Reply to "Comment on 'Undecidability in spacetime geometry via the AdS/CFT correspondence'"

Sameer Ahmad Mir,[1,2,*] Francesco Marino,[3,4,†] Arshid Shabir,[5,‡] Lawrence M. Krauss,[6,§] and Mir Faizal[5,7–9,‖]

[1]*Department of Physics, Jamia Millia Islamia, New Delhi 110025, India*
[2]*Department of Computer Sciences, Asian School of Business, Noida, Uttar Pradesh 201303, India*
[3]*CNR-Istituto Nazionale di Ottica, Via Sansone 1, I-50019 Sesto Fiorentino (FI), Italy*
[4]*INFN, Sezione di Firenze, Via Sansone 1, I-50019 Sesto Fiorentino (FI), Italy*
[5]*Canadian Quantum Research Center, 204-3002 32 Ave, Vernon, British Columbia V1T 2L7, Canada*
[6]*The Origins Project Foundation, Phoenix, Arizona 85028, USA*
[7]*Irving K. Barber School of Arts and Sciences, University of British Columbia Okanagan, Kelowna, British Columbia V1V 1V7, Canada*
[8]*Department of Mathematical Sciences, Durham University, Upper Mountjoy, Stockton Road, Durham DH1 3LE, United Kingdom*
[9]*Faculty of Sciences, Hasselt University, Agoralaan Gebouw D, 3590 Diepenbeek, Belgium*

(Received 16 September 2026; accepted 2 October 2026; published 27 October 2026)

DOI: 10.1103/vjcd-bk2j

I. INTRODUCTION

We make a correction to the bulk realization of saddle-selection undecidability presented in Ref. [1]. On the common circle-compactified boundary, the relevant zero-source competition is between the planar Euclidean anti–de Sitter (AdS) black brane and the AdS soliton, not the Poincaré quotient and the soliton. We derive their renormalized actions in arbitrary dimension, obtain the universal transition between the two saddles, derive the general leading displacement produced by stationary matter, and exhibit an explicit spatial-holonomy contribution that lifts the degeneracy at classical order. We show in top-down Aharony–Bergman–Jafferis–Maldacena (ABJM) and $\mathcal{N}=4$ super-Yang-Mills examples that this circle-sensitive branch response occurs at leading large-rank order. We state and prove an exact transfer theorem: for every uniformly computable holographic family in which the Cubitt–Pérez-García–Wolf phase dichotomy uniformly controls the sign of the limiting branch-action density difference within a globally complete real Euclidean saddle domain, a universal leading-order dominant-saddle algorithm would decide the corresponding undecidable phase problem. The corrected analysis replaces the published saddle assignment and isolates the precise dynamical criterion required for a complete holographic realization. Thus, although the bulk saddle realization requires these corrections, the central physical implication of Ref. [1] remains intact.

The Comment [2] identifies a small calculational mistake in the choice of competing zero-source fillings in Ref. [1]. Here, we demonstrate that this mistake can be easily corrected by reformulating the competing saddles within the same fixed boundary-value problem, without changing the logic of Ref. [1] or its main physical implication. It may be noted however that, after these corrections, a complete holographic realization requires a dynamical map from the microscopic phase to the holographic parameters controlling the branch-action difference, and this map remains to be constructed.

The Comment [2] is correct about the zero-source limit. At $\vartheta=0$, the deformations used in Ref. [1] vanish, so the scalar field is trivial and the Poincar'e quotient is not the dominant filling. In the low-temperature regime considered in the published work, the AdS soliton has lower action. However, this issue can be easily corrected by comparing the planar Euclidean black brane and the AdS soliton with the same boundary data. This changes the gravitational realization while leaving intact the computability implication of a uniform phase-to-action separation.

