

Scalar-assisted magnetogenesis during the radiation-dominated epoch

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We propose a novel mechanism to generate primordial magnetic fields (PMFs) strong enough to explain the observed cosmic magnetic fields. We employ a scalar field charged under $U(1)$ gauge symmetry with a nontrivial vacuum expectation value to provide an effective mass term to the electromagnetic field and thus break its conformal invariance. The primordial magneto-genesis takes place in the radiation dominated (RD) epoch, after the electroweak symmetry breaking (EWSB) phase. As a result, our mechanism is naturally free from the over-production of electric fields due to high conductivity in the RD epoch, and the baryon isocurvature problem which takes place only if magneto-genesis happens before the EWSB phase. In addition, we find that a significant amount of PMFs can be generated when the scalar field experiences a tachyonic phase. In this case, the scalar field is light and weakly coupled and has negligible energy density compared to the cold dark matter, hence the strong coupling problem and the backreaction problem are also absent. Therefore, our model is free from the above-mentioned problems that frequently appear in other primordial magneto-genesis scenarios.

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I. INTRODUCTION

Observations have confirmed the ubiquitous nature of magnetic fields [1]. Nonetheless, the origin of cosmic magnetic fields is still unclear. Observational signatures of magnetic fields in intergalactic void regions [2–5] imply a cosmological origin of large-scale magnetic fields, due to the difficulty of accounting for them in astrophysical mechanisms [6].

Understanding the origin of large-scale magnetic fields is a longstanding problem. The blazar observations suggest that the strength of magnetic fields with a coherence length of a few Mpc should be larger than 10^{-15} Gauss [7,8]. It is generically believed that primordial magneto-genesis before the structure formation sources the seed magnetic field, which, after astrophysical dynamo and compression amplification mechanisms, forms the large-scale magnetic field observed today, see [9,10] for a review. The strength of the seed magnetic field remains unclear due to uncertainties in the details of the dynamo mechanisms, which typically vary from 10^{-12} G to 10^{-22} G [11,12]. It is

difficult to produce such magnetic seeds through the primordial magneto-genesis process within classical electromagnetism and conventional cosmology [13].

Inflation is believed to be a major candidate for the production of a large-scale magnetic field. Over the past years, vast inflationary magneto-genesis models have been proposed in the literature [14–26]. All of these models share a key ingredient, namely, the breaking of conformal invariance [13]. One of the most well-studied ideas is the Ratra theory [27], in which a scalar field couples to the field strength $f(\phi)^2 F_{\mu\nu} F^{\mu\nu}$. Most of these models potentially suffer from the strong coupling problem [28] or the backreaction problem [29]. Additionally, a recent study reveals an overproduction of baryon isocurvature perturbations in any magneto-genesis scenario above the electroweak (EW) scale [30], arguably ruling out most inflationary magneto-genesis scenarios.

In this paper, we propose a novel mechanism free from all the above problems. We will instead consider a new scalar field charged under $U(1)$ gauge symmetry, and the conformal invariance is broken due to the effective mass term of the electromagnetic (EM) field generated due to a nontrivial vacuum expectation value (VEV) of the scalar. The magnetic field is then induced by the scalar currents in the early Universe. In contrast to the previous work we assume that the scalar field is very light and is extremely weakly coupled, and therefore, the scalar particles are not thermalized before the

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EW scale. Very weakly coupled fermion particles (milli charged particles) have been extensively discussed in the literature as a potential candidate for dark matter [31,32] and could arise as a low-energy limit of a new light $U(1)'$ gauge field which kinetically mixes with the Standard Model $U(1)$.

The baryon isocurvature problem becomes irrelevant as long as the magneto-genesis takes place below the EW scale. In the literature, the condition is conventionally fulfilled by working in a low-scale inflation scenario [33–36], namely the reheating temperature is way below the EW scale, at the cost of being contrary to the standard thermal history of the Universe. In our work, we instead assume that the magneto-genesis happens in the standard radiation dominated epoch.¹ The problem of baryon isocurvature can be evaded as long as the primordial magnetic field is generated after the electroweak symmetry breaking phase, without altering the standard thermal history of our Universe. Additionally, in the radiation dominated epoch, the Universe is effectively a plasma, in which the induced electric field is diluted away by the high conductivity. Thus, we will not need to worry about the overproduction of electric fields.

One may worry about the feasibility of producing sufficient magnetic fields in our setup, as the inflationary magnetogenesis in scalar QED theory is argued to be ineffective [38–42]. For instance, Ref. [42] reports a seed magnetic field of order 10^{-29} G on $\mathcal{O}(1)\text{Mpc}^{-1}$ scale from the coupling of inflaton with gauge field, which is marginally acceptable assuming a highly efficient astrophysical amplification mechanism. In the radiation dominated epoch things may become worse due to the existence of high conductivity. We will show that a significant number of primordial magnetic fields can be generated in the radiation dominated epoch without the backreaction problem, provided that the scalar field experiences a tachyonic growth phase. Thus, our formalism provides a viable mechanism for generating a large scale magnetic field without all problems mentioned above.

We organize the paper as follows. We present our model in Sec. II. The general formalism of induced magnetic field is discussed in Sec. III. In Sec. IV, we analyze the power spectrum of the magnetic field when the scalar field is tachyonically amplified. We explicitly calculate the magnetic field when the magneto-genesis takes place before and after the electron-positron annihilation in Secs. V and VI, respectively. We show that the seed magnetic field from the interaction with the scalar field can be compatible to observations and conclude in Sec. VII.

Throughout this manuscript, the scalar field ϕ takes mass dimension, and the fine structure constant is dimensionless such that the vector potential A_μ has mass dimension. The normal and conformal magnetic field shall carry $[M]^2$

dimension. The conformal time is denoted by τ . Unless specified, a prime shall denote the differentiation with respect to τ .

II. OUR MODEL

We work with the following action

$$S = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} \mathcal{R} - \frac{1}{2} (D_\mu \phi)^\dagger D^\mu \phi - V(\phi, \chi) + \mathcal{L}_\chi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{bg} \right], \quad (2.1)$$

with $D_\mu \equiv \partial_\mu - i\varrho A_\mu$ and $\mathcal{R}/2$ the Einstein-Hilbert action. The parameter ϱ is the effective coupling constant between the scalar field and the $U(1)$ gauge field. The term $\mathcal{L}_\chi$ represents the Lagrangian of an auxiliary scalar field χ , which we shall explain in Sec. IV B. The complex scalar field ϕ is coupled to the auxiliary field χ via the interaction

$$V(\phi, \chi) = \lambda^2 M_p \chi |\phi|^2. \quad (2.2)$$

The background geometry is described by the flat FLRW metric

$$ds^2 = -dt^2 + a(t)^2 dx_i dx^i = a(\tau)^2 (-d\tau^2 + dx_i dx^i), \quad (2.3)$$

where t is the cosmic time and $\tau \equiv \int dt/a$ the conformal time. As pointed out by [30], magneto-genesis scenarios above the electroweak scale are strongly restricted by the baryon isocurvature problem. In light of this result, we will work in the radiation-dominated (RD) epoch throughout this paper. The scale factor and the Hubble parameter $H \equiv \dot{a}/a$ are parametrized as follows:

$$a(\tau) = a_e \left(\frac{\tau}{\tau_e} \right), \quad H(\tau) = H_{\text{inf}} \frac{\tau_e^2}{\tau^2}, \quad \tau > \tau_e > 0, \quad (2.4)$$

where $\tau = \tau_e$ represents the beginning of the RD epoch and $a_e \equiv a(\tau_e)$ the scale factor at that time. We assume an instantaneous reheating process so the Hubble parameter at $\tau = \tau_e$ is identical to that in inflation epoch, which we denote as H_{inf} .

We will assume that both the scalar field ϕ and the auxiliary field χ are spectator fields whose energy density is negligible compared to other matter content in the early Universe. The background radiation is thus governed by other matter content, which we abbreviated as $\mathcal{L}_{bg}$. We will examine the backreaction from the scalar field ϕ and confirm this assumption in Secs. V and VI.

¹See also, e.g., Ref. [37].

III. FORMALISM FOR THE INDUCED MAGNETIC FIELD

A. Dynamical equations of motion

The equation of motion (EoM) for the vector potential A_μ is,

$$\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}F_{\lambda\sigma}g^{\alpha\lambda}g^{\mu\sigma}) + \frac{i}{2}\varrho(\phi\nabla^\mu\phi^* - \phi^*\nabla^\mu\phi) - \varrho^2 A^\mu\phi^*\phi = 0. \quad (3.1)$$

In our scenario, energy budget stored in the scalar field is converted into the gauge field sector, and therefore, throughout the period of our interest, the last term in the above equation is negligible compared to the second term. On the other hand, as we will show in Sec. VI, the resolution of the backreaction problem requires a small coupling constant ϱ , which can be another reason to neglect the ϱ^2 term. Adopting Weyl gauge $A_0 = 0$, the 0th and i th component of (3.1) simplify to

$$\partial^i A'_i = \frac{i\varrho}{2}a^2(\phi\phi^{*'} - \phi'\phi^*), \quad (3.2)$$

$$A''_i - \partial_j^2 A_i + \partial_i(\partial^j A_j) = \frac{i}{2}\varrho a^2(\phi\partial_i\phi^* - \phi^*\partial_i\phi). \quad (3.3)$$

Notice that a further choice of Coulomb gauge $\partial_i A^i = 0$ is incompatible with (3.2) due to the presence of the source term.

As we work in the RD epoch, the conductivity of our universe σ becomes important, which can be approximated as [43]

$$\sigma \simeq \frac{T}{\alpha^2 \ln(1/\alpha)}, \quad (3.4)$$

where T the temperature of the thermalized universe and $\alpha = 1/137$ the fine-structure constant. Equation (3.4) holds as long as Rutherford scattering dominates and determines the mean free path. Cosmological events such as e^+e^- annihilation at $T \simeq 0.1$ MeV, which we denote as $T_a \simeq 0.1$ MeV, lead to a sudden drop of σ [13],

$$\sigma = 10^{-10} \frac{m_e}{e^2} = 10^{-10} \frac{m_e}{4\pi\alpha} = 5.6 \times 10^{-13} \text{ GeV}. \quad (3.5)$$

As a comparison, shortly before the annihilation event we have

$$\sigma \simeq \frac{0.1 \text{ MeV}}{\alpha^2 \ln(1/\alpha)} = 3.8 \times 10^{-1} \text{ GeV}. \quad (3.6)$$

Namely, the conductivity drops more than ten orders of magnitude.

