

Thermal noise in optical coatings: A rigorous optomechanical comparison of nanolaminates and isotropic mixtures

Vincenzo Pierro^{1,2} , Vincenzo Fiumara^{3,2} , Veronica Granata^{4,2} , and Guerino Avallone^{5,*} 

¹*Department of Engineering DING, University of Sannio, I-82100 Benevento, Italy*

²*INFN Sezione di Napoli, Gruppo Collegato di Salerno, I-84084 Fisciano (Salerno), Italy*

³*Department of Engineering, University of Basilicata, I-85100 Potenza, Italy*

⁴*Dipartimento di Ingegneria Industriale, Elettronica e Meccanica (DIEM),
Università degli Studi di Roma Tre, I-00146 Roma, Italy*

⁵*Dipartimento di Fisica “E. R. Caianello,” Università di Salerno, I-84084 Fisciano (Salerno), Italy*



(Received 11 July 2026; accepted 2 September 2026; published 6 October 2026)

We investigate whether a perfectly ordered periodic nanolaminate is intrinsically less mechanically dissipative than an isotropic cosputtered mixture designed to provide the exact same optical function. Focusing on a benchmark room-temperature $\text{SiO}_2/\text{TiO}_2$ material system, we compare coating Brownian dissipation under strict parity of optical thickness and effective refractive index. The constituent amorphous phases are modeled using complex Young’s moduli and purely real Poisson’s ratios, corresponding to a single-loss-angle approximation within each phase. *A priori*, the outcome of this comparison is highly nontrivial due to competing topological effects: while the nanolaminate is optically optimal, its structural anisotropy forces in-plane shear deformations to operate at the highly dissipative Voigt upper bound, mechanically penalizing the structure compared to a well-mixed isotropic medium. The nanolaminate is treated deterministically through the exact long-wavelength optical (Wiener) and mechanical (Backus) effective-medium relations. Conversely, the isotropic mixture is described by realistic optical closures (Lorentz-Lorenz and Bruggeman) coupled to a complex mechanical self-consistent scheme. To assess the residual uncertainty associated with the unknown cosputtered morphology, we explore the rigorous optical and mechanical admissible regions in the complex plane. We demonstrate that, owing to the small intrinsic optical and mechanical losses of the $\text{SiO}_2/\text{TiO}_2$ pair, these rigorous admissible regions collapse into extremely narrow lenses, drastically restricting the microstructural uncertainty. Moreover, across all realistic isotropic closures, the nanolaminate systematically exhibits lower mechanical dissipation than the corresponding cosputtered mixture. Crucially, even when evaluated against the absolute thermodynamic lower bound of the isotropic state, this topological advantage remains strictly valid in the high-effective refractive index regime. We prove that the ordered architecture exploits the optical upper bound to drastically minimize the required volume fraction of the highly dissipative high-index phase, an effect that overwhelmingly overcompensates for its intrinsic mechanical anisotropy penalty.

DOI: [10.1103/v9rw-b3pf](https://doi.org/10.1103/v9rw-b3pf)

I. INTRODUCTION

The mechanical dissipation of optical coatings remains a central limitation in precision interferometry. In high-stability optical systems, such as ground-based gravitational-wave detectors (e.g., Virgo [1,2], LIGO [3,4], KAGRA [5,6]), coating Brownian noise directly dictates the fundamental displacement noise floor in the most

sensitive frequency bands [7,8]. Consequently, the search for improved coating materials is a highly active field of research in order to go a step further than the already significant results obtained since the first detection of gravitational waves [9].

While significant efforts have recently been directed toward exploring novel low-loss amorphous phases materials, such as titania doped silica (TiO_2 -doped SiO_2 [10–12]), titania doped germania (TiO_2 -doped GeO_2 [13]), and advanced ternary systems [14,15], experimentally optimizing the intrinsic loss angles of the constituent materials is only part of the solution. It is equally crucial to rigorously address the role of topology and microstructure in determining the final macroscopic dissipative response of the coating.

*Contact author: guavallone@unisa.it

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI.

The quest for optimal coating architectures has driven a decade-long debate within the gravitational-wave community regarding the best approach to synthesize materials with tailored refractive indices and low mechanical dissipation. Historically, the standard approach has been the codeposition (cosputtering) of different oxides to form a macroscopically homogeneous, amorphous isotropic mixture. However, an alternative and highly promising paradigm has been pioneered and extensively investigated: the use of perfectly ordered periodic nanolaminates, or 1D photonic superlattices [16–19].

Recent advancements in plasma-assisted electron-beam deposition [20] have proven that such ultrathin, perfectly ordered superlattices can be reliably manufactured with sub-nanometer precision [21], yielding metamaterials that strictly preserve their structural periodicity and amorphous nature of the nanolayered constituent materials even after high-temperature annealing [22], as confirmed using structural and optical characterization techniques similar to those employed for $\text{TiO}_2:\text{Ta}_2\text{O}_5$ coatings in Ref. [23].

By alternating subwavelength layers of pure low-index and high-index materials, it is possible to synthesize an effective metamaterial that matches the optical properties of a cosputtered mixture, while fundamentally altering its structural and mechanical topology. Over the last decade, several experimental and theoretical campaigns have demonstrated the unique advantages of Nanolayered Optical Films (NLOF). Crucially, embedding a high-index material (e.g., TiO_2) within ultrathin layers buffered by a glass former (e.g., SiO_2) geometrically frustrates crystallization. This allows the composite to withstand significantly higher annealing temperatures without developing scattering crystallites, which in turn drastically reduces the internal mechanical friction [24–27]. Furthermore, nanolayering has been shown to flatten and shift the low-temperature mechanical loss peaks (cryo-peaks) [28], making this architecture a prime candidate for third-generation cryogenic detectors like the Einstein Telescope [29] and Cosmic Explorer [30].

While the experimental and technological advantages of nanolaminates are increasingly well documented, their theoretical mechanical modeling has historically relied on simplified scalar approximations. Early estimates of coating thermal noise often treated the multilayer stack as an equivalent isotropic medium, attempting to describe its dissipation through a single effective loss angle ϕ_{eff} and a scalar Young’s modulus [31,32]. However, such scalar homogenizations fundamentally fail to capture the intrinsic mechanical anisotropy of the layered structure.

Under the Gaussian pressure profile dictated by Levin’s direct approach [33], a nanolaminate experiences a complex 3D stress field where in-plane shear and out-of-plane compressions dissipate energy at vastly different rates. Therefore, to definitively assess the thermal noise performance of these architectures, one must abandon scalar loss

angles and directly evaluate the elastic energy dissipated within the exact anisotropic stiffness tensor.

Building upon this extensive experimental and phenomenological background, the purpose of the present theoretical work is to definitively compare these two limiting architectures—the disordered isotropic mixture and the perfectly ordered nanolaminate—for the specific $\text{SiO}_2/\text{TiO}_2$ system under strict parity of optical design.

More precisely, we ask whether a nanolaminate is intrinsically less dissipative than an isotropic cosputtered mixture designed to exhibit the exact same effective refractive index and optical thickness. We focus specifically on coating Brownian thermal noise because this internal friction mechanism represents the dominant fundamental noise source limiting the astrophysical sensitivity of current ground-based gravitational-wave interferometers (see Ref. [34] for a comprehensive reference on the subject) in their most sensitive frequency band ($\sim 10\text{--}300$ Hz). The optomechanical comparison is performed using the specific benchmark room-temperature properties of the $\text{SiO}_2/\text{TiO}_2$ pair listed in Table I, assuming ideal sharp interfaces and homogeneous internal dissipation within the individual constituent amorphous phases. The constituent amorphous phases are modeled using complex Young’s moduli and strictly real Poisson’s ratios, thereby excluding a dissipative component of the Poisson’s ratios.

This comparison is nontrivial because the ordered layered structure is naturally anisotropic at the effective-medium level, whereas the isotropic cosputtered morphology is not uniquely defined and must be described through effective-medium closures and admissible microstructural

TABLE I. Mechanical and optical parameters of the constituent materials used in the theoretical bounding models. The silica parameters constitute representative benchmark values for high-quality sputtered amorphous silica [8,34]. The optical parameters assigned to the amorphous TiO_2 constituent were obtained from a refined analysis of the spectroscopic-ellipsometry data acquired during our experimental campaign on $\text{SiO}_2/\text{TiO}_2$ nanolayer samples [22,35]. The mechanical parameters assigned to TiO_2 are nominal room-temperature baseline inputs, consistent with the broad ranges considered for amorphous high-index coating materials in Ref. [34]. The values listed in this table should therefore not be interpreted as a parameter set measured jointly on a single specimen. The complex Young’s modulus is defined as $Y_i^* = Y_i(1 + i\phi_i)$, where Y_i is the storage modulus and ϕ_i is the mechanical loss angle of the i th material. The Poisson ratios ν_i are taken to be real. The complex refractive index is defined as $\tilde{n} = n + i\kappa$, where n is the real refractive index and κ is the extinction coefficient.

Material	Mechanical properties			Optical properties	
	Y [GPa]	ϕ	ν	n	κ
Silica (SiO_2)	72.0	0.5×10^{-4}	0.17	1.45	1.0×10^{-8}
Titania (TiO_2)	165.0	1.4×10^{-4}	0.28	2.47	1.0×10^{-3}

bounds [31,32]. The nanolaminate can be treated deterministically through the exact long-wavelength optical and mechanical effective-medium relations for layered media, namely the Wiener optical description [36,37] and the Backus mechanical averaging procedure [38].

By contrast, the isotropic cosputtered mixture requires a more indirect construction. The phase fraction is first inferred from realistic optical closures, such as the Lorentz-Lorenz (L-L) and Bruggeman-type models [39,40]. This fraction is then mapped into effective complex bulk and shear moduli through a nonlinear self-consistent mechanical scheme [41–43]. To assess the residual uncertainty associated with the unknown isotropic microstructure, we further consider rigorous optical and mechanical admissible regions in the complex plane. For the optical response, we rely on the exact bounds for complex dielectric permittivities developed by Bergman and Milton [44–47]. For the mechanical response, we employ the extension of the Hashin-Shtrikman variational principles to complex viscoelastic moduli formulated by Gibiansky, Milton, and Berryman [48–50].

