

Noninvertible symmetry as a solution to the strong CP problem in a GUT-inspired standard model

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We propose a three-zero texture for the down-quark mass matrix within a $SU(5)$ grand unified theory (GUT)-inspired Standard Model, enforced by a noninvertible selection rule originating from $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries. Assuming CP invariance at the high-energy scale, our framework solves the strong CP problem while reproducing a realistic quark mass matrix. The CP symmetry is spontaneously broken down at an intermediate scale via complex vacuum expectation values of new scalar fields η_i , which generate the CP phase in the Cabibbo-Kobayashi-Maskawa matrix without inducing the quantum chromodynamics (QCD) $\bar{\theta}$ term. The present model incorporates three generations of matter, Higgs, and additional scalars with a common structure under the noninvertible symmetry, which may be naturally embedded into $SO(10)$ GUTs at high energies.

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I. INTRODUCTION

Symmetries are an important principle in physics. In particular, symmetries are very important to determine allowed interactions and couplings among the elementary particles, that is, they provide us with consistent interactions and coupling selection rules. Therefore, group-theoretical symmetries were used in particle physics for a long time. However, the concept of symmetries has been, recently, generalized including noninvertible symmetries. (See Refs. [1–3] for reviews of noninvertible symmetries.) While the existence of topological operators is often formulated in terms of noninvertible symmetries, non-invertibility itself is not a property confined to symmetries in the strict sense. Rather, it can emerge as a more general algebraic frameworks such as hypergroups, as discussed for example in Refs. [4,5]. In this work, we refer to such phenomena as noninvertible selection rules, or, in a broader sense, as noninvertible symmetries.

Noninvertible symmetries were applied in particle physics in these years [6–12]. Among new symmetries, $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetry, whose selection rules are dictated by finite hypergroups, is very interesting [13–15]. This is because the new type of symmetry has opened a window in the particle physics to control the mass-matrix textures for quarks and leptons, which cannot be realized by the normal group-theory based symmetries [14,16,17].¹ In particular, it has been recently pointed out that the strong CP problem in quantum chromodynamics (QCD) can be solved owing to the noninverted selection rule [24] based on a CP invariance in the high-energy fundamental theory. In this paper we extend this Liang-Yanagida (LY) approach [24] in a $SU(5)$ -grand unified theory (GUT) inspired standard model (SM).

The CP problem in QCD is the discrepancy between the theoretically predicted $\mathcal{O}(1)$ CP -violating vacuum angle $\bar{\theta}$ and the experimental upper bound of $\bar{\theta} < \mathcal{O}(10^{-10})$ from electric dipole moment measurements of the neutron [25]. The CP invariance is the most straightforward solution to the strong CP problem. However, CP invariance must be spontaneously broken down at some intermediate scale to generate complex mass matrices for quarks in order to explain the CP violation in the Cabibbo-Kobayashi-Maskawa (CKM) matrix at the weak scale. The diagonalization of quark mass matrices induces a shift in the physical vacuum angle $\bar{\theta}$ away from zero unless the determinant of

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¹See Refs. [18–23] for other applications.

mass matrices for the up- and down-type quark, $\det[M_u M_d]$, is real. Thus, the strong CP problem arises in QCD even if CP is an exact symmetry at the fundamental level.

String theory is a promising candidate for fundamental theory and it has CP symmetry [26]. One possible scenario for spontaneous CP violation is moduli stabilization through compactifications. However, simple moduli stabilization leads to CP invariant vacua. In particular, if the theory has more symmetries, e.g., modular symmetry in addition to the CP symmetry, those symmetries tend to protect the CP symmetry from its violation. (See, e.g., Refs. [27–32].) Therefore, an alternative scenario is that CP is spontaneously broken down within the framework of four-dimensional (4D) effective field theory. Furthermore, we need a proper way for the spontaneous CP violation such that the CP violation appears in the weak CP phase, but not in $\det[M_u M_d]$.

The LY model is a proposal to address the above problem by enforcing $\det[M_u M_d]$ to be real through a noninvertible selection rule [24]. However, this solution seems not easy to be embedded in the grand unification, which motivates us in this paper to extend their approach in the framework of GUT-inspired SM.² Remarkably, the simpler noninvertible selection rule allows M_u to be taken as diagonal, thereby enabling the phenomenological observations to be accounted for with a reduced number of parameters.

It is extremely remarkable that the present extension to a $SU(5)$ -GUT inspired SM provides us with a very beautiful structure for the noninvertible selection rule. It seems to indicate the presence of some deep fundamental physics at high energies. $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries, which we use in this paper, can be originated from string compactification, e.g., $T^2/\mathbb{Z}_N$ orbifold compactification with the background magnetic flux N [13]. In particular, we can realize three-generation models by $T^2/\mathbb{Z}_N$ orbifold compactification with the background magnetic flux $N = 4$ or 5 [38]. In fact, we show in this letter that the $N = 5$ case provides us with a successful and very attractive noninvertible symmetry in the GUT inspired SM.

We assume, throughout this letter, the CP invariance at the high-energy fundamental level. And we further consider the CP symmetry is spontaneously broken down at some intermediate energy scale. It is extremely important in the LY approach that once the physical angle $\bar{\theta}$ is vanishing at some high energy scale it remains very small, $\bar{\theta} \simeq 10^{-16}$, at the weak scale [39].

This paper is organized as follows. In Sec. II, we give a brief review on $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries. In Sec. III,

we discuss our model to solve the strong CP problem. Section IV is devoted to our conclusions.