The evolution of A_μ in the presence of a high conductivity is given by [13],

$$A''_i + 4\pi\sigma a A'_i - \partial_j^2 A_i + \partial_i(\partial^j A_j) = \frac{i}{2}\varrho a^2(\phi\partial_i\phi^* - \phi^*\partial_i\phi). \quad (3.7)$$

B. Electric and magnetic fields

The electric and magnetic fields observed from a comoving observer $u^\nu = (1, \vec{0})$ are

$$E_\mu \equiv F_{\mu\nu}u^\nu = (0, -\dot{A}_i), \quad B_\mu \equiv \tilde{F}_{\mu\nu}u^\nu = \left(0, \frac{1}{a}\epsilon_i^{jk}\partial_j A_k\right). \quad (3.8)$$

In terms of 3D conformal EM fields, we have

$$\mathcal{E}_i \equiv aE_i = -A'_i, \quad \mathcal{B}_i \equiv aB_i = \epsilon_i^{jk}\partial_j A_k. \quad (3.9)$$

The EoM (3.2) gives

$$\partial^i \mathcal{E}_i = -\frac{i\varrho}{2}a^2(\phi\phi^{*'} - \phi'\phi^*), \quad (3.10)$$

and with the help of (3.3) we have

$$\begin{aligned} \mathcal{E}'_i = -A''_i &= 4\pi\sigma a A'_i - \partial_j^2 A_i + \partial_i(\partial^j A_j) \\ &- \frac{i}{2}\varrho a^2(\phi\partial_i\phi^* - \phi^*\partial_i\phi). \end{aligned} \quad (3.11)$$

Finally, the dynamical equations for $\mathcal{E}_i$ and $\mathcal{B}_i$ are

$$\mathcal{E}''_i + 4\pi\sigma a \mathcal{E}'_i - \partial^2 \mathcal{E}_i = S_{e,i}, \quad (3.12)$$

$$\mathcal{B}''_i + 4\pi\sigma a \mathcal{B}'_i - \partial^2 \mathcal{B}_i = S_{b,i}, \quad (3.13)$$

where

$$S_{e,i} \equiv i\varrho a^2[(\phi^{*'} + \mathcal{H}\phi^*)\partial_i\phi - (\phi' + \mathcal{H}\phi)\partial_i\phi^*], \quad (3.14)$$

$$S_{b,i} \equiv \frac{i}{2}\varrho a^2\epsilon_i^{jk}(\partial_j\phi\partial_k\phi^* - \partial_k\phi\partial_j\phi^*) = i\varrho a^2\epsilon_i^{jk}\partial_j\phi\partial_k\phi^*. \quad (3.15)$$

Notably, the electric fields get diluted in a plasma, so we can simply focus on the magnetic field. We will come back to this issue in Sec. VI C.

C. Dynamics of the magnetic field

The magnetic field in the Fourier domain is

$$\hat{\mathcal{B}}_i(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} [\mathcal{B}_{\vec{k}}(\tau)\hat{b}_{\vec{k}} + \mathcal{B}_{-\vec{k}}^*(\tau)\hat{b}_{-\vec{k}}^\dagger] \hat{e}_i(\hat{k}). \quad (3.16)$$

For simplicity we suppress the polarization state of magnetic field, and $\hat{e}_i$ now only indicates the “direction” of $\mathcal{B}$. The creation and annihilation operators satisfy

$$[\hat{b}_{\vec{k}_1}, \hat{b}_{\vec{k}_2}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2). \quad (3.17)$$

In the Fourier domain, the dynamical equation (3.12) becomes

$$\mathcal{B}_k'' + 4\pi\sigma a \mathcal{B}_k' + k^2 \mathcal{B}_k = S_{b,\vec{k}}(\tau), \quad (3.18)$$

with the source term being

$$S_{b,\vec{k}} \equiv \int d^3x S_{b,i} e^{-i\vec{k}\cdot\vec{x}} \equiv \int d^3x e^{-i\vec{k}\cdot\vec{x}} (iQa^2 \epsilon_i^{jk} \partial_j \phi \partial_k \phi^*). \quad (3.19)$$

The scalar field ϕ can be decomposed into a homogeneous background $\bar{\phi}(t)$ and a perturbative part $\delta\phi(\vec{x}, t)$,

$$\phi(\vec{x}, t) = \bar{\phi}(t) + \delta\phi(\vec{x}, t), \quad (3.20)$$

with

$$\delta\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} (\phi_{\vec{k}} \hat{a}_{\vec{k}} + \phi_{-\vec{k}}^* \hat{a}_{-\vec{k}}^\dagger), \quad (3.21)$$

where the creation and annihilation operators satisfy

$$[\hat{a}_{\vec{k}_1}, \hat{a}_{\vec{k}_2}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2). \quad (3.22)$$

The source term in the operator form becomes

$$\begin{aligned} \hat{S}_{b,\vec{k}} &= iQa^2 \int \frac{d^3p}{(2\pi)^3} (\vec{p} \times \vec{q}) (\phi_{\vec{p}} \hat{a}_{\vec{p}} + \phi_{-\vec{p}}^* \hat{a}_{-\vec{p}}^\dagger) \\ &\times (\phi_{\vec{q}} \hat{a}_{\vec{q}} + \phi_{-\vec{q}}^* \hat{a}_{-\vec{q}}^\dagger), \quad \vec{q} \equiv \vec{k} - \vec{p}, \end{aligned} \quad (3.23)$$

and the particular solution to the conformal magnetic field is

$$\vec{\mathcal{B}}_k(\tau) = \int_{\tau_0}^{\tau} g_k(\tau; \tilde{\tau}) \vec{S}_{b,\vec{k}}(\tilde{\tau}) d\tilde{\tau}, \quad (3.24)$$

where g_k is the Green’s function associated to (3.18). As we will show later, the fractional terms is much more important than the k^2 term, since the typical scale for intergalaxy magnetic field is

$$k_G/a_{\text{today}} = 1 \text{ Mpc}^{-1} = 6.4 \times 10^{-39} \text{ GeV}. \quad (3.25)$$

The dynamical equation then simplifies to

$$\mathcal{B}_k'' + 4\pi\sigma a \mathcal{B}_k' = S_{b,\vec{k}}(\tau). \quad (3.26)$$

D. Green’s function in different epochs

1. Before annihilation

Before the annihilation event, the electric conductivity is decided by (3.4). Since T scales as a^{-1} in the RD epoch, we define a new quantity

$$\gamma \equiv 2\pi\sigma a = \frac{2\pi}{\alpha^2 \ln(1/\alpha)} (aT)_{\text{RD}}, \quad (3.27)$$

which is constant in the RD epoch. Specifically, the value of aT can be evaluated at the radiation-matter equality epoch

$$(aT)_{\text{RD}} = a_{\text{eq}} T_{\text{eq}} = \frac{a_{\text{today}}}{z_{\text{eq}} + 1} \times 1 \text{ eV} \simeq 3.0 \times 10^{-13} \text{ GeV}, \quad (3.28)$$

and accordingly

$$\gamma = \frac{2\pi}{\alpha^2 \ln(1/\alpha)} (aT)_{\text{RD}} = 7.3 \times 10^{-9} \text{ GeV}. \quad (3.29)$$

It is easy to see that $\gamma \gg k_G$. Therefore, the equation of motion for the magnetic field simplifies to

$$\mathcal{B}_k'' + 2\gamma \mathcal{B}_k' = S_{b,\vec{k}}(\tau), \quad (3.30)$$

where γ is a constant evaluated in the Eq. (3.29), and the Green’s function simplifies to

$$g_k(\tau, \tau') \simeq \frac{\Theta(\tau - \tau')}{2\gamma} [1 - e^{-2\gamma(\tau - \tau')}] \simeq \frac{\Theta(\tau - \tau')}{2\gamma}. \quad (3.31)$$

2. After annihilation

After the annihilation event, we use

$$\begin{aligned} 4\pi\sigma a &= 4\pi\sigma H_{\text{inf}} a_e^2 \tau = 4\pi\sigma H_{\text{inf}} \tau \frac{(aT)_{\text{RD}}^2}{T_e^2} \\ &= \frac{4\pi^2 \sigma}{3M_p} \sqrt{\frac{g_*}{10}} (aT)_{\text{RD}}^2 \tau, \end{aligned} \quad (3.32)$$

where we have used the expression of energy density of background radiation

$$\rho_{\text{bg}} = 3H^2 M_p^2 = \frac{g_* \pi^2 T^4}{30}, \quad (3.33)$$

to write the H_{inf} as

$$H_{\text{inf}} = \frac{\pi \sqrt{g_*} T_e^2}{3\sqrt{10} M_p}. \quad (3.34)$$

For convenience, we define the following constant

$$\kappa \equiv \sqrt{\frac{2\pi^2\sigma}{3M_p}} \sqrt{\frac{g_*}{10}} (aT)_{\text{RD}}^2 = 3.0 \times 10^{-28} \times \left(\frac{g_*}{106.75}\right)^{\frac{1}{4}} \text{GeV}, \quad (3.35)$$

and one may verify that $\kappa \gg k_G/a_{\text{today}}$ and the dynamical equation simplifies to

$$\mathcal{B}_k'' + 2\kappa^2 \tau \mathcal{B}_k' \simeq S_{b,\vec{k}}(\tau). \quad (3.36)$$

The corresponding Green's function becomes

$$g_k(\tau, \tau') = \frac{\sqrt{\pi} e^{\kappa^2 \tau'^2}}{2\kappa} [\text{Erf}(\kappa\tau) - \text{Erf}(\kappa\tau')] \Theta(\tau - \tau'). \quad (3.37)$$

The conformal time is connected to the temperature T through

$$\tau = \frac{1}{aH} = \frac{T}{(aT)_{\text{RD}}} \frac{1}{H} = \frac{1}{(aT)_{\text{RD}}} \sqrt{\frac{10}{g_*}} \frac{3M_p}{\pi T},$$

$$d\tau = -\frac{1}{(aT)_{\text{RD}}} \sqrt{\frac{10}{g_*}} \frac{3M_p}{\pi T^2} dT, \quad (3.38)$$

Thus, the conformal time after the annihilation event satisfies

$$\tau > \frac{1}{(aT)_{\text{RD}}} \sqrt{\frac{10}{g_*}} \frac{3M_p}{\pi \times 0.1 \text{ MeV}}$$

$$= \sqrt{\frac{106.75}{g_*}} \times 1.2 \times 10^{35} \text{ GeV}^{-1}, \quad (3.39)$$

and we can see $\kappa\tau \gg 1$. The Green's function simplifies to

$$g_k(\tau, \tau') \simeq \frac{1}{2\kappa^2 \tau'} \left[1 - \frac{\tau'}{\tau} e^{-\kappa^2(\tau^2 - \tau'^2)} \right] \Theta(\tau - \tau'). \quad (3.40)$$

Additionally, the exponential term quickly shrinks to zero as long as τ differs from τ' even by a small ratio $\frac{\tau - \tau'}{\tau} > 10^{-8}$. We are justified to ignore that exponential term as long as the tachyonic phase is long enough. In terms of temperature, the Green's function simplifies to

$$g_k(T, T') \simeq \frac{1}{2\kappa^2} \frac{(aT)_{\text{RD}} \pi T' \sqrt{g_*}}{3\sqrt{10}M_p} \Theta(T' - T)$$

$$= \frac{4.7 \times 10^{23} T'}{[\text{GeV}]^2} \Theta(T' - T). \quad (3.41)$$

Notice that g_* within τ' and κ^2 cancel with each other and the Green's function is independent of g_* .