A central result of the present analysis is that, for the low-loss $\text{SiO}_2/\text{TiO}_2$ system, the exact admissible optical and mechanical regions collapse numerically into extremely narrow lenses. As a consequence, the rigorous bounds become very close to simple reduced geometric envelopes: short boundary segments in the optical plane and a narrow four-point Hashin-Shtrikman-Walpole (HSW)-like quadrilateral in the mechanical plane. This observation is important because it explains why the simplified full phase-space scan developed in this work remains quantitatively very close to the exact admissible set, despite not being presented as a fully rigorous coupled optical-mechanical extremal construction for a single realizable microstructure.

The paper is organized as follows. Section II describes the thermal noise model based on the Levin’s approach [33]. Section III establishes the ordered nanolaminate as the deterministic reference architecture and derives its effective optical and mechanical properties. Section IV introduces the realistic isotropic-mixture closures used to model cosputtered amorphous films. Section V discusses the rigorous admissible regions for the optical and mechanical effective properties and explains their low-loss reduction in the present system. Section VI combines these ingredients into the computational framework used for the full phase-space scan. In Sec. VII we present the dissipation comparison between the two architectures and shows that, within all realistic isotropic closures considered here, the nanolaminate remains less dissipative than the corresponding isotropic cosputtered mixture, especially in the high-index region. The paper in Sec. VII C summarizes the main findings and discusses the broader implications of this topological advantage for the design of next-generation gravitational-wave detectors. Finally, in Appendixes A

and B is mathematically demonstrated that for any target refractive index the nanolaminate requires a strictly smaller volume fraction of the highly dissipative high-index oxide (TiO_2) than any admissible isotropic morphology.

II. THERMAL NOISE EVALUATION VIA LEVIN’S APPROACH

The ultimate metric for comparing different coating architectures is their contribution to the Brownian thermal noise of the mirror. Among the pioneering contributors to the analysis of thermal noise in precision optical systems are Levin and Braginsky [33,51,52]. In the present work, we focus specifically on the coating Brownian contribution. According to Levin’s theorem, which provides an operational formulation derived from the fluctuation-dissipation theorem, the displacement spectral density is directly related to the time-averaged mechanical power dissipated when a virtual, harmonically oscillating pressure profile $p(r)$ (a *gedanken* force field) is applied to the mirror surface. We therefore compute the associated dissipated power W_{diss} under this virtual pressure.

For interferometric applications, the virtual pressure matches the intensity profile of the interrogating laser beam. While specific experimental metrology setups may employ non-Gaussian or mesa beam profiles [34], the fundamental Gaussian mode provides the canonical baseline for ground-based gravitational-wave detectors.

In the present calculations, we adopt the standard Gaussian profile:

$$p(r) = \frac{2F_0}{\pi w_0^2} \exp\left(-\frac{2r^2}{w_0^2}\right), \quad (1)$$

where F_0 is the virtual force amplitude and w_0 is the beam radius (i.e., the laser waist on the mirror).

For a thin coating of thickness d_{coat} deposited on a semi-infinite substrate, the complex strain energy density can be analytically integrated. As formalized in Refs. [13,53], the macroscopic mechanical response of any (transversely isotropic or isotropic) coating can be fully encapsulated by four effective parameters: A , B , D , and R_{mix} .

By defining the elastic moduli of the constituent materials as complex dynamic quantities, the mechanical dissipation is rigorously captured by the imaginary parts of these effective parameters. Assuming a perfectly elastic and lossless semi-infinite substrate, the total power dissipated within the coating is given by:

$$W_{\text{diss}} = \omega d_{\text{coat}} \left[\frac{1}{2} \text{Im}[A] I_{SD} + \frac{1}{2} \text{Im}[B] I_{\Delta} - \frac{1}{2} \text{Im}[D] I_T - R_{\text{mix}} I_{SDT} \right], \quad (2)$$

where $\omega = 2\pi f$ is the mechanical angular frequency. The terms I_{SD} , I_{Δ} , I_T , and I_{SDT} are geometric integrals over the

substrate deformations, defined by the Lamé coefficients of the substrate (λ_s , μ_s):

$$\begin{aligned} I_{SD} = I_{\Delta} &= \frac{F_0^2}{8\pi(\lambda_s + \mu_s)^2 w_0^2}, & I_T &= \frac{2F_0^2}{\pi w_0^2}, \\ I_{SDT} &= \frac{F_0^2}{2\pi(\lambda_s + \mu_s) w_0^2}. \end{aligned} \quad (3)$$

Note that a minus sign precedes $\text{Im}[D]$ because D acts as a compliancelike quantity, possessing an imaginary part of opposite sign compared to the stiffnesslike coefficients A and B . Furthermore, unlike the complex moduli A , B , and D , the cross-coupling term R_{mix} appears without the imaginary operator because it is inherently defined as a strictly dissipative quantity (specifically, the imaginary part of the ratio between off-diagonal and diagonal stiffness components, as detailed in subsequent sections).

The primary goal of the theoretical frameworks developed in the following sections is to accurately compute the parameters A , B , D , and R_{mix} for both the perfectly ordered nanolaminate (Sec. III) and the disordered isotropic mixture (Sec. IV), thereby enabling a direct and rigorous evaluation of Eq. (2).

III. EFFECTIVE PROPERTIES OF NANOLAMINATE

Before addressing the complex microgeometries of codeposited isotropic mixtures, we establish the theoretical baseline for perfectly ordered, periodically layered structures, commonly referred to as nanolaminates. We assume the individual layer thicknesses d_L and d_H are deeply subwavelength ($d_{L,H} \ll \lambda_0$ where λ_0 is the vacuum wavelength of the laser), allowing for a rigorous homogenization, while the total physical thickness of the stack d_{coat} is constrained by the macroscopic optical design. When the thickness of individual layers is significantly smaller than the wavelength of the probing field (optical or mechanical), the composite behaves macroscopically as a homogeneous, but anisotropic, effective medium. It should be emphasized that the anisotropy of the nanolaminate is purely structural: each constituent phase is locally isotropic, and the effective anisotropic response emerges solely from the periodic stacking geometry. As schematically shown in Fig. 1, the ordered coating is modeled as a periodic alternation of low- and high-index materials sublayers deposited on the substrate.

Recent morphological characterizations via scanning transmission electron microscopy and x-ray reflectivity have definitively proven that such ultrathin periodic structures can be physically realized with extremely sharp interfaces and negligible interdiffusion [22], fully justifying the application of exact stratified effective-medium relations.

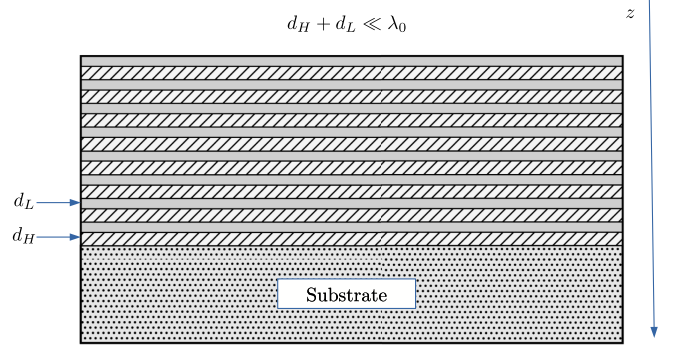


FIG. 1. Schematic representation of the ordered periodic nanolaminate deposited on a fused-silica substrate. The coating is modeled as an alternating sequence of low-index and high-index layers with period $d_H + d_L \ll \lambda_0$, so that the structure can be treated in the long-wavelength limit as an effective transversely isotropic medium. On the right-hand side of the figure, the z -axis is shown entering the laminated nanostructure and indicating the direction of the incident electromagnetic field.

A. Optical response of layered media

Optically, a nanolaminate composed of two isotropic phases behaves as a uniaxial crystal. The effective complex dielectric permittivity exhibits two distinct principal components depending on the polarization of the electric field relative to the layer interfaces. The classical optical limits for such stratified structures are defined by the Wiener bounds [36], extensively detailed in standard electromagnetic theory for layered media [37,54].

When the electric field is polarized parallel to the layers (in plane), the effective permittivity $\epsilon_{\parallel}$ is governed by the arithmetic mean:

$$\epsilon_{\parallel} = f_H \epsilon_H + f_L \epsilon_L \quad (4)$$

where $f_H = 1 - f_L$ is the volumetric fraction of high-refractive index medium (H). Conversely, when the electric field is polarized perpendicular to the layers (out of plane), the effective permittivity $\epsilon_{\perp}$ follows the harmonic mean (Reuss limit):

$$\epsilon_{\perp} = \left(\frac{f_H}{\epsilon_H} + \frac{f_L}{\epsilon_L} \right)^{-1}. \quad (5)$$

The exact analytical expressions (4) and (5) represent the extreme bounds of structural optical anisotropy, providing a deterministic reference against which the optical response of more disordered morphologies can be evaluated. Under standard (normal incidence) reflectometric measurements of thin films deposited on flat substrates—reproducing the normal incidence of laser light on the coating surface of end masses in interferometric gravitational-wave detectors—the primary optical response is determined by the in-plane refractive index, $\tilde{n}_{\parallel} = \sqrt{\epsilon_{\parallel}}$. Therefore, this effective index

dictates the physical thickness d_{coat} required to achieve a specific optical function, such as a high-reflectivity Bragg mirror.

For a target operating vacuum wavelength λ_0 (e.g., $\lambda_0 = 1064$ nm for current gravitational-wave interferometers), the quarter-wave optical thickness condition is strictly defined as $d_{\text{coat}} = \lambda_0 / (4\text{Re}[\tilde{n}_{\parallel}])$.

B. Mechanical Backus averaging and thermal noise parameters

Symmetrically to the optical domain, the mechanical response of a nanolaminate is highly direction dependent. A stratified medium composed of two isotropic materials (with stiffness components $c_{11}^{(i)}$, $c_{12}^{(i)}$, and $c_{44}^{(i)}$ for phases $i \in \{L, H\}$) macroscopically behaves as a transversely isotropic material with a vertical axis of symmetry.