Fix $\mathcal{B}_d = S^1_\beta \times \mathbb{R}^{d-2} \times S^1_{L_\theta}$ and compare the two fillings [3–5]. Removing $S^1_{L_\theta}$ from one row would compare different boundary theories. In a fermionic completion, both potentially contractible circles carry the same bounding spin structure. For $X \in \{\mathrm{bb}, \mathrm{sol}\}$ let $\chi_X = \tau$, θ be the contractible coordinate, $\ell_X = \beta, L_\theta$ its period,

*Contact author: sameerphst@gmail.com
†Contact author: francesco.marino@ino.cnr.it
‡Contact author: aslone186@gmail.com
§Contact author: lawrence@originsproject.org
‖Contact author: mirfaizalmir@googlemail.com

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

and $f_X(r) = 1 - (r_X/r)^d$, where smoothness fixes $r_X = 4\pi L^2/(d\ell_X)$. Both metrics are

$$ds_X^2 = \frac{r^2}{L^2}\left[f_X(r)d\chi_X^2 + \sum_{a\neq\chi_X} dy_a^2\right] + \frac{L^2 dr^2}{r^2 f_X(r)}. \quad (1)$$

Use $I_E = -(16\pi G_{d+1})^{-1}\int\sqrt{g}[R + d(d-1)/L^2] - (8\pi G_{d+1})^{-1}\int\sqrt{\gamma}K + I_{\rm ct}$. The flat-boundary counterterms suffice because the curvature and logarithmic terms vanish on this representative [6,7]. At $r = R_c$, $\sqrt{g} = r^{d-1}/L^{d-1}$ and $K = (\sqrt{f}/L)[d + rf'/(2f)]$. With $\mathcal{A}_d = \beta V_{d-2}L_\theta/(8\pi G_{d+1}L^{d+1})$, the regulated contributions are $I_{\rm bulk} = \mathcal{A}_d(R_c^d - r_X^d)$, $I_{\rm GHY} = -\mathcal{A}_d(dR_c^d - dr_X^d/2)$, and $I_{\rm ct} = \mathcal{A}_d(d-1)R_c^d\sqrt{1 - r_X^d/R_c^d}$. Their cutoff dependence cancels, leaving

$$I_X^{(0)} = -\frac{\beta V_{d-2}L_\theta}{16\pi G_{d+1}L^{d+1}}r_X^d, \quad \Delta\omega^{(0)} = \mathcal{C}_d(\beta^{-d} - L_\theta^{-d}), \quad (2)$$

where $\Delta\omega^{(0)} = \omega_{\rm sol}^{(0)} - \omega_{\rm bb}^{(0)}$ and $\mathcal{C}_d = (16\pi G_{d+1}L^{d+1})^{-1}(4\pi L^2/d)^d$. Thus the black brane dominates for $\beta < L_\theta$, the soliton dominates for $\beta > L_\theta$, and $T_0 = L_\theta^{-1}$. In particular, the published regime $\beta/L_\theta \gg 1$ lies on the soliton side, while the Poincaré quotient with $I_{\rm P} = 0$ lies above both saddles.

Equation (2) gives $\omega_{\rm bb}^{(0)} = -4\pi^2L^2/(27G_4\beta^3)$ in AdS_4 and $\omega_{\rm bb}^{(0)} = -\pi^3L^3/(16G_5\beta^4)$ in AdS_5, with β replaced by L_θ for the soliton. It also yields $s_{\rm bb} = d\mathcal{C}_dT^{d-1}$, $\varepsilon_{\rm bb} = (d-1)\mathcal{C}_dT^d$, $s_{\rm sol} = 0$, and $\varepsilon_{\rm sol} = -\mathcal{C}_dL_\theta^{-d}$, verifying the thermodynamic identities. Finite local counterterms cancel from $\Delta\omega$ because the two saddles have identical sources.

