vacuum moduli spaces of supersymmetric theories has a significant pedigree [24–26, 51–53]. The possibility of using the matter content and GIOs to solve for the vacuum moduli space of the MSSM was already considered as early as 1995. The authors of [50] write:

“There do exist mathematical algorithms well-known in algebraic geometry (e.g., the methods of resultant polynomials or Gröbner bases) which can in principle be used to decide whether such systems of simultaneous nonlinear equations have nontrivial solutions. Unfortunately, these methods are of no practical use at the level of complexity encountered here ... the required number of algebraic operations is demonstrably finite, but quite astronomical.”

Our interest in this problem began with [28, 29]. These works established that the vacuum moduli spaces of subsectors of the MSSM have identifiable geometry. For instance, restricting to the electroweak sector, where, consistent with the unbroken $SU(3)_C$ symmetry in nature, the VEVs of the squarks are set to zero *ab initio*, the vacuum moduli space is an affine cone over the Veronese surface, which is $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. (This structure is further discussed in [54].) Moreover, it was observed that these geometric features persist when we incorporate higher order non-renormalizable terms into the superpotential that preserve physically desirable properties such as R-parity, which ensures the stability of the proton and leaves a stable lightest supersymmetric particle after supersymmetry breaking as a candidate for the dark matter that today comprises some 25% of the energy density of the Universe. Based on these examples, the authors suggested that geometric structure in the vacuum is a precept for phenomenology and a selection principle for extensions of the Standard Model Lagrangian. The paper [27] investigated the structure of the vacuum moduli space of the electroweak sector scanning over the number of generations of particles and the number of pairs of Higgs doublets and made the observation that only when there are three generations are the vacuum moduli spaces toric. The followup [55] considered the role of R-parity in three generation models in greater detail. If we turn to the SQCD sector of the theory alone, the vacuum moduli space of the $SU(N)$ theory is Calabi-Yau for arbitrary numbers of flavors and colors, a fact deduced from the palindromicity of the numerator of the associated Hilbert series [30].

Proceeding to the full MSSM, [56] employs the plethystic program [57] to generate Hilbert series in the case where the superpotential is set to zero.⁸ In the following sections, we deduce the vacuum moduli space of the full MSSM with the superpotential (2.19) using Gröbner basis methods. Just as [50] noted, this is indeed an astronomical task, but one that is in fact tractable using Gröbner basis methods from computational algebraic geometry. The key is to exploit a very special algebraic aspect of the problem; a multigraded structure.

III. THE COMPONENTS OF THE MASTER SPACE

The superpotential we use is (2.19),

$$\begin{aligned} W_{\text{minimal}} = & C^0 \sum_{\alpha,\beta} H_\alpha \bar{H}_\beta \epsilon^{\alpha\beta} + \sum_{i,j} C_{ij}^1 \sum_{\alpha,\beta,a} Q_{a,\alpha}^i u_a^j H_\beta \epsilon^{\alpha\beta} \\ & + \sum_{i,j} C_{ij}^2 \sum_{\alpha,\beta,a} Q_{a,\alpha}^i d_a^j \bar{H}_\beta \epsilon^{\alpha\beta} \\ & + \sum_{i,j} C_{ij}^3 e^i \sum_{\alpha,\beta} L_\alpha^j \bar{H}_\beta \epsilon^{\alpha\beta}. \end{aligned} \quad (3.1)$$

⁸Standard Model flavor invariants are similarly calculated and discussed in [58].

In the initial part of the analysis, we do not consider the singlet right handed neutrinos in (2.21); we add neutrinos in Sec. VI.

A. Changing basis via the Yukawa couplings

A key simplification is to rewrite the superpotential using the following variables:

$$\tilde{e}^i := \sum_j C_{ji}^3 e^j, \quad \tilde{u}_a^i := \sum_j C_{ij}^1 u_a^j, \quad \tilde{d}_a^i := \sum_j C_{ij}^2 d_a^j. \quad (3.2)$$

By scaling variables, we may assume that $C^0 = 1$. We also assume as in [50] that these coupling matrices C^1 , C^2 , and C^3 are all invertible. The resulting superpotential in these variables is

$$\begin{aligned} W_{\text{minimal}} = & \sum_{\alpha,\beta} H_\alpha \bar{H}_\beta \epsilon^{\alpha\beta} + \sum_i \sum_{\alpha,\beta,a} Q_{a,\alpha}^i \tilde{u}_a^i H_\beta \epsilon^{\alpha\beta} \\ & + \sum_i \sum_{\alpha,\beta,a} Q_{a,\alpha}^i \tilde{d}_a^i \bar{H}_\beta \epsilon^{\alpha\beta} \\ & + \sum_i \tilde{e}^i \sum_{\alpha,\beta} L_\alpha^i \bar{H}_\beta \epsilon^{\alpha\beta}. \end{aligned} \quad (3.3)$$

The transformation $(C^1, C^2, C^3) \in GL(3, \mathbb{C}) \times GL(3, \mathbb{C}) \times GL(3, \mathbb{C})$ gives a linear action on $\mathbb{C}^{49}$.

The next proposition shows that the resulting change of variables does indeed preserve the linear span of each of the 28 GIO types appearing in the Appendix, and therefore we may change variables to the tilde versions, and the image will be linearly equivalent to the image under the original map.

Proposition 3.1. The group $GL(3, \mathbb{C}) \times GL(3, \mathbb{C}) \times GL(3, \mathbb{C})$ acts linearly on the span of each of the 28 GIO types appearing in the Appendix.

Proof. There are 28 cases, which we break into three classes:

Case 1: $LH, H\bar{H}, QQQL, QQQ\bar{H}, (QQQ)_4 LLH, (QQQ)_4 LH\bar{H}, (QQQ)_4 H\bar{H}\bar{H}$.

Case 2: $LLe, L\bar{H}e, (QQQ)_4 LLLe, (QQQ)_4 LL\bar{H}e, (QQQ)_4 L\bar{H}\bar{H}e, (QQQ)_4 H\bar{H}\bar{H}e$.

Case 3: $udd, QdL, QuH, Qd\bar{H}, QuQd, QuLe, uude, Qu\bar{H}e, dddLL, uuuee, QuQue, QQQQu, dddL\bar{H}, uudQdH, uudQdQd$.

The seven GIO types in Case 1 have no u , d or e in them, hence each individual GIO in each of these types is invariant under this group action.

For Case 2, neither u nor d appear, and since each GIO in these types has a linear factor of e_i , for some i , each type is visibly invariant under the group action.

For Case 3, we need only check that the action of C^1 on u , C^2 on d and C^3 on e leave the linear span of each of the 15 GIO types invariant. We check this computationally as follows. First, we work on each $GL(3, \mathbb{C})$ separately. Second, we create a base field K which is the fraction field of a polynomial ring whose nine variables are the entries in a 3×3 matrix. Then, we compute the linear span of the generators of the ideal consisting of all the GIOs of a given type, where we have acted on the respective u, d, e by this matrix. We compare this linear span (over K) with the original span and verify that they agree. ■

Example 3.2. The above computation is particularly transparent for the $dddLL$ type, because the action on the d variables is essentially multiplication of the 3×3 matrix $D = (d_a^i)$ by the matrix $C^2 = (C_{ij}^2)$, and since each $dddLL$ is a product of the determinant of D with a polynomial in the L variables, the action multiplies each polynomial by the determinant of the matrix C^2 , and hence the linear span of the set of $dddLL$ GIOs is invariant.

B. Equations for the master space

The master space is the zero set of the ideal J_W generated by the 49 partial derivatives of W_{minimal} .