The exact static effective stiffness tensor $\mathbf{C}^{NL}$ for this layered architecture was rigorously derived by Backus [38]. Utilizing the volumetric averaging operator $\langle x \rangle = f_H x_H + f_L x_L$, the effective elastic constants governing the nanolaminate are algebraically defined as follows.

The out-of-plane compressional stiffness (C_{33}) is strictly governed by the harmonic average of the individual longitudinal stiffnesses:

$$C_{33} = \langle c_{11}^{-1} \rangle^{-1}. \quad (6)$$

The coupling between out-of-plane and in-plane deformations is given by:

$$C_{13} = C_{33} \left\langle \frac{c_{12}}{c_{11}} \right\rangle. \quad (7)$$

The in-plane compressional and transverse stiffnesses (C_{11} and C_{12}) are mathematically more coupled, combining arithmetic averages of the plane-strain moduli with a correction proportional to the out-of-plane coupling:

$$C_{11} = \left\langle c_{11} - \frac{c_{12}^2}{c_{11}} \right\rangle + C_{33} \left\langle \frac{c_{12}}{c_{11}} \right\rangle^2 \quad (8)$$

$$C_{12} = \left\langle c_{12} - \frac{c_{12}^2}{c_{11}} \right\rangle + C_{33} \left\langle \frac{c_{12}}{c_{11}} \right\rangle^2. \quad (9)$$

Finally, the in-plane shear modulus (C_{66}) obeys the Voigt limit [55] (arithmetic mean):

$$C_{66} = \langle c_{44} \rangle. \quad (10)$$

To evaluate the dissipated power using Eq. (2), we map these tensorial components into the specific effective-medium parameters A , B , and D formalized for interferometric mirrors [53]:

$$A_{NL} = \frac{1}{2} \left(C_{11} + C_{12} - \frac{2C_{13}^2}{C_{33}} \right), \quad B_{NL} = C_{66}, \quad D_{NL} = \frac{1}{C_{33}}. \quad (11)$$

Additionally, the cross-coupling term R_{mix} is defined as the imaginary part of the ratio between the off-diagonal and diagonal out-of-plane stiffness components:

$$R_{\text{mix}}^{NL} = \text{Im} \left[\frac{C_{13}}{C_{33}} \right]. \quad (12)$$

Under the assumptions adopted in this work—namely, complex scalar Young's moduli and purely real Poisson's ratios for the constituent phases—this coupling coefficient vanishes identically for the nanolaminate architecture ($R_{\text{mix}}^{NL} = 0$) [53]. These mapped parameters establish the absolute theoretical limits of structural anisotropy for the composite. By evaluating the complex energy density using these Backus-derived coefficients, the nanolaminate architecture provides a deterministic minimum-dissipation baseline against which the behavior of disordered isotropic mixtures can be quantitatively compared.

Crucially, to account for the intrinsic mechanical dissipation of the constituent materials, we invoke the elastic-viscoelastic correspondence principle. The constituent stiffnesses are defined as complex dynamic moduli, $c_{ij}^* = c'_{ij} + ic''_{ij} = c'_{ij}(1 + i\phi_{ij})$, where ϕ_{ij} represents the specific mechanical loss angle of the phase.

Because the correspondence principle preserves the algebraic structure of the boundary conditions, the Backus averaging operators and the subsequent mapping into the A_{NL} , B_{NL} , and D_{NL} parameters are strictly valid in the complex plane. In our computational framework, these effective parameters are evaluated using full complex arithmetic. This avoids the truncation errors inherent in first-order Taylor expansions and allows the effective mechanical dissipation of the composite to be determined directly from the imaginary components of the final macroscopic parameters.

1. Validity of the quasistatic approximation

The exactness of the Backus averaging method relies on the condition $k_{\text{sound}} l' \ll 1$, where k_{sound} is the acoustic wave number and l' is the characteristic thickness of the layers. In our case, the frequency of interest for macroscopic mechanical perturbations is on the order of $f = 100$ Hz.

Given the macroscopic real part of Young's moduli of SiO_2 and TiO_2 (72 GPa and 165 GPa, respectively) and their standard densities, the acoustic wave velocity v in both phases exceeds 5000 m/s. Consequently, the acoustic wavelength at 100 Hz is extremely large ($\lambda_{\text{sound}} = v_{\text{sound}}/f \approx 50$ m).

For a typical nanolaminate with layer thicknesses l' in the nanometer regime ($10^{-9} - 10^{-8}$ m), the dimensionless

parameter $k_{\text{sound}} l'$ is on the order of 10^{-8} . Therefore, the elastic field variation across individual layers is negligible, placing the analysis deeply within the quasistatic regime. In this long-wavelength limit, the structural anisotropy is governed by the zero-order effective static moduli, and the Backus limits provide an exact analytical representation of the composite mechanics without dynamic scattering corrections.

IV. EFFECTIVE MEDIUM APPROXIMATIONS FOR REALISTIC ISOTROPIC MIXTURES

The characterization of codeposited amorphous thin films requires the estimation of effective properties from constituent phases. In experimental scenarios, where microstructural details are often inaccessible, effective-medium theories (EMTs) serve as the standard predictive framework [54]. These models relate the macroscopic response of the composite to the volume fractions of its constituents by assuming a specific idealized topology. This section outlines the analytical models employed to establish a realistic baseline for the optical and mechanical properties of the isotropic mixture.

A. Optical effective-medium models

In typical experimental scenarios, the effective real refractive index of the composite film (n_{eff}) is a known parameter, routinely measured via ellipsometry. To mathematically link the optical response to the mechanical one, we invert the classical optical EMTs to extract the effective volume fraction of the inclusions (f_H).

The first model considered is the L-L relation, mathematically equivalent to the Maxwell-Garnett approximation for vacuum-embedded inclusions. For a two-phase mixture with permittivities ϵ_H , ϵ_L , and effective permittivity $\epsilon_{\text{eff}} = \bar{n}_{\text{eff}}^2$, the L-L equation is expressed as

$$\frac{\epsilon_{\text{eff}} - 1}{\epsilon_{\text{eff}} + 2} = f_L \frac{\epsilon_L - 1}{\epsilon_L + 2} + f_H \frac{\epsilon_H - 1}{\epsilon_H + 2}, \quad (13)$$

where $f_H = 1 - f_L$ is the volumetric fraction of phase 2.

To account for a fully symmetric topology where neither phase is distinctly the host matrix, we also use the *Bruggeman symmetric effective-medium theory* [39]. For spherical inclusions, the standard Bruggeman equation imposes

$$f_L \frac{\epsilon_L - \epsilon_{\text{eff}}}{\epsilon_L + 2\epsilon_{\text{eff}}} + f_H \frac{\epsilon_H - \epsilon_{\text{eff}}}{\epsilon_H + 2\epsilon_{\text{eff}}} = 0. \quad (14)$$

Furthermore, to account for potential morphological anisotropies at the nanoscale, we implement a *generalized Bruggeman model* incorporating a depolarization (shape) factor L [40]:

$$f_L \frac{\epsilon_L - \epsilon_{\text{eff}}}{\epsilon_{\text{eff}} + L(\epsilon_L - \epsilon_{\text{eff}})} + f_H \frac{\epsilon_H - \epsilon_{\text{eff}}}{\epsilon_{\text{eff}} + L(\epsilon_H - \epsilon_{\text{eff}})} = 0. \quad (15)$$

The case $L = 1/3$ recovers the spherical case, $L < 1/3$ represents more prolate inclusions, and $L > 1/3$ represents more oblate inclusions. By numerically inverting these equations, we compute the specific volume fractions corresponding to a given target real refractive index. As rigorously proven in Appendix A, for any target refractive index, the high-index volume fraction required by the nanolaminate is strictly smaller than that of any admissible isotropic morphology.

B. Mechanical self-consistent scheme

Consistent with the treatment of the nanolaminate architecture, the constituent properties are introduced into the self-consistent scheme (SCS) as complex dynamic moduli. Following Barta's approach [43], this highly nonlinear coupled system is numerically solved in our framework using extended arithmetic precision in the complex plane. By avoiding linear approximations for the loss angles, the roots of these equations inherently yield both the real elastic constants (storage moduli) and the associated viscoelastic losses (loss moduli) of the effective isotropic medium.

Once the volume fraction is determined optically (as shown in Sec. IV A), the effective mechanical properties (complex bulk modulus K_{eff} and shear modulus μ_{eff}) of the isotropic mixture are evaluated using the SCS. Originally established independently by Hill [41] and Budiansky [42], the practical numerical formulation used here follows Barta [43]. Unlike simple mixing rules, the SCS implicitly couples the bulk and shear responses, making it suitable for densely packed, strongly interacting amorphous mixtures. The effective moduli are found by simultaneously solving

$$f_L \frac{K_L - K_{\text{eff}}}{K_{\text{eff}} + \alpha(K_L - K_{\text{eff}})} + f_H \frac{K_H - K_{\text{eff}}}{K_{\text{eff}} + \alpha(K_H - K_{\text{eff}})} = 0 \quad (16)$$

$$f_L \frac{\mu_L - \mu_{\text{eff}}}{\mu_{\text{eff}} + \beta(\mu_L - \mu_{\text{eff}})} + f_H \frac{\mu_H - \mu_{\text{eff}}}{\mu_{\text{eff}} + \beta(\mu_H - \mu_{\text{eff}})} = 0, \quad (17)$$

where the coupling coefficients α and β depend on the effective properties themselves:

$$\alpha = \frac{3K_{\text{eff}}}{3K_{\text{eff}} + 4\mu_{\text{eff}}}, \quad \beta = \frac{6(K_{\text{eff}} + 2\mu_{\text{eff}})}{5(3K_{\text{eff}} + 4\mu_{\text{eff}})}. \quad (18)$$

The system is solved directly in the complex plane, yielding both the real elastic constants and the associated viscoelastic losses of the effective isotropic medium.