For $d = 4$, the boundary $S_\beta^1 \times \mathbb{T}_R^2 \times S_{L_\theta}^1$ can accommodate the two spatial directions of the CPW model, while $S_{L_\theta}^1$ supplies the capped circle. In AdS_4 one may instead start from the one-dimensional theorem [8], provided a uniform holographic embedding is constructed. These results fix the common boundary, saddle pair, renormalized actions, and thermal ordering. The remaining task is the dynamical map from the microscopic phase to the branch-action difference.

II. STATIONARY MATTER AND BRANCH RESPONSE

A nonzero deformation is compared on fully backreacted stationary solutions with identical sources. For a scalar, $I_{\rm ren} = I_{\rm grav} + I_\Phi + I_{\Phi,{\rm ct}} + I_{\rm Leg}^{(\mathfrak{p})} + I_W$, and $\delta I_{\rm ren} = \int\sqrt{g_{(0)}}(P_{\mathfrak{p}} + \delta W/\delta Q_{\mathfrak{p}})\delta Q_{\mathfrak{p}}$. Both branches must obey the same stationary condition $P_{\mathfrak{p}} + \delta W/\delta Q_{\mathfrak{p}} = 0$. If the matter problem is invariant under $\tau \leftrightarrow \theta$, then $I_{\rm bb}(\beta, L_\theta) = I_{\rm sol}(L_\theta, \beta)$, and the degeneracy at $\beta = L_\theta$ is exact. A phase-sensitive split therefore requires circle-sensitive data or inequivalent regularity conditions.

For a common source λ coupled as $+\lambda\mathcal{O}$, stationarity gives $\partial_\lambda\Delta\omega = \langle\mathcal{O}\rangle_{\rm sol} - \langle\mathcal{O}\rangle_{\rm bb}$ and $\delta\Delta\omega = \int_0^\lambda d\lambda'\Delta\langle\mathcal{O}\rangle_{\lambda'}$. Expansion about $T_0 = L_\theta^{-1}$ gives $(T_c - T_0)/T_0 = -\delta\Delta\omega(T_0)/(d\mathcal{C}_dT_0^d) + O([\delta\Delta\omega/(\mathcal{C}_dT_0^d)]^2)$.

For $d > 2$, an elementary Maxwell sector displays the geometric response. Take $I_A = (4g_F^2)^{-1}\int\sqrt{g}F^2$ in the fixed-holonomy ensemble and $w_\theta = \oint A$. On the black brane, $A_\theta = w_\theta/L_\theta$ is regular and flat, hence $\delta\omega_{\rm bb} = 0$. In the trivial relative-flux sector of the undeformed soliton, regularity fixes $A_\theta(r) = (w_\theta/L_\theta)[1 - (r_0/r)^{d-2}]$, giving $\delta\omega_{\rm sol} = (d-2)r_0^{d-2}w_\theta^2/(2g_F^2L^{d-1}L_\theta^2) + O(w_\theta^4) > 0$. The induced metric begins at $O(w_\theta^2)$, but its order-w_θ^2 contribution to the on-shell gravitational action vanishes by stationarity at fixed sources.

The same calculation extends without freezing the scalar-dependent metric. Consider a neutral gauge sector $Z(\Phi)F^2/(4g_F^2)$ with $Z > 0$, no charged condensate or vector mass, and no active topological term. After factoring out the boundary coordinate volume, define on a homogeneous, diagonal, cohomogeneity-one zero-holonomy background $\mathcal{K}_{X,u}(r) = \sqrt{g_{X,u,0}}Z(\Phi_{X,u,0})g_{X,u,0}^{rr}g_{X,u,0}^{\theta\theta}$ and $\mathcal{J}_{{\rm sol},u} = \int_{r_{0,u}}^\infty dr/\mathcal{K}_{{\rm sol},u}(r)$, with $0 < \mathcal{J}_{{\rm sol},u} < \infty$. Soliton regularity and the Maxwell equation then give

$$\Delta\omega_u(T, w_\theta) = \Delta\omega_{u,0}(T) + \frac{w_\theta^2}{2g_F^2L_\theta^2\mathcal{J}_{{\rm sol},u}} + O(w_\theta^4), \quad (3)$$

where the spatial black-brane Maxwell connection is flat. The coefficient depends on the scalar profile and the complete zero-holonomy backreaction. It reduces to the preceding Maxwell coefficient when $Z = 1$ and the background is the pure AdS soliton. When equivalent large-gauge sectors labelled by n are present and no sector-dependent terms contribute, the leading classical envelope replaces w_θ^2 by $\min_n(w_\theta - 2\pi n/q)^2$; a fixed sector uses its chosen representative.