II. NONINVERTIBLE SELECTION RULES

In this section, we review noninvertible selection rules realized in $\mathbb{Z}_2$ gauging of $D_N \cong \mathbb{Z}_N \rtimes \mathbb{Z}_2$ [13,14]. The generator of $\mathbb{Z}_N$ symmetry obeys

$$a^N = e, \quad (2.1)$$

with e being the identity. Given all of the $\mathbb{Z}_N$ elements a^k ($k = 0, 1, \dots, N-1$) (mod N), the multiplication rule of two elements a^{k_1} and a^{k_2} obeys the conventional grouplike form

$$a^{k_1} a^{k_2} = a^{k_1+k_2}. \quad (2.2)$$

Then, let us focus on the outer automorphism of $\mathbb{Z}_2$ which is generated by the generator b with

$$b^2 = e, \quad ba^k b^{-1} = a^{-k}. \quad (2.3)$$

When we gauge this $\mathbb{Z}_2$, one can pick up the $\mathbb{Z}_2$ invariant conjugacy class of $D_N \cong \mathbb{Z}_N \rtimes \mathbb{Z}_2$, i.e.,

$$[g^{(k)}] = \{b^n a^k b^{-n} | n = 0, 1\} = \{a^k, a^{-k}\}, \quad (2.4)$$

which includes two elements except for $k = 0$ and $k = N/2$ for even N . We call this procedure as $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N \rtimes \mathbb{Z}_2$ which is represented as $\tilde{\mathbb{Z}}_N$ in what follows. Hence, the multiplication rule of two classes is different from the grouplike one

$$[g^{(k_1)}] \otimes [g^{(k_2)}] = [g^{(k_1+k_2)}] + [g^{(k_1-k_2)}], \quad (2.5)$$

which is commutable. Mathematically, our noninvertible selection rules correspond to hypergroups, in particular fusion algebras, which are concepts including group theory. These are mathematically well established concepts, and it is known that these are consistent with quantum field theory (see, e.g., Ref. [5]).

Let us assume that fields in 4D quantum field theory are labeled by a class $[g^{(k)}]$. Such a noninvertible selection rule follows the algebraic framework introduced in Refs. [4,5], in which noninvertibility arises in the 4D target spacetime. The present work adopts this viewpoint and applies it to 4D quantum field theory. This idea is also motivated by string compactifications. For instance, through the Kaluza-Klein reduction of higher-dimensional Yang-Mills theory on toroidal backgrounds, 4D massless fields have the global $\mathbb{Z}_N$ flavor symmetry in the context of type IIB magnetized D-brane models [40–42]. When we divide toroidal backgrounds by $\mathbb{Z}_2$ twist, the $\mathbb{Z}_N$ symmetry is broken, but $\mathbb{Z}_2$ invariant modes remaining in the effective action are labeled by a class $[g^{(k)}]$ on the $\mathbb{Z}_2$ orbifold [13]. In this case, the

²Recently, several models have been proposed to solve the problem by obtaining the real $\det[M_u M_d]$ based on a discrete grouplike symmetry [33–36], where supersymmetry (SUSY) plays an important role. Another approach [37] uses a six-dimensional spacetime with $\mathbf{T}^2/\mathbb{Z}_3$ orbifold compactification.

couplings of 4D fields $\phi_{k_1} \cdots \phi_{k_n}$ are allowed when the multiplication of corresponding classes $[g^{(k_1)}] \cdots [g^{(k_n)}]$ includes the class $[g^{(k=0)}]$, i.e., $\sum_i \pm k_i = 0 \pmod{N}$. Other string compactifications lead to these noninvertible selection rules in, e.g., 4D E_6 GUT in the context of heterotic string theory on Calabi-Yau threefolds with standard embedding [43]. Note that 4D field ϕ and its conjugate correspond to the same class $[g^{(k)}]$.

In the following, we discuss phenomenological applications of the noninvertible selection rules originating from $\tilde{\mathbb{Z}}_N$.³ In particular, we utilize the case with $N = 5$, where there exist three classes

$$\{[g^0], [g^1], [g^2]\} \quad (2.6)$$

obeying the following fusion rule:

$$\begin{aligned} [g^0] \otimes [g^0] &= [g^0], \\ [g^0] \otimes [g^1] &= [g^1] \otimes [g^0] = [g^1], \\ [g^0] \otimes [g^2] &= [g^2] \otimes [g^0] = [g^2], \\ [g^1] \otimes [g^1] &= [g^0] + [g^2], \\ [g^1] \otimes [g^2] &= [g^2] \otimes [g^1] = [g^1] + [g^2], \\ [g^2] \otimes [g^2] &= [g^0] + [g^1]. \end{aligned} \quad (2.7)$$

As mentioned in Introduction, $\mathbb{Z}_2$ gauging of $\mathbb{Z}_N$ symmetries can be realized in string compactification such as $T^2/\mathbb{Z}_2$ orbifold compactification with background magnetic flux N . In particular, magnetized orbifold models with $N = 5$ and 4 lead to three generations of matter fields [38].

Suppose that three generations of both left-handed and right-handed matter correspond to the three classes, $[g^0]$, $[g^1]$, and $[g^2]$. When the Higgs field corresponds to the class $[g^0]$, one can obtain the following Yukawa matrix:

$$\begin{pmatrix} \checkmark & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & \checkmark \end{pmatrix}, \quad (2.8)$$

where the check symbol $\checkmark$ denotes allowed coupling coefficient. That is the diagonal matrix. When the Higgs field corresponds to the class $[g^1]$, one can obtain the following Yukawa matrix:

$$\begin{pmatrix} 0 & \checkmark & 0 \\ \checkmark & 0 & \checkmark \\ 0 & \checkmark & \checkmark \end{pmatrix}. \quad (2.9)$$

³Other selection rules with and without gaugings are studied in Ref. [44], which will be left for future work.