Here, the subscripts L and H denote the low-index and high-index constituent phases, respectively. Their isotropic elastic properties are introduced through the complex bulk and shear moduli K_L , K_H , μ_L , and μ_H , obtained from the

corresponding Young's moduli Y_L^* , Y_H^* , and Poisson ratios ν_L , ν_H according to

$$K_i = \frac{Y_i^*}{3(1-2\nu_i)}, \quad \mu_i = \frac{Y_i^*}{2(1+\nu_i)}, \quad i \in \{L, H\}. \quad (19)$$

In the present work, the Young's moduli are treated as complex quantities, $Y_i^* = Y_i(1 + i\phi_i)$, ϕ_i being the mechanical loss angle, while the Poisson ratios are taken to be real.

A significant challenge in applying multiparameter viscoelastic formulations lies in the experimental determination of the bulk ($\phi_{K,i}$) and shear ($\phi_{\mu,i}$) loss angles separately. Many previous coating Brownian-noise models have adopted a single loss angle for each constituent, $\phi_{K,i} = \phi_{\mu,i} = \phi_i$, as a simplifying baseline. Generalized viscoelastic formulations, however, allow the two dissipation channels to be treated independently, and there is no first-principles reason requiring their equality [53,56]. Experimental studies aimed at separating bulk and shear losses provide evidence that the two channels may differ, but they have not yet produced a mutually consistent and transferable set of loss parameters. In particular, measurements on titania-doped tantala coatings [56,57] obtained different results for the relative magnitudes and frequency dependences of the bulk and shear losses. Moreover, no constituent-specific set of both $\phi_{K,i}$ and $\phi_{\mu,i}$ is available for the amorphous SiO_2 and TiO_2 materials considered here. We therefore adopt purely real constituent Poisson's ratios as a controlled single-loss-angle baseline in the absence of consistent experimental inputs.

From the resulting effective isotropic moduli K_{eff} and μ_{eff} , the stiffness tensor components are simply $C_{11}^{\text{ISO}} = K_{\text{eff}} + \frac{4}{3}\mu_{\text{eff}}$, $C_{12}^{\text{ISO}} = K_{\text{eff}} - \frac{2}{3}\mu_{\text{eff}}$, and $C_{66}^{\text{ISO}} = \mu_{\text{eff}}$.

Consequently, the parameters required for the thermal noise evaluation Eq. (2) are mapped as follows:

$$A_{\text{ISO}} = \frac{1}{2} \left(C_{11}^{\text{ISO}} + C_{12}^{\text{ISO}} - 2 \frac{(C_{12}^{\text{ISO}})^2}{C_{11}^{\text{ISO}}} \right), \quad B_{\text{ISO}} = C_{66}^{\text{ISO}},$$

$$D_{\text{ISO}} = \frac{1}{C_{11}^{\text{ISO}}}. \quad (20)$$

Unlike the nanolaminate case, the cross-coupling coefficient does not vanish for an isotropic mixture and is explicitly retained as a dissipative quantity:

$$R_{\text{mix}}^{\text{ISO}} = \text{Im} \left[\frac{C_{12}^{\text{ISO}}}{C_{11}^{\text{ISO}}} \right]. \quad (21)$$

V. RIGOROUS BOUNDS AND LOW-LOSS REDUCTION OF THE ADMISSIBLE REGIONS

The rigorous optical and mechanical bounds introduced in this section are not used merely as abstract extremal

results. Their main role is to quantify how much microstructural uncertainty remains for the isotropic cosputtered mixture once the constituent properties in Table I are fixed.

A. Theoretical bounds for the complex refractive index

To accurately model the optical response of the two-phase composite mixture (e.g., SiO_2 and TiO_2), it is necessary to determine its effective complex refractive index, $\tilde{n}_{\text{eff}} = n_{\text{eff}} + i\kappa_{\text{eff}}$, where n_{eff} is the real refractive index and κ_{eff} is the extinction coefficient. The complex refractive index is fundamentally related to the effective complex dielectric permittivity via $\epsilon_{\text{eff}} = \tilde{n}_{\text{eff}}^2$.

Since the exact microgeometry of the codeposited mixture is generally unknown, classical effective-medium approximations may yield inaccurate predictions, especially for optical losses. To overcome this limitation, we employ the rigorous bounding theory for the complex dielectric constant of two-component composites developed by Bergman [44,45] and Milton [46,47,58].

For a macroscopically homogeneous and isotropic three-dimensional composite with known volume fractions ($f_H, f_L = 1 - f_H$) and known constituent complex permittivities (ϵ_L, ϵ_H), the effective permittivity ϵ_{eff} is strictly confined within a closed, lens-shaped region $\mathcal{L}$ in the complex ϵ plane. The exact boundary of this region, formed by two circular arcs corresponding to the topological extremes where either the low-index or the high-index material acts as the host phase, can be explicitly parametrized by two real variables u and v (related to the geometric depolarization factors of the inclusions) as follows:

$$\epsilon_{\text{eff}}^{LH} = f_L \epsilon_L + f_H \epsilon_H - \frac{f_L f_H (\epsilon_L - \epsilon_H)^2}{3[u\epsilon_L + (1-u)\epsilon_H]}, \quad \text{for } \frac{f_H}{3} \leq u \leq 1 - \frac{f_L}{3}$$

$$\epsilon_{\text{eff}}^{HL} = f_L \epsilon_L + f_H \epsilon_H - \frac{f_L f_H (\epsilon_L - \epsilon_H)^2}{3[v\epsilon_H + (1-v)\epsilon_L]}, \quad \text{for } \frac{f_L}{3} \leq v \leq 1 - \frac{f_H}{3}. \quad (22)$$

Evaluating the first branch $\epsilon_{\text{eff}}^{LH}$ traces the primary circular arc, while evaluating the second branch $\epsilon_{\text{eff}}^{HL}$ traces the complementary arc. The two vertices where these arcs intersect correspond precisely to the exact analytical solutions for the *Doubly Coated Sphere* microgeometries, representing the theoretical extremes of the composite's optical response. A central topological feature, proven analytically in Appendix A and certified algebraically in Appendix B, is that this entire admissible region for isotropic media is strictly bounded away from the anisotropic Wiener upper bound, establishing a fundamental optical performance gap between the two architectures. The nonintersection between the Bergman-Milton boundary arcs and the Wiener limit is algebraically certified using the quantifier-elimination procedure detailed in Appendix B.

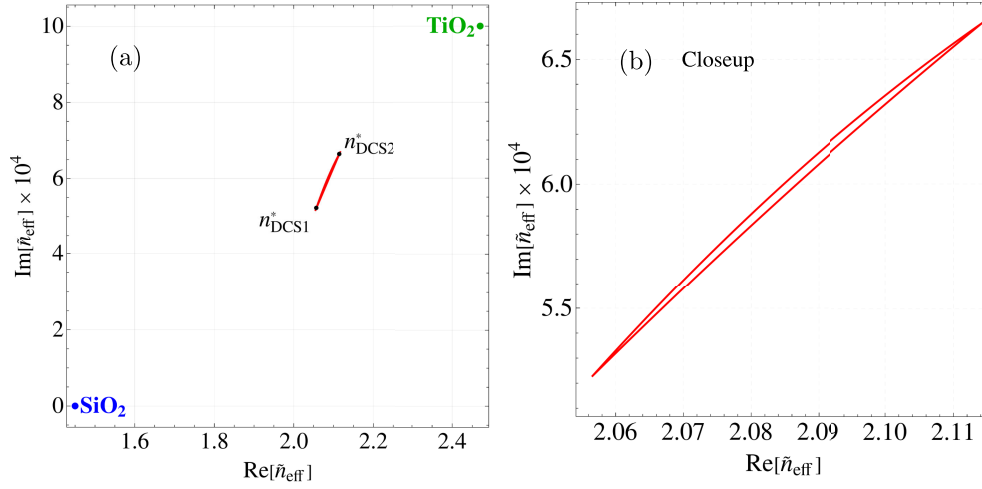


FIG. 2. Rigorous bounds for the effective complex refractive index ($\tilde{n}_{\text{eff}}$) of the $\text{SiO}_2/\text{TiO}_2$ composite. (a) Macroscopic view in the complex plane showing the constituent phases (Phase 1: blue label SiO_2 ; Phase 2: green label TiO_2) and the computed bounding region. Note the application of a 10^4 scaling factor on the imaginary axis to account for the intrinsically low optical losses. (b) A high-resolution closeup of the mapped Bergman-Milton region. The strictly permissible domain is confined within the red boundaries, which connect the two theoretical extremes corresponding to the doubly coated sphere geometries, denoted as n_{DCS1}^* and n_{DCS2}^* . The extreme narrowness of this lens-shaped region demonstrates that the effective optical losses of the composite are tightly constrained, rendering them largely independent of the specific microscopic morphology.

In our computational model, the exact boundary of the allowed region in the ϵ plane is calculated for a given volume fraction and subsequently mapped into the complex refractive-index plane by taking the principal square root, $\tilde{n} = \sqrt{\epsilon}$.

Figure 2 shows that, for the present $\text{SiO}_2/\text{TiO}_2$ system, the optical admissible region collapses into a very thin lens once the intrinsic extinction of the constituents is taken into account. In practice, once the real part of the refractive index is fixed, the admissible interval for the imaginary part becomes extremely narrow. This low-loss optical collapse is one of the reasons why the final full scan can be simplified without losing the main physical content.

The structure of the allowed region for the dielectric constant of an isotropic mixture in the complex plane allows us to formulate the analytical (general) results derived in Appendix A and the numerical evidence presented in Fig. 3, which elucidate that the nanolaminate architecture strictly minimizes the high-index volume fraction compared to any isotropic closure, a feature that remains consistently valid within the narrow rigorous bounds of the admissible region.

B. Theoretical bounds for the complex elastic moduli

The theoretical prediction of the mechanical response of a two-phase composite such as a $\text{SiO}_2/\text{TiO}_2$ mixture requires the determination of its effective macroscopic properties. While classical mixing rules provide simple arithmetic limits, they are inadequate for complex viscoelastic parameters, where the moduli possess both a real (storage) and an imaginary (loss) component. To accurately

constrain the effective complex bulk (K^*) and shear (μ^*) moduli without prior knowledge of the exact microgeometry, we employ the rigorous variational bounding method developed by Gibiansky, Milton, and Berryman [48–50].