Example 3.3. Notice that the variables $\tilde{u}_a^i$ only appear in one of the four summations in the superpotential. Thanks to the change of variables, the partial derivatives with respect to the $\tilde{u}$ variables have a particularly simple form:

$$\begin{pmatrix} -H_2 Q_{1,1,1} + H_1 Q_{1,1,2} \\ -H_2 Q_{1,2,1} + H_1 Q_{1,2,2} \\ -H_2 Q_{1,3,1} + H_1 Q_{1,3,2} \\ -H_2 Q_{2,1,1} + H_1 Q_{2,1,2} \\ -H_2 Q_{2,2,1} + H_1 Q_{2,2,2} \\ -H_2 Q_{2,3,1} + H_1 Q_{2,3,2} \\ -H_2 Q_{3,1,1} + H_1 Q_{3,1,2} \\ -H_2 Q_{3,2,1} + H_1 Q_{3,2,2} \\ -H_2 Q_{3,3,1} + H_1 Q_{3,3,2} \end{pmatrix}.$$

These equations are the 2×2 minors involving the first column of the 2×10 matrix,

$$\begin{pmatrix} H_1 & Q_{1,1,1} & Q_{1,2,1} & Q_{1,3,1} & Q_{2,1,1} & Q_{2,2,1} & Q_{2,3,1} & Q_{3,1,1} & Q_{3,2,1} & Q_{3,3,1} \\ H_2 & Q_{1,1,2} & Q_{1,2,2} & Q_{1,3,2} & Q_{2,1,2} & Q_{2,2,2} & Q_{2,3,2} & Q_{3,1,2} & Q_{3,2,2} & Q_{3,3,2} \end{pmatrix}.$$

Notice that the partial derivatives with respect to the $\tilde{d}_a^i$, $\tilde{e}^i$, and L_α^i variables behave similarly. It turns out that many of the 49 equations arise as in this previous example.

Consider the following matrices M_Q, M_{du}, M_H, M_L :

$$\begin{aligned} M_Q &:= \begin{pmatrix} Q_{1,1,1} & Q_{1,2,1} & Q_{1,3,1} & Q_{2,1,1} & Q_{2,2,1} & Q_{2,3,1} & Q_{3,1,1} & Q_{3,2,1} & Q_{3,3,1} \\ Q_{1,1,2} & Q_{1,2,2} & Q_{1,3,2} & Q_{2,1,2} & Q_{2,2,2} & Q_{2,3,2} & Q_{3,1,2} & Q_{3,2,2} & Q_{3,3,2} \end{pmatrix}, \\ M_{du} &:= \begin{pmatrix} \tilde{u}_{1,1} & \tilde{u}_{1,2} & \tilde{u}_{1,3} & \tilde{u}_{2,1} & \tilde{u}_{2,2} & \tilde{u}_{2,3} & \tilde{u}_{3,1} & \tilde{u}_{3,2} & \tilde{u}_{3,3} \\ -\tilde{d}_{1,1} & -\tilde{d}_{1,2} & -\tilde{d}_{1,3} & -\tilde{d}_{2,1} & -\tilde{d}_{2,2} & -\tilde{d}_{2,3} & -\tilde{d}_{3,1} & -\tilde{d}_{3,2} & -\tilde{d}_{3,3} \end{pmatrix}, \\ M_H &:= \begin{pmatrix} H_1 & \bar{H}_1 & H_1 & H_2 \\ H_2 & \bar{H}_2 & \bar{H}_1 & \bar{H}_2 \end{pmatrix}, \\ M_L &:= \begin{pmatrix} L_1^1 & L_1^2 & L_1^3 \\ L_2^1 & L_2^2 & L_2^3 \end{pmatrix}. \end{aligned}$$

Computation 3.4. The ideal $J_{W_{\text{minimal}}}$ defining the master space is generated by

- (i) The six monomials $\{\bar{H}_1 \tilde{e}^1, \bar{H}_1 \tilde{e}^2, \bar{H}_1 \tilde{e}^3, \bar{H}_2 \tilde{e}^1, \bar{H}_2 \tilde{e}^2, \bar{H}_2 \tilde{e}^3\}$.
- (ii) The four forms with a linear term in H_α or $\bar{H}_\alpha$: for each $\alpha \in \{1, 2\}$, we have

$$\begin{aligned} \bar{F}_\alpha &= \bar{H}_\alpha - \sum_{i,a} \tilde{u}_a^i Q_{a,\alpha}^i, \\ F_\alpha &= H_\alpha + \sum_{i,a} \tilde{d}_a^i Q_{a,\alpha}^i + \sum_i L_\alpha^i \tilde{e}^i. \end{aligned}$$

- (iii) Let $(M_H)_{\{i_1, i_2, \dots, i_k\}}$ denote columns $\{i_1, i_2, \dots, i_k\}$ of M_H . The remaining 39 generators for J_W come from the 2×2 minors of the following five matrices, but only those minors which involve the first column of each matrix. We use a vertical bar to denote concatenation,

$$\begin{aligned} N_1 &= (M_H)_1 | M_Q, \\ N_2 &= (M_H)_2 | M_Q, \\ N_3 &= (M_H)_3 | M_{du}, \\ N_4 &= (M_H)_4 | M_{du}, \\ N_5 &= (M_H)_2 | M_L. \end{aligned}$$

The first ingredient to understanding the vacuum moduli space is an analysis of the master space, which is provided by the next theorem.

Theorem 3.5. The master space has three components X_1, X_2 , and X_3 , which

- (i) are reduced and irreducible;
- (ii) are of dimensions 23, 27, 41, respectively;
- (iii) are defined by prime ideals I_i described explicitly below.

Proof. Define ideals I_1, I_2, I_3 as below:

$$I_1 = I_{2 \times 2}(M_1) + I_{2 \times 2}(M_2) + \langle \tilde{e}^1, \tilde{e}^2, \tilde{e}^3 \rangle + \langle F_1, F_2, \bar{F}_1, \bar{F}_2 \rangle, \text{ where}$$

$$M_1 = (M_H)_{1,2} | M_Q | M_L, \text{ and}$$

$$M_2 = (M_H)_{3,4} | M_{du},$$

$$I_2 = I_{2 \times 2}(N_1) + \langle \bar{H}_1, \bar{H}_2 \rangle + \langle \tilde{u}_a^i \rangle_{i,a} + \langle F_1, F_2 \rangle,$$

$$I_3 = \langle H_1, H_2, \bar{H}_1, \bar{H}_2 \rangle + \langle F_1, F_2, \bar{F}_1, \bar{F}_2 \rangle.$$

A direct computation shows that

$$J_W = I_1 \cap I_2 \cap I_3,$$

and that the dimensions are as described. It is possible to prove that the I_i are prime ideals directly. For example, for the ideal I_2 , we quotient by the $\bar{H}$ and $\tilde{u}$ variables. Vanishing of the F_i allows us to replace the H_i in the first column of N_1 with two bihomogeneous polynomials of type $dQ + Le$. Now simplify the resulting matrix using the M_Q columns to further reduce N_1 , to the matrix

$$\left(\sum_i L_1^i \tilde{e}^i, \sum_i L_2^i \tilde{e}^i \right)^T \Big| M_Q.$$

As the first column consists of two independent forms in variables distinct from the Q variables, the resulting ideal of 2×2 minors is prime.

Primality may also be verified computationally; this completes our analysis of the components of the master space. ■

IV. THE COMPONENTS M_1 AND M_2 OF THE VACUUM MODULI SPACE

We first analyze the image of the components X_1 and X_2 of the master space. It turns out on these two components that there are relatively few nonzero GIO types. In contrast, for X_3 the majority of the GIO types are nonzero, making it far more difficult to analyze, so we defer that analysis to Sec. V.