E. Power spectrum of magnetic field

The two-point correlation function of the magnetic field is

$$\langle \vec{\mathcal{B}}_{k_a}(\tau) \vec{\mathcal{B}}_{k_b}(\tau) \rangle = \int^\tau \int^\tau d\tau_a d\tau_b g_{k_a}(\tau; \tau_a) g_{k_b}(\tau; \tau_b)$$

$$\times \langle \vec{S}_{b,\vec{k}_a}(\tau_a) \vec{S}_{b,\vec{k}_b}(\tau_b) \rangle. \quad (3.42)$$

We first evaluate the two-point correlation function of the source term,

$$\langle \vec{S}_{b,\vec{k}_1}(\tau_a) \vec{S}_{b,\vec{k}_2}(\tau_b) \rangle$$

$$= -Q^2 a^4 \int \frac{d^3 p_a}{(2\pi)^3} \frac{d^3 p_b}{(2\pi)^3} (\vec{p}_a \times \vec{q}_a) (\vec{p}_b \times \vec{q}_b)$$

$$\times \phi_{\vec{p}_a}(\tau_a) \phi_{-\vec{q}_a}^*(\tau_a) \phi_{\vec{p}_b}(\tau_b) \phi_{-\vec{q}_b}^*(\tau_b) (2\pi)^6 \delta^{(3)}$$

$$\times (\vec{p}_a + \vec{k}_2 - \vec{p}_b) \delta^{(3)}(\vec{k}_1 + \vec{k}_2), \quad (3.43)$$

where we have used the fact

$$\vec{p}_a + \vec{q}_a = \vec{k}_1, \quad \vec{p}_b + \vec{q}_b = \vec{k}_2, \quad (3.44)$$

$$\delta^{(3)}(\vec{p}_a + \vec{k}_2 - \vec{p}_b) \delta^{(3)}(\vec{p}_b + \vec{k}_1 - \vec{p}_a)$$

$$= \delta^{(3)}(\vec{p}_a + \vec{k}_2 - \vec{p}_b) \delta^{(3)}(\vec{k}_1 + \vec{k}_2). \quad (3.45)$$

Following the conventional treatment of induced gravitational waves, we define the transfer function

$$\phi_{\vec{k}}(\tau) \equiv \mathcal{T}_k(\tau; \tau_*) \phi_{\vec{k}}(\tau_*), \quad (3.46)$$

where τ_* is a reference time. The scalar power spectrum is defined as

$$\langle \phi_{\vec{k}}(\tau) \phi_{\vec{k}'}(\tau) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \frac{2\pi^2}{k^3} P_\phi(k, \tau). \quad (3.47)$$

We also introduce the auxiliary variables

$$k \equiv |\vec{k}_1|; \quad v_a \equiv p_a/k, \quad v_b \equiv p_b/k, \quad (3.48)$$

such that

$$\langle \vec{S}_{b,\vec{k}_1}(\tau_a) \vec{S}_{b,\vec{k}_2}(\tau_b) \rangle = 4\pi^4 Q^2 a^4 (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2)$$

$$\times \int \frac{d^3 p_a}{(2\pi)^3} \frac{P_\phi(v_a k) P_\phi(v_b k)}{v_a^3 v_b^3 k^6}$$

$$\times |\vec{k}_1 \times \vec{p}_a|^2 \mathcal{T}_{k_a}(\tau_a; \tau_e) \mathcal{T}_{k_b}(\tau_b; \tau_e). \quad (3.49)$$

The power spectrum of conformal magnetic field at the time τ is defined as

$$\langle \vec{B}_{k_a}(\tau_a) \vec{B}_{k_b}(\tau_b) \rangle \equiv \frac{2\pi^2}{k_a^3} (2\pi)^3 \delta^{(3)}(k_a + k_b) P_B(k_a), \quad (3.50)$$

so we have

$$\begin{aligned} P_B(k) &\equiv \frac{k^3}{2\pi^2} \int^\tau \int^\tau d\tau_a d\tau_b g_k(\tau; \tau_a) g_k(\tau; \tau_b) \\ &\quad \times \langle \vec{S}_{b,\vec{k}}(\tau_a) \vec{S}_{b,\vec{k}'}(\tau_b) \rangle \\ &= \frac{1}{2} \varrho^2 k^4 \int dv_a dv_b \frac{|\vec{v}_a \times \vec{v}_b|^2}{v_a^2 v_b^2} P_\phi(v_a k) P_\phi(v_b k) |\mathcal{I}|^2, \end{aligned} \quad (3.51)$$

where $\mathcal{I}$ is the time integral

$$\mathcal{I}(\tau, \tau_*) \equiv \int^\tau d\tau' a^2(\tau') g_k(\tau; \tau') \mathcal{T}_{k_a}(\tau'; \tau_*) \mathcal{T}_{k_b}(\tau'; \tau_*). \quad (3.52)$$

The angular variable is determined by the implicit delta function, which appears when we integrate out $\vec{p}_b$,

$$v_b^2 = 1 + v_a^2 - 2v_a \cos \theta, \quad v_b \in (|1 - v_a|, 1 + v_a). \quad (3.53)$$

We further define auxiliary variables

$$s \equiv v_a + v_b, \quad d \equiv |v_a - v_b|, \quad (3.54)$$

and the power spectrum of the normal magnetic field becomes

$$\begin{aligned} P_B(k, \tau) &= \frac{P_B(k, \tau)}{a^4} = 4\varrho^2 \left(\frac{k}{a}\right)^4 \int_1^\infty ds \int_0^1 dd (1 - d^2) \\ &\quad \times (s^2 - d^2)^{-2} (s^2 - 1) \\ &\quad \times P_\phi\left(\frac{s+d}{2}k, \tau_*\right) P_\phi\left(\frac{s-d}{2}k, \tau_*\right) |\mathcal{I}(\tau; \tau_*)|^2, \end{aligned} \quad (3.55)$$

where τ_* is a certain reference time.

Sometimes, it would be convenient to write everything in terms of temperature. The time integral then becomes

$$\begin{aligned} \mathcal{I}(T; T_*) &\equiv \int^T -\frac{3\sqrt{10}}{\pi} g_*^{-\frac{1}{2}} g_k(T; T') M_p(aT)_{\text{RD}} \frac{dT'}{T'^4} \\ &\quad \times \mathcal{T}_{k_a}(T'; T_*) \mathcal{T}_{k_b}(T'; T_*), \end{aligned} \quad (3.56)$$

where we have used the following relation

$$d\tau' a^2(\tau') = -\frac{1}{(aT)_{\text{RD}}} \sqrt{\frac{10}{g_*}} \frac{3M_p}{\pi T'^2} dT' \times \frac{(aT)_{\text{RD}}^2}{T'^2}. \quad (3.57)$$

IV. AMPLIFICATION OF SCALAR FIELD THROUGH A TACHYONIC PHASE

A. The necessity of amplification on the scalar field

Let us start with a toy case where the ϕ field is simply a light scalar, i.e., $V(\phi) = \frac{1}{2}m^2|\phi|^2$ with $m \ll H_{\text{inf}}$. The scalar field acquires a nearly scale-invariant power spectrum in the inflation epoch

$$P_{\phi, \text{inf}} = \frac{H_{\text{inf}}^2}{4\pi}. \quad (4.1)$$

In the scalar QED theory, the scalar field ϕ couples to photons, which consequently induces a thermal mass for the scalar. The thermal mass can be estimated in the following. The magnitude of vector potential can be estimated by using

$$\rho_\gamma = \frac{1}{2}(\mathcal{E}^2 + \mathcal{B}^2) = \frac{1}{2}[(A'_i)^2 + (\epsilon^{jk} \partial_j A_k)^2] \equiv \frac{g_\gamma}{2} \omega^2 (A_i)^2, \quad (4.2)$$

where ρ_γ is the energy density of the background radiation. The structural constant g_γ comes from our estimation $A'_i \sim \omega A_i$ and $\partial_j A_k \sim k A_k = \omega A_k$. In principle, (4.2) is valid only for photons with a specific frequency, namely,

$$\frac{d\rho_\gamma}{dE} dE = \frac{g_\gamma(E)}{2} E^2 \frac{d(A_i(E))^2}{dE} dE. \quad (4.3)$$

In vacuum state, one has $\partial_j A_k = k A_k = \omega A_k$. If one further assumes the electric and magnetic fields contain similar energy of photon, then we see $g_\gamma = 2$. Although our case is much more complicated, $g_\gamma = 2$ would serve as a good estimation. Hereafter, we will take $g_\gamma = 2$ in the calculation.

For a gas of photons in equilibrium, the energy density per unit energy is

$$\frac{d\rho_\gamma}{dE} dE = \frac{8\pi}{(2\pi)^3} \frac{E^3 dE}{e^{E/T} - 1}, \quad (4.4)$$

so that

$$\rho_\gamma = \frac{T^4}{\pi^2} \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^2}{15} T^4. \quad (4.5)$$

Similarly, assuming g_γ varies slowly with respect to E , one has

$$\frac{d(A_i(E))^2}{dE} dE = \frac{2}{g_\gamma E^2} \frac{d\rho_\gamma}{dE} dE = \frac{2}{g_\gamma} \frac{EdE}{e^{E/T} - 1}, \quad (4.6)$$

which gives

$$\langle A_\mu A^\mu \rangle = \langle A_i A^i \rangle \sim \frac{\pi^2}{3g_\gamma} T^2. \quad (4.7)$$

Therefore, during the radiation epoch, the scalar field acquires a thermal mass $m_T^2 = \pi^2 \varrho^2 T^2/6$. The dynamical equation of the scalar field due to the thermal mass with $m_T^2 \gg m^2$ is given by

$$\phi_k'' + \frac{2}{\tau} \phi_k' + \left[k^2 + \frac{\pi^2}{6} \varrho^2 (aT)^2 \right] \phi_k = 0. \quad (4.8)$$

As we argued above, we can neglect the contribution of k^2 on the scale we are concerned with. For illustrative purposes, let us adopt one branch of general solution,

$$\phi_k \propto \frac{\sin\left(\frac{\pi}{\sqrt{6}} \varrho (aT)_{\text{RD}} \tau\right)}{\tau} \propto T \sin\left(\sqrt{\frac{15}{g_*}} \frac{\pi \varrho M_p}{T}\right). \quad (4.9)$$

Notice that the transfer function is insensitive to the relative coefficients of ϕ_k . Take the beginning of the RD epoch as

the reference time, namely $\tau_* = \tau_e$, we write the transfer function as

$$\mathcal{T}(T, T_*) = \frac{T \sin\left(\sqrt{\frac{15}{g_*}} \frac{\pi \varrho M_p}{T}\right)}{T_e \sin\left(\sqrt{\frac{15}{g_*}} \frac{\pi \varrho M_p}{T_e}\right)} = \frac{\sqrt{g_*} \sin\left(\sqrt{\frac{15}{g_*}} \frac{\pi \varrho M_p}{T}\right) T}{\sqrt{15} \pi \varrho M_p}, \quad (4.10)$$

where we have used the fact $\sin x \simeq x$ in the $|x| \ll 1$ limit. We see the scalar field is frozen as long as $m_T < aH$ and then starts to oscillate while decaying as T . Notably, the time integral and the power spectrum are scale invariant in the scale we are concerned with, so the momentum integral becomes trivial,

$$\int_1^\infty ds \int_0^1 dd (1-d^2)(s^2-d^2)^{-2}(s^2-1) = \frac{1}{2}. \quad (4.11)$$

We can easily calculate the magnetic field generated in our formalism. Before the annihilation event we have the Green's function (3.31), so

$$\begin{aligned} \mathcal{I}(T; T_e) &= - \int^T \frac{3\sqrt{10}}{\pi\gamma\sqrt{g_*}} M_p (aT)_{\text{RD}} \frac{dT' g_* \sin\left(\sqrt{\frac{15}{g_*}} \frac{\pi \varrho M_p}{T'}\right)^2}{T'^2 15\pi^2 \varrho^2 M_p^2} \\ &= - \frac{\sqrt{5g_*}}{20\sqrt{2}\pi^3 \gamma \varrho^2 M_p} (aT)_{\text{RD}} \left(-\frac{2}{T} + \frac{\sin\left(2\sqrt{\frac{15}{2}} \frac{\pi \varrho M_p}{T}\right)}{\sqrt{\frac{15}{2}} \pi \varrho M_p} \right) \\ &\simeq \frac{\sqrt{5g_*} (aT)_{\text{RD}} M_p}{\sqrt{2}\pi\gamma T} \times \begin{cases} \frac{1}{10\pi^2 \varrho^2 M_p^2} & \varrho M_p \gg T \\ \frac{1}{T^2} & \varrho M_p \ll T, \end{cases} \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} P_B(T) &= 2\varrho^2 \left(\frac{k}{a}\right)^4 P_{\phi, \text{inf}}^2 |\mathcal{I}(T; T_e)|^2 \\ &\simeq \left(\frac{k}{a}\right)^4 \frac{5g_* (aT)_{\text{RD}}^2 H_{\text{inf}}^4}{16\pi^4 \gamma^2 T^4} \times \begin{cases} \frac{T^2}{100\pi^4 \varrho^2 M_p^2} & \varrho M_p \gg T \\ \frac{\varrho^2 M_p^2}{T^2} & \varrho M_p \ll T, \end{cases} \end{aligned} \quad (4.13)$$

leading to a magnetic field strength today

$$B_{\text{today}} \simeq 1.4 \times 10^{-72} g_*^{\frac{1}{2}} \left(\frac{k/a_{\text{today}}}{1 \text{ Mpc}^{-1}}\right)^2 \frac{H_{\text{inf}}^2}{T^2} \times \begin{cases} \frac{T}{10\pi^2 \varrho M_p} & \varrho M_p \gg T \\ \frac{\varrho M_p}{T} & \varrho M_p \ll T \end{cases} [\text{G}], \quad (4.14)$$

whose upper bound is

$$\begin{aligned} B_{\text{today}} &\leq 1.5 \times 10^{-71} \left(\frac{g_*}{106.75}\right)^{\frac{1}{2}} \left(\frac{k/a_{\text{today}}}{1 \text{ Mpc}^{-1}}\right)^2 \frac{H_{\text{inf}}^2}{T_a^2} [\text{G}] \\ &\leq 1.5 \times 10^{-39} \left(\frac{g_*}{106.75}\right)^{\frac{1}{2}} \left(\frac{k/a_{\text{today}}}{1 \text{ Mpc}^{-1}}\right)^2 \left(\frac{H_{\text{inf}}}{10^{12} [\text{GeV}]}\right)^2 \left(\frac{T_a}{10^{-4} [\text{GeV}]}\right)^{-2} [\text{G}], \end{aligned} \quad (4.15)$$

which is not sufficient to explain the observations.

