

Pentaquarks made of light quarks and their admixture to baryons

Nicholas Miesch^{*,} Edward Shuryak,[†] and Ismail Zahed[‡]

*Center for Nuclear Theory, Department of Physics and Astronomy, Stony Brook University,
Stony Brook, New York 11794–3800, USA*



(Received 17 July 2025; accepted 29 August 2025; published 15 October 2025)

This paper is a continuation of our studies of multi-quark hadrons. The antisymmetrization of their wave functions required by Fermi statistics is nontrivial, as it mixes orbital, color, spin, and flavor structures. In our previous papers we developed a method to find them based on the representations of the permutation group, and derived the explicit wave functions for baryons excited to the first and second shells ($L = 1, 2$), tetraquarks $qq\bar{q}\bar{q}$ and hexaquarks ($6q$). Now we apply it to light pentaquarks ($qqqq\bar{q}$), in the S- and P-shells ($L = 0, 1$). Using Jacobi coordinates, one can use the hyperdistance approximation in 12-dimensional space. We further address the issue of “unquenching” of baryons, by considering their mixing with pentaquarks, via two channels, through the addition of σ -like or π -like $\bar{q}q$ pairs. This mixing is central for understanding of the observed flavor asymmetry of the antiquark sea, the amount of orbital motion issue as well as other nucleon properties.

DOI: [10.1103/vdp9-mmbj](https://doi.org/10.1103/vdp9-mmbj)

I. INTRODUCTION

A. Quark models of hadronic structure: Brief overview

Well before the experimental and theoretical development of quantum chromodynamics (QCD), in the early 1960s Gell-Mann and Levy [1] introduced the concept of chiral symmetry. Its spontaneous breaking leads to “constituent quarks,” as was shown by Nambu and Jona-Lasinio (NJL) [2]. The lowest energy mesonic states were three (nearly massless) pions and their chiral partner sigma, appearing in a chiral invariant potential depending solely on the invariant combination ($\vec{\pi}^2 + \sigma^2$).

The discovery of QCD and heavy quarkonia in the 1970s, shifted the discussion to gluon-exchange forces and confinement. Light quark spectroscopy was dominated by “bag models,” e.g., the MIT bag [3], the sigma bag [4], and the “little bag” [5], combining the MIT bag inside a pion-based skyrmion.

Recent developments in hadronic spectroscopy is due to studies of multi-quark hadrons. While discussed already in the 1960’s, they have received considerable experimental boost in the last decade. The clearest cases of multi-quark hadrons are charmed *tetraquarks* $\bar{c}c\bar{q}q$. The first “noncharmonium”

$X(3872)$, $J^P = 1^-$ state, presumed to be a “molecular” (deuteronlike) DD^* two-meson state, just below its threshold. More tetraquark states have been reported later, e.g., the doubly charmed $cc\bar{q}\bar{q}$ ones as well as fully charmed $\bar{c}\bar{c}cc$. Originally called X, Y, and Z etc for their unknown structures, the current particle data group (PDG) terminology splits $\bar{c}c\bar{q}q$ states into two categories. Those with quantum numbers $I = 1, J^P = 1^+$ (pionlike $\bar{q}q$) are obviously tetraquarks, now appropriately called $T_{\bar{c}c}$. For those with zero isospin, vectors $J^P = 1^-$ and pseudoscalars $J^P = 0^-$ (with vacuum quantum number of $\bar{q}q = \sigma$ added), Particle Data Group (PDG) still uses their generic notations, ψ, η_c respectively, which may be confusing. Yet at least four states listed as ψ in PDG 2024 are tetraquarks, as they do not fit to be charmonia levels.

Similarly, the addition of a $\bar{q}q$ with σ, π quantum numbers to qqq baryons would result in the pentaquarks. The history of their experimental and theoretical studies is full of puzzles we will not be able to cover it in full.

Isgur and collaborators [6,7] have argued that adding extra $\bar{q}q$ with σ (vacuum) quantum numbers require them to be in $^3P_0$ ($J = 0, S = 1, L = 1$) state, following earlier suggestions such as [8]. This step stressed why certain orbital motion in nucleons may be unavoidable.

Their pentaquarks were modeled via multiple baryon-meson states. Further studies along these lines were carried out by [9,10] and others, resulting in the “unquenched constituent quark model” (UCQM). These works lead to realization that all baryons, even the nucleons, do contain a significant admixture ($\sim 40\%$) of 5-quark states. (Original textbook results from the 3-quark constituent models (e.g., magnetic moments) could still be preserved due to

* Contact author: nicholas.miesch@stonybrook.edu

† Contact author: edward.shuryak@stonybrook.edu

‡ Contact author: ismail.zahed@stonybrook.edu

certain cancellations in the 5-quark sector, and new (e.g., orbital motion) can be explained.

Attempts to add a “pion cloud” to the nucleon followed a different direction. An extreme case was the “skyrmion” model, in which the qqq core is abolished and the baryons are entirely made of the semiclassical pion fields. Less extreme approaches add pions via various diagrams derived from effective chiral Lagrangians.

We will treat the addition of σ -like and π -like quark pairs on equal footing, as chiral symmetry suggests. Relations of the admixture operators for them go back to the 1950s, e.g., the Goldberger-Treiman relation and Gell-Mann-Levy “linear sigma model” [1] as reviewed, e.g., in [11].

B. Bridging hadronic spectroscopy and partonic observables on the light front

The alternate direction to hadronic structure originated in the 1970’s, and was based on data from deep inelastic scattering (DIS) experiments, providing parton distribution functions (PDFs). When expressed in terms of gluons and individual quark flavors, PDFs are not wave functions but *density matrices*, traced over all variables except the longitudinal momentum of a single parton. By definition, PDFs include all multiquark and multigluon sectors. Their off-forward generalizations in the form of generalized parton distributions (GPDs) are actively sought both theoretically and experimentally, and form the core program of the JLab facility and the future Electron Ion Collider (EIC).

Data analysis further revealed their dependence on scale of momentum transfer Q^2 , in agreement with the QCD predictions of anomalous dimensions and/or “perturbative evolution.” Unlike the chiral mechanism with the addition of $\bar{q}q$ pairs mentioned above, they are based on pQCD processes such as $g \rightarrow \bar{q}q$. While dominant at large $Q^2 \gg 1\text{GeV}^2$, they do not generate the quark sea flavor asymmetry, or quark orbital motion, which are the main focus of the present paper.

The main objective of our series of works is “bridging” hadronic spectroscopy (as defined with Hamiltonians and wave functions, either in the in CM frame or via light-cone formulation to the available parton data. The main idea, in the form of a “double-ark bridge,” is sketched in Fig. 1. The intermediate description at the scale $Q^2 \sim 1\text{GeV}^2$ should be derivable *both* from the left via “chiral evolution” (sigma-pion emissions), and from the right, via pQCD evolution. “Bridging” implies that these two different descriptions would eventually join smoothly at such intermediate scale. In particular, the mixed qqq and $qqq\bar{q}\bar{q}$ wave functions we will discuss in this work, would describe higher Fock components and the partonic observables, explaining puzzles associated with them.

Adding “pion clouds” to baryons via chiral diagrams (such as $p \rightarrow n\pi^+$) generates a flavor asymmetry. It has a long history, for a recent review see [12].

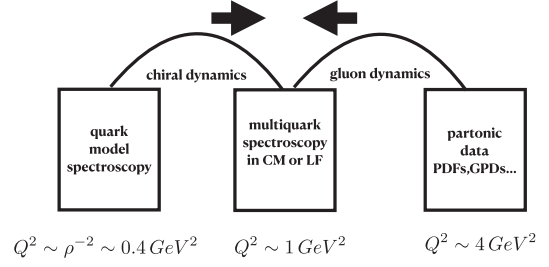


FIG. 1. Bridging quark spectroscopy and partonic data.

Our works in this direction started with [13], with the addition of a $\bar{q}q$ to a nucleon via instanton-induced 4-fermion t’Hooft Lagrangian. It was done in the light front (LF) formulation, and was able to reproduce not only the flavor asymmetry, but also the shape of the sea quark PDFs. Further discussion of the relative importance of the pion-based and t’Hooft-based mechanisms, in [14], concluded that their contributions are about equal. Hence both mechanisms are needed to explain the data.

Another (highly publicized) issue is the “spin puzzle.” The naive view that the nucleon spin is carried exclusively by the valence quarks was disproved by experiments. The most important additional contribution is believed to be the *orbital* motion in the nucleon. In our recent work [15] we have investigated whether it can be induced by a “deuteron-like” mechanism, via mixing with D – *shell* baryons. We concluded that it is highly unlikely, even by assuming some extreme tensor forces. In the current work, devoted to the admixture to pentaquark states, the data about orbital motion seems to be reproduced successfully.

For completeness, we note that other authors continue to work on 5-q Fock component of the nucleon as well, aiming to resolve the same issues. Perhaps the closest to our work is [16] in which states of the $L = 1$ pentaquark shell are discussed, including the mixing to the nucleon through the $^3P_0$ model. As the reader will see, our general construction is technically quite different, where the pentaquark wave functions are solely derived through the Fermi statistics, rather than combinations of the representations of the color-spin-flavor groups. This latter construction is discussed in Appendices for comparison and completeness. More importantly, and in contrast to [16], our mixing of the nucleon and pentaquark states will enforce chiral symmetry.

What is different in this paper in comparison to the existing literature, is that we will explicitly derive the wave functions of the pentaquark states, at the S-shell ($L = 0$) and the P – *shell* ($L = 1$). The key idea is that one has to enforce the Fermi statistics via representation of the permutation group S_4 . These wave functions are originally defined by tensors, with explicit orbital-color-spin-flavor indices. Then they are “flattened,” transformed to vectors in multidimensional space of all possible “monoms.” The individual coefficients are not numbers but functions of

coordinates, which can be naturally defined and manipulated by analytic tools provided by Wolfram *Mathematica*.

The kinematics of pentaquark wave functions (first in the CM and eventually in light-front (LF) formulations) starts with 4 Jacobi coordinates, after elimination of the center-of-mass motion. We further work with quarks of the same mass, all being light u , d , so the kinetic energy is a Laplacian operator with a hypercentral symmetry.

While the pentaquark masses and WFs are of interest by themselves, here we use them mainly as experimental manifestation in *baryon-pentaquark mixing*. In [14] we already introduced the phenomenological part of this problem, especially the so called flavor asymmetry of the proton/neutron antiquark sea. Yet the evaluation of its magnitude was done in “DGLAP-like” approximation, in which the probability to get an extra $\bar{q}q$ pair is calculated from the lowest order diagrams, ignoring any interactions or interferences between the produced and already existing quarks. Now we explore the mixing of the 3-q and 5-q sectors in a theoretically more satisfactory way.

II. PENTAQUARKS

A. Brief history

The experimental searches for light pentaquarks were performed for decades, but in so far have not resulted in well established results.

Looking for a state with a flavor-distinct—strange—antiquark, the experiment [17] announced in 2003 a $\Theta = uud\bar{s}$ resonance with mass and width

$$M_{\Theta} = 1.54 \pm 0.01 \text{ GeV}, \quad \Gamma_{\Theta} < 25 \text{ MeV}.$$

It was seen in few other experiments, but then was not found in other experiments with larger statistics. For a recent review of the experimental situation and current plans see, e.g., [18]. The chiral soliton model [19] has predicted such light Θ , as member of a antidecuplet family. Multiple other models were proposed, for recent discussion see, e.g., [20].

At that time, two of us had suggested a schematic quark-diquark symmetry [21], which suggested pentaquark structure approximated by a three-body object with two “good diquarks” $(ud)^2\bar{s}$, rotating in a P -wave. In such a model it should be approximately degenerate with P -wave excited decuplet baryons, suggesting higher mass $M \sim 1.9 \text{ GeV}$.

Needless to say, now we are far from such a naive model, with a full set of wave functions as derived below. While at this time there is no firm association of the derived states with identified resonances, we trust the general principles, Fermi statistics and the hyperdistance approximation. We simply follow the same path as in our previous studies of excited baryons, tetraquarks, and hexaquarks. These results are then used as an intermediate “stepping stones,” leading to understanding of 5-q Fock components of the nucleons.

As usual, understanding of complicated multi-quark wave functions starts with extreme quantum numbers, e.g., the channel with maximal spin $J^P = 5/2^+$. Before we discuss it theoretically, let us comment that perhaps it is the resonance $N^*(2000)5/2^+$, seen in channels such as $N\pi, N\sigma, \Delta\pi, \Lambda K^*$, see the large multichannel analysis in [22]. It is the second resonance with such quantum number, well above the first $5/2^+$, $N^*(1680)$ traditionally associated with the $qqq, L = 2$ D-shell state. (Furthermore, it is very close to the negative parity $N^*(2060)5/2^-$ state, perhaps its chiral partner.)

Finally, let us comment on pentaquark states with charm quarks. Our use of the hyperdistance approximation for fully charmed tetraquarks [23], does reproduce the splittings between these three states. Unfortunately it cannot be used for $uudc\bar{c}$ pentaquarks seen by LHCb, and therefore will not be discussed here.

B. Kinematics

The coordinates of five bodies, with the CM motion excluded, are described by four three-dimensional Jacobi coordinates,

$$\begin{aligned} \vec{X}_1 &= (15\sqrt{2}\vec{\alpha} + 5\sqrt{6}\vec{\beta} + 5\sqrt{3}\vec{\gamma} + 3\sqrt{5}\vec{\delta})/30, \\ \vec{X}_2 &= (-15\sqrt{2}\vec{\alpha} + 5\sqrt{6}\vec{\beta} + 5\sqrt{3}\vec{\gamma} + 3\sqrt{5}\vec{\delta})/30, \\ \vec{X}_3 &= (-10\sqrt{6}\vec{\beta} + 5\sqrt{3}\vec{\gamma} + 3\sqrt{5}\vec{\delta})/30, \\ \vec{X}_4 &= (-5\sqrt{3}\vec{\gamma} + \sqrt{5}\vec{\delta})/10, \\ \vec{X}_5 &= -(2/\sqrt{5})\vec{\delta} \end{aligned} \quad (1)$$

with the inverse relations as

$$\begin{aligned} \vec{\alpha} &= (\vec{X}_1 - \vec{X}_2)/\sqrt{2}, \\ \vec{\beta} &= (\vec{X}_1 + \vec{X}_2 - 2\vec{X}_3)/\sqrt{6} \\ \vec{\gamma} &= \sqrt{3}/6(\vec{X}_1 + \vec{X}_2 + \vec{X}_3 - 3\vec{X}_4), \\ \vec{\delta} &= \sqrt{5}/2(\vec{X}_1 + \vec{X}_2 + \vec{X}_3 + \vec{X}_4) \end{aligned} \quad (2)$$

The Jacobi coordinates are defined in a sequence, with the first two coinciding with the $\vec{\rho}, \vec{\lambda}$ traditionally used in baryon spectroscopy.

Each 3D vector can in turn be parametrized by its magnitude and two polar angles $\theta_i, \phi_i, i = 1, 2, 3$, and 4. The 4D space of their magnitudes $|\alpha|, |\beta|, |\gamma|, |\delta|$ are described by the *hyperdistance* in 12D

$$Y^2 = \vec{\alpha}^2 + \vec{\beta}^2 + \vec{\gamma}^2 + \vec{\delta}^2$$

and three new angles called $\theta_{1\chi}, \theta_{2\chi}, \phi_{\chi}$. In total there are Y , 8 “ordinary” angles and 3 “extraordinary” ones. The integration measure can be defined by a product of four 3D and one 4D solid angles

$$d\Omega_{11} = d\Omega_\alpha d\Omega_\beta d\Omega_\gamma d\Omega_\delta d\Omega_\chi \quad (3)$$

with $d\Omega_\chi = \sin^2(\theta_{2_\chi}) \sin(\theta_{1_\chi}) d\theta_{2_\chi} d\theta_{1_\chi} d\phi_\chi$ and the usual $d\Omega_i = \sin(\theta_i) d\theta_i d\phi_i$. The generic wave functions can be derived in a factorized form

$$\Psi = R_n(Y) F(\theta_{1_\chi}, \theta_{2_\chi}, \phi_\chi) \prod_{i=1}^4 Y_{l_i, m_i}(\theta_i, \phi_i)$$

Further details on the induced geometry—the metrics and Laplacian—are given in Appendix A.

In the nonrelativistic approximation to be used here, the kinetic energy is proportional to the Laplacian. The relativistic effects would modify the kinetic energy, and introduce spin-dependent forces. Since this paper is exploratory, we will ignore the former but include the latter perturbatively, in a spin-dependent part of the Hamiltonian. The nonrelativistic approximation would be removed in our next paper, devoted to pentaquarks on the light front.

C. The radial wave functions in hyperdistance approximation

We make the standard simplifying assumption that the ground state of pentaquarks made of the same-mass quarks, is approximately spherically symmetric. In Jacobi coordinates the kinetic energy is proportional to the Laplacian described in full in Appendix A. For a spherically symmetric function depending only on the hyperdistance, it is

$$K = -\frac{\Delta}{2m} = \left(-\frac{\partial^2}{\partial Y^2} - \frac{11}{Y} \frac{\partial}{\partial Y} + \Delta_\Omega \right) \frac{1}{2m} \quad (4)$$

The standard redefinition of the radial wave function, and Laplacian eigenvalues for $L = 0, 1$ states, are given in Appendix A.