A. The image M_1 of component X_1

Recall from the previous section that the ideal of X_1 is defined by

$$I_{2 \times 2}(M_1) + I_{2 \times 2}(M_2) + \langle \tilde{e}^1, \tilde{e}^2, \tilde{e}^3 \rangle + \langle F_1, F_2, \bar{F}_1, \bar{F}_2 \rangle.$$

Proposition 4.1. The image M_1 under all 973 GIOs is a line. More specifically:

- (1) Every GIO type except *udd* vanishes on the zero set of

$$I_{2 \times 2}(M_1) + \langle \tilde{e}^1, \tilde{e}^2, \tilde{e}^3 \rangle,$$

and consequently on the component X_1 .

- (2) The image of X_1 under the *udd* type has image a line in $\mathbb{C}^9 \subset \mathbb{C}^{973}$. In fact, since six of the nine *udd* GIOs vanish on X_1 , the line lies in a $\mathbb{C}^3$.

Proof. Part (1) is shown by a Macaulay2 computation, and that all but three of the nine *udds* on X_1 vanish. The same computation shows that the remaining three *udds* are all scalar multiples of each other, so define a line in the image space. ■

B. The image M_2 of component X_2

The analysis of M_2 is more complicated than that of X_1 , and requires new methods. Recall that by Theorem 3.5, the ideal I_2 of X_2 is defined by

$$I_2 = I_{2 \times 2}(N_1) + \langle \bar{H}_1, \bar{H}_2 \rangle + \langle \tilde{u}_a^i \rangle_{i,a} + \langle F_1, F_2, \bar{F}_1, \bar{F}_2 \rangle.$$

By reducing the 28 GIO types with respect to the ideal I_2 , we find that the only nonzero GIO types on X_2 are

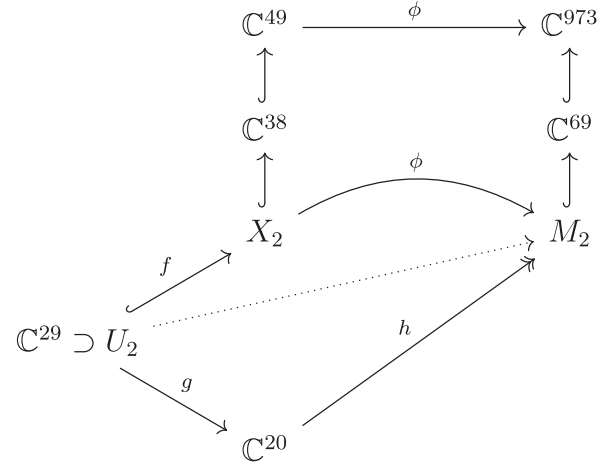
$$LH, \quad LLe, \quad QdL, \quad dddLL, \quad (QQQ)_4 LLe.$$

Furthermore, since $\bar{H}_a$ vanishes on X_2 , as do the $\tilde{u}_a^i$, the three LH elements may be written in terms of the *QdL* and *LLe* GIOs, so up to a change of coordinates, these three

elements are not needed in the map to $\mathbb{C}^{973}$. This leaves 69 nonvanishing GIOs.

Theorem 4.2. The image M_2 of X_2 under the map ϕ has dimension 15, is rational, and sits inside a linear space $\mathbb{C}^{69} \subset \mathbb{C}^{973}$. Furthermore, we compute explicit generators for the ideal of the image in $\mathbb{C}^{69} \subset \mathbb{C}^{973}$.

The diagram below is a roadmap of the high level steps in our analysis, showing the maps and describing the spaces that will be the key players.



- (1) The $\mathbb{C}^{38}$ is a reflection of the fact that the 11 variables $\tilde{u}_a^i$ and $\bar{H}_a$ vanish on X_2 .
- (2) The subset $U_2 \subset X_2$ is the open affine subset where $H_1 \neq 0$. On this open set, setting $t = H_2/H_1$, then $H_2 = tH_1$. Since the ideal of X_2 contains $I_{2 \times 2}(N_1)$, on U_2 we also have $Q_{i,j,2} = tQ_{i,j,1}$, for all i, j . So U_2 is (isomorphic to) a subvariety of $\mathbb{C}^{29}$, with the 29 variables

$$t, Q_{i,j,1} L_a^i \tilde{e}^i H_a \tilde{d}_a^i.$$

In particular, the $Q_{i,j,2}$ become redundant in describing U_2 .

- (3) The equations of U_2 in $\mathbb{C}^{29}$ are

$$F_1 = H_1 + \sum_{i,a} \tilde{d}_a^i Q_{a,1}^i + \sum_i L_1^i \tilde{e}^i,$$

$$F_2 - tF_1 = \sum_i (tL_1^i - L_2^i) \tilde{e}^i.$$

This visibly defines U_2 as a complete intersection in $\mathbb{C}^{29}$, or, by eliminating H_1 , as a hypersurface in $\mathbb{C}^{28}$.

- (4) The next step is to factor the GIO map from U_2 to M_2 through $\mathbb{C}^{20}$ in the diagram. The $\mathbb{C}^{20}$ arises from the fact that the GIOs factor on U_2 in a nice way, in terms of the following polynomials:
 - (a) The three $\tilde{e}^i$'s.
 - (b) The three 2×2 minors Δ_i of the L_a^i matrix.

- (c) The determinant $\det D$ of the 3×3 matrix $D = \tilde{d}_a^i$.
- (d) The determinant $\det Q$ of the 3×3 matrix $Q = Q_{i,a,1}$.
- (e) The nine entries $DQ_{i,j}$ of the product DQ .
- (f) The three elements $tL_i := tL_1^i - L_2^i$.

There are a total of twenty elements in the list above; these polynomials give a map g from the hypersurface $U_2 \subset \mathbb{C}^{28}$ to $\mathbb{C}^{20}$;

- (5) Since every GIO that is nonvanishing on U_2 factors into a product of elements in the list above, we obtain an induced map $h: \mathbb{C}^{20} \rightarrow \mathbb{C}^{69}$, such that the composition hg is equal to ϕf . Because of the factorization of the GIOs, the entries of h are monomials. The 69 GIOs factor (up to a constant scaling factor) as products of the 20 elements above as
 - (a) (LLe , 9 of these): $\Delta_i \cdot \tilde{e}^j$.
 - (b) (QdL , 27 of these): $DQ_{i,j} \cdot tL_i$.
 - (c) ($dddLL$, 3 of these): $(\det D) \cdot \Delta_i$.
 - (d) ($(QQQ)_4LLe$, 30 of these): $(\det Q) \cdot \tilde{e}^i \cdot m$, for the ten degree three monomials m in the tL_i .
- (6) The image of U_2 under g in $\mathbb{C}^{20}$ is defined by the following three equations.

$$\begin{pmatrix} tL_1 & tL_2 & tL_3 \end{pmatrix} \begin{pmatrix} \tilde{e}^1 & \Delta_1 \\ \tilde{e}^2 & \Delta_2 \\ \tilde{e}^3 & \Delta_3 \end{pmatrix} = 0,$$

$$\det(DQ) - \det D \det Q = 0.$$

Computation 4.3. The ideal of M_2 is defined by 6 linear forms, 816 quadratics, and 90 quartics. While this gives an explicit list of generators for the ideal of M_2 , the geometric structure of M_2 is more clearly understood via a birational projection

$$M_2 \xrightarrow{\pi} \mathcal{V}_2 \subseteq \mathbb{C}^{39}.$$

We tackle this in the next subsection.

C. A birational model of M_2 connecting to classical geometry

As we have seen, on X_2 , the LH GIOs can be written in terms of the LLe and QdL GIOs. A similar result holds for the $(QQQ)_4LLe$ GIOs on M_2 , except they are rational functions, not polynomials. For notational convenience, and consistency with the computations in §5, we denote the GIO types in use as

- (a) The 30 $(QQQ)_4LLe$ s as $q_1, \dots, q_{30}$.
- (b) The 27 QdL s as $a_1, \dots, a_{27}$.
- (c) The 9 LLe s as $b_1, \dots, b_9$.
- (d) The 3 $dddLL$ s as $i_1, \dots, i_3$.