For the complex bulk modulus, the admissible region shown in Fig. 4 is already extremely narrow. The closeup of Fig. 4 makes clear that the low constituent loss angles listed in Table I compress the rigorous lens into a thin strip, so that the exact admissible set is numerically very close to the straight segment joining the two HSW endpoints [48,59,60]. In practice, this means that the uncertainty associated with the isotropic cosputtered morphology is already strongly reduced at the level of the complex bulk response.

To avoid excessive notational clutter, in this subsection only we denote the low-index and high-index phases by the indices 1 and 2, respectively.

For the complex shear modulus, the problem is mapped into an auxiliary complex plane using the Y -transform [49]. The effective shear modulus μ^* is related to a complex variable y_μ (the Y -parameter, which encapsulates the microstructural geometric constraints) through the fractional linear transformation

$$\mu^* = \left[\sum_{i=1}^2 \frac{f_i}{\mu_i + y_\mu} \right]^{-1} - y_\mu. \quad (23)$$

Figure 5 shows that the exact 3D admissible region is not exactly the quadrilateral defined by the four complex HSW points μ_{11} , μ_{12} , μ_{21} , and μ_{22} .

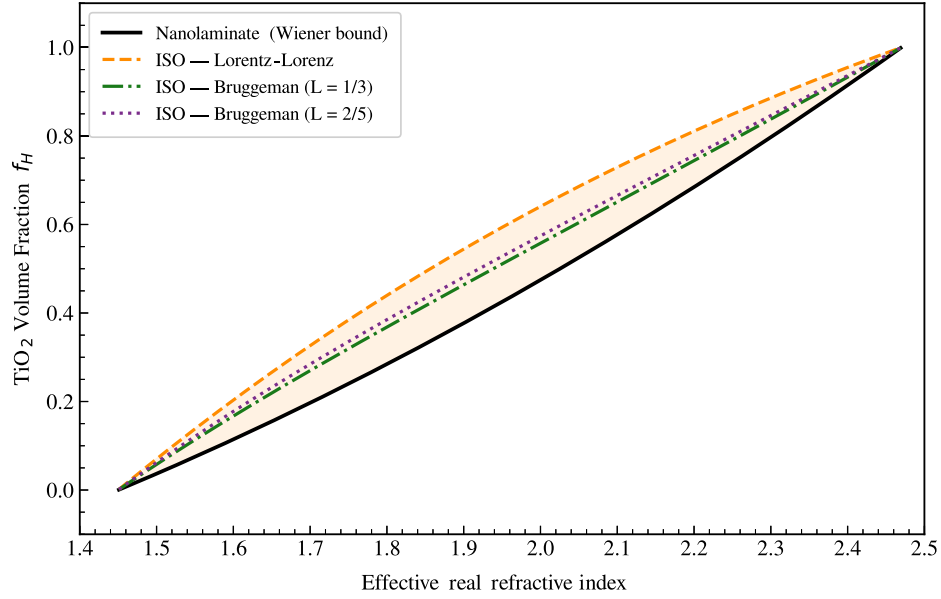


FIG. 3. Exact high-index (TiO_2) volume fraction f_H required to achieve a target real effective refractive index n_{eff} . The curves are computed using the constituent properties from Table I. The solid black line represents the perfectly ordered nanolaminate (Wiener upper bound), while the dashed and dotted lines correspond to standard isotropic effective-medium closures. The shaded area highlights the reduced volume fraction used by nanolaminated composite. (See Appendix A for the complete analytical proof of this systematic volume saving).

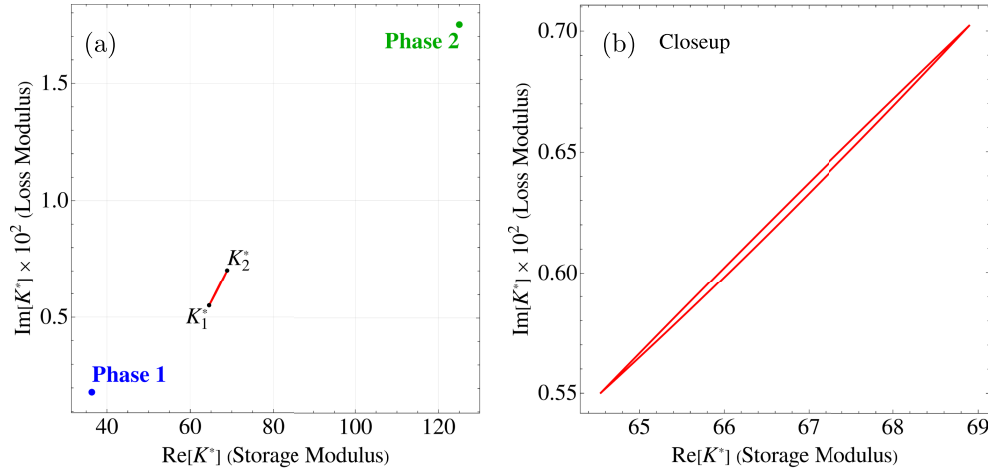


FIG. 4. Rigorous bounds for the effective complex bulk modulus K^* of the $\text{SiO}_2/\text{TiO}_2$ composite. (a) Macroscopic view of the complex plane showing the properties of the pure constituent phases (Phase 1: SiO_2 blue dot; Phase 2: TiO_2 green dot) and the bounding region (red) for a fixed volume fraction. The black dots, labeled K_1^* and K_2^* , represent the complex Hashin-Shtrikman-Walpole points. (b) A high-resolution closeup of the bounding region. Due to the extremely low mechanical loss tangents of the constituent materials ($\phi \sim 10^{-4}$), the bounding area degenerates into a highly elongated, eye-shaped lens. For practical modeling purposes, this microstructural uncertainty region is extremely narrow and can be well approximated by a straight line segment connecting the two vertices.

In this notation, the double indices represent the specific combinations of the constituent bulk and shear moduli corresponding to the theoretical doubly coated sphere microgeometries. The exact boundary exhibits a small outward convexity, as expected from the full Milton-Berryman construction in the auxiliary Y_μ plane [49,58]. However, for the present $\text{SiO}_2/\text{TiO}_2$ system, the difference

between the exact region and the polygon defined by the four extremal points is numerically very small because the loss angles are of order 10^{-4} (Table I). Therefore, while the four-point construction is not mathematically identical to the exact rigorous boundary, it provides an excellent low-loss approximation for the present materials system.

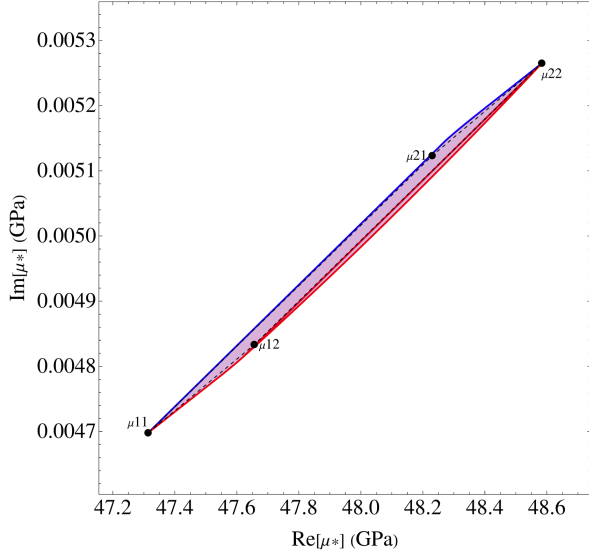


FIG. 5. Exact 3D bounding region for the effective complex shear modulus (μ^*) of the $\text{SiO}_2/\text{TiO}_2$ composite at a fixed volume fraction. The shaded purple area represents the strictly permissible domain for any realizable microgeometry. The solid blue and red curves denote the exact mapped inner and outer boundaries, respectively, computed via the continuous parametric algorithm in the complex Y_μ plane [49]. The four black dots represent the complex Hashin–Shtrikman–Walpole points (μ_{11} , μ_{12} , μ_{21} , μ_{22}). The dashed black line illustrates the simple quadrilateral connecting these vertices; notably, the exact rigorous bounds exhibit a slight outward convexity, demonstrating that the true permissible region is marginally larger than the polygon defined solely by the extreme points.

The key consequence of Figs. 2–4 is therefore the following. Strictly speaking, the final full scan is not presented as a fully rigorous coupled optical-mechanical extremal construction for a single realizable microstructure. However, in the specific low-loss $\text{SiO}_2/\text{TiO}_2$ case, the exact admissible domains collapse numerically into very narrow regions that are well approximated by short boundary segments in the optical plane and by the quadrilateral generated by the four mechanical HSW points. This is the reason why the simplified full phase-space scan remains physically meaningful and quantitatively informative for the present materials system.

VI. COMPUTATIONAL FRAMEWORK FOR THE FULL PHASE-SPACE SCAN

Having established the methods to compute the opto-mechanical properties for both architectures, this section outlines the numerical protocol used to generate the final dissipation comparison. The numerical algorithm, implemented in the Julia programming language [61], evaluates, for each prescribed effective real refractive index n_{eff} , the ratio between the dissipated power of the nanolaminate architecture (W_{NL}) and that of an isotropic mixed coating (W_{ISO}) under strict optical parity.