After the annihilation event $T_a = 0.1$ MeV, the conductivity drops. We set the reference time to be $T_* = T_a$ and get

$$\begin{aligned} \mathcal{I}(T; T_a) &\equiv \int_{T_a}^T -\frac{(aT)_{\text{RD}}^2}{\kappa^2} \frac{dT'}{T'} \frac{g_* \sin\left(\sqrt{\frac{15}{g_*}} \frac{\pi q M_p}{T'}\right)^2}{30\pi^2 q^2 M_p^2} \\ &= \frac{g_*(aT)_{\text{RD}}^2}{60\kappa^2 \pi^2 q^2 M_p^2} \left(\text{CosInt}\left(\sqrt{\frac{15}{g_*}} \frac{2\pi q M_p}{T_a}\right) - \text{CosInt}\left(\sqrt{\frac{15}{g_*}} \frac{2\pi q M_p}{T}\right) + \log \frac{T_a}{T} \right) \\ &\simeq 1.8 \times 10^{30} \frac{1}{\pi^2 q^2 M_p^2} \begin{cases} \frac{15\pi^2 q^2 M_p^2}{g_* T^2} - \frac{15\pi^2 q^2 M_p^2}{g_* T_a^2} & qM_p \ll T < T_a \\ \log \frac{T_a}{T} & qM_p \gg T_a > T \end{cases}. \end{aligned} \quad (4.16)$$

Accordingly

$$\begin{aligned} B_{\text{today}} &\simeq 10^{-30} \left(\frac{k/a_{\text{today}}}{1 \text{ Mpc}^{-1}} \right)^2 \frac{H_{\text{inf}}^2}{\pi^2 q^2 M_p^2} \times \begin{cases} \frac{15\pi^2 q^2 M_p^2}{g_* T^2} - \frac{15\pi^2 q^2 M_p^2}{g_* T_a^2} & qM_p \ll T \\ \log \frac{T_a}{T} & qM_p \gg T_a \end{cases} [\text{G}] \\ &\ll 10^{-30} \left(\frac{k/a_{\text{today}}}{1 \text{ Mpc}^{-1}} \right)^2 \frac{H_{\text{inf}}^2}{T_a M_p} \times \begin{cases} \frac{15}{g_*} \frac{T_a}{T} & qM_p \ll T \\ \log \frac{T_a}{T} & qM_p \gg T_a \end{cases} [\text{G}]. \end{aligned} \quad (4.17)$$

Assuming $H_{\text{inf}} = 10^{12}$ GeV, we have $H_{\text{inf}}^2/M_p T_a \sim 10^9$, which gives $B_{\text{today}} \ll 10^{-21} \frac{T_a}{T} [\text{G}]$. This is still insufficient to confront with observations unless we assign a ridiculously small T . We conclude that there must be an amplification for the scalar field to generate sufficient PMFs.

B. Setup and effective potential

As discussed before, we need to enhance the scalar field to generate sufficient PMFs. Let us introduce an auxiliary field χ which couples to the scalar ϕ , and the potential for the scalar field ϕ including the thermal mass

$$V(\phi) = \lambda^2 M_p \chi |\phi|^2 + \frac{\pi^2}{12} q^2 T^2 |\phi|^2. \quad (4.18)$$

We comment that the potential of ϕ shall include higher order terms, e.g., $\lambda_4^2 \phi^4$ for it to be bounded from below. Those terms have to be suppressed by the factor ϕ/M_p at the magneto-genesis epoch for the validation of the effective potential (4.18). On the other hand, the interaction term $\lambda^2 M_p \chi |\phi|^2$ has negligible contribution to the dynamics of χ field since λ is extremely small as we shall see later. Thus the dynamics of χ is approximately determined by $\mathcal{L}_\chi$.

We shall adopt some *Ansätze* for the auxiliary field χ . We assume that the χ field is initially in a local vacuum with $\langle \chi \rangle = -m^2/(\lambda^2 M_p) < 0$. Thus, when the universes cools down to a critical temperature

$$T_1 = \frac{2\sqrt{3}m}{\pi q}, \quad (4.19)$$

the effective mass of ϕ field becomes negative and ϕ will experience a tachyonic growth. The growth stop when χ transitions to the positive expectation value $\langle \chi \rangle = M^2/(\lambda^2 M_p)$. The timescale is controlled by the form of the potential for χ and the respective coupling constants. After the transition the scalar field ϕ will get a normal effective mass M and the tachyonic behavior terminates.

C. Dynamics of scalar field

Before the tachyonic phase, the scalar field ϕ locates at the true vacuum $\langle \phi \rangle = 0$ such that $\phi = \delta\phi$. The value of $\delta\phi$ can be estimated as $\langle \delta\phi \delta\phi \rangle = P_\phi$. When the temperature of the universe is high, the effective potential of ϕ is simply $V(\phi) \simeq \pi^2 q^2 T^2 \phi^2/12$ so that the thermal mass is given by $m_T^2 = \pi^2 q^2 T^2/6$. As long as $m_T < H$ the scalar field remains frozen. Using the relation (3.33), we get

$$m_T < H \rightarrow q < \sqrt{\frac{g_*}{15}} \frac{T}{M_p}, \quad (4.20)$$

as the criteria for the scalar field to be frozen.

One may wonder whether the scalar field is still frozen after the annihilation epoch, where the temperature drops below 0.1 MeV. We will evaluate the dynamics of the scalar field by assuming that it is always frozen before the tachyonic phase for the following two reasons. First, as we show in Sec. VIB, a sizable magnetic field originates from a typically small value $q \sim 10^{-25}$ and (4.20) holds even when T is of keV order. Second, from the computations in Secs. V and VI we confirm that the dynamics of scalar field has little impact on the magneto-genesis

process, since any information before the tachyonic phase is diluted by the exponential growth phase.

After reaching the critical temperature, the dynamical equation of ϕ becomes

$$\phi_k'' + \frac{2}{\tau} \phi_k' + \left[k^2 + \frac{\pi^2}{6} Q^2 (aT)^2 - 2m^2 a^2 \right] \phi_k = 0. \quad (4.21)$$

We define two auxiliary constants

$$\begin{aligned} c_1^2 &\equiv k^2 + \frac{\pi^2}{6} Q^2 (aT)^2, \\ c_2 &\equiv m \frac{a_e}{\tau_e} = \frac{\lambda m}{H_{\text{inf}} \tau_e^2} = \frac{\pi m}{3\sqrt{10} M_p} (a_{\text{eq}} T_{\text{eq}})^2 g_*^{-\frac{1}{2}} \\ &= 4.8 \times 10^{-46} \left(\frac{g_*}{106.75} \right)^{-\frac{1}{2}} m [\text{GeV}], \end{aligned} \quad (4.22)$$

where c_1 is of dimension $[M]$ and c_2 has dimension $[M]^2$, and the dynamical equation simplifies to

$$\phi_k'' + \frac{2}{\tau} \phi_k' + (c_1^2 - c_2^2 \tau^2) \phi_k = 0, \quad (4.23)$$

whose general solution is

$$\begin{aligned} \phi_k &= C_1 \frac{e^{-\frac{c_2^2 \tau^2}{2}}}{\tau} F_1 \left(\frac{1}{4} - \frac{c_1^2}{4c_2}, \frac{1}{2}, c_2 \tau^2 \right) \\ &+ C_2 \frac{e^{-\frac{c_2^2 \tau^2}{2}}}{\tau} \mathbf{H} \left(\frac{c_1^2}{2c_2} - \frac{1}{2}, \sqrt{c_2} \tau \right), \end{aligned} \quad (4.24)$$

where F_1 represents the Kummer confluent hypergeometric function and $\mathbf{H}$ represents the Hermitian polynomial. Notably, in the $\tau \rightarrow \infty$ limit, the branch of Hermitian polynomial approximates to a constant, and another branch has the following behavior

$$\frac{e^{-\frac{c_2^2 \tau^2}{2}}}{\tau} F_1 \left(\frac{1}{4} - \frac{c_1^2}{4c_2}, \frac{1}{2}, c_2 \tau^2 \right) \rightarrow \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{4} - \frac{c_1^2}{4c_2}\right)} \frac{e^{\frac{c_2^2 \tau^2}{2}}}{\tau} (c_2 \tau^2)^{-\frac{1}{4} - \frac{c_1^2}{4c_2}}. \quad (4.25)$$

Thus, the hypergeometric branch represents the exponential growing section since $\tau^2 \propto t$ and will be our focus. We calculate

$$\frac{c_1^2}{c_2} = \frac{4.2 \times 10^{19} Q^2 + 1.2 \times 10^{-32} \left(\frac{k}{1 \text{ Mpc}^{-1}} \right)^2}{m/[\text{GeV}]} \left(\frac{g_*}{106.75} \right)^{\frac{1}{2}}, \quad (4.26)$$

which will be important in the following calculations.

D. Matching condition and exponential amplification

In this section, we aim to fix the integration constant C_1 and C_2 in Eq. (4.24). First, the combination $c_2 \tau^2$ is a constant in inflationary epoch,

$$c_2 \tau^2 = m \frac{a}{\tau} \tau^2 = m a \tau = \frac{m}{H} \ll 1, \quad (4.27)$$

Further, this quantity scales as T^{-2} in the RD epoch, so

$$c_2 \tau_1^2 = \frac{m T_{\text{inf}}^2}{H T_1^2} = \frac{3\sqrt{10}\pi m M_p}{\sqrt{g_*} T_1 T_1} \quad (4.28)$$

where T_1 , defined by (4.19), is the critical temperature when the tachyonic phase begins and we denote the corresponding conformal time as τ_1 . Later in Sec. VIB, we will show that $m M_p \ll T_1^2$, enabling us to set $c_2 \tau_1^2 \ll 1$ and write (4.24) as

$$\phi_k(\tau) = \frac{C_1}{\tau} + \frac{C_2 \sqrt{\pi}}{2\sqrt{2}} \left(\frac{2}{\tau \Gamma(\frac{3}{4})} - \frac{\sqrt{c_2}}{\Gamma(\frac{3}{4})} \right) + \mathcal{O}(\tau), \quad \tau \sim \tau_1. \quad (4.29)$$

As the scalar field is frozen till the critical epoch the initial conditions are simply given by

$$\phi_k = \frac{H_{\text{inf}}}{2\sqrt{\pi}}, \quad \phi_k' = 0. \quad (4.30)$$

A direct match of the value of ϕ_k and ϕ_k' at the critical time τ_1 indicates that the τ^{-1} term must vanish, so

$$C_1 = -\frac{C_2 \sqrt{\pi}}{2\sqrt{2}} \frac{2}{\Gamma(\frac{3}{4})}, \quad -\frac{\sqrt{c_2} C_2 \sqrt{\pi}}{\Gamma(\frac{3}{4}) 2\sqrt{2}} = \frac{H_{\text{inf}}}{2\sqrt{\pi}}, \quad (4.31)$$

which gives

$$C_1 \simeq \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{3}{4})} \frac{H_{\text{inf}}}{\sqrt{\pi c_2}}. \quad (4.32)$$

We can understand the fact from an alternative perspective. At the beginning of the RD epoch, the scalar fluctuations we are concerned with are deeply outside the horizon and are approximately frozen. By frozen, we mean ϕ_k' should be much less than $\mathcal{H}^2$. The term proportional to τ^{-1} in ϕ_k would lead to a τ^{-2} term in ϕ_k' , which is comparable to $\mathcal{H}^2 \propto \tau^{-2}$. Therefore, the frozen scalar fluctuation leads to the disappearance of coefficients of the τ^{-1} term.