Computation 4.4. On X_2 , the $(QQQ)_4LLe$ GIOs are rational functions of the LLe , QdL , and $dddLL$

functions. We have relations for all $j = 1, \dots, 30$ and all $\ell = 1, 2, 3$,

$$q_j = \frac{b_k C(a_*)}{i_\ell},$$

where $C(a_*)$ is a determinantal cubic in the a variables, and k depends on ℓ and j .

By the above computation, to understand the component M_2 up to birational equivalence, it suffices to consider the image of X_2 under the map to $\mathbb{C}^{39}$ given by the GIOs of the three types LLe , QdL , and $dddLL$. This corresponds to the projection map $\pi: M_2 \subset \mathbb{C}^{69} \rightarrow \mathbb{C}^{39}$. Let $\mathcal{V}_2 := \pi(M_2)$ be this projection.

The remainder of this section is devoted to a proof that the geometry of $\mathcal{V}_2$ may be expressed as an incidence correspondence between Segre varieties, yielding a geometric picture of the structure of M_2 . The following matrices play a key role in our analysis:

- (i) $M_a := \begin{pmatrix} a_1 & a_4 & a_7 & a_{10} & a_{13} & a_{16} & a_{19} & a_{22} & a_{25} \\ a_2 & a_5 & a_8 & a_{11} & a_{14} & a_{17} & a_{20} & a_{23} & a_{26} \\ a_3 & a_6 & a_9 & a_{12} & a_{15} & a_{18} & a_{21} & a_{24} & a_{27} \end{pmatrix}.$
- (ii) $M_b := \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}.$
- (iii) $M_z := (i_1 \ i_2 \ i_3)$ (we use M_z rather than M_i as the latter looks like an index).
- (iv) $J := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$

Computation 4.5. The ideal of the $\mathcal{V}_2$ is minimally generated by 189 homogeneous quadratic equations, which may be partitioned into the five sets below:

- (1) The 108 2×2 minors of M_a .
- (2) The 18 2×2 minors of $M_b^T | M_z^T$.
- (3) The 9 entries of $M_z \cdot J \cdot M_a$.
- (4) The 27 entries of $M_b^T \cdot M_a$.
- (5) The 27 entries of $M_b \cdot J \cdot M_a$.

Lemma 4.6. The first ideal above defines the Segre variety $\Sigma_{2,8}$ of $\mathbb{P}^2 \times \mathbb{P}^8 \subseteq \mathbb{P}^{26}$, and the 2×2 minors of M_b^T define the Segre variety $\Sigma_{2,2} \subseteq \mathbb{P}^8$.

Proof. Let $\mathbf{a} = (a_0, \dots, a_n)$ be coordinates on $\mathbb{P}^n$ and similarly $\mathbf{b} = (b_0, \dots, b_m)$ be coordinates on $\mathbb{P}^m$. We can represent elements of $\mathbb{P}^{(n+1)(m+1)-1}$ by $(n+1) \times (m+1)$ matrices. The Segre variety $\Sigma_{n,m}$ is defined by the image of the map

$$(\mathbf{a}, \mathbf{b}) \mapsto \text{the rank one matrix } \mathbf{a}^T \mathbf{b} \in \mathbb{P}^{(n+1)(m+1)-1}.$$

The image consists of rank one matrices, and its ideal is generated by the 2×2 minors of a $(n+1) \times (m+1)$ matrix of coordinate variables on the image space. So, in the case at hand, $\Sigma_{2,8}$ is defined by the ideal of 2×2 minors

of M_a , and $\Sigma_{2,2}$ is defined by the ideal of 2×2 minors of M_b . For more details, see Harris [59] or Eisenbud [60]. ■

We close this section with the promised geometric interpretation of the ideal appearing in Computation 4.5.

Theorem 4.7. The affine variety $\mathcal{V}_2$ is the cone over the incidence correspondence

$$Y \subset \Sigma_{2,2} \times \Sigma_{2,8} \times \mathbb{P}^2,$$

defined by the remaining equations in the above computation that are not the equations of the Segre varieties. It consists of all points (thought of as row vectors),

$$((\theta_1, \theta_2), (\theta_3, \theta_4), \theta_5) \in (\mathbb{P}^2 \times \mathbb{P}^2) \times (\mathbb{P}^2 \times \mathbb{P}^8) \times \mathbb{P}^2,$$

such that $\theta_2 = \theta_5$ and $\theta_1 \perp \theta_3$ and $\theta_2 \perp \theta_3$. These conditions define an ideal of five homogeneous quadratic polynomials of codimension four. Therefore, Y is of dimension 12 and is rational and smooth, and the cone $\mathcal{V}_2$ is dimension 15, rational and smooth away from the two-dimensional vertex.

Proof. By Lemma 4.6, Y is a subset of the first displayed equation in the statement of the theorem. The 2×2 minors of $M_b^T | M_z^T$ involving the last column (which is θ_5^T) force θ_5 to be proportional to any row of M_b (M_b has rank one), accounting for the $\theta_2 = \theta_5$ equation. We have now accounted for all equations of types (1) and (2) above.

For the equations of type (3), the matrix $M_a = \theta_3^T \cdot \theta_4$, therefore $M_z \cdot J \cdot M_a$ yields the single equation,

$$\theta_5 \cdot J \cdot \theta_3^T = 0,$$

because $\theta_4 \neq 0$ and $\theta_5 \cdot J \cdot \theta_3^T \cdot \theta_4 = 0$.

For equations of type (4), we have

$$M_b^T \cdot M_a = \theta_2^T \cdot \theta_1 \cdot \theta_3^T \cdot \theta_4 = 0,$$

and therefore, reasoning as above,

$$\theta_1 \theta_3^T = 0.$$

For equations of type (5), we have

$$M_b \cdot J \cdot M_a = \theta_1^T \cdot \theta_2 \cdot J \cdot \theta_3^T \cdot \theta_4 = 0,$$

and therefore, reasoning as above,

$$\theta_2 \cdot J \cdot \theta_3^T = 0.$$

But $\theta_2 = \theta_5$, and this duplicates the equation of type (3) above.

$\theta_2 = \theta_5$ gives three homogeneous quadrics, and the two equations of type (3) and (4) each yield a homogeneous quadric, resulting in five quadrics. A computation confirms that the ideal generated by these five quadrics has codimension four. ■

Computation 4.8. Restricting M_2 by setting $Q = u = d = 0$ yields an irreducible complex affine variety of dimension five: a cone over the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2$.

This is precisely the variety which Sec. 3.3 of [27] shows defines the vacuum moduli space restricted to the electro-weak sector.

V. ANALYSIS OF THE M_3 COMPONENT OF THE VACUUM MODULI SPACE

Our goal is to understand the geometry of the image M_3 of component X_3 under the GIO map ϕ . In contrast to the analysis of M_1 and M_2 , there are 783 GIOs which do not vanish on X_3 , in the following 13 types:

$$\begin{aligned} & QdL, \quad udd, \quad uude, \quad uuuee, \quad dddLL, \quad LLe, \\ & QQQQL, \quad QuQd, \quad QuLe, \quad QQQQu, \quad QuQue, \\ & QQQ_4LLLe, \quad uudQdQd. \end{aligned}$$

Such a large number of GIOs makes the task of finding the image seem problematic, in particular, the ideal defining this image in $\mathbb{C}^{783}$ has so many generators that an explicit description is unenlightening.