Equation (3) gives a quantitative criterion. At one computable pair (T_*, w_θ), the complete right-hand side must lie below zero by a uniform margin in one microscopic phase and above zero by a uniform margin in the other, with the $O(w_\theta^4)$ remainder smaller than that margin. The Bausch–Cubitt–Watson construction [9] provides a related undecidable family with a zero-temperature local phase marker that distinguishes its two phases. A uniform holographic map must establish the corresponding finite-temperature response separation in Eq. (3); the zero-temperature marker alone does not do so. No bare source is allowed to contain the phase label itself.

More explicitly, set $\Sigma_u = \mathcal{J}_{{\rm sol},u}^{-1}$ and collect every additional zero-holonomy matter contribution and higher-order term into $\mathcal{R}_u$. If $\Delta\omega_u = \mathcal{C}_d(T^d - L_\theta^{-d}) + w_\theta^2\Sigma_u/(2g_F^2L_\theta^2) + \mathcal{R}_u$, choose fixed computable Σ_* and $T_*^d = L_\theta^{-d} - w_\theta^2\Sigma_*/(2g_F^2L_\theta^2\mathcal{C}_d) > 0$. Then $\Delta\omega_u(T_*, w_\theta) = w_\theta^2(\Sigma_u - \Sigma_*)/(2g_F^2L_\theta^2) + \mathcal{R}_u$. A uniform separation

$\Sigma_u \le \Sigma_* - \eta$ in one phase and $\Sigma_u \ge \Sigma_* + \eta$ in the other, together with $|\mathcal{R}_u| < w_\theta^2 \eta/(2g_F^2 L_\theta^2)$, produces opposite unique saddle labels at the same temperature. The uniform remainder bound is essential; pointwise perturbation theory is insufficient.

III. COMPUTABILITY-TRANSFER THEOREM

Let $C(u) = H_u$ be a total computable promised family of translation-invariant finite-range Hamiltonians whose alternatives $\mathcal{P}_-$ and $\mathcal{P}_+$ cannot be distinguished algorithmically [9–11]. Their fixed association with halting and nonhalting depends on the chosen construction and is immaterial below. Consider holographic families equipped with a total uniform computable map $\mathcal{E}: H_u \mapsto e_u$, where the finite code e_u specifies the rank-indexed boundary theory, ensemble, sources, spin structures, and renormalization prescription without querying the infrared phase.

Let $a_N \to \infty$ be a fixed, u-independent large-rank normalization, V_R the regulated noncompact boundary volume, and $I_X(u; N, R)$ the minimum renormalized action in branch class X. Define $i_X(u) = \lim_{R\to\infty}(\beta L_\theta V_R)^{-1} \lim_{N\to\infty} I_X(u; N, R)/a_N$ and $\delta i(u) = i_{\rm sol}(u) - i_{\rm bb}(u)$. Assume the limits exist, the declared real Euclidean saddle domain is globally complete with density $\min\{i_{\rm bb}, i_{\rm sol}\}$, and the minimizing class is unique. A sufficient separation condition is

$$\begin{aligned} &\delta i(u) = B\mathcal{M}(u) + \mathcal{R}(u), \qquad |\mathcal{R}(u)| < B\eta, \\ &\mathcal{M}(u) \le -\eta (u \in \mathcal{P}_-), \qquad \mathcal{M}(u) \ge \eta (u \in \mathcal{P}_+). \end{aligned} \tag{4}$$

for fixed, input-independent $B, \eta > 0$. The observable $\mathcal{M}(u)$ must arise from the computable specification of the theory. Equation (3) supplies a calculable classical branch-response coefficient; Eq. (4) states the phase-faithful property that its microscopic realization must establish.