When $c_2 \tau^2 > 1$, the exponential growth begins and we can use the asymptotic expression of the hypergeometric function to write

$$\phi_k \simeq \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4}-\frac{c_1^2}{4c_2})} \frac{H_{\text{inf}}}{e^{\frac{c_2}{2}\tau^2}} (c_2\tau^2)^{-\frac{3}{4}-\frac{c_1^2}{4c_2}}. \quad (4.33)$$

We define an auxiliary parameter

$$c_3^2 \equiv \frac{3\sqrt{10}\pi}{2\sqrt{g_*}} M_p m = 1.7 \times 10^{19} \left(\frac{g_*}{106.75} \right)^{-\frac{1}{2}} m [\text{GeV}], \quad (4.34)$$

so that $c_2\tau^2/2 = c_3^2/T^2$ and we can write

$$\phi_k \simeq 0.20 \times 2^{-\frac{3}{4}-\frac{c_1^2}{4c_2}} H_{\text{inf}} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{4}-\frac{c_1^2}{4c_2})} \left(\frac{T}{c_3} \right)^{\frac{3}{2}+\frac{c_1^2}{2c_2}} e^{\frac{c_2}{2}\tau^2}, \quad (4.35)$$

up to an overall phase factor, and in terms of T , one can easily check that ϕ_k has the dimension of mass. The validity of (4.35) starts to break down when χ deviates from the local vacuum. Since initially $\langle\phi\rangle = 0$, the new VEV of the scalar field is decided by the fluctuations with the longest wavelength, namely $\langle\phi\rangle = \phi_k(k=0)$. As we shall reveal in the following sections, $c_1^2 \ll c_2$ in the limit $k \rightarrow 0$, thus the reference slice (labeled by $T = T_2$) when the tachyonic phase terminates is indicated by

$$v = \frac{0.38}{\pi} H_{\text{inf}} \left(\frac{T_2}{c_3} \right)^{3/2} e^{\frac{c_2}{2}\tau^2}. \quad (4.36)$$

E. Scale dependence of scalar power spectrum

Finally, let us discuss the scale dependence of scalar power spectrum, which is crucial to the evaluation of momentum integral. The k -dependence of the power spectrum originates from $c_1(k)$, so we first write down the c_1 dependence of P_ϕ

$$P_\phi(c_1) \propto \frac{1}{\Gamma^2(\frac{1}{4}-\frac{c_1^2}{4c_2})} \left(\frac{T^2}{2c_3^2} \right)^{\frac{c_1^2}{2c_2}}. \quad (4.37)$$

It is straightforward to see that P_ϕ monotonously decreases with respect to the ratio c_1^2/c_2 and $P_\phi = 0$ when $c_1^2 = c_2$. This fact can be understood as follows. The exponential growth begins when $c_2\tau^2 > 1$. However, if in this case $c_1^2 > c_2$, then $m_{\text{eff}}^2 = c_1^2 - c_2\tau^2 > 0$. This means scalar fluctuations with $c_1^2 > c_2$ cannot feel the exponential growth, and the associated P_ϕ is negligible. We conclude that $P_\phi(k) \simeq 0$ when the wave number satisfies $c_1^2(k) \geq c_2$.

Thus, to determine the scale dependence of P_ϕ , it is important to emphasize the $c_1(k)$ relation. Utilizing (4.26), we find that c_1^2/c_2 is approximately a constant when $k \ll 5.9 \times 10^{25} \varrho \text{ Mpc}^{-1}$ and c_1^2/c_2 scales as k^2 when $k \gg 5.9 \times 10^{25} \varrho \text{ Mpc}^{-1}$.

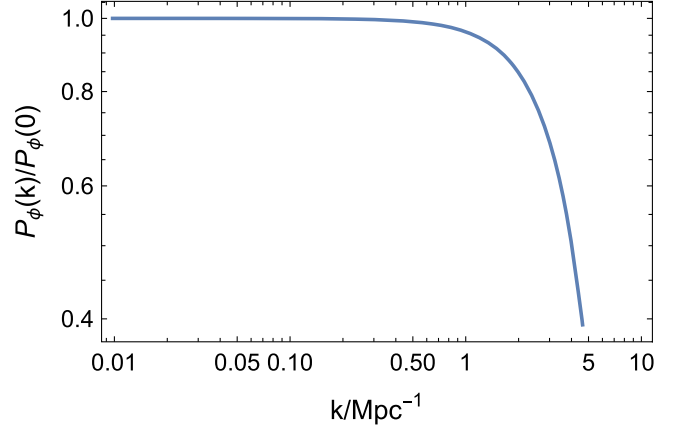


FIG. 1. Scalar power spectrum as a function of k . We take $\rho = 1.7 \times 10^{-26}$ and $m = 1.2 \times 10^{-30}$ (GeV) for illustrative purposes such that the critical scale is $k_c = 5.9 \times 10^{25} \varrho \text{ Mpc}^{-1} \simeq 1 \text{ Mpc}^{-1}$.

With all above considerations, we see that P_ϕ is approximately a constant before a critical scale, and decreases rapidly, and illustrate the scalar power spectrum in Fig. 1 to verify our conclusion. It is easy to see that the critical scale is determined by $k_c = 5.9 \times 10^{25} \varrho \text{ Mpc}^{-1}$, dependent solely on model parameter ϱ . Large scales with $k < k_c$ are approximately scale invariant with $P_\phi(k) \simeq v^2$, while the scalar power spectrum quickly diminishes on small scales with $k > k_c$. For simplicity, we may take $P_\phi(k) \simeq 0$ when $k > k_c$, and write

$$P_\phi = v^2 \Theta(k_c - k), \quad (4.38)$$

where Θ is the Heaviside function.

V. MAGNETO-GENESIS BEFORE THE ANNIHILATION

Now that we have successfully amplified the scalar field in the RD epoch, the next step is to calculate the induced magnetic field. In this section, we first work out the case when magneto-genesis happens before the e^+e^- annihilation epoch. In this section, we compute the magnetic field generated during the tachyonic phase before the annihilation event. As the evaluations of the magnetic field in different cases follow a similar logic, we will first illustrate the technique details extensively in Sec. VA, and include only main results in the subsequent cases.

A. Magnetic field generated during the tachyonic phase

1. Analytical expression of magnetic power spectrum

We set the reference slice of the time integral to be $T = T_2$, and the power spectrum of the induced magnetic field at $T = T_2$ is

$$P_B(k, T) = 4Q^2 \left(\frac{k}{a}\right)^4 \int_1^\infty ds \int_0^1 dd(1-d^2)(s^2-d^2)^{-2}(s^2-1) \\ \times P_\phi\left(\frac{s+d}{2}k, T_2\right) P_\phi\left(\frac{s-d}{2}k, T_2\right) |\mathcal{I}(T_2, T_1; T_2)|^2, \quad (5.1)$$

$$\mathcal{I}(T_2, T_1; T_2) \equiv \int_{T_1}^{T_2} -\frac{3\sqrt{10}}{2\pi\gamma\sqrt{g_*}} M_p(aT)_{\text{RD}} \frac{dT}{T^4} \mathcal{T}(T; T_2) \mathcal{T}(T; T_2). \quad (5.2)$$

There is a tricky point. The argument of Hypergeometric function ranges from $c_2\tau^2 \ll 1$ to $c_2\tau^2 \rightarrow \infty$, where the function exhibits different asymptotic behavior. It would be difficult to perform the time integral without any simplification of special functions. Nonetheless, before the exponential amplification, the generated magnetic field is too small to be compatible with observations. Thus, we can safely calculate the transfer function using the asymptotic form of the Hypergeometric function at $c_2\tau^2 \rightarrow \infty$. It is easy to see the transfer function during this period simplifies to

$$\mathcal{T}(T; T_2) = \frac{\phi_k(T)}{\phi_k(T_2)} \simeq \left(\frac{T}{T_2}\right)^{\frac{3}{2} + \frac{c_1^2}{2c_2}} e^{c_3^2(T^{-2} - T_2^{-2})}. \quad (5.3)$$

The time integral becomes

$$\begin{aligned} \mathcal{I} &= \int_{T_1}^{T_2} -\frac{3\sqrt{10}}{2\pi\gamma\sqrt{g_*}} M_p(aT)_{\text{RD}} \frac{dT}{T^4} \left(\frac{T}{T_2}\right)^{\frac{c_1^2}{c_2}-1} e^{2c_3^2(T^{-2} - T_2^{-2})} \\ &= \frac{3\sqrt{10}}{4\pi\gamma\sqrt{g_*}} M_p(aT)_{\text{RD}} \left[\left(\frac{-c_3^2}{2T^2}\right)^{\frac{c_1^2}{2c_2}} T e^{-\frac{2c_3^2}{T_2^2}} \left(\frac{T}{T_2}\right)^{\frac{c_1^2}{c_2}-1} \Gamma\left(\frac{c_1^2}{2c_2}, -\frac{2c_3^2}{T^2}\right) \right]_{T_1}^{T_2} \\ &\simeq \frac{3\sqrt{10}}{4\pi\gamma\sqrt{g_*}} \frac{M_p}{T_2^4} (aT)_{\text{RD}} \left[\frac{T_2^3}{2c_3^2} - \frac{T_1^3}{2c_3^2} \left(\frac{T_1}{T_2}\right)^{\frac{c_1^2}{c_2}-1} e^{2c_3^2(T_1^{-2} - T_2^{-2})} \right] \\ &= \frac{3\sqrt{10} \frac{M_p}{T_2^4} (aT)_{\text{RD}}}{8\pi\gamma T_2 c_3^2 \sqrt{g_*}} \times \left[1 - \left(\frac{T_1}{T_2}\right)^{\frac{c_1^2}{c_2}+2} e^{2c_3^2(T_1^{-2} - T_2^{-2})} \right] \\ &\simeq \frac{5.0 \times 10^{-6}}{mT_2} \times \left[1 - \left(\frac{T_1}{T_2}\right)^{\frac{c_1^2}{c_2}+2} e^{-2c_3^2/T_2^2} \right], \quad (5.4) \end{aligned}$$

where we used the fact $c_3 \gg T_1$ to admit the exponential amplification and also $T_1 \gg T_2$.