That is the so-called nearest neighbor interaction form. When the Higgs field corresponds to the class $[g^2]$, one can obtain the following Yukawa matrix:

$$\begin{pmatrix} 0 & 0 & \checkmark \\ 0 & \checkmark & \checkmark \\ \checkmark & \checkmark & 0 \end{pmatrix}. \quad (2.10)$$

These are interesting forms. We will discuss them in the next section. Note that these textures (2.9) and (2.10) cannot be obtained by conventional group-based symmetries and were often assumed by hand. However, one can derive these textures via noninvertible selection rules.

III. MODEL

Here, we study a solution to the strong CP problem. We start with the 4D CP invariant theory, assuming that the CP symmetry is exact in the underlying theory such as string theory and its compactifications. That is, we assume that the QCD phase θ is vanishing, and all of Yukawa couplings and other couplings are real in the high energy fundamental theory. However, we have CP -violating phases in the quark mass matrices at the electroweak energy scale. Thus, spontaneous CP violation should occur within the framework of 4D field theory at an intermediate energy scale. Then, the CP phase would appear in quark mass matrices through the Higgs sector. In general, that induces non-vanishing QCD phase $\bar{\theta}$ as

$$\bar{\theta} = \theta + \arg \det[M_u M_d], \quad (3.1)$$

where M_u and M_d denote the up- and down-type quark mass matrices, even if the bare QCD phase θ is vanishing. In general, the nonvanishing weak CP phase and $\arg \det[M_u M_d] = 0$ are not always consistent with each other. However, those are consistent in special mass matrices. In Ref. [45], such textures were studied. One of them is as follows,

$$M_u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & a & 0 \\ a' & b e^{i\phi} & c \\ 0 & c' & d \end{pmatrix}, \quad (3.2)$$

where a, a', b, c, c' and d are real, while the other textures in Ref. [45] correspond to asymmetric texture zeros in M_d . Here, we focus on the above texture, whose zero entries are symmetric. Note that the diagonal matrix for M_u can be easily obtained as Eq. (2.8) shown in the previous section. Also the texture for the above M_d is almost realized as in Eq. (2.9) except for the (2, 2) entry. The missing (2, 2) entry corresponds the allowed entry in both Eqs. (2.8) and (2.10).

We consider the quark sector as well as the lepton sector together as $SU(5)$ multiplets. That is, we consider three $SU(5)$ multiplets $\mathbf{\bar{5}}, \mathbf{10}$ involving the left-handed notations of quarks, lepton doublets

TABLE I. Charge assignments of matter fields under $\tilde{\mathbb{Z}}_5^{(1)} \times \tilde{\mathbb{Z}}_5^{(2)}$ and $\mathbb{Z}_3 \times \mathbb{Z}_2^{(2)} \times \mathbb{Z}_2^{(3)}$ symmetries.

	T	$\bar{F}$	H_1	H_2	H_3	η_1	η_2	η_3
$\tilde{\mathbb{Z}}_5^{(1)}$	$([g^0], [g^1], [g^2])$	$([g^0], [g^1], [g^2])$	$[g^0]$	$[g^1]$	$[g^2]$	$[g^0]$	$[g^1]$	$[g^2]$
$\tilde{\mathbb{Z}}_5^{(2)}$	$([g^0], [g^1], [g^2])$	$([g^0], [g^1], [g^2])$	$[g^0]$	$[g^1]$	$[g^0]$	$[g^0]$	$[g^1]$	$[g^1]$
$\mathbb{Z}_3$	$(1, 1, 1)$	$(1, 1, 1)$	2	1	1	1	1	0
$\mathbb{Z}_2^{(2)}$	Even	Even	Even	Even	Even	Even	Odd	Even
$\mathbb{Z}_2^{(3)}$	Even	Even	Even	Even	Even	Even	Even	Odd

$$T = \begin{pmatrix} 0 & u_b^c & -u_g^c & -u_r & -d_r \\ -u_b^c & 0 & u_r^c & -u_g & -d_g \\ u_g^c & -u_r^c & 0 & -u_b & -d_b \\ u_r & u_g & u_b & 0 & e^c \\ d_r & d_g & d_b & -e^c & 0 \end{pmatrix} = \begin{pmatrix} \epsilon^{\alpha\beta\gamma} \bar{U}_\gamma & -Q^{\alpha s} \\ Q^{\beta r} & \epsilon^{rs} \bar{E} \end{pmatrix},$$

$$\bar{F} = \begin{pmatrix} d_r^c \\ d_g^c \\ d_b^c \\ e \\ -\nu \end{pmatrix} = \begin{pmatrix} \bar{D}_\alpha \\ \epsilon_{rs} L^s \end{pmatrix}, \quad (3.3)$$

where $\{\alpha, \beta, \gamma\}$ and $\{r, s\}$ denote $SU(3)_C$ and $SU(2)_L$ indices, respectively.⁴

We also introduce three Higgs doublet fields H_1 and $H_{2,3}$ which belong to the $\bar{\mathbf{5}}$ representation of the $SU(5)$ GUT. They have the following Yukawa terms:

$$f_u TTH_1^\dagger + f_d \bar{F}TH_2 + f'_d \bar{F}TH_3. \quad (3.4)$$

For the purpose of simplifying the fermion charge assignments, we embed the SM fermions into representations of the $SU(5)$ grand unified group. We emphasize, however, that we do not attempt to unify all SM gauge interactions, nor do we assume complete GUT multiplets for the scalar sector. A consistent formulation of such a setup has been discussed in Refs. [46,47], and we refer to this framework as a GUT-inspired SM [46,48].