The potential energy is usually a sum of binary potentials, which are *not* spherically symmetric. For instance, the Cornell-like potential is projected to $\alpha, \beta, \gamma, \delta$ (all positive) fraction of a sphere, see Appendix A. Using it to solve the radial equation, we obtain the wave functions shown in Fig. 2. Note that for the ground state, the wave function peaks at $Y \approx 10 \text{ GeV}^{-1} \approx 2 \text{ fm}$. (Recall that Y^2 is *not* the squared interquark distance, but $1/5$ of sum of squared distances between all 10 pairs.)

The calculation of the radial hyperdistance wave functions for baryons and tetraquarks, was done in several papers (including ours [15,24]). The same procedure will be used for pentaquarks. The light quark mass was taken as $M = 0.35 \text{ GeV}$, and the potential is Cornell-type with proper averaging over angles. In order to avoid the unknown constant shift in the potential, we only use as predictions “splittings” (differences) between energies, e.g., for the five lowest $L = 0$ pentaquark masses, we have

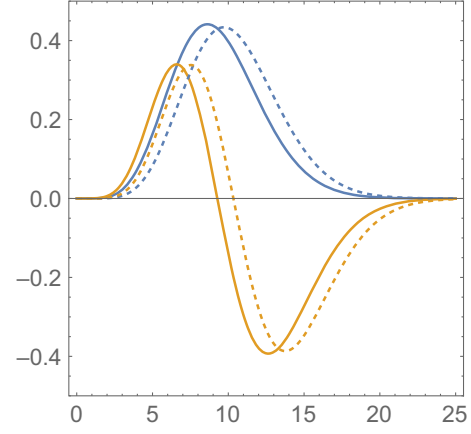


FIG. 2. The four curves refer to the wave functions from the radial Schrödinger equation for the pentastates, as $\psi(Y) * Y^{11/2}$ versus the hyperdistance Y in GeV^{-1} . The lowest S-shell penta-state is shown in solid-blue, and lowest in P-shell by dashed-blue (no node). The second S- and P-shell pentastates are shown in solid-orange and dashed-orange, both have one radial node.

$$E_n(L=0) - E_0(L=0) = 0.32, 0.62, 0.89, 1.16 \text{ GeV} \quad (5)$$

The $L = 1$ radial equation includes the extra centrifugal term in the Laplacian $10/Y^2$, the corresponding splitting is $E_2(L=1) - E_1(L=1) = 0.30 \text{ GeV}$. The corresponding wave functions are shown in Fig. 2.

D. Pentaquarks with maximal spin $S = 5/2$

Before we discuss the case at hand, the pentaquarks $q^4 \bar{q}$, let us remind the logic used in [24] for a related case of hexaquarks (q^6). As usual, one starts with the simplest cases, e.g., those of maximal possible spin, $S = 3$ for hexaquarks or $S = 5/2$ for pentaquarks. Recall that the $S = 3$ hexaquark is the only case for which a fully antisymmetric WF has actually been derived analytically in [25]. It was done by traditional means, adding up available color and flavor representations. The answer was written explicitly, as a sum of five parts with different color-flavor structures. This state is believed to be experimentally observed [26] as a resonance $d^*(2380)$ in reaction

$$p + n \rightarrow d + \pi^0 + \pi^0$$

Its width $\Gamma_{d^*} \approx 70 \text{ MeV}$ is significantly below that of Delta baryon $\Gamma_\Delta \approx 115 \text{ MeV}$, which was used against its interpretation as a $\Delta\Delta$ bound state. Also, for a “deuteronlike” interpretation of a $\Delta\Delta$ state, the binding needs to be $\approx 84 \text{ MeV}$, which is perhaps too large.

Similarly, the simplest pentaquarks must be those with maximal possible spin $5/2$. If the isospin is also maximal, the only function remaining is that of color, which cannot be antisymmetric since the color Young tableaux height is restricted to $N_c = 3$. Yet the antisymmetric pentaquark

states with $S = 5/2, I = 3/2$ and $S = 5/2, I = 1/2$ can be constructed, even by the usual procedure of combining color and flavor representations. Since its discussion is rather long, it is delegated to Appendix C.

The wave function for $S = 5/2, I = 1/2$ state is given in (C36). We translated it into the monom basis using Wolfram *Mathematica*, obtaining a 144-component wave function. Compared to the one from alternative derivation based on permutation group (to be explained next) we have observed full agreement.

Some properties of the maximal spin $S = 5/2$ states such as

$$\left\langle \sum_{i>j} \vec{\lambda}_i \vec{\lambda}_j \right\rangle = -40/3 \quad (6)$$

$$\left\langle \sum_{i>j} (\vec{\lambda}_i \vec{\lambda}_j) (\vec{S}_i \vec{S}_j) \right\rangle = -10/3 \quad (7)$$

obtained analytically, hold for other (much more complex) pentaquark states to be discussed below. Such numerical checks turned out to be very useful.

The distributions of flavor and color between the quark “core” and the antiquark are nontrivial. For instance, for the $S = 5/2, I = 1/2$ state the mean values of the isospin projections are

$$\langle I_z(q) \rangle = 1/4, \quad \langle I_z(\bar{q}) \rangle = -1/2 \quad (8)$$

(Note that the sum rule $4\langle I_z(q) \rangle + \langle I_z(\bar{q}) \rangle = 1/2$ holds as expected.)

E. Color-spin-flavor wave functions for S-shell ($L=0$) pentaquarks from permutation symmetry

Significant technical difficulty in building the theory of multiquark hadrons is a need to fulfill permutation antisymmetry of all quarks required by Fermi statistics. The traditional way of doing it is by adding a pair of quarks into all possible representations of all pertinent groups ($SU(3)$ for color, $SU(2)$ for spin and flavor, etc), and then adding next quark (or a pair), etc. How to do it is described in Appendix C.

The wave functions to be derived here follow from the novel technique we developed in [24]. For pentaquarks it is based on finding representations of the “quark core” using the permutation group S_4 . Let us start with the Table I giving the number of states of the “core,” as a function of isospin I and spin S . (As expected, the table is symmetric under their interchange.)

After the antiquark is added through standard procedure, one gets the number of pentaquark states given in the Table II. The antisymmetric representations of the permutation group for four light quarks S_4 , follows the procedure developed in [24]. Standard “monom space” for S-shell ($L=0$) pentaquarks has color-spin-flavor dimension

TABLE I. Number of states (per choice of I_z and S_z) for each combination of total spin and isospin for four quarks. Adding an antiquark can shift each state’s spin up or down by $1/2$.

IS	0	1	2
0	0	1	0
1	1	1	1
2	0	1	0

TABLE II. Spin S - isospin I table of antisymmetric pentaquark states with $L=0$. The integer numbers show the multiplicity of the states, the number in brackets are the dimensions of the good basis used to do calculations in each of the S, I sectors.

I/S	$1/2$	$3/2$	$5/2$
$1/2$	3 (75)	3 (60)	1 (15)
$3/2$	3(60)	3 (48)	1 (12)
$5/2$	1 (15)	1 (12)	0

$$d_{\text{monoms}} = 3^6 \times 2^5 \times 2^5 = 746496 \quad (9)$$

(note that the antiquark in color Young tableaux is represented by two squares, thus 3^6 .) In the case of P-shell ($L=1$) states (we will need for baryon mixing study below) it is additionally multiplied by 4, the number of Jacobi coordinates which appear linearly, the monom space dimension extends well over a million. It may appear that, even with the help of *Mathematica*, writing operators as matrices in this huge space would be impossible.

Fortunately, this is not so, for two reasons. First, for each of 4 subspaces (orbital, color, flavor, spin) one can define and operate in what we call the “good basis” based on pertinent Young tableaux. Therefore in practice one can define the permutation generators as matrices in spaces of dimensions much smaller than d_{monom}

$$N_{GB} = N_{GB}^{\text{color}} \times N_{GB}^{\text{spin}} \times N_{GB}^{\text{flavor}}$$

Two basic generators of the permutation groups are defined by the “*KronekerProduct*” operator in *Mathematica* of respective submatrices, and diagonalized. The wave functions we look for given by *antisymmetric* eigenvectors *common to both generators*. Since all $n!$ elements of S_n group can be constructed out of those two generators, such wave functions are antisymmetric for all permutation of the group.

For spherically symmetric S-shell ($L=0$) states there are color*spin*flavor subspaces: the antisymmetric states we found are listed in Table II. The number in brackets are dimensions N_{GB} of respective “good basis” in each case. While resulting permutation matrices are still too large to be presented here, their dimensions are much smaller than that of the monom space (9), and dealing with them creates no problem.

Consider as an example the pentaquarks with the maximal spin $S = 5/2$. The remaining nontrivial Young tableaux are those of color and isospin. Using the familiar $SU(2)$ addition of five spins one readily finds that there are 5 ways to add them to total isospin $I = 1/2$: thus the good basis for isospin has dimension 5. The Young tableaux for color adding to total color zero is made of two closed columns $\epsilon_{abc}\epsilon_{def}$, like for hexaquarks. Unlike that case, however, permutations should be done for quarks only, 4 indices $abcd$. This leads to a “good basis” of dimension 3, or $3 \times 5 = 15$ in total. The matrices of two generators of S_4 are then defined and diagonalized. One *common* eigenvector with eigenvalue -1 (antisymmetric) is found, out of 15. therefore we have found the *single* pentaquark state with $S = 5/2, I = 1/2$. (Recall that for hexaquarks with maximal spin it also was a single state.)

After all relevant antisymmetric states are found, the most efficient way to proceed is to return to the universal description in the monom basis of size (9) for all WFs. While vectors and operators (matrices of this size squared, $d_{\text{monoms}} \times d_{\text{monoms}}$) may appear frighteningly large, *Wolfram Mathematica* let us deal with them, with the usual notations, e.g., *vector*.matrix.vector* [27].

The actual number of nonzero components of the WFs is in fact much smaller than d_{monoms} , so these arrays are really quite sparse. Although we still cannot publish them in a paper, and only present average values of pertinent operators, it is still instructive to compare the number of nonzero terms in the monom representation. As anticipated, the smallest numbers are for maximal spin, $S = 5/2$. The one with $I = 3/2$ has only 252 nonzero terms, while that with $I = 1/2$ has 708 nonzero terms. Going to nonmaximal spin and then to $L = 1$ P-shell leads to a significant increase in the number of nonzero components, e.g., $13L = 1, S = 1/2, I = 1/2$ states we will use below (for mixing with the nucleons) have 187200 nonzero components (together).

For $I = 1/2$ and $S = 1/2, 3/2$ we found two triplets states each, as can also be derived based on Young tableaux (see Appendix C). The triplet states one gets are quite arbitrary, and only the eigenstates of certain physical operators make them physically distinct. The sum over ten pair operators $(\lambda\lambda) = \sum_{i>j} \vec{\lambda}_i \vec{\lambda}_j$ (for definition of antiquark color matrix see Appendix C) and $SS = \sum_{i>j} \vec{\sigma}_i \vec{\sigma}_j$ are related to the Casimirs of color and spin, so they are diagonal for these states. One needs a Hamiltonian, e.g., with one-gluon exchange color-spin forces summed over all 10 pairs

$$H_{\lambda\sigma} = -C_{\lambda\sigma} \sum_{i>j} (\vec{\lambda}_i \vec{\lambda}_j) (\vec{S}_i \vec{S}_j) \quad (10)$$

We evaluated the full Hamiltonian (matrix elements between all states) and diagonalized it. The obtained eigenvalues for seven of them for $I = 1/2$ ($S = 1/2$ triplet,

TABLE III. We list the properties of the triplet of $S = 1/2$, the triplet of $S = 1/2$ and single $S = 5/2$ states, all seven with the isospin $I = 1/2$ (the upper line in the previous table).

S	1/2	1/2	1/2	3/2	3/2	3/2	5/2
$\lambda\lambda$	-40/3	-40/3	-40/3	-40/3	-40/3	-40/3	-40/3
SS	-3/2	-3/2	-3/2	0	0	0	5/2
$H_{\lambda\sigma}/C_{\lambda\sigma}$	-4.66	-1.44	2.77	-3	1/3	10/3	10/3

$S = 3/2$ triplet, and $S = 5/2$ singlet) are given in Table III. As expected, the state with maximal spin $S = 5/2$ has also the largest energy. (If $N^*(2000)5/2^+$ happen to be indeed this pentaquark, others are then expected to be somewhat below 2 GeV in mass.)

(The alternative Hamiltonian, used, e.g., by [16], is the flavor-spin operator corresponding to pion exchanges, following Ripka and Glozman

$$H_{\tau\sigma} = -C_{\tau\sigma} \sum_{i>j} (\vec{\tau}_i \vec{\tau}_j) (\vec{S}_i \vec{S}_j) \quad (11)$$

Note that for spin we use $\vec{S} = \vec{\sigma}/2$ with a half, but for color and flavor we use Gell-Mann and Pauli matrices *without* a half. Having the state wave functions, we also calculated the Hamiltonian matrix for it. We do not present its eigenspectrum as it is not qualitatively different from that of color-spin Hamiltonian.)

Due to symmetry, the mean value of the S_z component is the same for all four quarks, but different for the antiquark. By construction, the mean values of $S_z(i)$ are related by the sum rule

$$\sum_{i=1..5} \langle S_z(i) \rangle = 4 \langle S_z^1 \rangle + \langle S_z^5 \rangle = S_z^{\text{total}} \quad (12)$$

The mean values of spins and isospins of the quarks and antiquark can be compared to the expectation values based on naive models. In particular, if the pentaquark is made of two spin and isospin zero diquarks [21], the $\langle S_z^q \rangle \approx \langle I_z^q \rangle \approx 0$ and all the spin and isospin are carried by the antiquark $\langle S_z^{\bar{q}} \rangle \approx \langle I_z^{\bar{q}} \rangle \approx 1/2$. Compared to results in the table one can see some states do have either zero mean spin or zero mean isospin on quarks, but both never occurs. It is one more example showing how naive such models really are.

The alternative naive model, in which the spin and isospin are divided equally between all five objects, also does not seem to be supported by this detailed analysis.

F. Pentaquarks of the $L = 1$ shell

Because the σ, π operators introducing pertinent mesons (to be used below to “unquench” the nucleons) necessarily include nonzero orbital momentum, the P-shell pentaquark states can mix with the nucleon.

TABLE IV. We list the antisymmetric pentaquark states with $L = 1$, spin S and isospin I . As in the previous Table II, the first integers are the numbers of corresponding states, the numbers in brackets show the dimension of the corresponding good basis.

I/S	$1/2$	$3/2$	$5/2$
$1/2$	13 (300)	11 (240)	3 (60)
$3/2$	11 (240)	10 (192)	3 (48)
$5/2$	3 (60)	3 (48)	1 (12)

For $L = 1$ pentaquarks, the dimension of “good states” increases by a factor 4 on top of what it was in S-shell, being just the number of Jacobi coordinate vectors $\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}$ appearing linearly. There are approximately that many more fully antisymmetric states found, see the list of the $L = 1$ states in Table IV.

The states that can mix with the nucleon should have $I = 1/2$, and either $S = 1/2$ or $S = 3/2$. The list of Fermi-statistics-compliant states is given in the Table IV, 24 in total [28].

Note that even the largest good basis with dimension 300, is still many orders of magnitude smaller than the monom basis of dimension 746496. The vectors are sparse but not too sparse: the 13 $S = 1/2$ states have 187200 nonzero matrix elements, 11 $S = 3/2$ states have 198000 nonzero matrix elements. P-shell states are much more complicated than the wave functions of the $L = 0$ shell, as the amplitude is no longer a number but an explicit function of 12 coordinates, e.g., eight angles $\theta_i, \phi_i, i = 1, 2, 3$, and 4 and four magnitudes. The operators of angular momenta contain differentiations over angles, as usual, which can be directly applied to our sparse vectors of very large dimension.

Some operators, like color

$$\sum_{i>j} \vec{\lambda}_i \vec{\lambda}_j \rightarrow -\frac{40}{3}$$

have the same expectation value for all states, owing to the [211] color representation of the Young tableau for the core.

The eigenvalues of the 13×13 color-spin Hamiltonians for the $L = 1$ pentaquark shell with $S = 1/2$ are shown in Fig. 3.

As we did for $L = 0$ pentaquark shell, we calculated the distribution of various physical quantities over the 4 Jacobi solid angles and the 5 bodies at hand. In order to not clutter the presentation, we only present those for the three select states, numbers 1, 7, 13 of the $S = 1/2$ set in Table V.

The four Jacobi coordinates have four pairs of angles θ_i, ϕ_i , and so we calculated the expectation values of the orbital momenta *squared*, in each of the four Jacobi solid angles

$$L_i^2 f = -\frac{1}{\sin[\theta_i]} \frac{\partial}{\partial \theta_i} \left[\sin[\theta_i] \frac{\partial f}{\partial \theta_i} \right] - \frac{1}{\sin[\theta_i]^2} \frac{\partial^2 f}{\partial \phi_i^2} \quad (13)$$

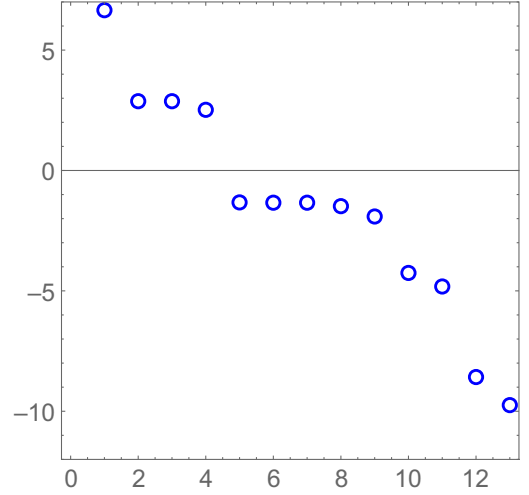


FIG. 3. Eigenvalues of the color-spin Hamiltonian for $L = 1, S = 1/2$ states, normalized to the matrix element of the potential $H_{\lambda\sigma}/C_{\lambda\sigma}$.