From the argument in Sec. IV E, the power spectrum P_ϕ can be nontrivial only if $c_1^2 \ll c_2$. Thus, we are justified to collect the contributions of momentum integral when this limit is satisfied, where (5.4) simplifies to

$$\mathcal{I} \simeq \frac{5.0 \times 10^{-6}}{mT_2} \times \left[1 - \left(\frac{T_1}{T_2}\right)^2 e^{-2c_3^2/T_2^2} \right]. \quad (5.5)$$

In addition, the expression of transfer function (5.3) is valid only if $T_1 < c_3$, otherwise (4.35) is not applicable anymore. Thus

$$0 < \left(\frac{T_1}{T_2}\right)^2 e^{-2c_3^2/T_2^2} \leq \left(\frac{c_3}{T_2}\right)^2 e^{-2c_3^2/T_2^2} \ll 1, \quad (5.6)$$

and we finally arrive at

$$\mathcal{I} \simeq \frac{5.0 \times 10^{-6}}{mT_2}, \quad (5.7)$$

which greatly simplifies the power spectrum to

$$P_B(k, T) \simeq \frac{10^{-10}}{m^2 T_2^2} Q^2 \left(\frac{k}{a}\right)^4 \int_1^\infty ds \\ \times \int_0^1 dd(1-d^2)(s^2-d^2)^{-2}(s^2-1) \\ \times P_\phi\left(\frac{s+d}{2}k, T_2\right) P_\phi\left(\frac{s-d}{2}k, T_2\right). \quad (5.8)$$

To proceed, we need to use the expression of P_ϕ , Eq. (4.38). Apparently, the momentum integral vanished when $k/k_c \geq 2$. In the case $k/k_c < 2$, the step function in (4.38) gives

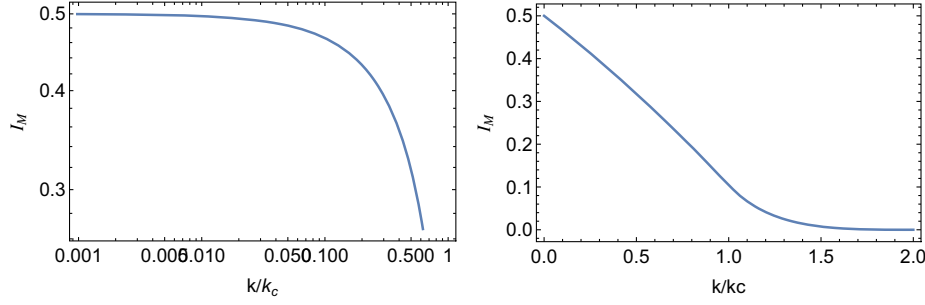


FIG. 2. Momentum integral as a function of k/k_c . We demonstrate $\mathcal{I}_M$ for the range $k/k_c < 1$ in the left panel and $k/k_c \leq 2$ in the right panel. It is evident that $\mathcal{I}_M$ remains roughly a constant when $k/k_c \leq 0.1$ and quickly shrinks to 0 when $k/k_c > 0.1$.

$$0 \leq \frac{k_1}{k_c} \leq 1, \quad 0 \leq \frac{k_2}{k_c} \leq 1 \rightarrow 0 \leq s \pm d \leq \frac{2k_c}{k}, \quad (5.9)$$

along with the constraints $s \geq 1$; we can translate the step function by a modified integration range of the momentum integral. Let us define

$$\mathcal{I}_M \equiv \int_1^\infty ds \int_0^1 dd (1-d^2)(s^2-d^2)^{-2}(s^2-1) \times \Theta(k_c - k_1)\Theta(k_c - k_2), \quad (5.10)$$

when $1 \leq k/k_c \leq 2$, we have

$$\mathcal{I}_M = \int_1^{\frac{2k_c}{k}} ds \int_0^{\frac{2k_c}{k}-s} dd (1-d^2)(s^2-d^2)^{-2}(s^2-1), \quad (5.11)$$

and when $0 \leq k/k_c \leq 1$, we have

$$\begin{aligned} \mathcal{I}_M &= \left(\int_1^{\frac{k_c}{k}} ds \int_0^{\min(s,1)} dd + \int_{\frac{k_c}{k}}^{\frac{2k_c}{k}} ds \int_0^{\min(1, \frac{2k_c}{k}-s)} dd \right) (1-d^2)(s^2-d^2)^{-2}(s^2-1) \\ &= \left(\int_1^{\frac{2k_c}{k}-1} ds \int_0^1 dd + \int_{\frac{2k_c}{k}-1}^{\frac{2k_c}{k}} ds \int_0^{\frac{2k_c}{k}-s} dd \right) (1-d^2)(s^2-d^2)^{-2}(s^2-1). \end{aligned} \quad (5.12)$$

We numerically evaluate $\mathcal{I}_M$ and depict the result in Fig. 2. It is evident that $\mathcal{I}_M \approx 1/2$ when $k/k_c \leq 0.1$, consistent with the analysis in Sec. IV A, and decreases to 0 when k/k_c grows. In the case $\mathcal{I}_M \approx 1/2$, the magnetic power spectrum is

$$P_B \simeq 5.0 \times 10^{-11} q^2 \left(\frac{k}{a} \right)^4 \frac{v^4}{\lambda^2 m^2 T_2^2}, \quad (5.13)$$

which is a highly blue spectrum with spectra index 4. When $k/k_c \geq 1$ it is difficult to compute the magnetic power spectrum analytically. We numerically evaluate P_B and show the result in Fig. 3. We find that the magnetic power spectrum has an approximate spectra index 4 when $k < k_c$, and almost vanishes on the scale $k > k_c$. Therefore, the full magnetic power spectrum can be approximately written as

$$P_B \simeq 5.0 \times 10^{-11} q^2 \left(\frac{k}{a} \right)^4 \frac{v^4}{\lambda^2 m^2 T_2^2} \Theta(k_c - k). \quad (5.14)$$

It would be convenient to write the resultant magnetic power spectrum in terms of ρ_ϕ , the energy density of scalar

field, to discuss the backreaction of both the scalar field and magnetic fields. First, from the action (2.1) and the effective potential (4.18), we see the scalar fields acquire an effective mass

$$m_{\text{eff}}^2 = \lambda^2 M_p \chi + \frac{\pi^2}{6} q^2 T^2 + q^2 A_\mu A^\mu. \quad (5.15)$$

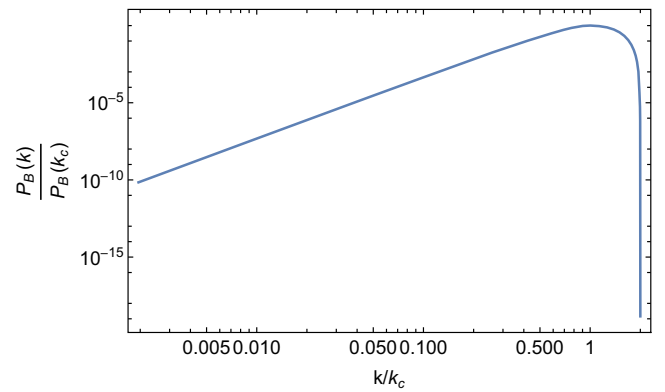


FIG. 3. Magnetic power spectrum rescaled by $P_B(k = k_c)$.

The term $q^2 A_\mu A^\mu$ represents the backreaction of the induced magnetic fields to the scalar field ϕ , which is second order and should be negligibly small. This can be verified by a simple estimation. The effective mass from the $A_\mu A^\mu |\phi|^2$ term is proportional to the energy density of induced magnetic fields ρ_B , which is of order $\mathcal{O}(10^{-72} [\text{GeV}]^4)$ to be compatible with blazar observation as we will show in Sec. VI C. On the other hand the temperature in radiation dominated epoch is $T > T_{\text{eq}} = 10^{-9} [\text{GeV}]$, thus the effective mass term $q^2 A_\mu A^\mu$ is much smaller than $\frac{\pi^2}{12} q^2 T^2$. In addition, after the tachyonic amplification, the auxiliary field χ rolls into the local vacuum $\langle \chi \rangle = M^2 / \lambda^2 M_p$, so the effective mass after tachyonic amplifications is

$$m_{\text{eff}}^2 \simeq M^2 + \frac{\pi^2}{6} q^2 T^2, \quad (5.16)$$

and the energy density of the scalar field is accordingly

$$\frac{d \ln \rho_\phi}{d \ln k} = P_\phi \times \begin{cases} \frac{\pi^2}{12} q^2 T^2 & qT \gg M \\ \frac{\lambda^2 M^2}{2} & qT \ll M \end{cases}. \quad (5.17)$$

When $qT \gg M$, ρ_ϕ scales as a^{-4} , and when $qT \ll M$, ρ_ϕ scales as a^{-3} , similarly to cold dark matter (CDM). We define

$$r \equiv \frac{\rho_\phi}{\rho_{\text{CDM}}}, \quad (5.18)$$

to be the ratio of ρ_ϕ versus ρ_{CDM} , the energy density of CDM. Now, r either decreases when $\rho T \gg M$ or remains constant when $qT \ll M$, so r reaches its maximum at $T = T_2$, the end of tachyonic phase. In addition, $\rho_{\text{CDM}} \leq \rho_{\text{bg}}$ in RD epoch. We conclude that $r(T = T_2) \ll 1$ is a sufficient condition for the energy density of ϕ to be negligibly smaller than the background one. This condition translates to

$$\begin{aligned} 1 \gg r(T_2) &\simeq \frac{P_\phi N}{3H_{\text{eq}}^2 M_p^2 (T_2/T_{\text{eq}})^3} \times \begin{cases} \frac{\pi^2}{6} q^2 T_2^2 & qT \gg M \\ M^2 & qT \ll M \end{cases} \\ &= \frac{5q^2 v^2 N}{2T_2 T_{\text{eq}}} \times \begin{cases} 1 & qT \gg M \\ \frac{6M^2}{\pi^2 q^2 T_2^2} & qT \ll M \end{cases}, \end{aligned} \quad (5.19)$$

where N is the e-folding number of inflation resulting from the momentum integration $d \ln k$ with respect to a scale-invariant power spectrum. In the following, we will use the shorthand notation $r \equiv r(T_2)$ as an abuse of notation. Note, that while the amplitude of the scalar field gets highly enhanced during the tachyonic phase, the scalar field energy density can still be highly subdominant $r \ll 1$ as long as the mass is sufficiently low $M \lesssim qT$ as the energy density scales as $r \propto q^2$. In particular, the

scaling is the same as for the magnetic power spectrum $P_B \propto q^2$. Indeed, in terms of r , the magnetic power spectrum is

$$\begin{aligned} P_B &= 2.0 \times 10^{-11} \left(\frac{k}{a} \right)^4 \frac{v^2}{m^2} \frac{r}{N} \frac{T_{\text{eq}}}{T_2} \Theta(k_c - k) \\ &\times \begin{cases} 1, & qT \gg M \\ \frac{\pi^2 q^2 T_2^2}{6M^2}, & qT \ll M \end{cases}. \end{aligned} \quad (5.20)$$

2. Coherence length and connection to observations

The key parameter in magneto-genesis is the coherence length, estimated by the wavelengths of magnetic fields averaged by the energy density,

$$\lambda_B \simeq \frac{\int \frac{d^3 k}{(2\pi)^3} \lambda_k \rho(k)}{\int \frac{d^3 k}{(2\pi)^3} \rho(k)} = \frac{\int \frac{d^3 k}{(2\pi)^3} \frac{2\pi a}{k} P_B(k)}{\int \frac{d^3 k}{(2\pi)^3} P_B(k)}. \quad (5.21)$$

Since the magnetic field scales as $P_B \propto (k/a)^4 \Theta(k_c - k)$, the coherence length is

$$\lambda_B \simeq \frac{\int_0^{k_c} 2\pi k^2 dk \left(\frac{k}{a} \right)^3}{\int_0^{k_c} k^2 dk \left(\frac{k}{a} \right)^4} = \frac{7}{6} \frac{2\pi a}{k_c}. \quad (5.22)$$