In order to realize the texture (3.2), we impose a direct product of two $\tilde{\mathbb{Z}}_5$ symmetries, i.e., $\tilde{\mathbb{Z}}_5^{(1)} \times \tilde{\mathbb{Z}}_5^{(2)}$ in our construction. Recalling that there are three classes for each $\tilde{\mathbb{Z}}_5$ shown in Eq. (2.6), we consider the following assignments for the three fermion matter fields and for the three Higgs bosons

$$\begin{aligned} T: & ([g^0][g^0], [g^1][g^1], [g^2][g^2]), \\ \bar{F}: & ([g^0][g^0], [g^1][g^1], [g^2][g^2]), \\ H_1: & [g^0][g^0], \\ H_2: & [g^1][g^1], \\ H_3: & [g^2][g^0]. \end{aligned} \quad (3.5)$$

It leads to the following Yukawa textures:

$$f_u = \begin{pmatrix} \checkmark & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & \checkmark \end{pmatrix}, \quad f_d = \begin{pmatrix} 0 & \checkmark & 0 \\ \checkmark & 0 & \checkmark \\ 0 & \checkmark & \checkmark \end{pmatrix},$$

$$f'_d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.6)$$

To realize these textures, we need to distinguish H_1 and $H_{2,3}$. Otherwise, $H_{2,3}$ are also allowed to couple with TT . Here, we impose the $\mathbb{Z}_3$ symmetry. We assume that all of $T, \bar{F}, H_{2,3}$ have the $\mathbb{Z}_3$ charge one, while H_1 has $\mathbb{Z}_3$ two. Then, the above mass matrices are realized. One more interesting point is that since T and $\bar{F}$ have the same charge, we can extend this model to $SO(10)$ GUT. The charge assignments are summarized in Table I.

The potential in the Higgs sector can be written by

$$\begin{aligned} V_H = & -m_1^2 H_1^\dagger H_1 - m_2^2 H_2^\dagger H_2 - m_3^2 H_3^\dagger H_3 \\ & + \lambda_1 (H_1^\dagger H_1)^2 + \lambda_2 (H_2^\dagger H_2)^2 + \lambda_3 (H_3^\dagger H_3)^2 \\ & + \lambda_{12} H_1^\dagger H_2^\dagger H_1 H_2 + \lambda_{13} H_1^\dagger H_3^\dagger H_1 H_3 + \lambda_{23} H_3^\dagger H_2^\dagger H_3 H_2 \\ & + \lambda'_{23} [(H_2^\dagger H_3)^2 + (H_3^\dagger H_2)^2], \end{aligned} \quad (3.7)$$

where all of coupling constants are real. When $\lambda'_{23} < 0$, the CP is not violated spontaneously.

Now, we introduce new scalar bosons, η_i for $i = 1, 2, 3$ to generate the spontaneous CP violation. Then, we have the following potential:

$$\begin{aligned} V_{H-\eta} = & \alpha_{23} (H_2^\dagger H_3 \langle \eta_3 \rangle + \text{H.c.}) + \alpha'_{23} (H_2^\dagger H_3 \langle \eta_3^\dagger \rangle \\ & + \text{H.c.}) + \alpha_{12} (H_1^\dagger H_2 \langle \eta_2 \rangle + \text{H.c.}), \end{aligned} \quad (3.8)$$

⁴The existence of right-handed neutrinos is not relevant to our discussions.

where α_{12} , α_{23} , and α'_{23} are real dimension one parameters. Notice here that the parameter $\alpha_{12} \simeq m^2/\langle\eta_2\rangle$ and $\alpha_{23}, \alpha'_{23} \simeq m^2/\langle\eta_3\rangle$. Here, the m should be less than a few hundred GeV, otherwise the Higgs bosons have too large masses. The parameters α_{12} , α_{23} , and α'_{23} are indeed extremely small since the vacuum expectation values (VEVs) of η_i should be very large like 10^{10} GeV. However, their smallness can be naturally explained since we have an enhanced $\mathbb{Z}_2^{(2)} \times \mathbb{Z}_2^{(3)}$ symmetry in the limit where they are vanishing. Under the $\mathbb{Z}_2^{(2)}$ ($\mathbb{Z}_2^{(3)}$) symmetry, the η_2 (η_3) transforms as odd and all other fields as even, as summarized in Table I. We impose the $\mathbb{Z}_2^{(2)} \times \mathbb{Z}_2^{(3)}$ symmetry in the η boson sector to control their potential as shown below.

If the vacuum expectation values $\langle\eta_i\rangle$ violate the CP symmetry, such a CP violation induces the CP violation in the Higgs sector. To allow the above couplings in Eq. (3.8), the classes of η_i must be assigned as

$$\eta_2: [g^1][g^1], \quad \eta_3: [g^2][g^1], \quad (3.9)$$

and η_2 and η_3 have $\mathbb{Z}_3$ charges, 1 and 0, respectively. In addition, we assign the classes of η_1 as follows,

$$\eta_1: [g^0][g^0], \quad (3.10)$$

and it has the $\mathbb{Z}_3$ charge one. In addition to $V_{H-\eta}$, we have the following terms:

$$- \sum_{i,j=1}^3 c_{ij} |\langle\eta_j\rangle|^2 H_i^\dagger H_i. \quad (3.11)$$

These are additional contributions to m_i^2 . We renormalize them to m_i^2 .