$i = 1, 2, 3, 4$. The symmetry between the quarks in the core leads their equality $\langle L^2(1) \rangle = \langle L^2(2) \rangle = \langle L^2(3) \rangle$, but the corresponding value for the antiquark—related to the fourth angle—is different. As seen from this table, the three selected states show a large variety of distributions. The first state in the second column is the one of the lowest energy. It has nearly no angular momentum on the antiquark, and nearly no spin on the quarks.

We also calculated mean values of z -component of $L_z(i), i = 1..4$ on sum states. All of our results reproduce accurately the respective sum rules

$$\sum_{i=1..4} \langle L^2(i) \rangle = 3 * L^2(1) + L^2(4) = 2 \quad (14)$$

$$\sum_{i=1..4} \langle L_z(i) \rangle = 3 * L_z(1) + L_z(4) = -1 \quad (15)$$

as expected for the $L = 1$ shell. (For completeness, we recall that the spin and isospin components satisfy different sum rules

TABLE V. Distribution of few physical quantities over Fermi-antisymmetric pentaquark states with $L = 1$, spin $S = 1/2$ and isospin $I = 1/2$. The operator $L^2(i)$ is the squared angular momentum, in the first and fourth Jacobi solid angles. The mean z -components of spin and isospin are given for the first quark and the 5th antiquark, respectively.

State number	1	7	13
$L^2(1)$	0.4965	0.6652	0.6547
$L^2(4)$	0.0258	0.0043	0.0358
$S_z(1)$	0.0963	0.0562	0.0931
$S_z(5)$	0.1146	0.2748	0.1272
$I_z(1)$	0.1523	0.0516	0.0545
$I_z(5)$	-0.1094	0.2934	0.2817

$$\sum_{i=1..5} \langle S_z(i) \rangle = 4 \times S_z(1) + S_z(5) = 1/2$$

$$\sum_{i=1..5} \langle I_z(i) \rangle = 4 \times T_z(1) + T_z(5) = 1/2$$

III. CAN PENTAQUARKS BE CHAOTIC?

Quantum many-body (and some few-body) systems are known to undergo transition into the so-called “quantum chaos” regime, for review see, e.g., [29]. This refers to atoms or nuclei, with several particles/holes near the filled shells (magic numbers). As a specific historical example, it was the atom of Cerium, with four valence electrons or 12 coordinates, which was shown by [30] that even the lowest states are chaotic.

Since this phenomenon is generic, one may also expect it to take place for multiquark systems. Pentaquarks, the subject of this paper, have 12 Jacobi coordinates, the same number as the Cerium atom. So one may expect to find similar “transition to chaos” in them. (One of us [13] has already discussed this issue for baryons, pentaquarks and their admixture in a particular light-cone model, with different wave functions and different set of basis states.)

The simplest manifestation of quantum chaos is *random* (Gaussian-like) Porter-Thomas distribution of the amplitudes of the (exact) Hamiltonian eigenstates, expanded in terms of some natural basis, e.g., of the single-body Hamiltonian. In Fig. 4 we show two distributions of the “monom coefficients” for our fixed-energy states.

The upper is for the $S = 5/2, I = 1/2$ S-shell state which we obtained both analytically and numerically. We recall that its wave function consists of three terms of different structure, times permutations. And the upper histogram confirms that there are only 2 values (up to a sign) of such amplitudes, repeated multiple times. Furthermore, recall that there are $4! = 24$ quark permutations, while occupancy in the histogram turns out to be $72 = 24 \times 3$. This state clearly is very regular, far from being chaotic.

The lower histogram is for one state from the P-shell $L = 1$ pentaquark. As one can see from it, the overall shape of the amplitude distribution is indeed similar to a Gaussian, although deviations from it are also clearly seen. The most obvious is a dip near zero value. Similar dip often is due to “finite size effects,” for example in Dirac quark eigenvalue spectrum in lattice or instanton studies of the QCD vacuum. However in this case the dip shape is well understood in chiral random matrix model [31,32], in terms of the box size and number of states. The analytically predicted distributions were well confirmed in multiple lattice studies. It is not clear to us what exactly creates the dip in this histogram. Let us just mention that the total number of nonzero coefficients 14400 in a monom basis of size 746496, so it is still rather sparse.

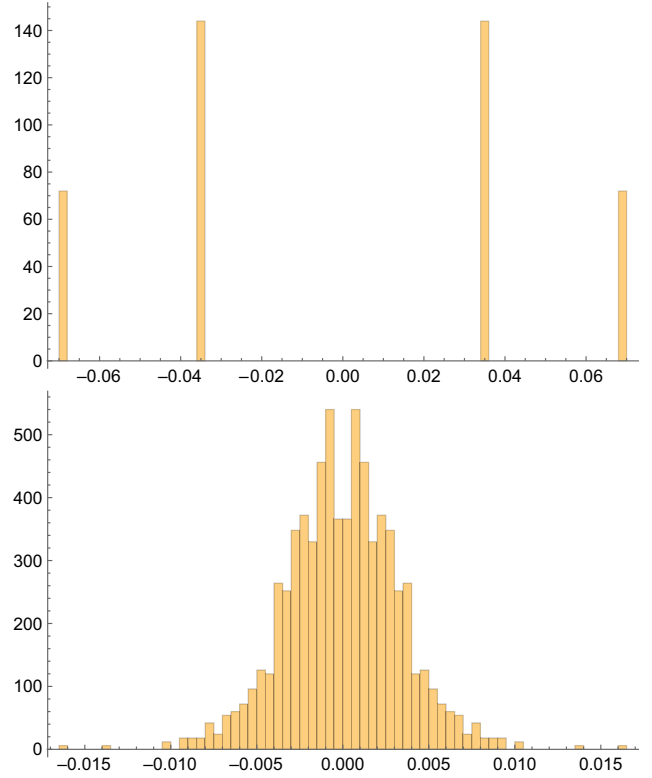


FIG. 4. Distribution of monom coefficients, for one of the S-shell state with $L = 0, S = 5/2, I = 1/2$ (upper plot) and one of the P-shell state $L = 1, S = 1/2, I = 1/2$ (lower plot).

Another feature of the lower histogram are regularly placed peaks. Perhaps those are an indication of a quantum system with the phase space in which both chaotic and regular motion are present, with the former being dominant.

Quasirandom coefficients of the wave function in a particular basis is of course only one of many observables one can study. Probability distributions over single-body properties (e.g., spatial distributions or those in single-body-energy) can be compared to predictions of some statistical models (e.g., those for Fermi gas with certain temperature and entropy). Indeed, a single many-body state can have a thermal description in terms of single-body observables. We defer such studies to future works.

Summarizing, we presented evidence that transition to quantum chaos in pentaquarks does happen in the second ($L = 1$) shell states.

IV. NUCLEON-PENTAQUARK MIXING

A. The “unquenching” of hadrons

The unquenching of mesons and baryons—in attempts to quantify the four-quark sector of mesons and the five-quark sector of baryons—has a long history but is still far from being completed.

From the 1960’s to about a decade ago the experimental and theoretical status of multiquark hadrons was assigned into

the general realm of “exotica,” which looked suspicious as lacking firmly established facts. It has all changed now. The revolutionary change perhaps has not yet settled in the minds of the community at large, so let us illustrate it by focusing on the pillar of QCD spectroscopy, heavy quarkonia.

While bottomonium spectroscopy is fully consistent with the $\bar{b}b$ interpretation, using a simple Cornell potential reproducing the states, already $\bar{c}c$ charmonia are different. The first “noncharmonium” state discovered was called $X(3872)$, $J^P = 1^-$ state. Originally called X , Y , Z etc for unknown structure, in the current particle data group (PDG) terminology they are split into two categories. “Axial vectors” (quantum numbers $I = 1$, $J^P = 1^+$) have nonzero isospin and therefore are considered “pion-added.” Since the 2024 PDG classification, they are identified as tetraquarks and are called $T_{\bar{c}c1}$. Yet for vectors $J^P = 1^-$ and pseudoscalars $J^P = 0^-$ (zero isospin or “ σ added”), the PDG uses still their generic notations, ψ , η_c respectively. This is unfortunate, especially in view of the observations that vectors and axials seem to form near-degenerate pairs, perhaps being “chiral partners.”

(Some charmed tetraquarks presumed to be “molecular” (deuteronlike) two-meson state (e.g., DD^*) just below the threshold, but not all. Lately many more tetraquark states are found, not only with $\bar{c}c$ but also cc pair, with even all-charm $\bar{c}\bar{c}cc$ tetraquark found at LHC. For the details on such tetraquarks and mixing we refer to [33] and references therein.)

Completing this detour to charmonia, let us make an important point: the number of pion-admixed and sigma-admixed known states seems to be equal. With four $T_{\bar{c}c1}$ and four “extra- ψ ” states unfit to be identified as $\bar{c}c$ levels, for which we observe a kind of $\pi - \sigma$ symmetry reminiscent of the original chiral symmetry. Chiral doubling in heavy-light systems was originally noted in [34–36] and since confirmed experimentally by the BABAR [37] and CLEO collaborations [38].

In general, the appearance of the $\sigma = f_0$ meson in hadronic reactions has been artificially suppressed over the years. To emphasize our point regarding this important chiral mixing, let us provide another little known case of the $(\pi - \sigma)$ symmetry. A list of branching ratios (from PDG24) of main-listing nucleon resonances, into both decay modes is given in the Table VI. While the pion branchings tend to be larger, this is actually a reflection of the larger phase space for $N\pi$ compared to $N\sigma$. The pertinent couplings, $g_{NN^*\pi}$ and $g_{NN^*\sigma}$, are basically equal within errors. This is what “naive” chiral symmetry suggests.

After this two brief detours to the PDG data, let us turn to the theory. The traditional strategy to evaluate the 5-q admixture to baryons consists of two steps:

- (i) formulation of certain “meson admixture operators” adding $\bar{q}q$ pair to a baryon,
- (ii) followed by projections of their product to “good tetraquark states” obeying general requirements such as Fermi statistics.

TABLE VI. Branching ratios of $N\pi$ and $N\sigma$ decays of $N^*(mass)J^P$ resonances (left column), from PDG24.

State	$N\pi$	$N\sigma$
$N^*(1440)1/2^+$	0.55–0.75	0.11–0.23
$N^*(1520)3/2^+$	0.55–0.65	< 0.1
$N^*(1535)1/2^-$	0.32–0.52	0.02–0.1
$N^*(1650)1/2^-$	0.5–0.7	0.02–0.18
$N^*(1675)5/2^-$	0.38–0.42	0.03–0.07
$N^*(1675)5/2^-$	0.38–0.42	0.03–0.07
$N^*(1680)5/2^+$	0.6–0.7	0.09–0.19
$N^*(1700)3/2^-$	0.07–0.17	0.02–0.14
$N^*(1710)1/2^+$	0.05–0.20	< 0.16
$N^*(1720)3/2^+$	0.08–0.14	0.02–0.14
$N^*(1875)3/2^-$	0.03–0.11	0.02–0.08
$N^*(1880)1/2^+$	0.03–0.31	0.08–0.40
$N^*(1895)1/2^-$	0.02–0.18	< 0.13
$N^*(1900)3/2^+$	0.01–0.20	0.01–0.07
$N^*(2060)5/2^-$	0.07–0.12	0.03–0.09
$N^*(2100)1/2^+$	0.08–0.32	0.14–0.35
$N^*(2120)3/2^-$	0.05–0.15	0.04–0.14
$N^*(2190)7/2^-$	0.1–0.2	0.03–0.09

Before discussing the technical implementation of steps (i) and (ii) let us mention some key facts about chiral symmetries and their breaking. Long before the discovery of QCD, Nambu-Jona-Lasinio [2] have introduced hypothetical a 4-fermion Lagrangian, and have shown that for a large enough coupling, it can break spontaneously the $SU(N_f)$ chiral symmetry. The discovery of fermion zero modes of instantons by t’ Hooft [39], had shown that QCD does generate nonperturbative multifermion $2N_f$ interactions. For two flavors ($N_f = 2$) the resulting 4-fermion Lagrangian can be written in many forms, in particular as

$$L_{\text{tHooft}} = G_{\text{tHooft}}(\sigma^2 + \vec{\pi}^2 - \eta'^2 - \vec{\delta}^2) \quad (16)$$

Here the binary quark-antiquark operators are given with respective mesonic names, as

$$\sigma = \bar{q}q, \quad \vec{\pi} = \bar{q}\vec{\tau}i\gamma_5q, \quad \eta' = \bar{q}i\gamma_5q, \quad \vec{\delta} = \bar{q}\vec{\tau}q$$

The first two attractive terms are the same as in the hypothetical Nambu-Jona-Lasinio Lagrangian. Note that the two last terms have negative (repulsive) sign, making a violation of the $U(1)_a$ chiral symmetry explicit. As shown in [40] and confirmed since by instanton phenomenology, the coupling constant G_{tHooft} is large enough to *break spontaneously* chiral $SU(2)_a$ symmetry. Also it was shown in review of correlation function in the QCD vacuum [41], that the t’ Hooft Lagrangian is indeed responsible for flavor mixing of all scalar and pseudoscalar mesons, making them remarkably different from others (vector, tensor...) channels

in which very small flavor mixing takes place (“Zweig rule”).

B. Admixture via ’t Hooft Lagrangian

Since both the NJL or ’t Hooft Lagrangian can turn one quark into $qq\bar{q}$, it seems logical to start with them, as the “admixture operators” for flavor mixing.

Let us start by recalling that the ’t Hooft Lagrangian is built out of instanton zero modes. For three flavors, it is a 6-fermion operator which can be represented as a flavor determinant. For two flavors u, d the ’t Hooft operator is 4-fermion one, which can be written with spin and isospin indices as follows

$$\epsilon_{f1f2}\epsilon_{g1g2}\left(\left(1-\frac{1}{2N_c}\right)(\bar{q}_{Lf1}q_{Rg1})(\bar{q}_{Lf2}q_{Rg2}) - \frac{1}{8N_c}(\bar{q}_{Lf1}\sigma_{\mu\nu}q_{Rg1})(\bar{q}_{Lf2}\sigma_{\mu\nu}q_{Rg2})\right) \quad (17)$$

where $L, R = (1 \pm \gamma_5)/2$ refer to the projected left-right spinors, and $\sigma_{\mu\nu} = (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/2$. The two epsilon symbols yield a determinant. Alternatively they can be substituted by flavor matrices via the Fierz-like identity

$$\epsilon_{f1f2}\epsilon_{g1g2} = (\hat{1} - \vec{\tau}_{f1g1}\vec{\tau}_{f2g2})/2$$

The $2N_f$ entries in the ’t Hooft vertex originate from a single instanton. Hence, the distance between them is restricted by the mean instanton size [40] to

$$r_\sigma \sim \rho \approx 1/3 fm$$

This size is comparable to the radii of the scalar σ and pseudoscalar π mesons, and defines many of their internal properties. However, the sizes of baryons and tetraquarks are much larger. This drastically reduces the effect of the

first-order contribution of the ’t Hooft Lagrangian, as we now show.

Let the 4 fermion coordinates (to be defined below) be $\vec{X}_i, i = 1, 2, 3, 4$. In coordinate representation, the ’t Hooft 4-fermion operator is proportional to the product of 4 zero modes

$$T_{\text{tHooft}} \sim \prod \gamma_\mu \partial_\mu \phi_0(\vec{X}_i - \vec{y}) \quad (18)$$

where $\vec{y}$ is the instanton center and the (full) zero mode is

$$\phi_0^{a\nu} = \psi_0 \frac{(1 - \gamma_5)}{2} \frac{x_\mu \gamma_\mu}{\sqrt{x^2}} U_{ab} \epsilon_{\nu b}$$

$$\psi_0 = \frac{\rho}{\pi} \frac{1}{(x^2 + \rho^2)^{3/2}} \quad (19)$$

with U being instanton orientation matrix in the $SU(3)$ color space.

(To understand how momenta in (18) are related, one has to express the zero modes via their Fourier transform $\tilde{\phi}_0(\vec{p})$)

$$\phi_0(\vec{X} - y) \rightarrow \int d^3 p \gamma_\mu p_\mu \tilde{\phi}_0(\vec{p}) e^{-i\vec{p}(\vec{X}-\vec{y})} \quad (20)$$

The integral over the instanton center $d\vec{y}$ produces overall momentum conservation, in the form $\delta(\sum_i \vec{p}_i)$. The expressions for pairwise product of $\tilde{\phi}_0(\vec{p})$ are well known, and yield “instanton form-factors” in many applications, but we will not need them here.)