The blazar observations suggest a preferred range of the strength of magnetic fields with a coherence length of a few Mpc [2,7,8],

$$B_{\text{today}} \sim 10^{-15} - 10^{-17} G, \quad \lambda_{B,\text{today}} \sim 1 \text{ Mpc}. \quad (5.23)$$

Thus, to be consistent with observations, we have

$$\frac{7}{6} \frac{2\pi a_{\text{today}}}{k_c} \sim 1 \text{ Mpc}, \quad \frac{k_c}{a_{\text{today}}} \simeq 1.4 \times 10^{-1} \text{ Mpc}^{-1}. \quad (5.24)$$

In light of the expression of $k_c = 5.9 \times 10^{25} q \text{ Mpc}^{-1}$, implying a suggested model parameter

$$q \simeq 2.3 \times 10^{-27} t \quad (5.25)$$

the integrated magnetic field strength is estimated by the root mean square of the magnetic power spectrum,

$$B_{\text{RMS}}^2 = \int \frac{dk}{k} P_B(k). \quad (5.26)$$

From the expression of the magnetic power spectrum (5.20), it is evident the magnetic power spectrum takes its maximum when $qT_2 \gg M$, namely the effective thermal mass being much larger than the normal mass. For convenience let us work in this case, where the induced magnetic field is,

$$\begin{aligned}
B^2 &= \int \frac{dk}{k} 2.0 \times 10^{-11} \left(\frac{k}{a}\right)^4 \frac{v^2}{m^2} \frac{r}{N} \frac{T_{eq}}{T_2} \Theta(k_c - k) \\
&= \int_0^{k_c} \frac{dk}{k} 2.0 \times 10^{-11} \left(\frac{k}{a}\right)^4 \frac{v^2}{m^2} \frac{r}{N} \frac{T_{eq}}{T_2} \\
&= 5.0 \times 10^{-12} \left(\frac{k_c}{a}\right)^4 \frac{v^2}{m^2} \frac{r}{N} \frac{T_{eq}}{T_2}. \tag{5.27}
\end{aligned}$$

Therefore, the integrated magnetic field today is

$$\begin{aligned}
B(\text{today}) &\leq 9.2 \times 10^{-65} \left(\frac{k_c/a_{\text{today}}}{0.14 \text{ Mpc}^{-1}}\right)^2 \left(\frac{r/N}{10^{-2}}\right)^{\frac{1}{2}} \frac{v}{m} \\
&\quad \times \left(\frac{T_2}{T_{eq}}\right)^{-\frac{1}{2}} [\text{G}]. \tag{5.28}
\end{aligned}$$

We adopt an extreme parameter set $v = M_p$, $T_1 = 10T_2$ and $H_{\text{inf}} = 10^{12}$ GeV, which surely makes an overestimation of B , then (4.36) tells $c_3/T_2 = 4.5$, which in turn gets the value of m . The resultant magnetic field is

$$\begin{aligned}
B(\text{today}) &< 2.9 \times 10^{-32} \left(\frac{k/a_{\text{today}}}{1 \text{ Mpc}^{-1}}\right)^2 \left(\frac{r/N}{10^{-2}}\right)^{\frac{1}{2}} \\
&\quad \times \left(\frac{T_2}{[\text{GeV}]}\right)^{-\frac{5}{2}} \left(\frac{g_*}{106.75}\right)^{-\frac{1}{2}} [\text{G}], \tag{5.29}
\end{aligned}$$

which has an upper bound $2.9 \times 10^{-22} [\text{G}]$ when we take $T_2 = T_a = 10^{-4} [\text{GeV}]$, marginally possible for seed magnetic fields. Notice that the g_* dependence emerges from the definition of c_3 (4.34). We conclude that the magnetic field generated during the tachyonic phase is too small to account for observations even after our overestimation.

B. Magnetic field generated after the tachyonic phase

There will be magneto-genesis after the tachyonic phase as well. To estimate the scalar power spectrum after $T = T_2$, we will simply assume that the transition of field χ to the new vacuum is instantaneous. Therefore, the scalar power spectrum after the transition is estimated as $P_\phi(T = T_2^+) \simeq v^2$. The transfer function becomes

$$\mathcal{T}(T; T_2) = \frac{\phi_k(T)}{\phi_k(T_2)} = \left(\frac{T}{T_2}\right)^n, \quad T_3 < T < T_2, \tag{5.30}$$

where T_3 is the termination of magneto-genesis and $n = 1$ for the $qT \gg \lambda M$ case while $n = 3/2$ for the $qT \ll \lambda M$ case. Here we ignore the oscillations and only focus on the decaying part of the transfer function. The time integral becomes

$$\mathcal{I}(T_3, T_2; T_2) = \int_{T_2}^{T_3} -\frac{3\sqrt{10}}{2\pi\gamma\sqrt{g_*}} M_p (aT)_{\text{RD}} \frac{dT}{T^{4-2n} T_2^{2n}}. \tag{5.31}$$

Assuming $T_3 \ll T_2$, we get

$$|\mathcal{I}(T_3, T_2; T_2)| \simeq \frac{3\sqrt{10}}{2\pi\sqrt{g_*}} \frac{M_p (aT)_{\text{RD}}}{\gamma T_2^3} \times \begin{cases} \frac{T_2}{T_3} & n = 1 \\ \log \frac{T_2}{T_3} & n = 3/2 \end{cases}, \tag{5.32}$$

and accordingly

$$P_B = \frac{45q^2}{g_*\pi^2} \left(\frac{k}{a}\right)^4 \frac{(aT)_{\text{RD}}^2 v^4 M_p^2}{\gamma^2 T_2^6} \times \begin{cases} \frac{T_2^2}{T_3^2} & n = 1 \\ \log^2 \frac{T_2}{T_3} & n = 3/2 \end{cases}. \tag{5.33}$$

In terms of r , the magnetic power spectrum becomes

$$\begin{aligned}
P_B &= \frac{18}{\pi^2 g_*} \left(\frac{k}{a}\right)^4 \frac{(aT)_{\text{RD}}^2 v^2 M_p^2 T_{eq}}{\gamma^2 T_2^3 T_3^2} \left(\frac{r}{N}\right) \\
&\quad \times \begin{cases} 1 & n = 1 \\ \frac{\pi^2 q^2 T_2^2 \log^2 \frac{T_2}{T_3}}{6\lambda^2 M^2 T_2^2/T_3^2} & n = 3/2 \end{cases}. \tag{5.34}
\end{aligned}$$

In the case $n = 3/2$, we have $q^2 T_2^2 \ll M^2$. Additionally, $\log(T_2/T_3)/(T_2/T_3) < 1$ since $T_2 > T_3$. We conclude that the induced magnetic field also reaches its maximum when $n = 1$, namely $qT > \lambda M$. So we will focus on the case $n = 1$, where the induced magnetic field is

$$\begin{aligned}
B &= \sqrt{\frac{9}{2\pi^2 g_*} \left(\frac{k_c}{a}\right)^4 \frac{(aT)_{\text{RD}}^2 v^2 M_p^2 T_{eq}}{\gamma^2 T_2^3 T_3^2} \left(\frac{r}{N}\right)} \\
&= 1.6 \times 10^{-23} \left(\frac{k_c/a_{\text{today}}}{0.14 \text{ Mpc}^{-1}}\right)^2 \left(\frac{r/N}{10^{-2}}\right)^{\frac{1}{2}} \left(\frac{v}{M_p}\right) \\
&\quad \times \left(\frac{T_2}{1 \text{ MeV}}\right)^{-\frac{3}{2}} \left(\frac{T_3}{0.1 \text{ MeV}}\right)^{-1} [\text{G}], \tag{5.35}
\end{aligned}$$

marginally acceptable even if we have adopted the extreme parameter set $v = M_p$ and $T_3 = T_a$. We conclude that our formalism cannot generate enough cosmic magnetic field before the e^+e^- annihilation epoch. Nonetheless, it is inspiring to see that we are quite close to sizable seed magnetic fields. Naturally, we would expect that after the annihilation epoch, the electric conductivity drops about ten orders of magnitude, and accordingly the induced magnetic field could be ten orders larger. Intuitively, a sufficient magnetic field can be generated if the magneto-genesis happens after the annihilation epoch. We will verify this assertion in the next section.

VI. MAGNETOGENESIS AFTER THE ANNIHILATION EPOCH

A. Magnetic field generated during the tachyonic phase

The magnetic field generated after the annihilation epoch can be evaluated in a same way. We also start with the magnetogenesis during the tachyonic phase, where

$$\mathcal{I}(T_2, T_1; T_2) \equiv - \int_{T_1}^{T_2} \frac{(aT)_{\text{RD}}^2}{2\kappa^2} \frac{dT}{T^3} \mathcal{T}_{k_a}(T; T_2) \mathcal{T}_{k_b}(T; T_2). \quad (6.1)$$

Implemented with the transfer function (5.3), we have

$$\begin{aligned} \mathcal{I}(T_2, T_1; T_2) &= \int_{T_1}^{T_2} -5.0 \times 10^{29} \left(\frac{g_*}{106.75} \right)^{-\frac{1}{2}} \frac{dT}{T_2^3} e^{\frac{2c_3^2}{T_2^2}} e^{-\frac{2c_3^2}{T^2}} \\ &\simeq \frac{1.2 \times 10^{29}}{c_3^2} \left(\frac{g_*}{106.75} \right)^{-\frac{1}{2}} = \frac{7.3 \times 10^9}{m[\text{GeV}]}. \end{aligned} \quad (6.2)$$

The power spectrum of the magnetic field is

$$P_B = 1.1 \times 10^{20} q^2 \frac{v^4}{m^2 [\text{GeV}]^2} \left(\frac{k}{a} \right)^4. \quad (6.3)$$

Given a typical coherence length $\lambda_B \simeq 1$ Mpc, we have $k_c/a_{\text{today}} = 0.14 \text{ Mpc}^{-1}$ and $q = 2.4 \times 10^{-27}$, the integrated magnetic field is

$$\begin{aligned} B(\text{today}) &= 5.1 \times 10^{-76} \left(\frac{q}{2.4 \times 10^{-27}} \right) \left(\frac{k_c/a_{\text{today}}}{0.14 \text{ Mpc}^{-1}} \right)^2 \\ &\times \frac{v^2}{m[\text{GeV}]} [\text{G}]. \end{aligned} \quad (6.4)$$

We shall also check the backreaction effect of the magnetic field. The induced magnetic field gives negligible backreactions if $r/N \ll 1$. After a quick examination we find that the magnetic field today also reaches its maximum when $qT_2 \gg \lambda M$, where

$$r/N = \frac{5q^2 v^2}{2T_2 T_{\text{eq}}}. \quad (6.5)$$

The value of m has to be determined through (4.36). Since we assume the tachyonic phase happens after the annihilation epoch, we must have $T_1 \leq 0.1$ MeV. We set $H_{\text{inf}} = 10^{12}$ GeV and $T_1 = 0.1$ MeV and organize the corresponding magnetic field from different model parameters in Table I. Note that the g_* dependence is implicitly in the ratio v/m according to (4.36). We see that a sizable PMF can be generated during the tachyonic phase with different model parameters. Specifically, seeds of magnetic fields with $B \geq 10^{-21}$ [G] can be produced for $v \geq 10^{12}$ GeV and appropriate T_2 during the tachyonic phase. In addition, it is suggested that the value of v shall exceed 10^{14} GeV to generate a PMF of $\mathcal{O}(10^{-16})$ [G] size during the tachyonic phase.