Theorem. Let $e_u = \mathcal{E}(C(u))$ be a uniformly computable holographic family encoding the undecidable phase alternatives $\mathcal{P}_-$ and $\mathcal{P}_+$. If the declared real Euclidean saddle domain is globally complete with a unique minimizing class and the limiting branch-action difference $\delta i(u) = i_{\rm sol}(u) - i_{\rm bb}(u)$ uniformly has opposite signs in the two phases, then no Turing machine halts on every promised code e_u and returns the unique globally dominant leading saddle class.

Proof. Let $\mathcal{U} \coloneqq \mathcal{P}_- \sqcup \mathcal{P}_+$ be the admissible input domain, and let $\mathcal{S} \coloneqq \{\text{sol}, \text{bb}\}$ denote the two branch-class labels, where sol denotes the AdS-soliton class and bb the planar Euclidean AdS-black-brane class. Suppose, for contradiction, that a Turing machine A halts on every code e_u with $u \in \mathcal{U}$ and returns the unique globally dominant leading saddle class in $\mathcal{S}$. Set $F \coloneqq \mathcal{E} \circ C$. Since C and $\mathcal{E}$ are total computable maps, F is total and computable on $\mathcal{U}$, with $F(u) = e_u = \mathcal{E}(C(u))$. The assumed uniform sign separation gives $\delta i(u) < 0$ for $u \in \mathcal{P}_-$ and $\delta i(u) > 0$ for $u \in \mathcal{P}_+$. For the sufficient separation criterion used here, these inequalities follow explicitly from Eq. (4). Explicitly, if $u \in \mathcal{P}_-$, then $\mathcal{M}(u) \le -\eta$ and $|\mathcal{R}(u)| < B\eta$, so $\delta i(u) = B\mathcal{M}(u) + \mathcal{R}(u) \le -B\eta + |\mathcal{R}(u)| < 0$. Hence $i_{\rm sol}(u) < i_{\rm bb}(u)$. If $u \in \mathcal{P}_+$, then $\mathcal{M}(u) \ge \eta$ and $|\mathcal{R}(u)| < B\eta$, so $\delta i(u) \ge B\eta - |\mathcal{R}(u)| > 0$, and hence $i_{\rm bb}(u) < i_{\rm sol}(u)$. Because the declared saddle domain is globally complete with leading action density $\min\{i_{\rm bb}(u), i_{\rm sol}(u)\}$ and has a unique minimizing class, it follows that $A(e_u) = \text{sol}$ if and only if $u \in \mathcal{P}_-$, while $A(e_u) = \text{bb}$ if and only if $u \in \mathcal{P}_+$. Define the finite decoding map $d: \mathcal{S} \to \{-, +\}$ by $d(\text{sol}) = -$ and $d(\text{bb}) = +$. The composition $D \coloneqq d \circ A \circ F = d \circ A \circ \mathcal{E} \circ C$ is therefore computable and halts on every $u \in \mathcal{U}$. Moreover, $D(u) = -$ if and only if $u \in \mathcal{P}_-$, and $D(u) = +$ if and only if $u \in \mathcal{P}_+$. Thus D decides the phase dichotomy $(\mathcal{P}_-, \mathcal{P}_+)$ on its entire admissible domain, contradicting its assumed undecidability. Therefore, no universal leading-order dominant-saddle algorithm exists. ∎

Together with the separation hypotheses, the corrected saddle competition suffices for the reduction. Equation (4) assigns opposite unique branch labels to the two microscopic phases, and the theorem converts any universal saddle-selection algorithm into a decision procedure for that phase problem. The required ingredients are a uniform phase transducer, the renormalized branch actions, and control of all competing saddles. This answers the Comment's second question without placing the phase label in a source or inferring it from a one-loop determinant.