The scan is performed over the interval $n_L \leq n_{\text{eff}} \leq n_H$. For each iteration step (i.e., for each target n_{eff}), the algorithm executes the following sequence:

- (1) *Optical thickness fixing*: The physical thickness of both coatings is constrained by the quarter-wave condition at the design wavelength λ_0 :

$$d_{\text{coat}} = \frac{\lambda_0}{4n_{\text{eff}}}. \quad (24)$$

- (ii) *Nanolaminate evaluation*: The optical volume fraction f_H^{NL} is extracted using the stratified Wiener relation. This fraction is fed into the Backus averaging formulas in the complex plane to yield A_{NL} , B_{NL} , and D_{NL} . Because the nanolaminate is perfectly symmetric in plane, the cross-coupling term vanishes ($R_{\text{mix}}^{NL} = 0$). The dissipated power W_{NL} is then computed via Eq. (2).
- (3) *Isotropic mixture evaluation*: The volume fraction f_H^{ISO} is computed using three distinct optical closures: L-L, spherical Bruggeman, and generalized Bruggeman ($L = 0.40$). For each fraction, the non-linear self-consistent scheme is numerically solved in the complex plane to extract K_{eff} and μ_{eff} . These are mapped to A_{ISO} , B_{ISO} , D_{ISO} , and the coupling term $R_{\text{mix}}^{\text{ISO}}$ (which is strictly nonzero for isotropic mixtures), allowing the computation of W_{ISO} via Eq. (2).
- (4) *Admissible envelope construction*: Independent of the specific EMT models, the rigorous Bergman-Milton optical bounds are inverted to obtain the mathematically permissible interval $[f_{H,\text{min}}^{\text{opt}}, f_{H,\text{max}}^{\text{opt}}]$ for the given real n_{eff} . The code scans this interval, evaluating the complex mechanical vertices dictated by the HSW bounds. The minimum and maximum dissipation values obtained define the absolute rigorous envelope for the isotropic mixed coating.

For each n_{eff} , the code outputs the dissipation ratio W_{NL}/W_{ISO} for the realistic EMT closures, alongside the upper and lower bounds of the rigorous envelope. The results of this full phase-space scan are presented in Fig. 6 and discussed in the next section.

VII. RESULTS AND DISCUSSION

The purpose of this section is to translate the effective-medium and bounding results of the previous sections into a direct dissipation comparison between the two coating architectures. We first analyze the realistic isotropic-mixture models introduced in Sec. IV and compare them with the nanolaminate baseline through the full phase-space scan of Fig. 6. We then examine the wider admissible envelope to assess how strongly the conclusion depends on the unknown isotropic microstructure, ultimately discussing the physical mechanisms driving the observed advantage of the ordered layered topology.

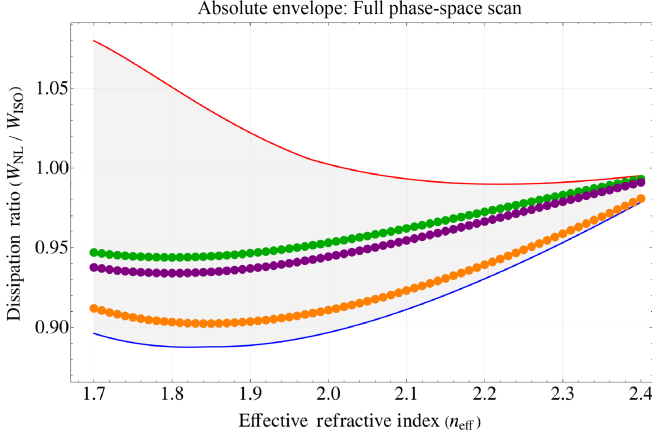


FIG. 6. Full phase-space scan of the dissipation ratio W_{NL}/W_{ISO} as a function of the target real effective refractive index n_{eff} . The red and blue curves delimit the optically admissible mechanical envelope obtained by scanning the isotropic mixed coating over the range of phase fractions allowed by the Bergman-Milton optical bounds and over the corresponding Hashin-Shtrikman (HS)/Hashin-Shtrikman-Walpole (HSW)-like mechanical vertices. The orange points correspond to the realistic isotropic mixture obtained by combining the L-L optical closure with the complex self-consistent mechanical solution. The green points show the analogous result obtained with the spherical Bruggeman optical closure, while the purple points correspond to the generalized Bruggeman closure with shape parameter $L = 0.40$. Values below unity indicate that the nanolaminate dissipates less power than the corresponding isotropic mixed coating. In the realistic closures considered here, the nanolaminate remains consistently less dissipative over the full interval of n_{eff} explored, while the broader red-blue band shows that this advantage is not absolute over the entire admissible phase space.

A. Thermal-noise comparison at parity of optical design

The final result of the analysis is summarized in Fig. 6, which reports the dissipation ratio W_{NL}/W_{ISO} as a function of the target real effective refractive index n_{eff} . The red and blue curves delimit the optically admissible mechanical envelope, while the orange, green, and purple points correspond to the three realistic isotropic closures introduced in Sec. IV A.

The primary result is that all realistic closures lie strictly below unity over the full interval of n_{eff} explored here. This indicates that, across all realistic isotropic-mixture models considered, the nanolaminate is consistently less dissipative than the corresponding cosputtered isotropic coating. Among these closures, the L-L model yields the lowest values of W_{NL}/W_{ISO} , suggesting that the most finely dispersed isotropic morphology considered here is also the most dissipative relative to the ordered layered architecture.

The spherical Bruggeman and generalized Bruggeman closures are less unfavorable to the isotropic mixture, yet they still preserve a clear dissipative advantage for the nanolaminate. In the intermediate refractive-index region

($n_{eff} \sim 1.8$ – 2.0), the reduction in dissipation is on the order of a few percent for Bruggeman-type models, approaching the 10% mark for closures more favorable to the nanolaminate architecture, such as the L-L model.

B. Interpretation of the admissible envelope

Figure 6 also illustrates that this advantage is not mathematically absolute over the entire theoretically admissible phase space. The upper red branch remains above unity at low and intermediate n_{eff} , implying that highly specific, idealized, or extremal isotropic microstructures could theoretically compete with or exceed the nanolaminate in that corner of the admissible domain.

However, the crucial point is that the rigorous Bergman-Milton and Gibiansky-Milton-Berryman admissible regions in the complex plane collapse into extremely narrow lenses due to the intrinsically low mechanical and optical losses of the constituent materials ($\phi \sim 10^{-4}$, $\kappa \sim 10^{-8}$ – 10^{-3}). This geometric degeneration drastically restricts the microstructural uncertainty, proving that our phase-space scan remains quantitatively robust and tightly bounds the exact admissible set. Consequently, all physically realistic isotropic closures lie well inside this narrow envelope and remain strictly below unity across the full scan. The superiority of the nanolaminate is therefore not an isolated artifact of a single mixing rule, but a stable, structurally driven feature.

C. The topological trade-off

The physical origin of this result resolves a highly nontrivial topological trade-off. *A priori*, it is not obvious that a nanolaminate should outperform an isotropic mixture. From a purely mechanical standpoint, the nanolaminate is actually disadvantaged: its parallel layered structure forces the in-plane shear deformations—excited by the Gaussian laser pressure—to operate at the Voigt upper bound, which maximizes viscoelastic energy dissipation. An isotropic mixture, conversely, distributes shear strain more efficiently.

However, the nanolaminate possesses a critical optical advantage: it operates at the Wiener upper bound for the in-plane dielectric permittivity. This extreme optical efficiency allows the nanolaminate to reach the target real refractive index using a significantly lower volume fraction of the TiO_2 compared to an isotropic mixture (as rigorously proven and algebraically certified in Appendixes A and B). Our calculations show that, for the material parameters considered here, this drastic reduction in the volume of lossy material overwhelmingly overcompensates for the mechanical anisotropy penalty, resulting in a net reduction of internal friction.

This theoretical demonstration of a purely topological advantage perfectly complements the known experimental and technological benefits of NLOF. As extensively investigated within the gravitational-wave community [18,24,25],

the geometric confinement inherent in subwavelength NLOF frustrates the crystallization of high-index oxides. This allows the coatings to withstand significantly higher annealing temperatures without developing scattering defects, which in turn leads to a further, substantial reduction in mechanical loss. Furthermore, nanolayering flattens and shifts the low-temperature mechanical loss peaks [28], addressing one of the most critical bottlenecks for cryogenic interferometry.

As characterized in Ref. [35], nanoscale confinement produces a slight reduction and a weak thickness dependence of the refractive index of the constituent oxides owing to interface and density effects; the experimentally derived indices adopted here therefore incorporate these nanoscale optical effects.

The numerical baseline reported in Table I combines established benchmark parameters for the low-index silica constituent [8,34], nominal room-temperature mechanical parameters for amorphous titania, and optical parameters obtained from a refined analysis of the spectroscopic-ellipsometry data acquired on our nanolayer samples, supported by their structural characterization [22,35]. The quantitative dissipation ratios reported in this work are therefore conditional on this particular baseline parameter set. The optical component of our argument is independent of the mechanical loss parameters. Because the nanolaminate operates at the Wiener optical upper bound, it reaches the prescribed effective refractive index using a strictly smaller volume fraction of the high-index phase, $f_H^{\text{NL}} < f_H^{\text{ISO}}$, as rigorously proven in Appendix A. For the benchmark dissipative contrast adopted here, $\phi_H > \phi_L$, the reduction of the high-index volume fraction also reduces the amount of the more dissipative constituent. This provides a physically motivated heuristic explanation for the lower dissipation obtained for the realistic isotropic closures considered in this work, but it does not constitute a general robustness proof. Within the low-loss approximation, a common rescaling of both constituent loss angles is expected to modify the dissipations of the two architectures by approximately the same factor and therefore to have a much smaller effect on their ratio. By contrast, independent variations of the constituent loss angles, Young's moduli, or Poisson's ratios may modify $W_{\text{NL}}/W_{\text{ISO}}$. These observations provide only a qualitative sensitivity argument and should not be interpreted as a substitute for a systematic parametric analysis.

VIII. CONCLUSIONS AND OUTLOOK

In this work, we introduced a fully coupled optomechanical framework to evaluate the thermal noise dissipation of perfectly ordered periodic nanolaminates compared to disordered isotropic cosputtered mixtures. By enforcing a strict quarter-wave optical thickness constraint and targeting the exact same effective real refractive index, we ensured an unbiased and physically consistent

comparison between the two architectures for the baseline $\text{SiO}_2/\text{TiO}_2$ system.