A. Overview of the strategy

Instead of finding the defining equations for the image in $\mathbb{C}^{783}$, our plan is to find a projection of M_3 to a smaller dimensional base space Y , which preserves certain essential features of M_3 , such as dimension and such that the resulting projection map from M_3 to Y is generically (i.e., on an open dense subset of M_3) one-to-one.

We use a fast probabilistic algorithm to determine the dimension of M_3 , which a computation shows to be 29, as expected for the quotient by the group action.

To identify the most economical (i.e., in a target space where we can understand the image) projection, we systematically search through all small subsets (e.g., two or three or four sets of the 13 GIO types that are non-vanishing on X_3) to pinpoint those having the image of X_3 of the expected dimension 29.

It turns out to be the case that there no subsets of pairs of GIO types having image dimension 29, but there are 81 subsets of three GIO types having image dimension 29.

Our next aim is to identify one of these sets where we can prove that the projection map from M_3 to Y is generically one-to-one.

Computation 5.1. There are no sets involving only one or two GIO types such that the image of X_3 under the maps determined by these types has image of dimension 29. There are 81 triples of GIO types with 29 dimensional image, and we list several of the smallest sets in Table II:

In terms of the number of polynomials it contains, the smallest GIO set above is $\{QdL, udd, uuuee\}$. However, a computation shows that while it has a 29-dimensional image, the map from M_3 to it is generically two-to-one, so this set is not suited to our purposes. It turns out that the set of GIO types $\{QdL, udd, uude\}$ yields a manageable set of

TABLE II. Types of GIO triples with 29 dimensional image on component X_3 .

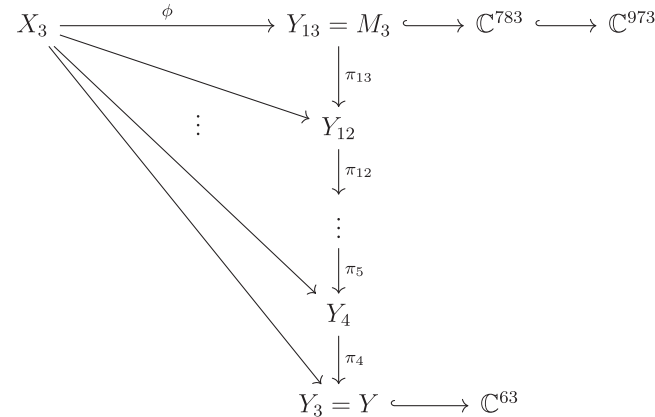
Types	Total number of GIOs
$\{QdL, udd, uuuee\}$	42
$\{QdL, uude, dddLL\}$	57
$\{QdL, uude, uuuee\}$	60
$\{QdL, udd, uude\}$	63
$\vdots$	$\vdots$

equations, and we can prove that the projection is generically one-to-one. From a physics point of view, this set is preferable because the udd and $uude$ types correspond to the (SUSY) neutron and hydrogen atom, and so have natural physical interpretations.

To show that the resulting map $\pi: M_3 \rightarrow Y$ is birational our method is to divide and conquer: starting with our base set of three GIO types, with image $Y = Y_3$, we add a single GIO type to our base set and consider the projection map from the image Y_4 to $Y_3 = Y$, and show that this is birational. Then we iterate the process, adding one GIO type at a time as in Fig. 3, showing each of these projections is birational, and consequently so is their composition.

Choosing the order of this sequence of projections is a delicate process, because a poor choice at one step makes the determination of birationality of that step computationally exorbitant. To overcome this, we will use a multi-grading on the GIO types to identify the order in which to add the GIO types.

There is a natural $\mathbb{Z}^7$ grading on the polynomial ring R with $\{u, d, L, Q, e, H, \bar{H}\}$ corresponding to $\{\mathbf{e}_1, \dots, \mathbf{e}_7\}$. Each of the 28 GIO types is homogeneous with respect to this grading. However, on X_3 , the H_α and $\bar{H}_\alpha$ vanish, so we have a $\mathbb{Z}^5$ (rather than $\mathbb{Z}^7$) grading. An additional constraint is that the equations defining X_3 are not graded, as two of the equations are of the form $\sum dQ + \sum Le$. So we further restrict from the $\mathbb{Z}^5$ -grading to a $\mathbb{Z}^4$ -grading as in Table III below.

FIG. 3. The tower of birational projections from M_3 to the base Y .TABLE III. The $\mathbb{Z}^4$ -grading on component X_3 .

u	d	L	Q	e
1	0	0	0	0
0	1	0	0	1
0	0	1	0	-1
0	0	0	1	1

By construction, the image M_3 of X_3 is also $\mathbb{Z}^4$ -graded. In fact I_{X_3} and the ideal of every intermediate Y_i in the tower of projections in Fig. 3 are all $\mathbb{Z}^4$ -graded.

B. Equations for the base space Y using the GIO types $QdL, uude, udd$

Recall that $Y = Y_3$ is the image of X_3 under the map defined by the set of base GIO types $QdL, udd, uude$ and serves as the foundation for the tower of the projections in Fig. 3. We now describe the ideal of $Y = Y_3$, i.e., the polynomial relations on these three types of GIOs, modulo the equations defining X_3 .

For ease of notation in this section, we will rename the 13 types of GIOs which are nonvanishing on X_3 with variables in Table IV. Notice that we avoid certain letters that are used elsewhere in the paper (e.g., d, e). Explicitly, we rename the 27 QdL variables as $\{a_0, \dots, a_{26}\}$, the nine udd variables as $\{c_0, \dots, c_8\}$, and the 27 $uude$ variables as $\{h_0, \dots, h_{26}\}$. Notice that these 63 variables are the coordinates on $Y_3 \subset \mathbb{C}^{63}$.

Now define matrices $M_{a1}, M_{a3}, M_{h1}, M_{h2}, M_{h3}, M_C, v_a, v_c, J$ as below:

$$M_{a1} := \begin{pmatrix} a_0 & a_3 & a_6 & a_9 & a_{12} & a_{15} & a_{18} & a_{21} & a_{24} \\ a_1 & a_4 & a_7 & a_{10} & a_{13} & a_{16} & a_{19} & a_{22} & a_{25} \\ a_2 & a_5 & a_8 & a_{11} & a_{14} & a_{17} & a_{20} & a_{23} & a_{26} \end{pmatrix},$$

TABLE IV. Renaming the 13 GIO types for component X_3 ; there are a total of 783 polynomials in these 13 types. A table describing all 28 GIO types is given as an Appendix.

Type	Variable	Number
QdL	a	27
LLe	b	9
udd	c	9
$uuuee$	g	6
$uude$	h	27
$dddLL$	i	3
$QQQL$	j	24
$QuQue$	k	108
$QuLe$	m	81
$QuQd$	n	81
$QQQQu$	p	54
$(QQQ)_4LLLe$	q	30
$uudQdQd$	r	324

$$M_{a3} := \begin{pmatrix} a_0 & a_1 & a_2 & a_9 & a_{10} & a_{11} & a_{18} & a_{19} & a_{20} \\ a_3 & a_4 & a_5 & a_{12} & a_{13} & a_{14} & a_{21} & a_{22} & a_{23} \\ a_6 & a_7 & a_8 & a_{15} & a_{16} & a_{17} & a_{24} & a_{25} & a_{26} \end{pmatrix},$$

$$M_{h1} := \begin{pmatrix} h_0 & h_3 & h_6 & h_9 & h_{12} & h_{15} & h_{18} & h_{21} & h_{24} \\ h_1 & h_4 & h_7 & h_{10} & h_{13} & h_{16} & h_{19} & h_{22} & h_{25} \\ h_2 & h_5 & h_8 & h_{11} & h_{14} & h_{17} & h_{20} & h_{23} & h_{26} \end{pmatrix},$$