The potential of η_i can be given by

$$\begin{aligned} V_\eta = & -\mu_3 \eta_3 \eta_3^\dagger - \mu'_3 (\eta_3^2 + \eta_3^{\dagger 2}) + \xi_3 (\eta_3 \eta_3^\dagger)^2 + \xi'_3 (\eta_3^4 + \eta_3^{\dagger 4}) + \xi'_{33} (\eta_3^2 + \eta_3^{\dagger 2}) \eta_3 \eta_3^\dagger \\ & - \mu_1 \eta_1 \eta_1^\dagger - \mu_2 \eta_2 \eta_2^\dagger + \xi_1 (\eta_1 \eta_1^\dagger)^2 + \xi_2 (\eta_2 \eta_2^\dagger)^2 + \xi_{12} (\eta_1 \eta_1^\dagger) (\eta_2 \eta_2^\dagger) \\ & + \sum_{i=1,2} [\xi_{3i} \eta_3 \eta_3^\dagger + \xi'_{3i} (\eta_3^2 + \eta_3^{\dagger 2})] \eta_i \eta_i^\dagger \\ & - \beta (\eta_1^3 + \eta_1^{\dagger 3}) - \beta' (\eta_1 \eta_2^2 + \eta_1^\dagger \eta_2^{\dagger 2}) - \xi'_{12} [(\eta_1^\dagger \eta_2)^2 + (\eta_1 \eta_2^\dagger)^2], \end{aligned} \quad (3.12)$$

where all of the coefficients are real. Here, we have imposed a new symmetry $\mathbb{Z}_2^{(2)} \times \mathbb{Z}_2^{(3)}$. We remark that η_2 (η_3) transforms as odd and all other fields as even under the $\mathbb{Z}_2^{(2)}$ ($\mathbb{Z}_2^{(3)}$) symmetry. This is the symmetry that can explain naturally the smallness of the parameters α_{12} and α_{23} in Eq. (3.8). Notice that this $\mathbb{Z}_2^{(2)} \times \mathbb{Z}_2^{(3)}$ symmetry is approximate one in the total Lagrangian, but we consider it is exact in the η sector at the tree level.

The potential terms in the first line include only η_3 . This potential was studied in Refs. [24,37] and it was shown that the CP symmetry is spontaneously broken. (The phase depends also on the VEVs of η_i due to the couplings in the third line.) That is, the vacuum expectation value $\langle\eta_3\rangle$ has nonvanishing phase. The above potential has an additional term with the coefficients ξ'_{3i} . We denote $\eta_i = |\eta_i| e^{i\varphi_i}$. Then, the potential terms relevant to the phase φ_3 are written by

$$\begin{aligned} V_{3\varphi_3} = & -2\mu'_3 |\eta_3|^2 \cos 2\varphi_3 + 2\xi'_3 |\eta_3|^4 \cos 4\varphi_3 + 2|\eta_3|^2 \sum_{i=1,2,3} \xi'_{3i} |\eta_i|^2 \cos 2\varphi_3 \\ = & -2|\eta_3|^2 \cos 2\varphi_3 \left(\mu'_3 - \sum_{i=1,2,3} \xi'_{3i} |\eta_i|^2 \right) + 2\xi'_3 |\eta_3|^4 (2 \cos^2 2\varphi_3 - 1). \end{aligned} \quad (3.13)$$

Similar to Refs. [24,37], the potential minimum corresponds to nonvanishing phase $\varphi_3 \neq 0, \pi$ and $\cos 2\varphi \neq 1$ for generic parameters.

The phases of η_1 and η_2 appear only in the last two terms of V_η . These terms are written by

$$\begin{aligned} & -2\beta |\eta_1|^3 \cos 3\varphi_1 - 2\beta' |\eta_1| |\eta_2|^2 \cos(\varphi_1 + 2\varphi_2) \\ & - 2\xi'_{12} |\eta_1|^2 |\eta_2|^2 \cos(2\varphi_1 - 2\varphi_2). \end{aligned} \quad (3.14)$$

When all of β , β' , and ξ'_{12} are positive, the potential minimum corresponds to

$$\varphi_1 = \varphi_2 = \frac{2\pi m}{3}, \quad (3.15)$$

with $m = 0, 1, 2$.

Now let us study the CP phases of the Higgs vacuum expectation values H_i , which are denoted by $|H_i|e^{i\phi_i}$. The potential terms in $V_{H-\eta}$ are important. We use the $U(1)$ -hypercharge rotation so that $\langle H_2 \rangle$ is real. Since $\langle \eta_3 \rangle$ has nonvanishing phase, the VEV $\langle H_3 \rangle$ must have nonvanishing phase because of the terms with the coefficients α_{23} and α'_{23} in $V_{H-\eta}$. Indeed, the relevant term can be written by

$$2\alpha_{23}|H_2||H_3|\cos(\varphi_3 + \phi_3) + 2\alpha'_{23}|H_2||H_3|\cos(\varphi_3 - \phi_3). \quad (3.16)$$

Thus, $\langle H_3 \rangle$ has a nonvanishing phase $\phi_3 \neq 0$ depending on parameters as well as φ_3 . For example, when $\alpha_{23}, \alpha'_{23} < 0$ and $|\alpha_{23}| \gg |\alpha'_{23}|$ ($|\alpha'_{23}| \gg |\alpha_{23}|$), the potential minimum corresponds to $\phi_3 \approx -\varphi_3$ ($\phi_3 \approx \varphi_3$), where the value of φ_3 depends on parameters.