By evaluating the exact Wiener-Backus limits for the layered medium against realistic effective medium theories (Lorentz-Lorenz and Bruggeman closures coupled with a complex self-consistent scheme) and mapping their absolute variational bounds, we demonstrated that the nanolaminate systematically exhibits lower thermal noise dissipation ($W_{\text{NL}}/W_{\text{ISO}} < 1$).

The physical mechanism driving this performance is a favorable topological trade-off. The nanolaminate operates at the Wiener upper bound for the in-plane dielectric permittivity, maximizing optical efficiency. As mathematically proven and algebraically certified in Appendixes A and B, this efficiency allows the ordered structure to reach the target refractive index using a strictly lower volume fraction of the highly dissipative high-index phase. We showed that this reduction in the volume of lossy material more than compensates for the inherent mechanical penalty associated with in-plane shear deformations.

Nonetheless, we must acknowledge certain physical limitations of the present idealized theoretical framework. First, our thermodynamic analysis assumes that the constituent amorphous phases (SiO_2 and TiO_2) preserve their independent structural, optical, and dissipative identities when codeposited or layered. In real cosputtered mixtures, nanoscale structural rearrangements, localized coordination changes, or chemical interactions between the different oxides might alter their intrinsic loss angles, potentially causing the composite to depart from the predicted thermodynamic bounds. Second, our nanolaminate model relies on perfectly sharp, steplike interfaces between the alternating subwavelength layers. In practice, a subnanometer interdiffusion (intermixing) zone at the boundaries inevitably occurs during deposition and subsequent thermal annealing. Although experimental characterizations show that such interdiffused regions are extremely narrow [17], their presence could introduce local gradient-index optical effects and shear stress relaxation states that are neglected in our exact Wiener-Backus treatment. Nevertheless, these idealizations do not undermine the physical validity of our approach; rather, they establish a rigorous theoretical benchmark—an ultimate performance limit to strive for in the practical development of composite optical coatings.

A further limitation of the present analysis is the use of a single loss angle for each constituent phase, implemented through purely real Poisson's ratios. Introducing independent bulk and shear loss angles, $\phi_{K,i} \neq \phi_{\mu,i}$, would generally result in complex constituent Poisson's ratios and would require the viscoelastic bounds and the corresponding Brownian dissipations to be recomputed. In the zero-loss limit, the admissible complex-modulus regions reduce to the classical Hashin-Shtrikman-Walpole intervals on the real axis. Independent imaginary components generally open these intervals into two-dimensional regions bounded

by circular arcs or, for the general shear-modulus problem, by intersections of circular bounding regions [48,49]. Since the loss angles relevant to the present optical coatings are small, these admissible regions are expected to remain narrow in absolute terms, although their width, orientation, and detailed geometry may change.

We conjecture that, for the benchmark $\text{SiO}_2/\text{TiO}_2$ system and the realistic isotropic closures considered here, the inequality $W_{NL}/W_{ISO} < 1$ remains valid under sufficiently small departures from the single-loss-angle approximation, provided that all constituent bulk and shear loss angles remain in the low-loss regime. A quantitative assessment of this conjecture and, more generally, of the robustness of the present conclusions requires a systematic parametric investigation of the constituent optical and mechanical properties, supported by reliable constituent-specific experimental data, particularly independent measurements of the bulk and shear loss angles. This combined theoretical, computational, and experimental extension represents an important direction for future work.

Under this perspective, these theoretical findings indicate that synthesizing optical materials as highly ordered 1D superlattices provides a fundamental, structurally driven reduction in internal friction. Combined with their experimentally proven ability to frustrate crystallization and mitigate low-temperature mechanical loss peaks, nanolaminates emerge as a highly promising architectural pathway, setting a clear design target for the development of ultralow noise mirrors in next-generation (3G) gravitational-wave observatories, such as the Einstein Telescope [29] and Cosmic Explorer [30].

ACKNOWLEDGMENTS

The authors are deeply grateful to I.M. Pinto, who provided the original inspiration for this research, for his continuous encouragement and invaluable technical insights throughout this work. We also extend our sincere appreciation to G. Cagnoli and E. Cesarini for their constructive feedback. Furthermore, we acknowledge the Virgo CRD and LIGO Optics Working Groups for the stimulating scientific exchanges that contributed to this research. This work was partially supported by INFN Sezione di Napoli, Gruppo Collegato di Salerno.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: MINIMUM HIGH-INDEX FRACTION

This appendix proves a statement stronger than the usual intermediate-index property of passive mixtures. Specifically, in the passive low-loss complex regime relevant to the optical oxides considered here, the Wiener-parallel nanolaminate reaches any strictly interior prescribed real refractive index with a strictly smaller high-index volume fraction than any admissible macroscopically isotropic morphology made of the same two constituents.

We denote the complex permittivities of the high- and low-index phases by

$$\varepsilon_H = \varepsilon'_H + i\varepsilon''_H, \quad \varepsilon_L = \varepsilon'_L + i\varepsilon''_L, \quad (\text{A1})$$

with

$$\varepsilon'_H > \varepsilon'_L > 0, \quad \varepsilon''_H, \quad \varepsilon''_L \geq 0. \quad (\text{A2})$$

We also set

$$f_L = 1 - f_H, \quad \Delta := \varepsilon_H - \varepsilon_L = \Delta' + i\Delta'', \quad \Delta' = \varepsilon'_H - \varepsilon'_L > 0. \quad (\text{A3})$$

No sign assumption is imposed on Δ'' . In particular, the proof below remains valid even if the high-index phase is optically less lossy than the low-index phase.

We measure the optical-loss strength by

$$\eta_{\max} := \max \left\{ \frac{\varepsilon''_L}{\varepsilon'_L}, \frac{\varepsilon''_H}{\varepsilon'_H}, \frac{|\varepsilon''_H - \varepsilon''_L|}{\varepsilon'_H - \varepsilon'_L} \right\}. \quad (\text{A4})$$

The sign-independent low-loss hypothesis used in this appendix is

$$\eta_{\max} < \frac{1}{4}. \quad (\text{A5})$$

The factor $1/4$ is a sign-independent safety margin controlling possible upward excursions of the Bergman-Milton lens when Δ'' is allowed to have either sign. For the $\text{SiO}_2/\text{TiO}_2$ data used in this paper, this hypothesis is satisfied by a very large margin.

The nanolaminate in-plane permittivity is the Wiener upper value

$$\varepsilon_k(f_H) = f_L \varepsilon_L + f_H \varepsilon_H = \varepsilon_L + f_H \Delta. \quad (\text{A6})$$

For the same volume fraction, the effective permittivity of any macroscopically isotropic two-phase composite lies in the Bergman-Milton admissible lens $\mathcal{L}(f_H)$. Its boundary consists of the two arcs

$$\begin{aligned}\varepsilon_b^{LH}(u) &= \varepsilon_k(f_H) - \frac{f_L f_H \Delta^2}{3[u\varepsilon_L + (1-u)\varepsilon_H]}, \quad \text{where } \frac{f_H}{3} \leq u \leq 1 - \frac{f_L}{3}, \\ \varepsilon_b^{HL}(v) &= \varepsilon_k(f_H) - \frac{f_L f_H \Delta^2}{3[v\varepsilon_H + (1-v)\varepsilon_L]}, \quad \text{where } \frac{f_L}{3} \leq v \leq 1 - \frac{f_H}{3}.\end{aligned}\quad (\text{A7})$$

These are the same boundary arcs used in the main text, reported here for ease of reference.

Lemma A1. Polar identity for the real refractive index. Let $z = x + iy$, with $x > 0$ and $y \geq 0$, and let the square root be taken on the principal branch. Then

$$2(\text{Re}\sqrt{z})^2 = |z| + x. \quad (\text{A8})$$

Proof. Write $z = |z|e^{i\theta}$, with $\theta \in [0, \pi/2)$. Then

$$\sqrt{z} = \sqrt{|z|}e^{i\theta/2}, \quad \text{Re}\sqrt{z} = \sqrt{|z|}\cos\frac{\theta}{2}.$$

Therefore

$$2(\text{Re}\sqrt{z})^2 = 2|z|\cos^2\frac{\theta}{2} = |z|(1 + \cos\theta) = |z| + x.$$

Lemma A2. Strict monotonicity of the nanolaminate index. Under the low-loss hypothesis (A5), the function

$$F_{NL}(f_H) := \text{Re}\sqrt{\varepsilon_k(f_H)} \quad (\text{A9})$$

is strictly increasing for $0 < f_H < 1$.

Proof. Let

$$\varepsilon_k(f_H) = x_k + iy_k = |\varepsilon_k|e^{i\theta_k}, \quad \theta_k \in [0, \pi/2).$$

By Lemma A1,

$$2F_{NL}^2 = |\varepsilon_k| + x_k.$$

Since $d\varepsilon_k/df_H = \Delta$, differentiation gives

$$\begin{aligned}4F_{NL} \frac{dF_{NL}}{df_H} &= \frac{d|\varepsilon_k|}{df_H} + \frac{dx_k}{df_H} \\ &= \Delta' \cos\theta_k + \Delta'' \sin\theta_k + \Delta' \\ &= \Delta'(1 + \cos\theta_k + \eta \sin\theta_k),\end{aligned}\quad (\text{A10})$$

where

$$\eta := \frac{\Delta''}{\Delta'}.\quad (\text{A11})$$

By (A4), $|\eta| \leq \eta_{\max} < 1$. Moreover,

$$1 + \cos\theta_k > \sin\theta_k \quad \text{for } \theta_k \in [0, \pi/2).$$

Hence

$$1 + \cos\theta_k + \eta \sin\theta_k \geq 1 + \cos\theta_k - |\eta| \sin\theta_k > 0.$$

Since $\Delta' > 0$ and $F_{NL} > 0$, Eq. (A10) gives $dF_{NL}/df_H > 0$. ■

Lemma A3. Dominant horizontal separation of the boundary. Let $0 < f_H < 1$. Denote by $\varepsilon_k = f_L \varepsilon_L + f_H \varepsilon_H$ the Wiener upper permittivity, and let $\varepsilon_b \in \partial\mathcal{L}(f_H)$ be any point on the Bergman-Milton boundary. If

$$\varepsilon_k = x_k + iy_k, \quad \varepsilon_b = x_b + iy_b,$$

and

$$\delta x := x_k - x_b, \quad \delta y := y_b - y_k,$$

then, under the low-loss hypothesis (A5),

$$\delta x > 0, \quad |\delta y| \leq \frac{3\eta_{\max}}{1 - 3\eta_{\max}^2} \delta x < 4\eta_{\max} \delta x. \quad (\text{A12})$$

Equivalently, the boundary of the isotropic admissible lens remains separated from the Wiener point by a strictly positive real gap, while its vertical excursion is only a low-loss correction of order $\eta_{\max}$.