$$M_{h2} := \begin{pmatrix} h_0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \\ h_9 & h_{10} & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} & h_{17} \\ h_{18} & h_{19} & h_{20} & h_{21} & h_{22} & h_{23} & h_{24} & h_{25} & h_{26} \end{pmatrix},$$

$$M_{h3} := \begin{pmatrix} h_0 & h_1 & h_2 & h_9 & h_{10} & h_{11} & h_{18} & h_{19} & h_{20} \\ h_3 & h_4 & h_5 & h_{12} & h_{13} & h_{14} & h_{21} & h_{22} & h_{23} \\ h_6 & h_7 & h_8 & h_{15} & h_{16} & h_{17} & h_{24} & h_{25} & h_{26} \end{pmatrix},$$

$$M_C := \begin{pmatrix} c_5 & 0 & c_2 & 0 & 0 & 0 & c_6 & 0 & c_7 & 0 & 0 & 0 & -c_8 & 0 & c_8 & 0 \\ -c_4 & 0 & -c_1 & 0 & 0 & 0 & 0 & c_6 & 0 & 0 & c_8 & 0 & c_7 & c_7 & 0 & 0 \\ c_3 & 0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & c_7 & 0 & c_8 & -c_6 & -c_6 & c_6 & c_6 \\ 0 & 0 & 0 & c_2 & c_8 & 0 & -c_3 & 0 & -c_4 & 0 & 0 & 0 & -c_5 & -c_5 & 0 & c_5 \\ 0 & 0 & 0 & -c_1 & -c_7 & 0 & 0 & -c_3 & 0 & 0 & -c_5 & 0 & c_4 & 0 & -c_4 & -c_4 \\ 0 & 0 & 0 & c_0 & c_6 & 0 & 0 & 0 & 0 & -c_4 & 0 & -c_5 & -c_3 & 0 & 0 & 0 \\ 0 & c_5 & 0 & 0 & 0 & c_8 & c_0 & 0 & c_1 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 \\ 0 & -c_4 & 0 & 0 & 0 & -c_7 & 0 & c_0 & 0 & 0 & c_2 & 0 & 0 & 0 & c_1 & 0 \\ 0 & c_3 & 0 & 0 & 0 & c_6 & 0 & 0 & 0 & c_1 & 0 & c_2 & 0 & 0 & 0 & c_0 \end{pmatrix},$$

$$v_a := (a_0 + a_{12} + a_{24} a_1 + a_{13} + a_{25} a_2 + a_{14} + a_{26}),$$

$$v_c := (c_2 - c_1 c_0 c_5 - c_4 c_3 c_8 - c_7 c_6)$$

$$J := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Computation 5.2. The polynomials in the list below minimally generate the ideal I_Y .

- (1) The 84 elements of type aaa are the 3×3 minors of M_{a1} .
- (2) The 108 elements of type hh are the 2×2 minors of M_{h1} .
- (3) The 9 elements of type ah are the entries of the product $v_a M_{h1}$.

- (4) The 3 elements of type ac are the entries of the product $M_{a1} v_c^T$.
- (5) The 48 elements of type hc are the entries of the product $M_{h1} M_C$.
- (6) The 18 elements of type ah^2 : These are quite complex, and we describe them in some detail below:
 - (a) Nine of the 18 equations of type (6) come from products of the form

$$M_{a1} \cdot \Lambda^2(M_{h2_{ijk}} \cdot J),$$

$$\text{for } (ijk) \in \{(036), (147), (258)\},$$

and we flatten the 3×3 matrix $\Lambda^2(M_{h2_{ijk}} \cdot J)$ into a 9×1 column vector.

- (b) The remaining 9 equations of type (6) come from products of the form

$$M_{a1} \cdot [\Lambda^2(J \cdot M_{h3_{ij}}), \Lambda^2(J \cdot M_{h3_{ik}}), \Lambda^2(J \cdot M_{h3_{k-j+i,k}})], \quad \text{for } (ijk) \in \{(047), (058), (158)\},$$

and the matrix of the three concatenated Λ^2 terms is written as a 9×1 column vector.

The indexing of the minors of the Λ^2 matrices is chosen to be consistent with that used by the `MSSM.m2` package.

All of these have geometric interpretations. For example, the 3×3 entries of M_a define the secant variety to the Segre variety $\Sigma_{2,8}$, while the 48 entries of type hc can be interpreted as taking Plücker coordinates on the 18 maximal minors of the 6×3 matrix representing $uuudd$, but excluding the two pure minors uuu and ddd .

C. Birationality of the maps in the tower of GIOs

We now describe the projections π_i and the relations which define the maps. It is important to note that we are working on X_3 , and hence can reduce relations modulo the ideal of X_3 . We begin with a “proof of concept” using the most simple relation:

Computation 5.3. Each GIO in the $uuuee$ type may be expressed as a rational function in the udd and $uude$ GIOs. Recall that we have renamed the six $uuuee$ GIOs as $\{g_0, \dots, g_5\}$, the 9 udd GIOs as $\{c_0, \dots, c_8\}$, and the 27 $uude$ GIOs as $\{h_0, \dots, h_{26}\}$. In this notation, we obtain that on X_3 we have

$$\begin{aligned} g_0 &= \frac{6h_{15}h_{21} - 6h_{12}h_{24}}{c_8}, \\ g_1 &= \frac{6h_{15}h_{22} - 6h_{12}h_{25}}{c_8}, \\ g_2 &= \frac{6h_{16}h_{22} - 6h_{13}h_{25}}{c_8}, \\ g_3 &= \frac{6h_{15}h_{23} - 6h_{12}h_{26}}{c_8}, \\ g_4 &= \frac{6h_{16}h_{23} - 6h_{13}h_{26}}{c_8}, \\ g_5 &= \frac{6h_{17}h_{23} - 6h_{14}h_{26}}{c_8}. \end{aligned}$$

The key to finding these relations is to observe that in multidegree $(4, 4, -2, 4)$ there are monomials of type $g \cdot c$ corresponding to $(uuuee)(udd)$, and of type h^2 corresponding to $(udde)(uude)$. By exploiting this multigrading, we are able to determine an order for the projections in the tower of Fig. 3 that allows us to compute the relations. The equations displayed above are the simplest example; the expressions as rational functions become increasingly complex for GIO types with more members.

Computation 5.4. The main result of this section is the description of the maps appearing in Fig. 3, which are displayed in Table V. The bottommost (hence final) projection in the tower is

TABLE V. Maps in the tower of Fig. 3. Base GIO types are $QdL = a$, $udd = c$, $uude = h$.

Map	GIO eliminated	Relations in degrees
π_{13}	$QuQue = k$	$k = k, kj = pm$
π_{12}	$(QQQ)_4LLLe = q$	$q = ja + jb, qa = jab, qab = jaab$
π_{11}	$QQQQu = p$	$p = p, pi = jac$
π_{10}	$QuLe = m$	$m = m, mcc = cah$
π_9	$uudQdQd = r$	$r = ah, rb = aah$
π_8	$QuQd = n$	$n = n, ni = aac$
π_7	$QQQL = j$	$ji = aaa$
π_6	$dddLL = i$	$ih = bcc$
π_5	$LLe = b$	$b = a, bha = haa$
π_4	$uuuee = g$	$gc = hh$

$$Y_4 \xrightarrow{\pi_4} Y_3 = Y,$$

and is exactly the map described in Computation 5.3.

There are a number of instances where a set of GIOs, when restricted to X_3 , is not minimal. For example, the type $QuQd$ consists of 81 GIOs. However, when restricting to X_3 , nine of these GIOs become linearly dependent. These linear relations are reflected in the table by “ $n = n$ ” for $QuQd$; this phenomenon occurs in several other types. Some GIO variables may be expressed linearly in terms of other GIO variables; the “ $b = a$ ” for LLe reflects that there are linear dependencies between the a and b variables. Finally, some GIO variables can be expressed as quadrics in other GIO variables, such as “ $q = ja + jb$ ” for $(QQQ)_4LLLe$ and “ $r = ah$ ” for $uudQdQd$.