The notations for the coordinates in the projection integral $\langle P|T|B \rangle$ are explained in Fig. 5. The pentaquark wave function depends on five coordinates $P(\vec{X}_i), i = 1..5$ while that of the baryon on three $B(\vec{X}'_n), n = 1, 2, 3$. Let the transition operator describes the interaction of quark number labeled by 3 with a quark-antiquark pair labeled by 4-5,

$$T_{\text{tHooft}} \sim \int \left(\prod d^3 X_i \right) \left(\prod d^3 X'_i \right) \delta(\vec{X}_1 - \vec{X}'_1) \delta(\vec{X}_2 - \vec{X}'_2) P(\vec{X}) B(\vec{X}') T \quad (21)$$

$$T = \int d\vec{y} \psi_0(\vec{X}'_3 - \vec{y}) \psi_0(\vec{X}_3 - \vec{y}) \psi_0(\vec{X}_4 - \vec{y}) \psi_0(\vec{X}_5 - \vec{y}) \quad (22)$$

The 4-point function T can be estimated by the standard expansion-of-exponent method

$$\psi_0(\vec{X}) \sim \exp\left(-\frac{3}{2\rho^2} \vec{X}^2\right)$$

and, after integration over $\vec{y}$, we obtain the vertex function in a Gaussian form

$$T \sim \exp\left(-\frac{9}{8\rho^2}(\vec{X}_1^2 + \vec{X}_2^2 + \vec{X}_3^2 + \vec{X}_4^2) + \frac{3}{4} \sum_{i>j} (\vec{X}_i \vec{X}_j)\right) \quad (23)$$

More simplifications follow from the “quasilocal” approximation, using the fact that instanton size is small $\rho \ll |X_i|$. If we assume that all 4 coordinates in the ’t Hooft vertex coincide

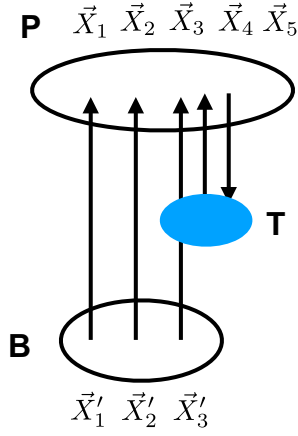


FIG. 5. Pentaquark-baryon-meson overlap structure, with the definition of the relevant in-out coordinates for the quarks.

$$\vec{X}'_3 = \vec{X}_3 = \vec{X}_4 = \vec{X}_5$$

then the three Jacobi coordinates should vanish

$$\vec{\beta} = \vec{\gamma} = \vec{\delta} = 0$$

leaving only one nonzero $\vec{\alpha} \neq 0$. Therefore, the overlap integral should scale as a large power of the small parameter

$$\langle P|T|B \rangle \sim \left(\frac{\rho}{R}\right)^9 \quad (24)$$

(where R is the pentaquark size) and therefore is very small. We conclude that the mixing induced by the first order in 't Hooft quasilocal operator is too small, and is unable to compete with the nonlocal production of a quark pair in the $\sigma, \vec{\pi}$ states (viewed as a coherent iteration of the 't Hooft interaction in the QCD instanton vacuum) as described below.

C. “Soft additions” of the $\bar{q}q$ pairs

Mixing of states with the same global quantum numbers is a phenomenon well known all over atomic and nuclear physics. The old-fashion perturbation theory to first order

$$\begin{aligned} \psi(B) &= N[|B\rangle + \Delta\psi] \\ \Delta\psi &= \sum_n \frac{\langle P_n^* | \tilde{T} | B \rangle}{M_{P_n} - M(\bar{q}q) - M(B)} |P_n\rangle \end{aligned} \quad (25)$$

tells us that the matrix element of the mixing operator $\tilde{T}$ gets measured against the energy splitting between the in-out states. Therefore, in some cases mixing can be substantial even if *both* numerators and denominators are small. Perhaps the best known effect of this type in atoms

and nuclei, is the mixing of opposite parity states due to neutral currents in weak interactions.

The spontaneous breaking of chiral symmetry by the NJL or 't Hooft interactions is due to high-order bubble chains of $\bar{q}q$ “exploding” in the instanton vacuum, reaching infinite distance for massless pions and producing nonzero VEV $\langle \bar{q}q \rangle \neq 0$. The 4-fermion t'Hooft vertex (shown in Fig. 5 by a blue ellipse) gets split into two

$$\sigma^2 + \vec{\pi}^2 \rightarrow \sigma(x) \times \sigma(y) + \vec{\pi}(x) \times \vec{\pi}(y)$$

chirally symmetric contributions. The sigma insertion gets disjoint even for arbitrarily well-separated points x, y , with the same quark condensate everywhere. The pion insertion yields the pion propagator, or a Yukawa-type potential between the coordinates $\vec{X}_3$ and the mean of the coordinates $(\vec{X}_4 + \vec{X}_5)/2$ in the notations of Fig. 5. The technical details on how this happens in the instanton vacuum can be found, e.g., in the reviews [11,42] and references therein.

In summary, we will use the operators of the “soft $\bar{q}q$ admixture,” a sum of the “sigma”-scalar with vacuum quantum numbers for the pair, and the “pion”-pseudoscalar for the pair. Each $\bar{q}q$ vertex is near-local, with the instanton-induced size. Their implementation will be carried below.

Before we detail our analysis, let us comment on the historic development and the relation between them. $SU(N_f)$ chiral symmetry implies that each state has a chiral copy. This includes the nucleon state and its chiral copies induced by the unitary rotation $\exp(\sigma + i\gamma_5(\vec{\tau}\vec{\pi}))$. Even after its spontaneous breaking (adding nonzero VEV to σ), this transformation leads multiple chiral relations connecting the sigma and pion couplings, e.g., the famed Goldberger-Treiman relation $g_A m_N = g_{\pi NN} f_\pi$.

The admixture of a $\bar{q}q$ pair with vacuum (sigma) quantum numbers referred to as the $^3P_0$ ($S=1, L=1, J=0$) model, was suggested in the 1970s (e.g., [8]), and actively discussed in the 1990s. In particular, Isgur and collaborators [43] started quantitative studies aimed at “unquenching” the nucleon, mostly by adding strange $\bar{s}s$ and charm $\bar{c}c$ components. They also raised the question of whether a large $\bar{q}q$ admixture is compatible with some of the successes of the qqq models, such as the values of their magnetic moments.

The same mixing operator was used in subsequent “baryon unquenching” works, such as [10]. They also used the same baryon-meson set of states in the oscillator basis, following on the work in [43]. While these states do not really represent pentaquarks, they allow for a simple enforcement of Fermi statistics.

Another direction of research focused on the “pion cloud” of the nucleons. It is based on chiral dynamics, with pertinent Lagrangians and diagrams. The pion-based admixture is discussed mainly in connection to the issue of *flavor asymmetry of the antiquark sea*, $\bar{u}(x) \neq \bar{d}(x)$. A particularly popular approach defines the admixture as

$N\pi, \Delta\pi$ states in the nucleon, with the pion possessing certain momentum.

It has been pointed out by Garvey [44], that the magnitude of the flavor asymmetry in this simple model is *equal* to another puzzling observable, the magnitude of the *quark orbital momentum* (QOM) inside the nucleon. While this model have not described these two observables quantitatively, its predictions are of the right magnitude. These works affirmed the general conclusion that the nucleon's 5-q sector is *not* small.

The issue of quark orbital motion inside the nucleon has been addressed from a different point of view in our recent work [15], where we studied the possible admixture of the second D-shell ($J^P = 2^+$) N^* resonances. However, we concluded that such admixture to the nucleon should remain small, even if the tensor forces are large.

D. Sigma-induced admixture of pentaquarks to baryons

As noted long ago, the scalar quantum number of a quark-antiquark pair requires spin $S = 1$ and orbital momentum $L = 1$ added together to $J^P = 0^+$. This mechanism is believed to be the leading reason for the observed quark orbital motion in the nucleons. It is this important physical observation supplemented by the strictures of chiral symmetry and Fermi statistics, that have motivated our current P-wave analysis of the pentaquark states.

Multiple studies of “baryon unquenching” (e.g., [10]) have described the admixed 5-q states as a “baryon plus a meson” state. These states were described as the $N + \pi$ or $\Delta + \pi$ states, with rather large pion momenta. While the search for baryon resonances are done this way, and while a loop diagram with a virtual pion does lead to some flavor asymmetry of the antiquark sea, we do not see why one should neglect, e.g., $N + \sigma$ channels (see below). More generally, why not look at all pertinent pentaquark states.

Let us start with the sigma-induced admixture operator studied in many works since the 1970s. Its form in momentum space is [10]

$$T = -3\gamma_0 \int d\vec{p}_4 d\vec{p}_5 \delta(\vec{p}_4 + \vec{p}_5) \times e^{-r_\sigma^2(\vec{p}_4 - \vec{p}_5)^2/6} Y_1(\vec{p}_4 - \vec{p}_5) \chi_{45}^{\text{color, flavor, spin}} \quad (26)$$

Note that $\vec{p}_1, \vec{p}_2, \vec{p}_3$ refer to the quark momenta in a baryon, and $\vec{p}_4, \vec{p}_5$ those in the scalar $q\bar{q}$ pair, or Jacobi coordinates $\vec{\alpha}, \vec{\beta}$, and $\vec{\delta}$, respectively. The parameter r_T is the radius of the region inside the baryon, where the vacuum $q\bar{q}$ pair appears, with $r_T \sim 0.3$ fm. This value is fully consistent with its instanton-induced origin, and the typical size of the instantons in instanton liquid model.

We will use the coordinate version of (26) with minor changes (e.g., the exponent reads $e^{-(\vec{X}_4 - \vec{X}_5)^2/r_\sigma^2/2}$). The color-flavor parts of the wave function are $\chi_{45}^{\text{color, flavor, spin}} \sim \delta^{f1f2} \epsilon_{c4, c5, c6}$. The coordinate and spin must

be convoluted to produce a total angular momentum $J_{45} = 0$ due to the scalar operator $\vec{S} \cdot \vec{L}$ in the operator T . The emission of a sigma is tagged to the 3-quark, as our pentaquark wave functions are all properly antisymmetrized. Those are *projected* on the product of the vacuum pair production operator T (26), and the baryon wave function B

$$A_{\text{mix}}^n = \langle P_n^* | T | B \rangle \quad (27)$$

Note that only pentaquarks with the first power of coordinate $\vec{\delta}$ in their wave function contribute, since the vacuum operator T is linear in the first angular harmonics of this vector. Thus, the orbital admixture in baryons arises from their mixture to pentaquarks.

Let us start with the radial integrals first, and determine how many “pentaquark shells” may contribute to this projection. Note that since we have appropriately antisymmetrized the pentaquarks, there is no need to use cumbersome and very restrictive oscillator basis to do it, as done in the literature. We define the radial part as

$$A_{\text{mix}}^n(\text{radial}) = \int \alpha^2 d\alpha \beta^2 d\beta \gamma^2 d\gamma \delta^2 d\delta \times R_n \left(\sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2} \right) \times \tilde{T}(\delta) B \left(\sqrt{\alpha^2 + \beta^2} \right) \quad (28)$$

where $\tilde{T}$ is the Fourier transform of the vacuum pair operator T , R_n are the respective solutions of the radial equation in 12-d space, and B is the ground state baryon solution in 6D. The results for the radial modes $n = 1..5$ normalized to the first one are as follows:

$$A_{\text{mix}, 1..5}(\text{radial}) = 1, 0.044, 0.203, 0.0340, 0.075 \quad (29)$$

suggesting that the sum should converge relatively well. Note that the overlaps for odd n —those with odd number of nodes in the wave functions—are smaller than those with even n . In practice, perhaps keeping $n = 1$ and $n = 3$ would be enough.

The energy splittings or gaps for $n = 1..5$ from the radial Schrodinger equation, are listed in (5). While they may appear to be small compared to the overall mass scale $M \sim 2$ GeV, the denominator of the first order perturbation

$$\sum \frac{A_{\text{mix}, n}}{Gap_n + M - M_n - m_\sigma}$$

is actually $M - M_n - m_\sigma \approx 0.5$ GeV rather than just M . Therefore Isgur’s “completeness approximation” in which the denominator dependence on n is neglected, does not hold well. These gaps help in the convergence of the sum.

With gaps included, (29) changes to

$$1., 0.040, 0.173, 0.027, 0.056$$

(We used a sigma meson of $M(\bar{q}q) \approx 0.5$ GeV, and for the baryon mass we used the spin-average $N - \Delta$ mass $M(B) \approx 1$ GeV. Note that for pentaquarks we also do not include the spin-dependent forces).

To complete the task, one has to calculate the overlaps of the coordinate-color-flavor-spin wave functions. Finally, the actual magnitude of the pentaquarks admixture is proportional to the overall parameter γ_0 in the vacuum production operator T . While its magnitude can be estimated using the instanton liquid parameters, below we choose to fit empirically to the observed nucleon “sea.”

E. Pion-induced admixture to nucleons

The motivation for adding a “pion-induced admixture” (also known as a “pion cloud”) has been detailed above, so we now proceed to its implementation.

One important point is change of the spatial parity in the transition operator. While the sigma-induced operator T_σ includes the spin-orbit ($\vec{S}\vec{L}$) combination with positive parity $P = +1$, the pion-induced one includes a different spin-momentum combination

$$O_{Sp} = (\vec{S}\vec{P}) \quad (30)$$

with negative parity $P = -1$. Note that the pion (unlike the sigma) is a Nambu-Goldstone mode, so its interaction with any hadron must vanish in the long wavelength $\vec{P} \rightarrow \vec{0}$ limit, as is the case here.

A standard way of implementing this, is to use the momentum representation of the wave functions for fixed $\vec{p}$, evaluate the pertinent overlap, and then Fourier back to coordinate space to make use of the pentaquark wave functions. This way is straightforward, but very cumbersome.

Here we follow a different approach, whereby $\hat{\vec{P}} = -i\partial/\partial\vec{x}$ in the coordinate representation. The operator (30) in the coordinate representation, is made first to act on the pentaquark side. For $L = 1$ it is a complicated combination of orbit-color-spin-isospin prefactors

$$F_{n,j} = \vec{a}_{n,j}\vec{\alpha} + \vec{b}_{n,j}\vec{\beta} + \vec{c}_{n,j}\vec{\gamma} + \vec{d}_{n,j}\vec{\delta} \quad (31)$$

linear in Jacobi coordinates $\vec{\alpha}...$ The indices n label the 24 pertinent pentaquark states, while the index j runs over the pertinent monoms, about $O(10^4)$ of them. (We note that acting by a derivative on a multicomponent “sparse array” of functions $F_{n,j}$ is straightforward in *Mathematica*.)

The derivative also acts on the radial WF $R(Y)$, a function of the hyperdistance Y . This leads to a combination of the form

$$\left(\vec{S}\frac{\partial F_{n,j}}{\partial\vec{\delta}}\right)R(Y) + \left(\vec{S}\frac{\vec{\delta}}{Y}\right)F_{n,j}R'(Y) \quad (32)$$

in which each contribution picks terms containing only the 4th Jacobi vector $\vec{\delta}$ in F . After averaging over the directions of $\vec{\delta}$, both terms are proportional to $d_{n,j}$ (which is a multidimensional tensor in color-spin-flavor indices, different for each of the states n). It is then multiplied by $\vec{S} = \vec{S}_4 + \vec{S}_5$ spin operator, and projected to the product of baryon-meson wave functions, over all color-spin-flavor indices.

The second term in (32) (with a derivative R' of radial pentaquark WFs) picks a coefficient $1/12$. Evaluating the integral over $\alpha, \beta, \gamma, \delta$ as in (28), we find that its ratio to the first term is only ≈ -0.054 , a relatively small correction.

We have evaluated the projected coefficients or “mixture amplitudes” C_n in the proton wave function shift $\Delta\psi_\pi$ in (25), for all $n = 1..24$ and $L = 1$ pentaquark states. The results for the $S = I = 1/2$ are listed in Table VII. Note that the pentaquark states should have the proton $S = I = 1/2$ quantum numbers, thus the baryon-meson isospin can be either $p\pi^0$ or $n\pi^+$.

Although the pion-projected admixture wave function $\Delta\psi_\pi$ in (25) is evaluated using the pentaquark P-shell basis, there are qualitative differences between the dependence on the labels n and j . The former is by no means a random set, as is shown in Fig. 6, while the states themselves seems to be quasirandom, see Fig. 4.

In Fig. 6 we show the distribution of the admixture amplitudes to the nucleon in $\Delta\psi_\pi$, from a set of pentaquark states. (These are the same $13S = 1/2, I = 1/2$ states with energies shown in Fig. 3.) What this histogram shows is far from random, with four states contributing more than the remaining 9. (Note that the admixture probability is the amplitude squared.)

While we have not investigated why this happens, one can perhaps still use this feature to simplify the model of $\bar{q}q$ admixture by keeping only few states.

In summary, while the pentaquark states may be rather chaotic by themselves (see Sec. III), the 5-q $\Delta\psi$ admixture

TABLE VII. Projection coefficients C_n for $13S = I = 1/2$ and $11S = 1/2, I = 3/2$ pentaquark states with $L = 1$, the same as Fig. 6.