Next, we study the phase of $\langle H_1 \rangle$ and we write its phase by ϕ_1 . The term with the coefficient α_{12} in $V_{H-\eta}$ is written by

$$2\alpha_{12}|H_2||H_1|\cos(\varphi_2 - \phi_1). \quad (3.17)$$

Suppose that $\alpha_{12} < 0$. Then, the potential minimum corresponds to

$$\phi_1 = \varphi_2 = \frac{2\pi m}{3}. \quad (3.18)$$

Obviously, if $\phi_1 = 0$, the quark masses in the up sector are real. Even if the phase is nonvanishing, we find the phase factor of $\det M_u$ is $e^{2\pi m i}$ because 3×3 matrix. As a result, we can realize the texture of Eq. (3.2) with $\arg \det[M_u] = 0$ and nonvanishing $\phi = \phi_3$ in the $(2, 2)$ entry of M_d .

Now we are at the point to discuss quantum corrections of the above argument. It is known that quantum loop corrections violate the noninvertible symmetry selection rule. (See, e.g., Refs. [4,5,15].) In fact, one loop diagrams with intermediate H_2 bosons induce unwanted operators, $H_1^\dagger H_3 \eta_2^\dagger \eta_3$ and $H_1^\dagger H_3 \eta_2^\dagger \eta_3^\dagger$, which shift the phase of H_1 . However, the coefficients are proportional to $\alpha_{12}\alpha_{23}\xi_{32}/m_\eta^2$ for $H_1^\dagger H_3 \eta_2^\dagger \eta_3$ and $\alpha_{12}\alpha_{23}\xi'_{32}/m_\eta^2$ for $H_1^\dagger H_3 \eta_2^\dagger \eta_3^\dagger$ appearing in Eqs. (3.8) and (3.12), where m_η denotes a (typical) mass scale of the η_i sector, up to a loop factor. The corrections must be small compared with Eq. (3.8) enough to realize the experimental constraint $\bar{\theta} < 10^{-10}$. That requires the ratio $\alpha_{23}\xi_{23}\langle \eta_3 \rangle / m_\eta^2 < 10^{-10}$. For example, when $\xi_{23} \sim 1$, $\alpha_{23} \simeq (100 \text{ GeV})^2 / \langle \eta_3 \rangle$, and $\langle \eta_3 \rangle \sim m_\eta$, we need $\langle \eta_3 \rangle \sim m_\eta \gg 10^6 \text{ GeV}$.

There are other correction terms due to loop effects. However, if we use couplings between the $\eta_{2,3}$ and the Higgs $H_{1,2,3}$ in Eq. (3.8), the loop amplitudes are always suppressed by their small coupling constants α_{ij} . Therefore, we consider only the interactions in Eq. (3.12) which has the $\mathbb{Z}_2^{(3)}$ symmetry, where η_3 is $\mathbb{Z}_2^{(3)}$ odd and η_1 and η_2 are $\mathbb{Z}_2^{(3)}$ even. Furthermore, we have an accidental exchange symmetry between η_3 and $\eta_3^\dagger$. Together with both symmetries, we conclude loop-induced effective operators that are relevant for the phase determination of the η_1 is $(\eta_1^3 + \eta_1^{\dagger 3})(\eta_3^2 + \eta_3^{\dagger 2})$. It is remarkable that the phase of the η_1 does not shift from $2n\pi/3$ even if the η_3 has an arbitrary nonvanishing phase φ_3 .

We have not discussed, in this paper, the possible cutoff Λ suppressed operators such as $(1/\Lambda)\eta_1^3\eta_3^2 + \text{H.c.}$, since they depend largely on the physics at the cutoff scale. However, if $|\langle \eta_i \rangle|/\Lambda < 10^{-10}$, the experimental constraint on the physical vacuum angle $\bar{\theta}$ is satisfied.

IV. CONCLUSIONS

We have constructed a 3 zero texture for the down-type quark mass matrix M_d and a real diagonal texture for the up-type quark mass matrix, based on a noninvertible selection rule, $\tilde{\mathbb{Z}}_5^{(1)} \times \tilde{\mathbb{Z}}_5^{(2)}$, in the $SU(5)$ -GUT inspired SM. Together with the assumption of CP invariant in the high-energy fundamental theory, we solve the strong CP problem in QCD [24], since $\det[M_u M_d]$ is real. The mass texture we have constructed in this paper is exactly the same as one of the mass matrices $M_d^{(1)}$ in [45] which is consistent with all of the CKM phenomenology.

We have started with the CP invariant 4D theory, and the CP symmetry is assumed to be spontaneously broken down at some intermediate energy scale. Such CP violation appears in the Higgs potential and one of the Higgs fields acquires a complex VEV generating the complex CKM matrix. However, we need a proper texture such that the CP violation does not appear in the QCD phase $\bar{\theta}$. We have constructed a consistent texture for the quark mass matrices using the noninvertible coupling selection rules derived from $\mathbb{Z}_2$ gauging of $\mathbb{Z}_5$ symmetries.⁵

The fermion matter sector in our model corresponds exactly to $SU(5)$ multiplets [48], and they can be embedded into $SO(10)$ multiplets, because **10** and $\bar{\mathbf{5}}$ multiplets are assigned to have the same charges of the noninverted symmetry. Our model includes three Higgs fields $H_i (i = 1-3)$ and extra scalar fields $\eta_i (i = 1-3)$ as well as three generations of matter fields. That is, all of the

⁵As pointed out in Ref. [24], quantum corrections to the Yukawa coupling matrices can generate a complex phase in $\det[M_d M_u]$ at the two-loop level, but the resulting shift in the vacuum angle, $\bar{\theta}$, is well below the current experimental bound $\bar{\theta} < 10^{-10}$.

mode numbers are equal to three. Furthermore, their structures of the first $\tilde{Z}_5$ class are the same, i.e., $([g^0], [g^1], [g^2])$, for the fermion matters. It would be interesting to study the unification of the matter sector, the Higgs sector, and the η sector.⁶ However, it may be very difficult to generate the desired breaking of such a unification model down to the SM.