VI. ADDING NEUTRINOS TO THE SUPERPOTENTIAL

Up to this point, we have considered the superpotential without the singlet right-handed neutrinos, as in (2.19):

$$\begin{aligned} W_{\text{minimal}} &= C^0 \sum_{\alpha, \beta} H_\alpha \bar{H}_\beta \epsilon^{\alpha\beta} + \sum_{i,j} C_{ij}^1 \sum_{\alpha, \beta, a} Q_{a,\alpha}^i u_a^j H_\beta \epsilon^{\alpha\beta} \\ &\quad + \sum_{i,j} C_{ij}^2 \sum_{\alpha, \beta, a} Q_{a,\alpha}^i d_a^j \bar{H}_\beta \epsilon^{\alpha\beta} \\ &\quad + \sum_{i,j} C_{ij}^3 e^i \sum_{\alpha, \beta} L_\alpha^j \bar{H}_\beta \epsilon^{\alpha\beta}. \end{aligned}$$

We now add in neutrino terms, as in (2.21)

$$W_{\text{MSSM}} = W_{\text{minimal}} + W_{\text{neutrinos}},$$

where

$$W_{\text{neutrinos}} = \sum_{i,j} C_{ij}^4 \nu^i \nu^j + \sum_{i,j} C_{ij}^5 \nu^i \sum_{\alpha, \beta} L_\alpha^j H_\beta.$$

A. Relating the superpotentials W_{minimal} and W_{MSSM}

For brevity, let $H\bar{H}$, QuH , $Qd\bar{H}$, $L\bar{H}e$ denote the summands in W_{minimal} , and $\nu\nu$, νLH the summands in $W_{\text{neutrinos}}$. Then the master space equations for W_{MSSM} satisfy

$$\begin{aligned} \frac{\partial W_{\text{MSSM}}}{\partial Q} &= \frac{\partial W_{\text{minimal}}}{\partial Q}, \\ \frac{\partial W_{\text{MSSM}}}{\partial u} &= \frac{\partial W_{\text{minimal}}}{\partial u}, \\ \frac{\partial W_{\text{MSSM}}}{\partial d} &= \frac{\partial W_{\text{minimal}}}{\partial d}, \\ \frac{\partial W_{\text{MSSM}}}{\partial e} &= \frac{\partial W_{\text{minimal}}}{\partial e}, \\ \frac{\partial W_{\text{MSSM}}}{\partial \bar{H}} &= \frac{\partial W_{\text{minimal}}}{\partial \bar{H}}, \\ \frac{\partial W_{\text{MSSM}}}{\partial H} &= \frac{\partial W_{\text{minimal}}}{\partial H} + \frac{\partial \nu LH}{\partial H}, \\ \frac{\partial W_{\text{MSSM}}}{\partial L} &= \frac{\partial W_{\text{minimal}}}{\partial L} + \frac{\partial \nu LH}{\partial L}, \\ \frac{\partial W_{\text{MSSM}}}{\partial \nu} &= \frac{\partial W_{\text{neutrinos}}}{\partial \nu}. \end{aligned} \quad (6.1)$$

1. Comparison to results on the electroweak sector

As a quick calibration, we restrict the master space of W_{MSSM} to the electroweak sector, which corresponds to setting (for all indices)

$$Q = u = d = 0 \text{ in } W_{\text{MSSM}}.$$

Let $W_{\text{MSSM}}^{\text{res}}$ denote the specialization above, and compute the Jacobian ideal of $W_{\text{MSSM}}^{\text{res}}$.

Computation 6.1. The restricted master space $W_{\text{MSSM}}^{\text{res}}$ has two components, defined by the vanishing locus of ideals P_1 and P_2 , where (again, all indices present)

$$\begin{aligned} P_2 &= \left\langle Q, u, d, \nu, \bar{H}, H_1 + \sum_i L_1^i e^i, H_2 + \sum_i L_2^i e^i, I_{2 \times 2}[H|L] \right\rangle \\ &= I_2 + \langle Q, u, d, \nu \rangle, \\ P_3 &= \left\langle Q, u, d, \nu, \bar{H}, H, \sum_i L_1^i e^i, \sum_i L_2^i e^i \right\rangle \\ &= I_3 + \langle Q, u, d, \nu \rangle. \end{aligned}$$

Here $I_{2 \times 2}[H|L]$ denotes the 2×2 minors of the 2×4 matrix whose first column is $[H_1, H_2]^T$ and remaining three columns are the 2×3 matrix of the L variables.

On the locus $V(P_2)$, all the GIOs vanish, while on $V(P_3)$, all the GIO types except LLe vanish. A computation confirms that the image of the restricted master space is the Veronese surface, as shown in Sec. 5 of [27].

B. The master space and vacuum moduli space for W_{MSSM}

Assuming that the coupling matrices are invertible, note that the last equality in (6.1),

$$\frac{\partial W_{\text{MSSM}}}{\partial \nu} = \frac{\partial W_{\text{neutrinos}}}{\partial \nu},$$

shows that each neutrino ν^i may be expressed as a bihomogeneous quadratic equation in the L , H variables.

In particular, the ν_i are expressed parametrically in terms of the L , H variables. There is also a deformation resulting from

$$\begin{aligned} \frac{\partial W_{\text{MSSM}}}{\partial H} &= \frac{\partial W_{\text{minimal}}}{\partial H} + \frac{\partial \nu LH}{\partial H}, \\ \frac{\partial W_{\text{MSSM}}}{\partial L} &= \frac{\partial W_{\text{minimal}}}{\partial L} + \frac{\partial \nu LH}{\partial L}. \end{aligned}$$

However, these deformations are fairly simple, because the two partials,

$$\frac{\partial \nu LH}{\partial H} \quad \text{and} \quad \frac{\partial \nu LH}{\partial L},$$

yield bihomogeneous polynomials of type νL and νH .

Computation 6.2. The Jacobian ideal of W_{MSSM} defines the master space of the superpotential with neutrinos. It has three irreducible components corresponding to prime ideals J_i below; in the notation from Sec. III, the corresponding ideals are

$$\begin{aligned} I_1 &= J_1, \\ I_2 &\subseteq J_2, \\ I_3 &= J_3. \end{aligned}$$

The codimension of $V(J_2)$ in $V(I_2)$ is two, so $V(J_2)$ is of dimension 25. Furthermore, the variables ν^i vanish on each of the components of the master space for W_{MSSM} . Hence, the master spaces for W_{MSSM} and W_{minimal} may both be regarded as embedded in the same ambient space. The result above shows that for the corresponding vacuum moduli spaces, the images of two of the three components of the master spaces coincide.

The ideal of J_2 is minimally generated by 14 linear forms and 80 quadrics; using the notation of Sec. III we have

$$J_2 = \langle \bar{H}, u, \nu, I_{2 \times 2}[(M_H)_1 | M_Q | M_L], F_1, F_2 \rangle.$$

Note that J_2 differs from I_2 only in the addition of $|M_L|$ to (matrix defining) the ideal of 2×2 minors, and that setting $Q = u = d = 0$ in J_2 yields the ideal P_2 .

Computation 6.3. The image

$$V(J_2) \rightarrow \mathbb{C}^{976}$$

under the map determined by the full set of all GIOs is zero.

Proof. Compute the restriction of each GIO type to $V(J_2)$, note they all vanish. ■

As $V(J_2) \subset V(I_2)$ we might have naively expected that the image of $V(J_2)$ would reflect the Veronese surface appearing after Computation 6.1. But in fact the Veronese surface is the image of the restriction of $V(I_2) \cap V(J_3)$ to the electroweak sector with neutrinos added (i.e., $Q = u = d = \nu = 0$).