C_n	1	2	3	4	5	6
	−.0998	.1230	−.0177	.2269	.1067	−.0829
7	8	9	10	11	12	13
−.2159	.1099	.0410	.0328	.0763	.0036	.0700
1'	2'	3'	4'	5'	6'	7'
−.00546	−.1373	−.2893	.3924	.2756	.2522	.039740
8'	9'	10'	11'			
.0145	.1070	−.0288	−.00546			

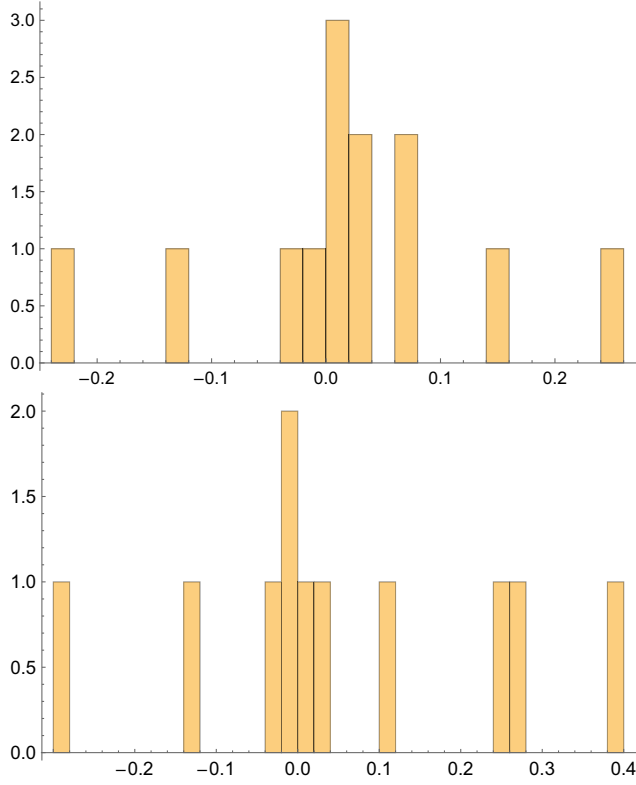


FIG. 6. Distribution of pion-induced $\Delta\psi_\pi$ coefficients of pentaquark states to a nucleon, from $13S = I = 1/2$ (upper) and $11S = 1/2, I = 3/2$ (lower) pentaquark states from the P-shell ($L = 1$).

to baryons is clearly *not* chaotic, with few states $\sim O(4)$ dominating all 24 states in the P-shell.

V. OBSERVABLES

By now, we have defined a *unique* WF in the 5-q sector admixed to the nucleon

$$\Delta\psi = \sum_n C_n |P_n\rangle \quad (33)$$

both for the sigma and pion-induced operators. Yet the total physical result is proportional to the triple couplings $g_{NP}^{\sigma,\pi}$ which are so far left undefined, but can be fitted to pertinent observables.

The most accurately measured one is the flavor asymmetry of the “quark sea.” Let us remind the reader how this is obtained. In Fig. 7 we show the data from E866 experiment [45] for $x(\bar{d} - \bar{u})$ parton distributions, with some parametrizations of the structure functions. From this we can deduce a difference in the number of antiquarks

$$\int_0^1 dx (\bar{d} - \bar{u}) = 0.118 \pm 0.012 \quad (34)$$

with most error stemming from small x .

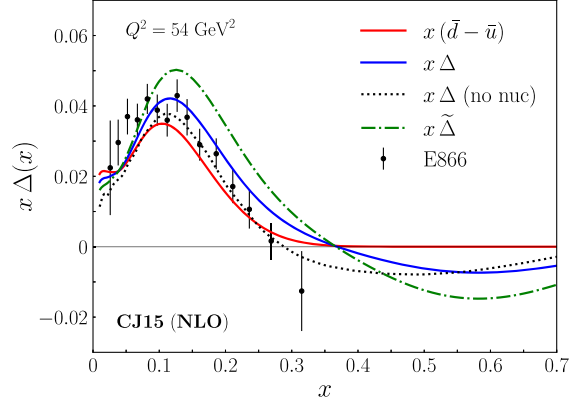


FIG. 7. Flavor asymmetry of the light antiquark sea, from [46]. Experimental points are from E866 experiment, and they are compared to parametrization of the combination $x(\bar{d} - \bar{u})$ shown by the red curve. (Ignore other curves.)

(Note that the total momentum carried by this admixture, given by the integral under the curve in this figure,

$$\int_0^1 dx x(\bar{d} - \bar{u}) \approx 0.006 \quad (35)$$

may appear to be more accurate. But, as seen in Fig. 7, it is unreliable since this integral can be changed by $x > 1/3$ region in which no actual measurements were made.)

Using our results for the weighted 5q admixture $\Delta\psi$ wave function, we can calculate the antiquark asymmetry

$$\frac{\langle \Delta\psi \tau_q^3 \Delta\psi \rangle}{\langle \Delta\psi \Delta\psi \rangle} \approx 0.335 \quad (36)$$

If the probability of the 5q configuration in the nucleon is P_{5q} , then the weighted isospin asymmetry is

$$\langle \bar{d} - \bar{u} \rangle = 0.335 \frac{P_{5q}}{1 + P_{5q}} \quad (37)$$

Comparing this to the experimental value (34), we find $P_{5q} \sim 0.5$.

The contribution to the axial charge of the nucleon g_A is related to the operator $\langle \sum_{j=1..5} \sigma_3(j) \tau_3(j) \rangle$, for which the mean value over $\Delta\psi$ is 0.249 in our analysis. A comparison to the accurately known experimental value

$$g_A = 1.267 = \frac{5/3 + 0.249 * P_{5q}}{1 + P_{5q}} \quad (38)$$

yields $P_{5q} \approx 0.4$.

As discussed in the Introduction, the second major observable is the orbital motion inside the nucleon. The 5q part carry $L = 1$ by construction, and the 3q part $L = 0$. The spin sum rule requirement yields also $P_{5q} \approx 0.4$.

Finally, there are of course the *magnetic moments* of the proton, neutron and hyperons, as well as proton and neutron form-factors measured both in elastic channels and in transitions to particular resonances. We expect to discuss those in future works, in which the pentaquark theory and its admixture will be taken to the light front formulation.

VI. SUMMARY AND DISCUSSION

In our study of the light quark pentaquarks we used novel methods in the construction of the representations of the permutation groups, in this case of *four* quarks S_4 . Using a “brute force” approach based on tensor product of permutation generators in orbital-color-spin-flavor spaces, we were able to construct fully anti-symmetric wave functions of the $L = 0$ and $L = 1$ pentaquarks, explicitly in the monom space of dimension $d_{\text{monoms}} = 3^6 \times 2^5 \times 2^5 = 746496$. The actual number of nonzero terms in the $L = 1$ shell states is of order $O(10^4)$. Note that the pertinent amplitudes are in this case not numbers but functions of 12 coordinates. Still, with Wolfram *Mathematica*, one can work with those straightforwardly, e.g., using various spin and differential operators in their traditional notations. For some $L = 0$ pentaquark states we also were able to obtain wave functions analytically, by more standard Young tableaux representations. Needless to say, we have checked that results of both approaches do agree.

We presented evidences that the P-shell $L = 1$ pentaquark states are already displaying some evidences for the “quantum chaos” phenomenon, namely a near-Gaussian coefficient distribution. It is amusing that in the hadronic world we find it first for the pentaquarks with a quark core of 4 and 12 Jacobi coordinates, the same number as for a Cerium atom with four valence electrons [30].

In the Introduction, and in particular in Fig. 1, we have illustrated our main goal, “to bridge” hadronic spectroscopy with partonic observables, by constructing an intermediate “unquenched” description of hadrons at a resolution $Q^2 \sim 1\text{GeV}^2$. It is supposed to complement the classic description of light baryons as qqq states by an additional $\bar{q}q$ pair, described as admixing with pentaquarks.

The distinction between charmonia $\bar{c}c$ and tetraquarks $\bar{c}c\bar{q}q$, has been developed in the last decade, with at least 4 tetraquarks in which $\bar{q}q = \pi$ and other 4 in which $\bar{q}q = \sigma$.

Multiple hadronic transitions between charmonia and tetraquarks are observed, providing pertinent information about their admixtures.

The problem we studied in this paper, the mixing between light quark baryons and pentaquarks, is similar in spirit. Admittedly, it is much more complicated and much less developed. Yet we were able to surpass the technical difficulties, and derive the *unique* 5-q part of the nucleon wave function.

Our use of the pentaquark states is done “in bulk,” meaning that the wave functions for these states are used as a (reasonably complete) basis for admixture calculation. We argued that the admixture operators T should include both of the σ and the $\pi\bar{q}q$ operators, as it is the case for $\bar{c}c\bar{q}q$ tetraquarks.

The information about 5-q component of the baryons come mostly from features of the “antiquark sea,” experimentally available in the light front partonic phenomenology. In Sec. V we have shown that the isospin asymmetry of the sea, the magnitude of g_A and that of the orbital angular momentum in the nucleon, all indicate that the probability of 5q Fock component is large, $P_{5q} \sim 0.4$. One may anticipate even stronger mixing of baryon resonances, including those which so far are associated with the qqq sector alone. We hope to address these issues in future publications.

ACKNOWLEDGMENTS

This work is supported by the Office of Science, U.S. Department of Energy under Contract No. DE-FG-88ER40388. It is also supported in part within the framework of the Quark-Gluon Tomography (QGT) Topical Collaboration, under Contract No. DE-SC0023646.

DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: PENTAQUARK VARIABLES

The Jacobi coordinates have been defined in (1) in terms of four 3D vectors $\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}$. For each vector, we use the usual two polar angles $\theta_i, \phi_i, i = 1..4$. The remaining variables are the hyperdistance Y and 3 angles in the 4D space, called $\theta_{2\chi}, \theta_{1\chi}, \phi_\chi$.

$$\vec{\alpha} = Y * \sin(\theta_{2\chi}) * \sin(\theta_{1\chi}) * \cos(\phi_\chi) \times \{\sin(\theta_\alpha) \cos(\phi_\alpha), \sin(\theta_\alpha) \sin(\phi_\alpha), \cos(\theta_\alpha)\} \quad (\text{A1})$$

$$\vec{\beta} = Y * \sin(\theta_{2\chi}) * \sin(\theta_{1\chi}) * \sin(\phi_\chi) \times \{\sin(\theta_\beta) \cos(\phi_\beta), \sin(\theta_\beta) \sin(\phi_\beta), \cos(\theta_\beta)\}$$

$$\vec{\gamma} = Y * \sin(\theta_{2\chi}) * \cos(\theta_{1\chi}) \times \{\sin(\theta_\gamma) \cos(\phi_\gamma), \sin(\theta_\gamma) \sin(\phi_\gamma), \cos(\theta_\gamma)\}$$

$$\vec{\delta} = Y * \cos(\theta_{2\chi}) \times \{\sin(\theta_\delta) \cos(\phi_\delta), \sin(\theta_\delta) \sin(\phi_\delta), \cos(\theta_\delta)\} \quad (\text{A2})$$

The notations for these 12 variables are

$$y = \{Y, \theta_\alpha, \phi_\alpha, \theta_\beta, \phi_\beta, \theta_\gamma, \phi_\gamma, \theta_\delta, \phi_\delta, \theta_{2\chi}, \theta_{1\chi}, \phi_\chi\} \quad (\text{A3})$$

The metric tensor is diagonal with its elements (coefficients of $dy_i^2, i = 1..12$) given by

$$g_{ii} = \{1, Y^2 \cos(\phi_\chi)^2, Y^2 \cos(\phi_\chi)^2 \sin(\theta_{1\chi})^2 \sin(\theta_{2\chi})^2 \sin(\theta_\alpha)^2, \\ Y^2 \sin(\theta_{1\chi})^2 \sin(\theta_{2\chi})^2 \sin(\phi_\chi)^2, Y^2 \sin(\theta_{1\chi})^2 \sin(\theta_{2\chi})^2 \sin(\theta_\beta)^2 \sin(\phi_\chi)^2, \\ Y^2 \cos(\theta_{1\chi})^2 \sin(\theta_{2\chi})^2, Y^2 \cos(\theta_{1\chi})^2 \sin(\theta_{2\chi})^2 \sin(\theta_\gamma)^2, Y^2 \cos(\theta_{2\chi})^2, \\ Y^2 \cos(\theta_{2\chi})^2 \sin(\theta_\delta)^2, Y^2, Y^2 \sin(\theta_{2\chi})^2, Y^2 \sin(\theta_{1\chi})^2 \sin(\theta_{2\chi})^2\} \quad (\text{A4})$$

and the volume element

$$\sqrt{\det(g)} = Y^{11} \cos(\theta_{1\chi})^2 \cos(\theta_{2\chi})^2 \cos(\phi_\chi)^2 \sin(\theta_{1\chi})^4 \sin(\theta_{2\chi})^7 \sin(\theta_\alpha) \sin(\theta_\beta) \sin(\theta_\gamma) \sin(\theta_\delta) \sin(\phi_\chi)^2 \quad (\text{A5})$$

The Laplace-Beltrami operator is

$$-\bar{\nabla}^2 = \frac{\partial}{\partial Y^2} + \frac{11}{Y} \frac{\partial}{\partial Y} + \frac{1}{Y^2} \left[2 \cot(\phi_\chi) \csc(\theta_{1\chi})^2 \csc(\theta_{2\chi})^2 \frac{\partial}{\partial \phi_\chi} \right. \\ - 2 \csc(\theta_{1\chi})^2 \csc(\theta_{2\chi})^2 \tan(\phi_\chi) \frac{\partial}{\partial \phi_\chi} + \csc(\theta_{1\chi})^2 \csc(\theta_{2\chi})^2 \frac{\partial^2}{\partial \phi_\chi^2}, \\ + 4 \cot(\theta_{1\chi}) \csc(\theta_{2\chi})^2 \frac{\partial}{\partial \theta_{1\chi}} - 2 \csc(\theta_{2\chi})^2 \tan(\theta_{1\chi}) \frac{\partial}{\partial \theta_{1\chi}} \\ + (\csc(\theta_{2\chi})^2 \frac{\partial^2}{\partial \theta_{1\chi}^2} + (7 \cot(\theta_{2\chi}) \frac{\partial}{\partial \theta_{2\chi}} - (2 \tan(\theta_{2\chi}) \frac{\partial}{\partial \theta_{2\chi}} + \frac{\partial^2}{\partial \theta_{2\chi}^2} \\ + \csc(\theta_\delta)^2 \sec(\theta_{2\chi})^2 \frac{\partial^2}{\partial \phi_\gamma^2} + \cot(\theta_\delta) \sec(\theta_{2\chi})^2 \frac{\partial}{\partial \theta_\gamma} \\ + (\sec(\theta_{2\chi})^2 \frac{\partial^2}{\partial \theta_\gamma^2} + \csc(\theta_{2\chi})^2 \csc(\theta_\gamma)^2 \sec(\theta_{1\chi})^2 \frac{\partial}{\partial \phi_\gamma} \\ + \cot(\theta_\gamma) \csc(\theta_{2\chi})^2 \sec(\theta_{1\chi})^2 \frac{\partial}{\partial \theta_\gamma} + \csc(\theta_{2\chi})^2 \sec(\theta_{1\chi})^2 \frac{\partial^2}{\partial \theta_\gamma^2} \\ + \csc(\theta_{1\chi})^2 \csc(\theta_{2\chi})^2 \csc(\theta_\beta)^2 \csc(\phi_\chi)^2 \frac{\partial^2}{\partial \phi_\beta^2} + \cot(\theta_\beta) \csc(\theta_{1\chi})^2 \csc(\theta_{2\chi})^2 \csc(\phi_\chi)^2 \frac{\partial}{\partial \theta_\beta} \\ + \csc(\theta_{1\chi})^2 \csc(\theta_{2\chi})^2 \csc(\phi_\chi)^2 \frac{\partial^2}{\partial \theta_\beta^2} + \csc(\theta_{1\chi})^2 \csc(\theta_{2\chi})^2 \csc(\theta_\alpha)^2 \sec(\phi_\chi)^2 \frac{\partial^2}{\partial \phi_\alpha^2} \\ \left. + \cot(\theta_\alpha) \sec(\phi_\chi)^2 \frac{\partial}{\partial \theta_\alpha} + \sec(\phi_\chi)^2 \frac{\partial^2}{\partial \theta_\alpha^2} \right] \quad (\text{A6})$$

is proportional to kinetic energy, provided all quarks have the same masses.