IV. TOP-DOWN BRANCH CHECKS

The ABJM and $\mathcal{N} = 4$ examples below show that circle holonomy changes the classical branch competition at leading large-rank order. They test this gravitational response; the microscopic map remains governed by Eq. (4).

In the M-theory/ABJM truncation at zero temporal chemical potential, take the AdS-soliton-connected branch and let $x = \varphi_\theta/\varphi_{\max}$, where φ_θ is the boundary spatial holonomy, $\varphi_{\max} = 4\pi L/\sqrt{3}$, and $B(x) = 1 – 3x^2/2 + (1 - x^2)^{3/2}$. Among the specified homogeneous black-brane and soliton branches, the fully backreacted two-derivative classical comparison is $\Delta\omega_{\rm ABJM} = 2\pi^2 L^2 [2(TL_\theta)^3 - B(x)]/(27 G_4 L_\theta^3)$, with $\beta_c = L_\theta [2/B(x)]^{1/3}$ for $|x| < \sqrt{3}/2$ [12,13]. The regular soliton branch has $|x| \le 1$. The transition reaches $T_c = 0$ at $|x| = \sqrt{3}/2$; for $\sqrt{3}/2 < |x| \le 1$, the black brane wins at every finite temperature. For the $k = 1$ ABJM dual, $L^2/G_4 = (2\sqrt{2}/3) N^{3/2}$ [14,15], so the action split is $O(N^{3/2})$.

The $\text{AdS}_5 \times S^5$ reduction gives $G_5 = \pi L^3/(2N_c^2)$ [16], while the current normalization of Ref. [17] gives $1/g_5^2 = N_c^2/(16\pi^2 L)$. For dimensionless w_θ and $F = A_\theta'(r) dr \wedge d\theta$, the Chern–Simons term is inactive. Through

$O(w_\theta^2)$ the result is $\Delta\omega_{\mathcal{N}=4} = N_c^2\{2\pi^2[(TL_\theta)^4 - 1] + w_\theta^2 + O(w_\theta^4)\}/(16L_\theta^4)$ and $T_cL_\theta = 1 - w_\theta^2/(8\pi^2) + O(w_\theta^4)$. The neutral and holonomy terms are both $O(N_c^2)$. Type-IIB-upliftable AdS_5 solitons with scalar-dependent gauge couplings and multiple regular branches are known explicitly [18]. In both examples the holonomy term scales with the same large-rank power as the neutral action. These embeddings realize the classical circle response in top-down settings while leaving the uniform microscopic map as the condition stated in Eq. (4).

V. REMAINING FORMULA-LEVEL POINTS

The following points are independent of the central $\vartheta = 0$ correction. The Brown-York tensor of the AdS_4 soliton is anisotropic: in Fefferman-Graham gauge $g^{\mathrm{sol}}_{(3)ij} = (r_0^3/3L^6)\mathrm{diag}(1,1,-2)$ in the order (τ, x, θ), whereas $g^{\mathrm{bb}}_{(3)ij} = (r_h^3/3L^6)\mathrm{diag}(-2,1,1)$, with $\langle T_{ij}\rangle = 3L^2 g_{(3)ij}/(16\pi G_4)$ [6,7]. Both tensors are traceless, and the soliton has negative Lorentzian energy density. The statement $K = -3/L$ in the Supplemental Material of Ref. [1] uses the opposite normal convention from its Gibbons-Hawking evaluation. The AdS soliton is not a freely acting quotient of Poincaré AdS because the S^1_θ orbit degenerates at the tip.