In Sec. III we showed that the primary decomposition of the Jacobian ideal of W_{minimal} is the intersection of three prime ideals. It turns out that the primary decomposition of the Jacobian ideal of W_{MSSM} is more complicated, however, the radical of this ideal is still an intersection of three prime ideals. It would be interesting to understand the geometric and physical implications of this dichotomy.

C. Software and future directions

Computations were performed using the Macaulay2 package MSSM.m2, which will be part of the next distribution of Macaulay2, available at [61]. Our work here opens up doors to a number of followup questions on the geometric structure of the vacuum moduli space; we are in the process of

extending our computational toolkit to handle other variants of the superpotential.

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The authors declare no conflict of interest in this research.

DATA AVAILABILITY

The data that support the findings of this article are openly available [61].

APPENDIX: GAUGE INVARIANT OPERATORS IN THE MSSM

TABLE VI. The 976 generators of gauge invariant operators for the MSSM. For $QQQQu$, a naïve count gives 72, but there are 18 linear relations, reducing the number of independent operators.

Type	Explicit Sum	Index	Number
ν	ν^i	$i = 1, 2, 3$	3
LH	$L_\alpha^i H_\beta \epsilon^{\alpha\beta}$	$i = 1, 2, 3$	3
$H\bar{H}$	$H_\alpha \bar{H}_\beta \epsilon^{\alpha\beta}$		1
udd	$u_a^i d_b^j d_c^k \epsilon^{abc}$	$i, j = 1, 2, 3; k = 1, \dots, j-1$	9
LLe	$L_\alpha^i L_\beta^j e^k \epsilon^{\alpha\beta}$	$i, k = 1, 2, 3; j = 1, \dots, i-1$	9
QdL	$Q_{a,\alpha}^i d_a^j L_\beta^k \epsilon^{\alpha\beta}$	$i, j, k = 1, 2, 3$	27
QuH	$Q_{a,\alpha}^i u_a^j H_\beta \epsilon^{\alpha\beta}$	$i, j = 1, 2, 3$	9
$Qd\bar{H}$	$Q_{a,\alpha}^i d_a^j \bar{H}_\beta \epsilon^{\alpha\beta}$	$i, j = 1, 2, 3$	9
$L\bar{H}e$	$L_\alpha^i \bar{H}_\beta \epsilon^{\alpha\beta} e^j$	$i, j = 1, 2, 3$	9
$QQQL$	$Q_{a,\beta}^i Q_{b,\gamma}^j Q_{c,\alpha}^k L_\delta^l \epsilon^{abc} \epsilon^{\beta\gamma} \epsilon^{\alpha\delta}$	$i, j, k, l = 1, 2, 3; i \neq k, j \neq k, j \leq i, (i, j, k) \neq (3, 2, 1)$	24
$QuQd$	$Q_{a,\alpha}^i u_a^j Q_{b,\beta}^k d_b^l \epsilon^{\alpha\beta}$	$i, j, k, l = 1, 2, 3$	81
$QuLe$	$Q_{a,\alpha}^i u_a^j L_\beta^k e^l \epsilon^{\alpha\beta}$	$i, j, k, l = 1, 2, 3$	81
$uude$	$u_a^i u_b^j d_c^k e^l \epsilon^{abc}$	$i, j, k, l = 1, 2, 3; j < i$	27

(Table continued)

TABLE VI. (Continued)

Type	Explicit Sum	Index	Number
$QQQ\bar{H}$	$Q_{a,\beta}^i Q_{b,\gamma}^j Q_{c,\alpha}^k \bar{H}_\delta \epsilon^{abc} \epsilon^{\beta\gamma} \epsilon^{a\delta}$	$i, j, k = 1, 2, 3; i \neq k, j \neq k,$ $j \leq i, (i, j, k) \neq (3, 2, 1)$	8
$Qu\bar{H}e$	$Q_{a,\alpha}^i u_a^j \bar{H}_\beta e^k \epsilon^{\alpha\beta}$	$i, j, k = 1, 2, 3$	27
$dddLL$	$d_a^i d_b^j d_c^k L_\alpha^m L_\beta^n \epsilon^{abc} \epsilon_{ijk} \epsilon^{\alpha\beta}$	$m, n = 1, 2, 3, n < m$	3
$uuuee$	$u_a^i u_b^j u_c^k e^m e^n \epsilon^{abc} \epsilon_{ijk}$	$m, n = 1, 2, 3, n \leq m$	6
$QuQue$	$Q_{a,\alpha}^i u_a^j Q_{b,\beta}^k u_b^m e^n \epsilon^{\alpha\beta}$	$i, j, k, m, n = 1, 2, 3;$ antisymmetric $\{(i, j), (k, m)\}$	108
$QQQQQu$	$Q_{a,\beta}^i Q_{b,\gamma}^j Q_{c,\alpha}^k Q_{f,\delta}^m u_f^n \epsilon^{abc} \epsilon^{\beta\gamma} \epsilon^{a\delta}$	$i, j, k, m, n = 1, 2, 3; i \neq k, j \neq k,$ $j \leq i, (i, j, k) \neq (3, 2, 1)$	54
$dddL\bar{H}$	$d_a^i d_b^j d_c^k L_\alpha^m \bar{H}_\beta \epsilon^{abc} \epsilon_{ijk} \epsilon^{\alpha\beta}$	$m = 1, 2, 3$	3
$uudQdH$	$u_a^i u_b^j d_c^k Q_{f,\alpha}^m d_f^n H_\beta \epsilon^{abc} \epsilon^{\alpha\beta}$	$i, j, k, m, n = 1, 2, 3; j < i$	81
$(QQQ)_4 LLH$	$(QQQ)_4^{\alpha\beta\gamma} L_\alpha^m L_\beta^n H_\gamma$	$m, n = 1, 2, 3, n \leq m$	6
$(QQQ)_4 LH\bar{H}$	$(QQQ)_4^{\alpha\beta\gamma} L_\alpha^m H_\beta \bar{H}_\gamma$	$m = 1, 2, 3$	3
$(QQQ)_4 H\bar{H}\bar{H}$	$(QQQ)_4^{\alpha\beta\gamma} H_\alpha \bar{H}_\beta \bar{H}_\gamma$		1
$(QQQ)_4 LLLe$	$(QQQ)_4^{\alpha\beta\gamma} L_\alpha^m L_\beta^n L_\gamma^p e^q$	$m, n, p, q = 1, 2, 3, n \leq m, p \leq n$	30
$uudQdQd$	$u_a^i u_b^j d_c^k Q_{f,\alpha}^m d_f^n Q_{g,\beta}^p d_g^q \epsilon^{abc} \epsilon^{\alpha\beta}$	$i, j, k, m, n, p, q = 1, 2, 3;$ $j < i$, antisymmetric $\{(m, n), (p, q)\}$	324
$(QQQ)_4 LL\bar{H}e$	$(QQQ)_4^{\alpha\beta\gamma} L_\alpha^m L_\beta^n \bar{H}_\gamma e^p$	$m, n, p = 1, 2, 3, n \leq m$	18
$(QQQ)_4 L\bar{H}\bar{H}e$	$(QQQ)_4^{\alpha\beta\gamma} L_\alpha^m \bar{H}_\beta \bar{H}_\gamma e^n$	$m, n = 1, 2, 3$	9
$(QQQ)_4 \bar{H}\bar{H}\bar{H}e$	$(QQQ)_4^{\alpha\beta\gamma} \bar{H}_\alpha \bar{H}_\beta \bar{H}_\gamma e^m$	$m = 1, 2, 3$	3

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