For spherically symmetric states $\psi = R[Y]$, only two terms with Y derivative remain. The standard radial wave function redefinition

$$R \rightarrow \phi(Y)/Y^{11/2}$$

eliminates the first derivative, but leads to a quasicentrifugal term

$$-Laplacian = -\phi(Y)'' + \frac{99}{4} \frac{\phi(Y)}{Y^2} \quad (\text{A7})$$

For the $L = 1$ states with the first power of coordinates such as $\vec{\alpha}$, the (angle averaged) expression becomes

$$-Laplacian = -\phi(Y)'' + \left(\frac{99}{4} + 10\right) \frac{\phi(Y)}{Y^2} \quad (A8) \quad \chi_2 = \pi/2 - \epsilon. \text{ We choose } \epsilon = 0.02 \text{ and obtain}$$

The interaction between quarks are functions of interquark distances, and thus not spherically symmetric in 12 dimensions. As customary, we evaluate them via their angular averaging. As a typical case let us take the quark pair 1-2, and note that the distance between them is $r_{12} = \sqrt{2}\alpha$. We recall that in the 4D space of lengths $\alpha, \beta, \gamma, \delta$ we have introduced three angles $\chi_2, \chi_1, \chi_\phi$, and used them to carry the averaging the of Cornell-like potential as follows

$$\langle r_{12} \rangle = \frac{\int \sqrt{2} Y \cos(\chi_2) \cos(\chi_2)^2 d\chi_2}{\int \cos(\chi_2)^2 d\chi_2} = \frac{4\sqrt{2}}{3\pi} Y \approx 0.60Y \quad (A9)$$

The Coulomb term diverges logarithmically and needs regularization, so the corresponding integration is done till

$$\left\langle \frac{1}{r_{12}} \right\rangle \approx \frac{3.3}{Y} \quad (A10)$$

APPENDIX B: METHOD FOR FINDING GENERAL PENTAQUARK STATES

The pentaquark wave functions used in this paper were generated using the method described in [24]. However, a modification must be made to the treatment of pentaquarks described there. Instead of describing the four quark core and adding on the antiquark, it is possible to jump straight to considering all five particles as long as the *symmetry group* is still S_4 . The modified form of the method's "PermuteInBasis" function, now with input n for number of indecies and input m for the order of the symmetry group takes the form

```
(*Basic permutation generators*)
Permute[n_, m_] :=
  PermutationMatrix[#, n] & /@ GroupGenerators[SymmetricGroup[m]]

(*Permute the first m digits in base dim with n total indices*)
ColumnSwap[n_, m_, dim_, i_, x_] :=
  FromDigits[Permute[n, m][[i]]. IntegerDigits[x, dim, n], dim]
(* needed after flattening all tensors
and i = 1 is 12 while i = 2 is cycle generator *)
(*Rank m permutation matrix acting on C^(dim^n)*)
BigPermute[n_, m_, dim_, i_] :=
  SparseArray[
    Table[{x+1, ColumnSwap[n, m, dim, i, x] + 1} -> 1, {x, 0,
      dim^n-1}]]

(*Gives the representation of both permutation generators in the basis\
spanned by index rearrangements of the tensor T. \
Now just the first m indices of T*)
PermuteInBasis[T_, m_] := Module[{n, nm, dim, tensorList, basis},
  n = TensorRank[T];
  (*For color nm should be m,
  for spin it should be n-this has to do with which symmetry group's r\
epresentation the basis is in*)

  If[n-m == 1, nm = n, nm = m];
  dim = Length[T];
  tensorList =
  Flatten /@ (Transpose[T, PermutationList[#, n]] & /@
    GroupElements[SymmetricGroup[nm]]);
  basis = DeleteCases[Orthogonalize[tensorList], {0 ..}];
  {SparseArray[basis. BigPermute[n, m, dim, 1]. Transpose[basis]],
  SparseArray[basis. BigPermute[n, m, dim, 2]. Transpose[basis]]}
]
```

and the important "GoodBasis" function that gives the basis for the representation of S_m in monoms is

```
(*Gives just the basis associated with T instead*)
GoodBasis[T_, m_] := Module[{n, nm, dim, tensorList, basis},
  n = TensorRank[T];
```

```

(*For spin just goodbasis m doesn't actually matter*)
If[n-m == 1, nm = n, nm = m];
tensorList=
  Flatten /@ (Transpose[T, PermutationList[#, n]] & /@
    GroupElements[SymmetricGroup[nm]]);
SparseArray[DeleteCases[Orthogonalize[tensorList], {0 ..}]]
]

```

The input T to these functions is one Young Tableau with the correct symmetry; for color the natural choice is

```
pentaColor=TensorProduct[LeviCivita[3],LeviCivita[3]]
```

and for each possible total and directional spin/isospin

```

u = {1, 0}; d = {0, 1};
pentaspin12 = {TensorProduct[d, LeviCivitaTensor[2],
  LeviCivitaTensor[2]],
  TensorProduct[u, LeviCivitaTensor[2], LeviCivitaTensor[2]]};
pentaspin32 = {TensorProduct[TensorProduct[d, d, d],
  LeviCivitaTensor[2]],
  TensorProduct[Symmetrize[TensorProduct[d, d, u]],
  LeviCivitaTensor[2]],
  TensorProduct[Symmetrize[TensorProduct[d, u, u]],
  LeviCivitaTensor[2]],
  TensorProduct[Symmetrize[TensorProduct[u, u, u]],
  LeviCivitaTensor[2]]};
pentaspin52 = {TensorProduct[d, d, d, d, d],
  Symmetrize[TensorProduct[d, d, d, d, u]],
  Symmetrize[TensorProduct[d, d, d, u, u]],
  Symmetrize[TensorProduct[d, d, u, u, u]],
  Symmetrize[TensorProduct[d, u, u, u, u]],
  TensorProduct[u, u, u, u, u]};

```

Because the color wave function for a pentaquark has *six* indices (composed of combinations of $\epsilon_{ijk}\epsilon_{lmn}$), for color n is 6 while for spin and isospin it is 5. In both cases however, m is still 4. This doesn't cause any issues, because the matrices found with `PermuteInBasis` all belong to representations the same group S_m and therefore can always be Kronecker multiplied together into a new representation.

The coordinate parts of the wave functions can be treated essentially the same as [24], except the permutations matrices acting on them are now n -dimensional matrices of m -dimensional permutations:

```

(*Transforms to modified jacobi coordinates*)
Jacobi[n_] :=
  Inverse[DiagonalMatrix[
    Table[Sqrt[i + 1]/Sqrt[i], {i, 1, n}]]]. (Table[
    Join[Table[1/i, i], Table[0, n-i]], i, 1, n] +
    DiagonalMatrix[Table[-1, n-1], 1])
(*Permutation matrices in jacobi basis*)
JacobiPermute[n_, m_, i_] :=
  Transpose@
    SparseArray[(Jacobi[n].Permutates[n, m][[i]].Inverse[Jacobi[n]])[[
      1 ;; n-1, 1 ;; n-1]]]

```

Clearly an explicit example is in order here. In Table II we have seen that the simplest states are for maximal spin $S = 5/2$ (to be discussed in the next subsection) while the *most complex* case is for $S = I = 1/2$, with the largest dimension of the “good basis” $3 \times 5 \times 5 = 75$ for color, spin and isospin, respectively. These dimensions can readily be understood in terms of Young tableaux, for example 3 for color corresponds to three of them shown in (C2).

Two generators of the permutation group S_4 are: (1) G_1 is the interchange of particles $1 \leftrightarrow 2$; (2) G_2 is a cycling one, moving all indices by one. For the most complex example of $L = 0$ pentaquarks at hand, with $S = 1/2, I = 1/2, S_z = -1/2$, these two permutation generators take the form

$$\begin{aligned}
G_1 &= \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \otimes \begin{vmatrix} 1/2 & 0 & \sqrt{3}/2 & 0 & 0 \\ 0 & 1/2 & 0 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 0 & -1/2 & 0 & 0 \\ 0 & \sqrt{3}/2 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \otimes \begin{vmatrix} 1/2 & 0 & \sqrt{3}/2 & 0 & 0 \\ 0 & 1/2 & 0 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 0 & -1/2 & 0 & 0 \\ 0 & \sqrt{3}/2 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \\
G_2 &= \begin{vmatrix} 1/3 & 2\sqrt{2}/3 & 0 \\ -\sqrt{2}/3 & 1/6 & \sqrt{3}/2 \\ \sqrt{2}/3 & -1/2\sqrt{3} & 1/2 \end{vmatrix} \otimes \begin{vmatrix} -1/4 & -\sqrt{3}/4 & \sqrt{3}/4 & 3/4 & 0 \\ \sqrt{3}/4 & -1/4 & 1/4 & -1/4\sqrt{3} & \sqrt{2}/3 \\ -\sqrt{3}/4 & -3/4 & -1/4 & -\sqrt{3}/4 & 0 \\ 3/4 & -\sqrt{3}/4 & -1/4\sqrt{3} & 1/12 & -\sqrt{2}/3 \\ 0 & 0 & \sqrt{2}/3 & -\sqrt{2}/3 & -1/3 \end{vmatrix} \\
&\otimes \begin{vmatrix} -1/4 & -\sqrt{3}/4 & \sqrt{3}/4 & 3/4 & 0 \\ \sqrt{3}/4 & -1/4 & 1/4 & -1/4\sqrt{3} & \sqrt{2}/3 \\ -\sqrt{3}/4 & -3/4 & -1/4 & -\sqrt{3}/4 & 0 \\ 3/4 & -\sqrt{3}/4 & -1/4\sqrt{3} & 1/12 & -\sqrt{2}/3 \\ 0 & 0 & \sqrt{2}/3 & -\sqrt{2}/3 & -1/3 \end{vmatrix}
\end{aligned}$$

One can then proceed to do the tensor product, after which simultaneously diagonalize these two generators (e.g., with the `AntiSymmetricStates` method), and find 3 common antisymmetric vectors for both 75-dimensional matrices. Those are three pentaquark states. Returning from “good basis” notations back to large monom space, one have their wave functions explicitly.

APPENDIX C: THE PENTAQUARK STATES BUILT VIA EXPLICIT YOUNG TABLEAUX

For readers who found the previous subsection to be too quick, let us show how one builds the states using Young tableaux explicitly. The space-color-spin-flavor representations of the pentaquark states $[qqqq\bar{q}]$ with light u, d flavors are involved, owing to the antisymmetrization of the core $[qqqq]$. In this Appendix, we briefly detail their

construction using representations of the permutation group of S_4 in addition to the standard unitary representations. The construction makes extensive use of the character representations of S_4 , to extract the pertinent representations. It is a natural extension of the construction for the select set of S_3 representations introduced by Isgur and Karl [47,48], and extended later to pentaquarks in [16,49–52]. The results are in overall agreement with the monom construction detailed above.

1. Color and flavor representations

The color singlet wave function of $q^4\bar{q}$ follows from the combination of the color triplet of q^4 with the color antitriplet of $\bar{q}$. Since the total wave function is antisymmetric in color, the corresponding Young tableau follows from the product

$$1_C = \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \equiv C[222] \right) = \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \end{array} \equiv C[211] \right) \otimes \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \equiv C[11] \right) \quad (C1)$$

with the quark-antiquark color identifications for the Young tableaux $C[222] = [q^4\bar{q}]_C$, $C[211] = [q^4]_C$ and $C[11] = [\bar{q}]_C$. The color representation $C[211]$ is 3-degenerate mixed representation

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array}
\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array}
\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array}
\quad (C2)$$

To guarantee the antisymmetry of the full color-space-spin-flavor wave function, the space-spin-flavor wave function $[q^4]_{[LS]}$ for the identical quarks is valued in the conjugate representation of its color counterpart $[q^4]_C$,

$$[q^4]_{LSF} = \overline{[q^4]_C} = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \equiv LSF[31] \quad (C3)$$

is 3-degenerate as well

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}
\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}
\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}
\quad (C4)$$

2. Jacobi-like representations of S_4

Before proceeding to the representations of S_4 we first recall the Jacobi coordinates for 5 particles, which will be used to generalize the Isgur-Karl representations of the 3 particles wave functions to 5 particles. Another form of Jacobi coordinates we use is

$$\begin{aligned}
\vec{\alpha} &= \frac{1}{\sqrt{2}} \vec{r}_{12} \\
\vec{\beta} &= \frac{1}{\sqrt{6}} (\vec{r}_{13} + \vec{r}_{23}) \\
\vec{\gamma} &= \frac{1}{\sqrt{12}} (\vec{r}_{14} + \vec{r}_{24} + \vec{r}_{34}) \\
\vec{\delta} &= \frac{1}{\sqrt{20}} (\vec{r}_{15} + \vec{r}_{25} + \vec{r}_{35} + \vec{r}_{45})
\end{aligned} \quad (C5)$$

with $\vec{r}_{1,2,3,4}$ referring to the coordinates of the 4 quarks, and $\vec{r}_5$ to the coordinate of the fifth antiquark. The CM coordinate is given by

$$\vec{R} = \frac{1}{5} (\vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \vec{r}_4 + \vec{r}_5) \quad (C6)$$

3. Color representations

The first 2 Young tableaux in (C2) are readily identified with the two irreducible Jacobi-like $\rho = \alpha$, $\lambda = \beta$ representations of S_3 which are parts of the cycles in S_4 , and which we have previously used [24]. More specifically,

$$C[211]_\alpha = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} \quad (C7)$$

$$C[211]_\beta = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \quad (C8)$$

$$C[211]_\gamma = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \quad (C9)$$

Note that the removal of box 4 in the first two tableaux, yields the mixed symmetry representation for three identical quarks $[q^3]_M$ which is 2-degenerate, as per the Isgur-Karl representation [24]. The last Young tableau is commensurate with the Jacobi-like γ , as the new irreducible representation of S_4 .

To make more explicit the color representations $[211]_{\xi=\alpha,\beta,\gamma}$, we use the standard $C = R, G, B$ and $\bar{C} = \bar{R}, \bar{G}, \bar{B}$ for the quark and antiquark colors, respectively. From the Young tableau color representation, it follows that the singlet pentaquark state follows from pairing the valence antiquark color with the detached Young box, to neutralize the color. More explicitly, we have

$$C[211]_\xi = \frac{1}{\sqrt{3}} (C[221]_\xi(R)\bar{R} + C[221]_\xi(G)\bar{G} + C[221]_\xi(B)\bar{B}) \quad (C10)$$

with

$$C[211]_\alpha(C) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} \begin{array}{|c|c|} \hline R & C \\ \hline G & \\ \hline B & \\ \hline \end{array}
\quad C[211]_\beta(C) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \begin{array}{|c|c|} \hline R & C \\ \hline G & \\ \hline B & \\ \hline \end{array}
\quad C[211]_\gamma(C) = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 & \\ \hline 2 & \\ \hline \end{array} \begin{array}{|c|c|} \hline R & C \\ \hline G & \\ \hline B & \\ \hline \end{array} \quad (C11)$$

The explicit forms of the Jacobi color states, are obtained by using the pertinent projection operators of the permutation group of S_4 , on each of the $[211](R, G, B)$ representations [53]. With this in mind, we have explicitly for $C = R$

$$\begin{aligned}
C[211]_\alpha(R) &= \frac{1}{\sqrt{48}} (3|RGRB\rangle - 3|GRRB\rangle - 3|RBGR\rangle + 3|BRRG\rangle - |RGBR\rangle \\
&\quad + |GRBR\rangle + |RBGR\rangle - |BRGR\rangle + 2|GBRR\rangle - 2|BGRR\rangle) \\
C[211]_\beta(R) &= \frac{1}{\sqrt{16}} (2|RRGB\rangle - 2|RRBG\rangle - |RGRB\rangle - |GRRB\rangle + |RBGR\rangle \\
&\quad + |BRRG\rangle + |RGBR\rangle + |GRBR\rangle - |RBGR\rangle - |BRGR\rangle) \\
C[211]_\gamma(R) &= \frac{1}{\sqrt{6}} (|BRGR\rangle + |RGBR\rangle + |GBRR\rangle - |RBGR\rangle - |GRBR\rangle - |BGRR\rangle)
\end{aligned} \tag{C12}$$

The other color representations with $C = G, B$ in (C10), are obtained similarly, with the results

$$\begin{aligned}
C[211]_\alpha(G) &= \frac{1}{\sqrt{48}} (3|RGGB\rangle - 3|GRGB\rangle - 3|BGGR\rangle + 3|GBGR\rangle - |RGBG\rangle \\
&\quad + |GRBG\rangle - |GBRG\rangle + |BGRG\rangle + 2|BRGG\rangle - 2|RBGG\rangle) \\
C[211]_\beta(G) &= \frac{1}{\sqrt{16}} (2|GGBR\rangle - 2|GGRB\rangle + |RGGB\rangle + |GRGB\rangle - |RGBG\rangle \\
&\quad - |GRBG\rangle + |GBRG\rangle + |BGRG\rangle - |GBGR\rangle - |BGGR\rangle) \\
C[211]_\gamma(G) &= \frac{1}{\sqrt{6}} (|RGBG\rangle - |GRBG\rangle - |RBGG\rangle + |BRGG\rangle + |GBRG\rangle - |BGRG\rangle)
\end{aligned} \tag{C13}$$

and

$$\begin{aligned}
C[211]_\alpha(B) &= \frac{1}{\sqrt{48}} (3|BRBG\rangle - 3|BBBG\rangle + 3|GBBR\rangle - 3|BGBR\rangle + |RBGB\rangle \\
&\quad - |BRGB\rangle - |GBRB\rangle + |BGRB\rangle + 2|RGBB\rangle - 2|GRBB\rangle) \\
C[211]_\beta(B) &= \frac{1}{\sqrt{16}} (2|BBRG\rangle - 2|BBGR\rangle + |RBGB\rangle + |BRGB\rangle - |GBRB\rangle \\
&\quad - |BGRB\rangle - |BBBG\rangle - |BRBG\rangle + |GBBR\rangle + |BGBR\rangle) \\
C[211]_\gamma(B) &= \frac{1}{\sqrt{6}} (|RGBB\rangle - |GRBB\rangle - |RBGB\rangle + |BRGB\rangle + |GBRB\rangle - |BGRB\rangle)
\end{aligned} \tag{C14}$$

It is readily checked that the colored core states (C12)–(C14) are orthonormalized for each color sector R, G , and B . We note that (C12) is in agreement with [51,53].