Still, there are theoretical constraints that shall be considered. First, (4.36) indicates a lower bound of v once H_{inf} , T_1 and T_2 are given. For a typical value $H_{\text{inf}} = 10^{12}$ GeV, $T_1 = 0.1$ MeV and $T_2 = 0.01$ MeV, v has a lower bound of order 10^{10} GeV. Additionally, the tachyonic growth can happen only if $c_1^2/c_2 \ll 1$. Using (4.26), this condition translates into $4.2 \times 10^{19} q^2 [\text{GeV}]/m \ll 1$ for large scale fluctuations. Since we fixed $q = 2.4 \times 10^{-27}$ to ensure $\lambda_B = 1$ [Mpc], this condition further leads to the constraints $m \gg 2.4 \times 10^{-34} [\text{GeV}]$. We confirm that all parameters in Table I meet this criteria. Furthermore, we find that $r/N \ll 10^{-10}$ due to the smallness of q . We conclude that the energy density of scalar field is much smaller than that of CDM in our scenario, so the scalar field ϕ shall neither introduce severe backreaction issues nor change the predictions of the standard Λ CDM paradigm.

B. Magnetic field generated after the tachyonic phase

After the tachyonic phase, the time integral is

$$\begin{aligned} \mathcal{I}(T_3, T_2; T_2) &= \int_{T_2}^{T_3} - \frac{(aT)_{\text{RD}}^2}{2\kappa^2} \frac{dT}{T^{3-2n} T_2^{2n}} \\ &= \frac{(aT)_{\text{RD}}^2}{2\kappa^2} \times \begin{cases} T_2^{-2} \ln \frac{T_2}{T_3} & qT \gg M \\ T_2^{-3} (T_2 - T_3) & qT \ll M \end{cases}. \end{aligned} \quad (6.6)$$

TABLE I. Numerical result for our magneto-genesis scenario with different values of v and T_2 . Here, we set $H_{\text{inf}} = 10^{12}$ GeV, $g_* = 106.75$ and $T_1 = 0.1$ MeV.

Model parameters			Auxiliary variables		Observables	
$m/[\text{GeV}]$	$T_2/[\text{GeV}]$	q	$v/[\text{GeV}]$	r/N	$B/[\text{G}]$	$\lambda_B/[\text{Mpc}]$
1.7×10^{-29}	10^{-5}	2.4×10^{-27}	10^{12}	1.4×10^{-15}	3.0×10^{-23}	1.0
1.7×10^{-31}	10^{-6}	2.4×10^{-27}	10^{12}	1.4×10^{-14}	3.0×10^{-21}	1.0
3.4×10^{-29}	10^{-5}	2.4×10^{-27}	10^{13}	1.4×10^{-13}	1.5×10^{-21}	1.0
3.4×10^{-31}	10^{-6}	2.4×10^{-27}	10^{13}	1.4×10^{-12}	1.5×10^{-19}	1.0
4.9×10^{-29}	10^{-5}	2.4×10^{-27}	10^{14}	1.4×10^{-11}	1.0×10^{-19}	1.0
4.9×10^{-31}	10^{-6}	2.4×10^{-27}	10^{14}	1.4×10^{-10}	1.0×10^{-17}	1.0

For $T_3 \ll T_2$ the result is not very sensitive on the precise value of T_3 and both cases are of the same order. For simplicity, we neglect the relative difference as it will be smaller than the errors resulting from our assumptions leading to

$$\begin{aligned} P_B &\simeq 5.0 \times 10^{59} \left(\frac{g_*}{106.75} \right)^{-1} q^2 \left(\frac{k}{a} \right)^4 v^4 T_2^{-4} \\ &\simeq 1.8 \times 10^{-151} \left(\frac{g_*}{106.75} \right)^{-1} \left(\frac{q}{2.4 \times 10^{-27}} \right)^2 \\ &\quad \times \left(\frac{k/a_{\text{today}}}{0.14 \text{ Mpc}^{-1}} \right)^4 \frac{v^4}{T_2^4} [\text{GeV}]^4. \end{aligned} \quad (6.7)$$

The integrated magnetic field is

$$\begin{aligned} B(\text{today}) &= 3.4 \times 10^{-56} \left(\frac{g_*}{106.75} \right)^{-\frac{1}{2}} \left(\frac{q}{2.4 \times 10^{-27}} \right) \\ &\quad \times \left(\frac{k_c/a_{\text{today}}}{0.14 \text{ Mpc}^{-1}} \right)^2 \frac{v^2}{T_2^2} [\text{G}]. \end{aligned} \quad (6.8)$$

Once we fix $\rho = 2.4 \times 10^{-27}$ to get $\lambda_B = 1 \text{ Mpc}$, the magnetic field is solely determined by the ratio v/T_2 . To get an intergalactic magnetic field with strength $B = \mathcal{O}(10^{-16})[\text{G}]$, it is suggested that $v/T_2 \simeq 10^{20}$. We organize the magnetic field from (6.8) in Table II. Notably, for the parameter set $v = 10^{14} \text{ MeV}$, $T_2 = 10^{-6} \text{ MeV}$, the induced magnetic field after the tachyonic phase is about 30 times larger than that during the tachyonic phase. Thus, we may safely ignore the contributions during the tachyonic phase and conclude that sizable PMF of $\mathcal{O}(10^{-16})[\text{G}]$ can be produced after the tachyonic phase with reasonable parameter sets. It should be emphasized that we have set $g_* = 106.75$ in the numerical evaluation, however, this value may be altered in different cosmic epochs and that would bring us a factor of order $\mathcal{O}(1) \sim \mathcal{O}(10)$ error, which does not invalidate our magneto-genesis scenario.

C. Discussions on conceptual issues

In this section, we discuss several potential conceptual problems in our magneto-genesis scenario.

First, we come to the potential overproduction of electric fields. One may worry about the overproduction of electric field due to the drop of electric conductivity. This can be argued as follows. In plasma physics, one adopts the Debye

length to characterize the electrostatic effect. We may assume that on scales much larger than Debye length, the universe is in a quasineutral state and the electric field is negligible. The lower limit of the Debye length is estimated as

$$\lambda_D(T) \geq \sqrt{\frac{\epsilon_0 k_B T}{n_e q_e^2}}, \quad (6.9)$$

where ion terms are dropped for simplicity. After annihilation the number density of electrons is

$$\begin{aligned} n_e &\simeq 10^{-10} n_\gamma = 10^{-10} \times \int \frac{8\pi}{(2\pi)^3} \frac{E^2 dE}{e^{E/T} - 1} \\ &= 2.4 \times 10^{-11} T^3. \end{aligned} \quad (6.10)$$

We remind the readers that we are working in Planck units such that $\hbar = k_B = c = 1$. This leads to

$$\lambda_D = \frac{6.7 \times 10^5}{T} = 8.3 \times 10^4 \left(\frac{T}{0.1 \text{ MeV}} \right)^{-1} \text{ m}, \quad (6.11)$$

which is much smaller than galaxy scale. Thus, we are justified to work in this phase without the problem of the overproduction of the electric field.

Then, we discuss whether the total magnetic energy density remains subdominant to the background energy density. Since in our scenario P_B is highly blue on large scales, and it peaks at $k \simeq k_c$, we may estimate the magnetic energy density by collecting the contributions with $k \leq k_c$,

$$\begin{aligned} \rho_B &\simeq \int_0^{k_c} d \ln k \rho_k = \int_0^{k_c} d \ln k P_B(k) \\ &= \int_0^{k_c} \frac{dk}{k} \left(\frac{k}{k_c} \right)^4 P_B(k_c) = \frac{P_B(k_c)}{4}. \end{aligned} \quad (6.12)$$

Using (6.7), along with the numerical results in Table II, we see that when the integrated magnetic field $B \simeq 3.4 \times 10^{-16}[\text{G}]$, the model parameters satisfy $v/T_2 = 10^{20}$, and $P_B(k_c) = 1.8 \times 10^{-71}[\text{GeV}]^4$. As a result, the energy density of magnetic field ρ_B is of $\mathcal{O}(10^{-72})[\text{GeV}]^4$. On the other hand, the background energy density at matter-radiation equilibrium epoch is $\rho_{\text{eq}} \simeq 9.7 \times 10^{-38}[\text{GeV}]^4$. Thus, $\rho_B \ll \rho_{\text{eq}}$ and we verify that the total magnetic energy density is subdominant to the background one.

TABLE II. Numerical result for our magneto-genesis scenario with different value of v and T_2 . Here, we set $H_{\text{inf}} = 10^{12} \text{ GeV}$, $g_* = 106.75$ and $T_1 = 0.1 \text{ MeV}$.

Model parameters			Auxiliary variables		Observables	
$m/(\text{GeV})$	$T_2/(\text{GeV})$	q	$v/(\text{GeV})$	r/N	$B/(\text{G})$	$\lambda_B/(\text{Mpc})$
6.3×10^{-29}	10^{-5}	2.4×10^{-27}	10^{15}	1.4×10^{-9}	3.4×10^{-16}	1.0
4.9×10^{-31}	10^{-6}	2.4×10^{-27}	10^{14}	1.4×10^{-10}	3.4×10^{-16}	1.0
3.4×10^{-33}	10^{-7}	2.4×10^{-27}	10^{13}	1.4×10^{-11}	3.4×10^{-16}	1.0

Utilizing the above result, we can also compute the ratio ρ_B/ρ_ϕ . Notably, the energy density of the magnetic field decays no slower than the scalar-field energy density, thus the ratio ρ_B/ρ_ϕ reaches its maximum at the end of the tachyonic phase, $T = T_2$, and we compute

$$\frac{\rho_B(T_2)}{\rho_\phi(T_2)} = \frac{\rho_B(T_2)}{\rho_\gamma(T_2)} \frac{\rho_\gamma(T_2)}{\rho_\phi(T_2)} \sim 10^{-19} r^{-1}, \quad (6.13)$$

which is quite small. For instance, in the second row of Table 2, $T = 10^{-6}$ GeV and $r = 1.4 \times 10^{-10}$, thus in this case $\frac{\rho_B}{\rho_\phi}(T = T_2) \simeq 10^{-9}$. We conclude that the energy density of induced magnetic fields is way smaller than the energy density of scalar fields.

Last, let us briefly comment on the nonlinear evolution of the magnetic field which has been extensively discussed in [44–46]. During radiation domination the magnetic field can get significantly damped at the peak frequency while the coherence length grows. However, the nonlinear interaction only becomes important if the Alfvén crossing time of a mode is smaller than the Hubble horizon, i.e.

$$kV_A(k)\tau \geq 1 \quad (6.14)$$

where

$$V_A(k) = \sqrt{\frac{d\rho_B}{d \log k} \frac{1}{\rho_\gamma + p_\gamma}}. \quad (6.15)$$

Using (6.7) the modes at the peak frequency k_c are still far outside the nonlinear regime during radiation matter equality. After recombination the nonlinear interaction only provides logarithmic correction, which can be neglected for our purposes [44]. Note that our case differs from standard magnetogenesis where the nonlinear effects are important (see for instance [47]) as the modes are enhanced on deep superhorizon scales.

VII. CONCLUSION

We propose a novel mechanism of primordial magnetogenesis that takes place in the standard radiation dominated epoch after the electroweak symmetry breaking phase to evade the baryon isocurvature problem and the overproduction of electric fields. The conformal invariance of the EM field is broken because of its effective mass term that originated from the interaction with a scalar field. The scalar field is assumed to be very light and is extremely weakly coupled, and thus the strong coupling problem is absent. We find that sufficient magnetic fields capable to account for seed magnetic fields can be produced without the backreaction problem provided that the scalar field experiences a tachyonic growth phase. Therefore, our mechanism provides a viable mechanism for generating a large scale magnetic field without all above problems.

This work highlights the possibility of primordial magnetogenesis in the radiation dominated epoch and motivates further exploration along this line. Notably, in our minimal setup, the spectra index of the magnetic field power spectrum $P_B(k)$ is always 4. As future astrophysical observations may reveal more information about the spectra index, it would be interesting to generalize our models and explore the allowed spectra index in future works.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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