The three doublet Higgs bosons are one of crucial predictions at the weak scale in the present model. In particular, two Higgs bosons couple to the down-type quark and charged lepton sectors and hence it causes the flavor changing neutral current events in the down-type quark and the charged lepton sector. This is a smoking gun in the present model.

We have not studied details in the lepton sector in this paper. However, it is a straightforward task to include the

lepton sector if we introduce three right-handed neutrinos, $N_{Ri} (i = 1-3)$. We will construct the model and discuss the CP violation in the neutrino oscillation and the baryon-number creation through the leptogenesis [49] in a future publication [50].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

⁶In general, string compactification leads to several Higgs fields. In addition, the matter sector and Higgs sector are combined to one sector, e.g., E_6 27 multiplets in heterotic string theory with standard embedding.

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- [1] T.D. Brennan and S. Hong, Introduction to generalized global symmetries in QFT and particle physics, [arXiv:2306.00912](#).
 - [2] S. Schafer-Nameki, ICTP lectures on (non-)invertible generalized symmetries, *Phys. Rep.* **1063**, 1 (2024).
 - [3] S.-H. Shao, What's done cannot be undone: TASI lectures on non-invertible symmetries, [arXiv:2308.00747](#).
 - [4] J.J. Heckman, J. McNamara, M. Montero, A. Sharon, C. Vafa, and I. Valenzuela, Fate of stringy noninvertible symmetries, *Phys. Rev. D* **110**, 106001 (2024).
 - [5] J. Kaidi, Y. Tachikawa, and H.Y. Zhang, On a class of selection rules without group actions in field theory and string theory, *SciPost Phys.* **17**, 169 (2024).
 - [6] Y. Choi, H. T. Lam, and S.-H. Shao, Noninvertible global symmetries in the standard model, *Phys. Rev. Lett.* **129**, 161601 (2022).
 - [7] C. Cordova and K. Ohmori, Noninvertible chiral symmetry and exponential hierarchies, *Phys. Rev. X* **13**, 011034 (2023).
 - [8] C. Cordova, S. Hong, S. Koren, and K. Ohmori, Neutrino masses from generalized symmetry breaking, *Phys. Rev. X* **14**, 031033 (2024).
 - [9] Y. Choi, M. Forslund, H. T. Lam, and S.-H. Shao, Quantization of axion-gauge couplings and noninvertible higher symmetries, *Phys. Rev. Lett.* **132**, 121601 (2024).
 - [10] C. Cordova, S. Hong, and L.-T. Wang, Axion domain walls, small instantons, and non-invertible symmetry breaking, *J. High Energy Phys.* **05** (2024) 325.
 - [11] C. Cordova, S. Hong, and S. Koren, Non-invertible Peccei-Quinn symmetry and the massless quark solution to the strong CP problem, *Phys. Rev. X* **15**, 031011 (2025).
 - [12] A. Delgado and S. Koren, Non-invertible Peccei-Quinn symmetry, natural 2HDM alignment, and the visible axion, *J. High Energy Phys.* **02** (2025) 178.
 - [13] T. Kobayashi and H. Otsuka, Non-invertible flavor symmetries in magnetized extra dimensions, *J. High Energy Phys.* **11** (2024) 120.
 - [14] T. Kobayashi, H. Otsuka, and M. Tanimoto, Yukawa textures from non-invertible symmetries, *J. High Energy Phys.* **12** (2024) 117.
 - [15] S. Funakoshi, T. Kobayashi, and H. Otsuka, Quantum aspects of non-invertible flavor symmetries in intersecting/magnetized D-brane models, *J. High Energy Phys.* **04** (2025) 183.
 - [16] T. Kobayashi, Y. Nishioka, H. Otsuka, and M. Tanimoto, More about quark Yukawa textures from selection rules without group actions, *J. High Energy Phys.* **05** (2025) 177.
 - [17] T. Kobayashi, H. Otsuka, M. Tanimoto, and H. Uchida, Lepton mass textures from non-invertible multiplication rules, *J. High Energy Phys.* **08** (2025) 189.
 - [18] M. Suzuki and L.-X. Xu, Phenomenological implications of a class of non-invertible selection rules, [arXiv:2503.19964](#).
 - [19] T. Kobayashi, H. Okada, and H. Otsuka, Radiative neutrino mass models from non-invertible selection rules, *J. High Energy Phys.* **12** (2025) 111.
 - [20] T. Kobayashi, H. Mita, H. Otsuka, and R. Sakuma, Matter symmetries in supersymmetric standard models from non-invertible selection rules, [arXiv:2506.10241](#).
 - [21] T. Nomura and O. Popov, No-group scotogenic model, [arXiv:2507.10299](#).