The printed heat-kernel factors in Ref. [1] also require correction. Combining $(4\pi s)^{-2}$ with the additional circle factor and the proper-time measure ds/s produces $s^{-7/2}$ and a $K_{5/2}$ integral, whereas the asserted four-dimensional power produces K_2. The calculation replaces a fixed bulk mass m_Φ by a branch-dependent m_{eff} without deriving that identification, and the Bessel-order inequality used in Supplemental Material of Ref. [1] has the wrong direction [19]. The quoted massive Casimir expression vanishes as $m_* \to 0$ because $m_*^2 K_1(\beta m_*) \sim m_*$; one free boson in $2+1$ dimensions instead has $f(\beta, m) = -[x\mathrm{Li}_2(e^{-x}) + \mathrm{Li}_3(e^{-x})]/(2\pi\beta^3)$, where $x = \beta m$, and $f(\beta, 0) = -\zeta(3)/(2\pi\beta^3)$.

Finally, in the Supplemental Material of Ref. [1], the amplifier does not activate as written: with $\delta \ll L_k^q$, changing s_k from zero to one costs $L_k^q - \delta > 0$ in both branches. Any repair must overcome this positive activation cost while preserving the hierarchy and reestablishing the gap bounds. That separate construction is not needed for the transfer theorem stated here.

VI. CONCLUSION

The corrected comparison uses the black brane and AdS soliton with identical boundary data and gives the neutral transition at $\beta = L_\theta$. The backreacted holonomy coefficient supplies a phase-sensitive branch response, while the transfer theorem converts a uniform sign separation into undecidability of dominant-saddle selection. ABJM and $\mathcal{N} = 4$ show that this response occurs at leading classical order in top-down settings. The small calculational error in the published saddle assignment is thereby removed, and the precise conditions required for a full realization of the undecidability mechanism are made explicit. However, this correction does not change the physical implications of Ref. [1].

ACKNOWLEDGMENTS

We thank Agus Hasan Budiyanto for identifying the saddle-selection issue addressed here.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

[1] S. A. Mir, F. Marino, A. Shabir, L. M. Krauss, and M. Faizal, Phys. Rev. D **113**, L021904 (2026).
[2] A. H. Budiyanto, preceding comment, Phys. Rev. D **114**, 088901 (2026).
[3] G. T. Horowitz and R. C. Myers, Phys. Rev. D **59**, 026005 (1998).
[4] S. Surya, K. Schleich, and D. M. Witt, Phys. Rev. Lett. **86**, 5231 (2001).
[5] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).
[6] S. de Haro, K. Skenderis, and S. N. Solodukhin, Commun. Math. Phys. **217**, 595 (2001).
[7] K. Skenderis, Classical Quantum Gravity **19**, 5849 (2002).
[8] J. Bausch, T. S. Cubitt, A. Lucia, and D. Pérez-García, Phys. Rev. X **10**, 031038 (2020).
[9] J. Bausch, T. S. Cubitt, and J. D. Watson, Nat. Commun. **12**, 452 (2021).
[10] T. S. Cubitt, D. Pérez-García, and M. M. Wolf, Nature (London) **528**, 207 (2015).
[11] T. S. Cubitt, D. Pérez-García, and M. M. Wolf, Forum Math. Pi **10**, e14 (2022).
[12] J. P. Gauntlett and O. Varela, Phys. Rev. D **76**, 126007 (2007).
[13] A. Anabalón and S. F. Ross, J. High Energy Phys. 07 (2021) 015.
[14] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, J. High Energy Phys. 10 (2008) 091.
[15] N. Drukker, M. Mariño, and P. Putrov, Commun. Math. Phys. **306**, 511 (2011).

[16] M. Henningson and K. Skenderis, J. High Energy Phys. 07 (1998) 023.
[17] G. Policastro, D. T. Son, and A. O. Starinets, J. High Energy Phys. 09 (2002) 043.
[18] A. Anabalón, H. Nastase, and M. Oyarzo, J. High Energy Phys. 05 (2024) 217.
[19] NIST Digital Library of Mathematical Functions, Sec. 10.37, https://dlmf.nist.gov/10.37.