4. Spin, flavor: $[q^4]_{S,F}$

The spin flavor configurations of $[q^4]$ follow from the standard $SU(2)$ Young tableaux,

$$[q^4]_S = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \equiv S[4] \oplus S[31] \oplus S[22] \tag{C15}$$

and similarly for $[q^4]_F$ with $S \rightarrow F$,

a. Maximum weight representations for $[q^4]_{S,F}$

We now construct explicitly the 4 quark states with maximum spin that appear in (C15) by using the standard procedures for $SU(2)_S$ Young tableaux. For the completely symmetric tableau, the state is unique

$$S[4]^{22} = \begin{array}{|c|c|c|c|} \hline \uparrow & \uparrow & \uparrow & \uparrow \\ \hline \end{array} = \uparrow\uparrow\uparrow\uparrow \tag{C16}$$

For the mixed tableau,

$$S[31]^{11} = \begin{array}{|c|c|c|} \hline \uparrow & \uparrow & \uparrow \\ \hline \downarrow & & \\ \hline \end{array} \rightarrow S[31]_{\alpha,\beta,\gamma}^{11} \quad (C17)$$

there are 3 properly symmetrized states. These states are readily obtained from the Jacobi-like ρ , λ Isgur-Karl spin states discussed in [24], by tagging an additional spin state. In the α , β , and γ labeling used here, they are

$$\begin{aligned} S[31]_{\beta}^{11} &= \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow\uparrow \\ S[31]_{\alpha}^{11} &= \frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)\uparrow \\ S[31]_{\gamma}^{11} &= \frac{1}{\sqrt{12}}(3\uparrow\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow\uparrow - \uparrow\downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow\uparrow) \end{aligned} \quad (C18)$$

The explicit construction of $S[22]^{00}$ follows a similar reasoning, and

$$S[22]^{00} = \begin{array}{|c|c|} \hline \uparrow & \uparrow \\ \hline \downarrow & \downarrow \\ \hline \end{array} \rightarrow S[22]_{\alpha,\beta}^{00} \quad (C19)$$

with 2 symmetrized states. Note that the maximum representations for the core $[q^4]_F$ follow a similar reasoning for $SU(2)_F$ with u , d light flavors.

b. Maximum weight representations for $([q^4] \otimes \bar{q})_{S,F}$

The maximum weight configuration for the spin in pentastates are obtained by recoupling through pertinent Clebsch-Gordon coefficients,

$$S_5[4] \left[\begin{smallmatrix} 5 & 5 \\ 2 & 2 \end{smallmatrix} \right] = |\uparrow\uparrow\uparrow\uparrow\bar{\uparrow}\rangle \quad (C20)$$

$$S_5[4] \left[\begin{smallmatrix} 3 & 3 \\ 2 & 2 \end{smallmatrix} \right] = \sqrt{\frac{4}{5}}|\uparrow\uparrow\uparrow\uparrow\bar{\downarrow}\rangle - \sqrt{\frac{1}{20}}(|\uparrow\uparrow\uparrow\downarrow\bar{\uparrow}\rangle + |\uparrow\uparrow\downarrow\uparrow\bar{\uparrow}\rangle + |\uparrow\downarrow\uparrow\uparrow\bar{\uparrow}\rangle + |\downarrow\uparrow\uparrow\uparrow\bar{\uparrow}\rangle) \quad (C21)$$

and

$$\begin{aligned} S_5[22]_{\alpha} \left[\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix} \right] &= \sqrt{\frac{1}{4}}(|\uparrow\downarrow\uparrow\downarrow\bar{\uparrow}\rangle - |\downarrow\uparrow\uparrow\downarrow\bar{\uparrow}\rangle - |\uparrow\downarrow\downarrow\uparrow\bar{\uparrow}\rangle + |\downarrow\uparrow\downarrow\uparrow\bar{\uparrow}\rangle) \\ S_5[22]_{\beta} \left[\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix} \right] &= \sqrt{\frac{1}{12}}(2|\uparrow\uparrow\downarrow\downarrow\bar{\uparrow}\rangle - |\uparrow\downarrow\uparrow\downarrow\bar{\uparrow}\rangle - |\downarrow\uparrow\uparrow\downarrow\bar{\uparrow}\rangle - |\uparrow\downarrow\downarrow\uparrow\bar{\uparrow}\rangle - |\downarrow\uparrow\downarrow\uparrow\bar{\uparrow}\rangle + 2|\downarrow\downarrow\uparrow\uparrow\bar{\uparrow}\rangle) \end{aligned} \quad (C22)$$

and

$$\begin{aligned} S_5[31]_{\alpha} \left[\begin{smallmatrix} 3 & 3 \\ 2 & 2 \end{smallmatrix} \right] &= \sqrt{\frac{1}{2}}(|\uparrow\downarrow\uparrow\uparrow\bar{\uparrow}\rangle - |\downarrow\uparrow\uparrow\uparrow\bar{\uparrow}\rangle) \\ S_5[31]_{\beta} \left[\begin{smallmatrix} 3 & 3 \\ 2 & 2 \end{smallmatrix} \right] &= \sqrt{\frac{1}{6}}(2|\uparrow\uparrow\downarrow\uparrow\bar{\uparrow}\rangle - |\uparrow\downarrow\uparrow\uparrow\bar{\uparrow}\rangle - |\downarrow\uparrow\uparrow\uparrow\bar{\uparrow}\rangle) \\ S_5[31]_{\gamma} \left[\begin{smallmatrix} 3 & 3 \\ 2 & 2 \end{smallmatrix} \right] &= \sqrt{\frac{1}{12}}(3|\uparrow\uparrow\uparrow\downarrow\bar{\uparrow}\rangle - |\uparrow\uparrow\downarrow\uparrow\bar{\uparrow}\rangle - |\uparrow\downarrow\uparrow\uparrow\bar{\uparrow}\rangle - |\downarrow\uparrow\uparrow\uparrow\bar{\uparrow}\rangle) \end{aligned} \quad (C23)$$

and

$$\begin{aligned} S_5[31]_{\alpha} \left[\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix} \right] &= \sqrt{\frac{1}{3}}(|\uparrow\downarrow\uparrow\uparrow\bar{\downarrow}\rangle - |\downarrow\uparrow\uparrow\uparrow\bar{\downarrow}\rangle) - \sqrt{\frac{1}{12}}(|\uparrow\downarrow\uparrow\downarrow\bar{\uparrow}\rangle + |\uparrow\downarrow\downarrow\uparrow\bar{\uparrow}\rangle - |\downarrow\uparrow\uparrow\downarrow\bar{\uparrow}\rangle - |\downarrow\uparrow\downarrow\uparrow\bar{\uparrow}\rangle) \\ S_5[31]_{\beta} \left[\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix} \right] &= \frac{1}{3}(2|\uparrow\uparrow\downarrow\uparrow\bar{\downarrow}\rangle - |\uparrow\downarrow\uparrow\uparrow\bar{\downarrow}\rangle - |\downarrow\uparrow\uparrow\uparrow\bar{\downarrow}\rangle) \\ &\quad - \frac{1}{6}(2|\uparrow\uparrow\downarrow\downarrow\bar{\uparrow}\rangle - |\uparrow\downarrow\uparrow\downarrow\bar{\uparrow}\rangle - |\downarrow\uparrow\uparrow\downarrow\bar{\uparrow}\rangle + |\uparrow\downarrow\downarrow\uparrow\bar{\uparrow}\rangle + |\downarrow\uparrow\downarrow\uparrow\bar{\uparrow}\rangle - 2|\downarrow\downarrow\uparrow\uparrow\bar{\uparrow}\rangle) \\ S_5[31]_{\gamma} \left[\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix} \right] &= \sqrt{\frac{1}{18}}(3|\uparrow\uparrow\downarrow\uparrow\bar{\downarrow}\rangle - |\downarrow\uparrow\uparrow\uparrow\bar{\downarrow}\rangle - |\uparrow\uparrow\downarrow\uparrow\bar{\downarrow}\rangle - |\uparrow\uparrow\uparrow\downarrow\bar{\downarrow}\rangle) \\ &\quad - \sqrt{\frac{1}{18}}(|\uparrow\uparrow\downarrow\downarrow\bar{\uparrow}\rangle + |\uparrow\downarrow\uparrow\downarrow\bar{\uparrow}\rangle + |\downarrow\uparrow\uparrow\downarrow\bar{\uparrow}\rangle - |\uparrow\downarrow\downarrow\uparrow\bar{\uparrow}\rangle - |\downarrow\uparrow\downarrow\uparrow\bar{\uparrow}\rangle - |\downarrow\downarrow\uparrow\uparrow\bar{\uparrow}\rangle) \end{aligned} \quad (C24)$$

Note that the antiquark spin satisfies $\bar{\uparrow} = -i\sigma^2\uparrow = \downarrow$ and $\bar{\downarrow} = -i\sigma^2\downarrow = -\uparrow$. Finally, we note that the maximum weight representations for the flavor contributions follow from a similar reasoning, with the substitutions $\uparrow \rightarrow u, \downarrow \rightarrow d$.

5. S-state pentaquarks

To proceed to the LSF representations, we will first focus on the ground pentaquark or S-states with $L = 0$, and then proceed to detail the extension of the construction to the P-states which is more involved.

a. Spin-flavor mixed representations: $SF[31]$

For the ground state wave function with $L = 0$, the 4-quarks spin-flavor mixed representation conjugate to the 4-quarks color representation is also 3-degenerate, with the Jacobi-like nomenclature

$$SF[31]_\alpha = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array} \quad (C25)$$

$$SF[31]_\beta = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array} \quad (C26)$$

$$SF[31]_\gamma = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \quad (C27)$$

These mixed representations follow from the product representations

$$[q^4]_S \otimes [q^4]_F \quad (C28)$$

with a mixed $[31]_{S,F}$ net component. It follows from the character representations of S_4 that the product representations (C28) using (C15) are

$$\begin{aligned} S[4] \otimes F[31] &= SF[31] \\ S[31] \otimes F[4] &= SF[31] \\ S[31] \otimes F[31] &= SF[4] \oplus SF[31] \oplus SF[22] \\ S[31] \otimes F[22] &= SF[31] \\ S[22] \otimes F[22] &= SF[4] \oplus SF[22] \end{aligned} \quad (C29)$$

and the spin-flavor exchanged products

$$\begin{aligned} F[4] \otimes S[31] &= SF[31] \\ F[31] \otimes S[31] &= SF[4] \oplus SF[31] \oplus SF[22] \\ F[31] \otimes S[22] &= SF[31] \\ F[31] \otimes S[4] &= SF[31] \end{aligned} \quad (C30)$$

hence

$$\begin{aligned} SF[31] \subset S[4] \otimes F[31], S[31] \otimes F[4], S[31] \otimes F[31], S[31] \\ \otimes F[22], S[22] \otimes F[31] \end{aligned} \quad (C31)$$

The properly antisymmetrized $[31]_{FS}$ Jacobi core states are

$$\begin{aligned} (SF[31]:[31]_F \otimes [4]_S)_{\alpha,\beta,\gamma} &= F[31]_{\alpha,\beta,\gamma} S[4] \\ (SF[31]:[4]_F \otimes [31]_S)_{\alpha,\beta,\gamma} &= F[4] S[31]_{\alpha,\beta,\gamma} \end{aligned} \quad (C32)$$

and

$$\begin{aligned} (SF[31]:[31]_F \otimes [31]_S)_\alpha &= \frac{1}{\sqrt{3}} (F[31]_\alpha S[31]_\beta + \alpha \leftrightarrow \beta) + \frac{1}{\sqrt{6}} (F[31]_\alpha S[31]_\gamma + \alpha \leftrightarrow \gamma) \\ (SF[31]:[31]_F \otimes [31]_S)_\beta &= \frac{1}{\sqrt{3}} (F[31]_\alpha S[31]_\alpha - \alpha \rightarrow \beta) + \frac{1}{\sqrt{6}} (F[31]_\gamma S[31]_\beta + \gamma \leftrightarrow \beta) \\ (SF[31]:[31]_F \otimes [31]_S)_\gamma &= \frac{1}{\sqrt{6}} (F[31]_\alpha S[31]_\alpha + \alpha \rightarrow \beta) - \frac{2}{\sqrt{6}} F[31]_\gamma S[31]_\gamma \end{aligned} \quad (C33)$$

and

$$\begin{aligned} (SF[31]:[22]_F \otimes [31]_S)_\alpha &= \frac{1}{2} (F[22]_\alpha S[31]_\beta + \alpha \leftrightarrow \beta) - \frac{1}{\sqrt{2}} F[22]_\alpha S[31]_\gamma \\ (SF[31]:[22]_F \otimes [31]_S)_\beta &= \frac{1}{2} (F[22]_\alpha S[31]_\alpha - \alpha \rightarrow \beta) - \frac{1}{\sqrt{2}} F[22]_\beta S[31]_\gamma \\ (SF[31]:[22]_F \otimes [31]_S)_\gamma &= -\frac{1}{\sqrt{2}} (F[22]_\alpha S[31]_\alpha + \alpha \rightarrow \beta) \end{aligned} \quad (C34)$$

and

$$\begin{aligned}
(SF[31]:[31]_F \otimes [22]_S)_\alpha &= \frac{1}{2} (S[22]_\alpha F[31]_\beta + \alpha \leftrightarrow \beta) - \frac{1}{\sqrt{2}} S[22]_\alpha F[31]_\gamma \\
(SF[31]:[31]_F \otimes [22]_S)_\beta &= \frac{1}{2} (S[22]_\alpha F[31]_\alpha - \alpha \rightarrow \beta) - \frac{1}{\sqrt{2}} S[22]_\beta F[31]_\gamma \\
(SF[31]:[31]_F \otimes [22]_S)_\gamma &= -\frac{1}{\sqrt{2}} (S[22]_\alpha F[31]_\alpha + \alpha \rightarrow \beta)
\end{aligned} \tag{C35}$$

b. Pentaquarks, S-shell ($L=0$) and maximal spin $S=5/2$

In this work we consider the case of pentaquarks formed of only the light u, d flavors of equal mass. The ground state wave function carries $L=0$ or space symmetric S-state. As a result, the totally antisymmetric color-spin-flavor wave function in the S-state reads

$$\begin{aligned}
\Psi_P^S &= \frac{\varphi_{00L}}{\sqrt{3}} ([C[211]_\beta SF[31]_\alpha - C[211]_\alpha SF[31]_\beta \\
&\quad + C[211]_\gamma SF[31]_\gamma] C[11] SF[1])
\end{aligned} \tag{C36}$$

with $\varphi_{00L}(R)$ valued in terms of the hyperspherical distance R . Note that the reduction of (C36) to the 3 particle subspace is of the antisymmetric form

$$C_\beta SF_\alpha - C_\alpha SF_\beta$$

familiar from the singlet $\mathbf{1}_A$ Isgur-Karl representation for

$$\begin{aligned}
\varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right] &= \frac{\varphi_{00L}}{\sqrt{3}} |\uparrow\uparrow\uparrow\uparrow\bar{\uparrow}\rangle \left(C[211]_\beta \frac{1}{\sqrt{2}} (ud - du) uu\bar{u} - C[211]_\alpha \frac{1}{\sqrt{6}} (2uud - udu - duu) u\bar{u} \right. \\
&\quad \left. + C[211]_\gamma \frac{1}{\sqrt{12}} (3uuud - duuu - uduu - uudu) \bar{u} \right)
\end{aligned} \tag{C37}$$

with the color entries $C[211]_{\xi=\alpha,\beta,\gamma}$ given in (C10). After averaging over the spatial part, one readily checks that the individual isospin contributions $i=1, 2, 3$, and 4 in the core are equal as they should

TABLE VIII. Pentaquark states with $L=0$, spin S , and isospin I .

I/S	1/2	3/2	5/2
1/2	3	3	1
3/2	3	3	1
5/2	1	1	0

the nucleon odd-parity excited state (see Eq. (A4) in [24]). (C36) is in agreement with the result in [51].

Following the strategy used previously for 6-q hexoquark states, we start with the maximal spin case, in which the spin part of the wave function is trivially symmetric. Now it means that spin is $S=5/2$. The remaining wave functions are the $SU(3)$ color and $SU(2)$ flavor (isospin), and one may think that following the usual representation theory based on Young tableaux it can be done analytically.

Since the state with maximal spin *and* isospin $S=I=5/2$ cannot produce antisymmetric state consistent with Fermi statistics, we have two such states possible, with $I=3/2$ or $I=1/2$.

In Table VIII we list the $L=0$ pentaquark states, with the degeneracy associated to the pertinent spin-isospin entry, in agreement with the monom construction above.