- [22] J. Chen, C.-Q. Geng, H. Okada, and J.-J. Wu, A radiative lepton model in a non-invertible fusion rule, [arXiv:2507.11951](#).
- [23] H. Okada and Y. Shigekami, Three-loop induced neutrino mass model in a non-invertible symmetry, [arXiv:2507.16198](#).
- [24] Q. Liang and T. T. Yanagida, Non-invertible symmetry as an axion-less solution to the strong CP problem, [Phys. Lett. B **868**, 139706 \(2025\)](#).
- [25] C. Abel *et al.*, Measurement of the permanent electric dipole moment of the neutron, [Phys. Rev. Lett. **124**, 081803 \(2020\)](#).
- [26] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies and Phenomenology* (Cambridge University Press, Cambridge, 1988).
- [27] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi, and H. Uchida, CP violation in modular invariant flavor models, [Phys. Rev. D **101**, 055046 \(2020\)](#).
- [28] T. Kobayashi and H. Otsuka, Challenge for spontaneous CP violation in Type IIB orientifolds with fluxes, [Phys. Rev. D **102**, 026004 \(2020\)](#).
- [29] K. Ishiguro, T. Kobayashi, and H. Otsuka, Spontaneous CP violation and symplectic modular symmetry in Calabi-Yau compactifications, [Nucl. Phys. **B973**, 115598 \(2021\)](#).
- [30] P. P. Novichkov, J. T. Penedo, and S. T. Petcov, Modular flavour symmetries and modulus stabilisation, [J. High Energy Phys. **03** \(2022\) 149](#).
- [31] V. Knapp-Perez, X.-G. Liu, H. P. Nilles, S. Ramos-Sanchez, and M. Ratz, Matter matters in moduli fixing and modular flavor symmetries, [Phys. Lett. B **844**, 138106 \(2023\)](#).
- [32] T. Higaki, T. Kobayashi, K. Nasu, and H. Otsuka, Spontaneous CP violation and partially broken modular flavor symmetries, [J. High Energy Phys. **09** \(2024\) 024](#).
- [33] S. Antusch, M. Holthausen, M. A. Schmidt, and M. Spinrath, Solving the strong CP problem with discrete symmetries and the right unitarity triangle, [Nucl. Phys. **B877**, 752 \(2013\)](#).
- [34] F. Feruglio, A. Strumia, and A. Titov, Modular invariance and the QCD angle, [J. High Energy Phys. **07** \(2023\) 027](#).
- [35] S. T. Petcov and M. Tanimoto, A_4 modular invariance and the strong CP problem, [Eur. Phys. J. C **84**, 914 \(2024\)](#).
- [36] J. T. Penedo and S. T. Petcov, Finite modular symmetries and the strong CP problem, [J. High Energy Phys. **10** \(2024\) 172](#).
- [37] Q. Liang, R. Okabe, and T. T. Yanagida, Three-zero texture of quark-mass matrices as a solution to the strong CP problem, [Phys. Lett. B **859**, 139123 \(2024\)](#).
- [38] H. Abe, T. Kobayashi, and H. Ohki, Magnetized orbifold models, [J. High Energy Phys. **09** \(2008\) 043](#).
- [39] J. R. Ellis and M. K. Gaillard, Strong and weak CP violation, [Nucl. Phys. **B150**, 141 \(1979\)](#).
- [40] H. Abe, K.-S. Choi, T. Kobayashi, and H. Ohki, Non-Abelian discrete flavor symmetries from Magnetized/Intersecting brane models, [Nucl. Phys. **B820**, 317 \(2009\)](#).
- [41] M. Berasaluce-Gonzalez, P. G. Camara, F. Marchesano, D. Regalado, and A. M. Uranga, Non-Abelian discrete gauge symmetries in 4D string models, [J. High Energy Phys. **09** \(2012\) 059](#).
- [42] F. Marchesano, D. Regalado, and L. Vazquez-Mercado, Discrete flavor symmetries in D-brane models, [J. High Energy Phys. **09** \(2013\) 028](#).
- [43] J. Dong, T. Kobayashi, R. Nishida, S. Nishimura, and H. Otsuka, Coupling selection rules in heterotic Calabi-Yau compactifications, [J. High Energy Phys. **09** \(2025\) 012](#).
- [44] J. Dong, T. Jeric, T. Kobayashi, R. Nishida, and H. Otsuka, On discrete gauging and non-invertible selection rules, [arXiv:2507.02375](#).
- [45] M. Tanimoto and T. T. Yanagida, Occam's razor in quark mass matrices, [Prog. Theor. Exp. Phys. **2016**, 043B03 \(2016\)](#).
- [46] M. Ibe, S. Shirai, M. Suzuki, and T. T. Yanagida, Novel GUT with apparently complete $SU(5)$ multiplets, [Phys. Rev. D **100**, 055024 \(2019\)](#).
- [47] T. Hotta, K. I. Izawa, and T. Yanagida, Dynamical models for light Higgs doublets in supersymmetric grand unified theories, [Phys. Rev. D **53**, 3913 \(1996\)](#).
- [48] K. Nakayama, F. Takahashi, and T. T. Yanagida, A theory of extra radiation in the universe, [Phys. Lett. B **697**, 275 \(2011\)](#).
- [49] M. Fukugita and T. Yanagida, Baryogenesis without grand unification, [Phys. Lett. B **174**, 45 \(1986\)](#).
- [50] T. Kobayashi, H. Otsuka, M. Tanimoto, and T. T. Yanagida, GUT-motivated non-invertible symmetry as a solution to the strong CP problem and the neutrino CP -violating phase, [arXiv:2510.01680](#).