For a direct comparison to the monom construction above, we explicit here the S-pentas states with isospin $\frac{1}{2}$ and spin $\frac{5}{2}$ content

$$\varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right]^\dagger I_i^3 \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right] = \frac{1}{4} \tag{C38}$$

Similarly, the total 3-spin, total 3-isospin, and total isospin are

$$\begin{aligned}
\varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right]^\dagger \left(\sum_{i=1}^5 S_i^3 \right) \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right] &= \frac{5}{2} \\
\varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right]^\dagger \left(\sum_{i=1}^5 I_i^3 \right) \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right] &= \frac{1}{2} \\
\varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right]^\dagger \left(\sum_{i=1}^5 (\vec{I}_i)^2 \right) \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right] &= \frac{3}{4}
\end{aligned} \tag{C39}$$

the 12-pair color and isospin correlations are

$$\begin{aligned}\varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right]^\dagger (\lambda_1 \cdot \lambda_2) \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right] &= \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right]^\dagger \frac{1}{3} \left(\bar{C}_4^2 - \frac{16}{3} \right) \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right] = -\frac{4}{3} \\ \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right]^\dagger (\tau_1 \cdot \tau_2) \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right] &= \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right]^\dagger \left(\frac{1}{3} \bar{I}_4^2 - 1 \right) \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right] = -\frac{1}{3}\end{aligned}\quad (C40)$$

and the spin-spin correlations are

$$\varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right]^\dagger \left(\sum_{i < j} S_i \cdot S_j \right) \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right] = \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right]^\dagger \left(\frac{1}{2} \bar{S}^2 - \frac{15}{8} \right) \varphi_P^S \left[\frac{11}{22}, \frac{55}{22} \right] = \frac{5}{2}\quad (C41)$$

c. Pentaquarks in P-shell $L=0$ and $FS[31]$ core

The pentaquark P-states commensurate with the nucleon state $\frac{1}{2}^+$, are more numerous and involved in their LSF arrangements. The P-state pentaquark with an S-state symmetric core and an antiquark in a P-state, is

$$\begin{aligned}\Psi_{Pm_L}^A &= \frac{\varphi_{m_L}}{\sqrt{3}} ([C[211]_\beta ([4]_L [31]_{SF})_\alpha - C[211]_\alpha ([4]_L [31]_{SF})_\beta \\ &\quad + C[211]_\gamma ([4]_L [31]_{SF})_\gamma] C[11]SF[1])\end{aligned}\quad (C42)$$

with the short hand notation

$$([4]_L [31]_{SF})_{\alpha,\beta,\gamma} = [4]_L ([31]_{SF})_{\alpha,\beta,\gamma}\quad (C43)$$

Note that the mixed Jacobi SF representations follow from 3 distinct arrangements

$$\begin{aligned}[4]_L ([31]_{FS} : [31]_F \otimes [31]_S) \\ [4]_L ([31]_{FS} : [22]_F \otimes [31]_S) \\ [4]_L ([31]_{FS} : [31]_F \otimes [22]_S)\end{aligned}\quad (C44)$$

making the state (C42) for fixed m_L , 3-degenerate. The ensuing and explicit Jacobi states are given in (C33)–(C35).

d. P-states: $L=1$ and $FS[4]$ core

From parity considerations, it follows that for the core $[q^4]$ only $L[4]$ with the unpaired $\bar{q}$ in a P-state, or $L[31]$ with the unpaired $\bar{q}$ in an S-state are allowed. Using the character assignments for finite groups, and the tetrahedral representations of S_4 , it follows that for 2 flavors, the mixed representations are

$$\begin{aligned}L[4] \otimes SF[31] &= LSF[31] \\ L[31] \otimes SF[31] &= LSF[4] \oplus LSF[31] \oplus LSF[22] \\ L[31] \otimes SF[22] &= LSF[31] \\ L[31] \otimes SF[4] &= LSF[31]\end{aligned}\quad (C45)$$

The pentaquark with a spatial P-core but with a symmetric FS-core, to which we can tag an antiquark in an S-state, is given by

$$\begin{aligned}\Psi_P^P &= \frac{\tilde{\varphi}_{1m_L}}{\sqrt{3}} ([C[211]_\beta ([31]_L [4]_{FS})_\alpha - C[211]_\alpha ([31]_L [4]_{FS})_\beta \\ &\quad + C[211]_\gamma ([31]_L [4]_{FS})_\gamma] C[11]SF[1])\end{aligned}\quad (C46)$$

with $\tilde{\varphi}_{1m_L}(R)$ valued in terms of the hyper-spherical distance R . The symmetric FS-core is composed of 2 mixed Jacobi configurations

$$\begin{aligned}([31]_L ([4]_{FS} : [31]_F [31]_S))_{\alpha,\beta,\gamma} \\ ([31]_L ([4]_{FS} : [22]_F [22]_S))_{\alpha,\beta,\gamma}\end{aligned}\quad (C47)$$

e. P-states: $L=1$ and $FS[31]$ core

There are 3 mixed Jacobi configurations with a spatial P-core but with an asymmetric FS-core

$$\begin{aligned}([31]_L ([31]_{FS} : [31]_F [31]_S))_{\alpha,\beta,\gamma} \\ ([31]_L ([31]_{FS} : [22]_F [31]_S))_{\alpha,\beta,\gamma} \\ ([31]_L ([31]_{FS} : [31]_F [22]_S))_{\alpha,\beta,\gamma}\end{aligned}\quad (C48)$$

6. Spin-color-flavor operators and Hamiltonians

In the S-state the spin-orbit and tensor interactions vanish, with only the spin-color exchange from the perturbative gluon exchange, and the emergent 't Hooft flavor induced interactions adding to the central kinetic and confining interactions. We now consider them sequentially.

a. Color interaction

The simplest pair color interaction in the pentaquark state reads

$$\langle \Psi_P^S | \sum_{i<j} \lambda_i \cdot \lambda_j | \Psi_P^S \rangle = \langle \Psi_P^S | \left(2\vec{C}_5^2 - \frac{40}{3} \right) | \Psi_P^S \rangle = -\frac{40}{3} \quad (\text{C49})$$

where the Casimir operator $\vec{C}_5^2 = \sum_{i=1}^5 \lambda_i^2/4$ is null in the color singlet pentaquark state. Note that when the exchange involves the antiquark labeled by 5, the replacement of the Gelman matrices $\lambda_5 \rightarrow -\lambda^*$ is assumed (and similarly $\sigma_5 \rightarrow -\sigma^*$ for the spin below).

b. Perturbative color-spin interaction

We use the same protocol when evaluating the color-spin interaction induced by 1-gluon exchange in the pentaquark S-state, for light quarks of mass m_Q . The pair interactions

$$\mathbb{V}_{1g}(1, 2, 3, 4, 5) = \sum_{i<j \leq 5} V_{1g}(r_{ij}) \frac{\lambda_i \cdot \lambda_j \sigma_i \cdot \sigma_j}{m_Q^2} \quad (\text{C50})$$

are sandwiched between the pertinent color-spin wave functions in (C36).

Assuming the short ranged V_{1g} to be about constant, we can simplify the color-spin interaction by noting that the 4 quark core $[q^4]$ is antisymmetric under the spin-flavor-color permutations

$$P_{ij}^S P_{ij}^F P_{ij}^C [q^4] = -[q^4] \quad (\text{C51})$$

with $i, j = 1, 2, 3, 4$ and

$$\begin{aligned} P_{ij}^S &= \frac{1}{2} (1 + \sigma_i \cdot \sigma_j) \\ P_{ij}^F &= \frac{1}{2} (1 + \tau_i \cdot \tau_j) \\ P_{ij}^C &= \frac{1}{2} \left(\frac{2}{3} + \lambda_i \cdot \lambda_j \right) \end{aligned} \quad (\text{C52})$$

hence the identity

$$-\sum_{i<j} \lambda_i \cdot \lambda_j \sigma_i \cdot \sigma_j = -\frac{32}{3} + \frac{4}{3} \vec{S}_4^2 + 4\vec{I}_4^2 + 2\vec{C}_4^2 \quad (\text{C53})$$

Here

$$\begin{aligned} \vec{S}_4^2 &= \left(\sum_{i=1}^4 \frac{1}{2} \vec{\sigma}_i \right)^2 \\ \vec{I}_4^2 &= \left(\sum_{i=1}^4 \frac{1}{2} \vec{\tau}_i \right)^2 \\ \vec{C}_4^2 &= \left(\sum_{i=1}^4 \frac{1}{2} \vec{\lambda}_i \right)^2 \end{aligned} \quad (\text{C54})$$

are respectively, the spin, isospin and color Casimirs for the completely antisymmetric 4-quark configuration $[q^4]$. We further note that the color Casimir for the antisymmetric core is that of the fundamental color triplet representation with $\vec{C}_4^2 \rightarrow \frac{4}{3}$, hence

$$-\sum_{i<j} \lambda_i \cdot \lambda_j \sigma_i \cdot \sigma_j \rightarrow -8 + \frac{4}{3} \vec{S}_4^2 + 4\vec{I}_4^2 \quad (\text{C55})$$

c. Color-spin shift from the core

The pentaquark states are configured in a core plus valence antiquark $[q^4 \bar{q}]$. The expectation value of (C55) depends solely on the net spin and isospin of the core irrespective of the $\xi = \alpha, \beta$, and γ assignments from permutation symmetry, hence for (C33)–(C35) we have respectively

$$\begin{aligned} \langle \Psi_P^S | \mathbb{V}_{1g} | \Psi_P^S \rangle_4 &= -\frac{V_{1g}}{3m_Q^2} \sum_{\xi=\alpha,\beta,\gamma} \left(SF_5[31]_\xi \right. \\ &\quad \times \left. \left(-8 + \frac{4}{3} \vec{S}_4^2 + 4\vec{I}_4^2 \right) SF_5[31]_\xi \right) \end{aligned} \quad (\text{C56})$$

d. Color-spin shift from the valence

The remaining spin-color exchange operator between the pairs $i5$ (i-quark and 5-antiquark) with $i = 1, 2, 3, 4$ in the core, can be obtained by evaluating

$$\lambda_4 \cdot \lambda_5 \sigma_4 \cdot \sigma_5 = 2 \left(\vec{C}_{4+5}^2 - \vec{C}_4^2 - \vec{C}_5^2 \right) \sigma_4 \cdot \sigma_5 \quad (\text{C57})$$

in terms of the pertinent Casimirs for the 4–5 particles, and then using the permutation symmetry of the core to deduce the other combinations through the identity

$$\begin{aligned} \lambda_i \cdot \lambda_5 \sigma_i \cdot \sigma_5 &= P_{i4}^S P_{i4}^C [\lambda_4 \cdot \lambda_5 \sigma_4 \cdot \sigma_5] P_{i4}^C P_{i4}^S \\ &\rightarrow P_{i4}^F [\lambda_4 \cdot \lambda_5 \sigma_4 \cdot \sigma_5] P_{i4}^F = \lambda_4 \cdot \lambda_5 \sigma_4 \cdot \sigma_5 \end{aligned} \quad (\text{C58})$$

for $i = 1, 2$, and 3 . The rightmost result holds only for expectation values in pentastates with an antisymmetric core, and follows from the commutation of the flavor matrices with the spin-color matrices.

The shift induced by (C57) in the nucleonlike pentaquark states, splits into

$$\begin{aligned}
\langle \Psi_P^S | \mathbb{V}_{1g} | \Psi_P^S \rangle_5 &= \frac{4V_{1g}}{3m_Q^2} [(C_5[211]_\alpha \lambda_4 \cdot \lambda_5 C_5[211]_\alpha) \times (SF_5[31]_\beta \sigma_4 \cdot \sigma_5 SF_5[31]_\beta) \\
&\quad + (C_5[211]_\beta \lambda_4 \cdot \lambda_5 C_5[211]_\beta) \times (SF_5[31]_\alpha \sigma_4 \cdot \sigma_5 SF_5[31]_\alpha) \\
&\quad + (C_5[211]_\gamma \lambda_4 \cdot \lambda_5 C_5[211]_\gamma) \times (SF_5[31]_\gamma \sigma_4 \cdot \sigma_5 SF_5[31]_\gamma)] \quad (C59)
\end{aligned}$$

where only the diagonal contributions are nonzero due to the diagonal form of the color matrix element. Indeed, the color contributions in (C59) are readily obtained by noting that the 4-5-particles are either in an octet in the α, β color configuration, or a singlet in the γ color representation, hence

$$\begin{aligned}
C_5[211]_{\alpha\beta} \lambda_4 \cdot \lambda_5 C_5[211]_{\alpha\beta} &= 2(C_{8_c}^2 - C_{3_c}^2 - C_{3_c}^2) \\
&= 2\left(3 - \frac{4}{3} - \frac{4}{3}\right) = \frac{2}{3} \\
C_5[211]_\gamma \lambda_4 \cdot \lambda_5 C_5[211]_\gamma &= 2(C_{1_c}^2 - C_{3_c}^2 - C_{3_c}^2) \\
&= 2\left(0 - \frac{4}{3} - \frac{4}{3}\right) = -\frac{16}{3} \quad (C60)
\end{aligned}$$

where $C_{X_c}^2$ are the color Casimirs for the pertinent color representation X_c of $SU(3)_c$. Recall that for the $SU(3)_c$ color representation X_c with Dynkin index $D(p, q)$ (with p -quarks and q -antiquarks), the quadratic Casimir is

$$C_{X_c}^2 = \frac{1}{3}(p^2 + q^2 + 3p + 3q + pq) \quad (C61)$$

The recoupling of the 4-5-spins in (C59) is more involved, and require the explicit spin configurations for the pentaquark states. For the maximum weight, the Jacobi spin states are given in (C20)–(C24).

$\mathbf{L} = \mathbf{0}, \mathbf{I} = \frac{1}{2}, \mathbf{S} = \frac{5}{2}$ *pentastate*

For $L = 0$ there is 1 pentastate with $IJ = \frac{1}{2}\frac{5}{2}$. The 45-spin contribution (C59) for $\xi = \alpha, \beta$, and γ amounts to

$$\begin{aligned}
SF_5[31]_\xi \sigma_4 \cdot \sigma_5 SF_5[31]_\xi &= (S[4]_\sigma \sigma_4 \cdot \sigma_5 S[4]_\sigma)(F[31]_\xi F[31]_\xi) \\
&= 1 \quad (C62)
\end{aligned}$$

and similarly for $4 \rightarrow 1, 2$, and 3 thanks to the symmetry of the spin combination $S[4]$. The combination of (C60) and (C62) in (C59), yields

$$\left\langle \Psi_P^S \left[\frac{15}{22} \right] | \mathbb{V}_{1g} | \Psi_P^S \left[\frac{15}{22} \right] \right\rangle_5 = \frac{4V_{1g}}{3m_Q^2} \left(2 \times \frac{2}{3} - \frac{16}{3} = -4 \right) \quad (C63)$$

The final result for the perturbative gluon exchange (C50) in this pentastate, is the sum of the 4-core contribution

$$\left\langle \Psi_P^S \left[\frac{15}{22} \right] | \mathbb{V}_{1g} | \Psi_P^S \left[\frac{15}{22} \right] \right\rangle_4 = -\frac{V_{1g}}{3m_Q^2} \sum_{\xi=\alpha,\beta,\gamma} (S[4]F[31]_\xi \left(-8 + \frac{4}{3}\vec{S}_4^2 + 4\vec{I}_4^2 \right) S[4]F[31]_\xi)_4 = \frac{V_{1g}}{m_Q^2} (-8) \quad (C64)$$

plus the 5-remainder valence contribution (C63),

$$\left\langle \Psi_P^S \left[\frac{15}{22} \right] | \mathbb{V}_{1g} | \Psi_P^S \left[\frac{15}{22} \right] \right\rangle_{4+5} = \frac{4V_{1g}}{3m_Q^2} (-4 - 6 = -10) \quad (C65)$$

in agreement with the result derived using the monon basis analysis listed in the main text.

$\mathbf{L} = \mathbf{0}, \mathbf{I} = \frac{1}{2}, \mathbf{S} = \frac{3}{2}$ *pentastate*:

For $L = 0$, there are 3 pentastates with $IJ = \frac{1}{2}\frac{3}{2}$ assignments with the following cores as given in (C33)–(C35)

$$C[211](S[31]F[31]), C[211](S[31]F[22]), C[211](S[4]F[31]) \quad (C66)$$

The color-spin contributions from the cores are respectively

$$\langle \Psi_P^S | \mathbb{V}_{1g} | \Psi_P^S \rangle_4 = -\frac{V_{1g}}{3m_Q^2} \sum_{\xi=\alpha,\beta,\gamma} \left(SF_5[31]_\xi \left(-8 + \frac{4}{3}\vec{S}_4^2 + 4\vec{I}_4^2 \right) SF_5[31]_\xi \right) = \frac{V_{1g}}{m_Q^2} \left(-\frac{8}{3}, +\frac{16}{3}, -8 \right) \quad (C67)$$

Once combined with the valence contributions, the results are in agreement with the monon basis analysis listed in the main text.

$\mathbf{L} = \mathbf{0}, \mathbf{I} = \frac{1}{2}, \mathbf{S} = \frac{1}{2}$ *pentastate*:

For $L = 0$, there are 3 pentastates with $IJ = \frac{1}{2}\frac{1}{2}$ assignments with the following cores

$$\begin{aligned} &C[211](S[31]F[31]), C[211](S[31]F[22]), \\ &C[211](S[22]F[31]) \end{aligned} \quad (\text{C68})$$

The color-spin contributions from the cores are respectively

$$\begin{aligned} &\langle \Psi_P^S | \mathbb{V}_{1g} | \Psi_P^S \rangle_4 \\ &= -\frac{V_{1g}}{3m_Q^2} \sum_{\xi=\alpha,\beta,\gamma} \left(SF_5[31]_\xi \left(-8 + \frac{4}{3} \bar{S}_4^2 + 4\bar{I}_4^2 \right) SF_5[31]_\xi \right) \\ &= \frac{V_{1g}}{m_Q^2} \left(-\frac{8}{3}, +\frac{16}{3}, 0 \right) \end{aligned} \quad (\text{C69})$$

Again, once combined with the valence contributions, the results are in agreement with the monon basis analysis listed in the main text.

e. Nonperturbative 't Hooft interactions

The nonperturbative 't Hooft interactions induced by rescattering of light quark zero modes through instantons

and anti-instantons is

$$\begin{aligned} &\mathbb{V}_{TH}(1, 2, 3, 4, 5) \\ &= \sum_{i,j \leq 5} V_{TH}(r_{ij})(1 - \tau_i \cdot \tau_j)(1 - a\sigma_i \cdot \sigma_j) \end{aligned} \quad (\text{C70})$$

with

$$V_{TH} = -\frac{1}{4}|\kappa_2|A_{2N} = -|\kappa_2|\frac{(2N_c - 1)}{2N_c(N_c^2 - 1)} \quad (\text{C71})$$

for zero size instantons, and $a = \frac{1}{2N_c - 1} \rightarrow \frac{1}{5}$ which is seen to vanish in the large N_c limit. In this limit, (C70) can be readily evaluated in the nucleonlike pentaquark states

$$\begin{aligned} \langle \Psi_P^S | \mathbb{V}_{TH} | \Psi_P^S \rangle &= \frac{V_{TH}}{3} \sum_{\xi=\alpha,\beta,\gamma} SF_5[31]_\xi \\ &\times \left(\sum_{i,j \leq 5} (1 - \tau_i \cdot \tau_j) \right) SF_5[31]_\xi \\ &= V_{TH}(26, 26, 26) \end{aligned} \quad (\text{C72})$$

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