Proof. Both boundary arcs in (A7) can be written in the unified form

$$\varepsilon_b = \varepsilon_k - \frac{f_L f_H \Delta^2}{D}, \quad (\text{A13})$$

where

$$D = 3[u\varepsilon_L + (1-u)\varepsilon_H] \quad \text{or} \quad D = 3[v\varepsilon_H + (1-v)\varepsilon_L]. \quad (\text{A14})$$

The parameters u and v range over the intervals stated in (A7). In both cases, the coefficients of $D/3$ are non-negative and sum to one. Thus $D/3$ is a convex combination of the two constituent permittivities. Consequently, writing $D = D' + iD''$,

$$D' > 0, \quad 0 \leq D'' \leq \eta_{\max} D'. \quad (\text{A15})$$

Define

$$\eta := \frac{\Delta''}{\Delta'}, \quad \gamma := \frac{D''}{D'}, \quad C := \frac{f_L f_H D' (\Delta')^2}{|D|^2} > 0. \quad (\text{A16})$$

Then, by (A4) and (A15),

$$|\eta| \leq \eta_{\max}, \quad 0 \leq \gamma \leq \eta_{\max}. \quad (\text{A17})$$

Using (A13) and separating real and imaginary parts gives the exact identities

$$\delta x = C(1 - \eta^2 + 2\gamma\eta), \quad (\text{A18})$$

and

$$\delta y = C[\gamma(1 - \eta^2) - 2\eta]. \quad (\text{A19})$$

From (A17),

$$\delta x \geq C(1 - 3\eta_{\max}^2) > 0, \quad (\text{A20})$$

and, since $|\eta| < 1$,

$$\begin{aligned} |\delta y| &\leq C(\gamma(1 - \eta^2) + 2|\eta|) \\ &\leq C(\eta_{\max} + 2\eta_{\max}) = 3\eta_{\max}C. \end{aligned} \quad (\text{A21})$$

Combining (A20) and (A21) yields

$$|\delta y| \leq \frac{3\eta_{\max}}{1 - 3\eta_{\max}^2} \delta x.$$

Since $\eta_{\max} < 1/4$, the prefactor satisfies

$$\frac{3}{1 - 3\eta_{\max}^2} < 4,$$

and the final inequality in (A12) follows. $\blacksquare$

Lemma A4. From boundary separation to refractive-index separation. For every $0 < f_H < 1$ and every boundary point $\varepsilon_b \in \partial\mathcal{L}(f_H)$,

$$\operatorname{Re}\sqrt{\varepsilon_b} < \operatorname{Re}\sqrt{\varepsilon_k(f_H)}. \quad (\text{A22})$$

Proof. Write

$$\varepsilon_k = x_k + iy_k, \quad \varepsilon_b = x_b + iy_b,$$

and define

$$\delta x = x_k - x_b, \quad \delta y = y_b - y_k.$$

We first prove the identity

$$\begin{aligned} &2[(\operatorname{Re}\sqrt{\varepsilon_k})^2 - (\operatorname{Re}\sqrt{\varepsilon_b})^2](|\varepsilon_k| + |\varepsilon_b|) \\ &= \delta x(|\varepsilon_k| + |\varepsilon_b| + x_k + x_b) - (y_b^2 - y_k^2). \end{aligned} \quad (\text{A23})$$

By Lemma A1,

$$2[(\operatorname{Re}\sqrt{\varepsilon_k})^2 - (\operatorname{Re}\sqrt{\varepsilon_b})^2] = (|\varepsilon_k| + x_k) - (|\varepsilon_b| + x_b).$$

Since $\delta x = x_k - x_b$, this becomes

$$2[(\operatorname{Re}\sqrt{\varepsilon_k})^2 - (\operatorname{Re}\sqrt{\varepsilon_b})^2] = |\varepsilon_k| - |\varepsilon_b| + \delta x.$$

Multiplying by the positive quantity $|\varepsilon_k| + |\varepsilon_b|$, we obtain

$$\begin{aligned} &2[(\operatorname{Re}\sqrt{\varepsilon_k})^2 - (\operatorname{Re}\sqrt{\varepsilon_b})^2](|\varepsilon_k| + |\varepsilon_b|) \\ &= (|\varepsilon_k| - |\varepsilon_b|)(|\varepsilon_k| + |\varepsilon_b|) + \delta x(|\varepsilon_k| + |\varepsilon_b|). \end{aligned}$$

Using

$$(|\varepsilon_k| - |\varepsilon_b|)(|\varepsilon_k| + |\varepsilon_b|) = |\varepsilon_k|^2 - |\varepsilon_b|^2,$$

and

$$|\varepsilon_k|^2 = x_k^2 + y_k^2, \quad |\varepsilon_b|^2 = x_b^2 + y_b^2,$$

we get

$$\begin{aligned} |\varepsilon_k|^2 - |\varepsilon_b|^2 &= x_k^2 - x_b^2 + y_k^2 - y_b^2 \\ &= \delta x(x_k + x_b) - (y_b^2 - y_k^2). \end{aligned}$$

Therefore,

$$\begin{aligned} &2[(\operatorname{Re}\sqrt{\varepsilon_k})^2 - (\operatorname{Re}\sqrt{\varepsilon_b})^2](|\varepsilon_k| + |\varepsilon_b|) \\ &= \delta x(x_k + x_b) - (y_b^2 - y_k^2) + \delta x(|\varepsilon_k| + |\varepsilon_b|) \\ &= \delta x(|\varepsilon_k| + |\varepsilon_b| + x_k + x_b) - (y_b^2 - y_k^2), \end{aligned}$$

which proves (A23).

By Lemma A3, $\delta x > 0$. If $y_b \leq y_k$, then, since boundary points are passive, $y_b \geq 0$, and of course $y_k \geq 0$. Thus

$$y_b^2 - y_k^2 \leq 0,$$

and the right-hand side of (A23) is strictly positive.

It remains only to consider the upward-bulge case $y_b > y_k$. Then $\delta y > 0$, and Lemma A3 gives

$$0 < \delta y < 4\eta_{\max}\delta x. \quad (\text{A24})$$

Moreover, by the definition of $\eta_{\max}$,

$$y_k = f_L \varepsilon_L'' + f_H \varepsilon_H'' \leq \eta_{\max}(f_L \varepsilon_L' + f_H \varepsilon_H') = \eta_{\max} x_k. \quad (\text{A25})$$

The Bergman-Milton lens lies in the right half-plane under the present passive low-loss assumptions, so $x_b > 0$. Hence

$$0 < \delta x < x_k. \quad (\text{A26})$$

Combining (A24)–(A26),

$$y_b + y_k = 2y_k + \delta y < 2\eta_{\max}x_k + 4\eta_{\max}\delta x < 6\eta_{\max}x_k. \quad (\text{A27})$$

Therefore

$$y_b^2 - y_k^2 = \delta y(y_b + y_k) < 24\eta_{\max}^2 x_k \delta x. \quad (\text{A28})$$

On the other hand,

$$|\varepsilon_k| + |\varepsilon_b| + x_k + x_b = (|\varepsilon_k| + x_k) + (|\varepsilon_b| + x_b) \geq 2x_k. \quad (\text{A29})$$

Substituting (A28) and (A29) into (A23) gives

$$\begin{aligned} & \delta x(|\varepsilon_k| + |\varepsilon_b| + x_k + x_b) - (y_b^2 - y_k^2) \\ & > 2x_k \delta x - 24\eta_{\max}^2 x_k \delta x = 2x_k \delta x(1 - 12\eta_{\max}^2). \end{aligned} \quad (\text{A30})$$

Since $\eta_{\max} < 1/4$, the factor $1 - 12\eta_{\max}^2$ is positive. Hence the right-hand side of (A23) is strictly positive also in the upward-bulge case. Thus

$$(\text{Re}\sqrt{\varepsilon_k})^2 > (\text{Re}\sqrt{\varepsilon_b})^2.$$

Both real parts are positive, and (A22) follows. ■

Lemma A5. Dominance over the whole isotropic admissible lens. For every $0 < f_H < 1$ and every admissible isotropic effective permittivity $\varepsilon_{\text{eff}}^{\text{ISO}} \in \mathcal{L}(f_H)$,

$$\text{Re}\sqrt{\varepsilon_{\text{eff}}^{\text{ISO}}} < \text{Re}\sqrt{\varepsilon_k(f_H)}. \quad (\text{A31})$$

Proof. The principal square root is analytic in the right half-plane. Hence

$$\Phi(z) := \text{Re}\sqrt{z}$$

is harmonic there. The Bergman-Milton lens $\mathcal{L}(f_H)$ is compact and lies in the right half-plane under the passive low-loss assumptions above. Therefore, by the maximum principle for harmonic functions, the maximum of Φ over $\mathcal{L}(f_H)$ is attained on $\partial\mathcal{L}(f_H)$. Lemma A4 shows that every boundary value is strictly smaller than $\text{Re}\sqrt{\varepsilon_k(f_H)}$. The same strict inequality therefore holds throughout the whole admissible lens. ■

Theorem A6. Strict inequality of the required high-index volume fractions. Let n_* be a strictly interior target real refractive index, $n_L < n_* < n_H$, attained by both architectures with volume fractions in $(0,1)$. Let f_H^{NL} be the high-index fraction required by the nanolaminate and let f_H^{ISO} be the high-index fraction required by an admissible macroscopically isotropic morphology:

$$\text{Re}\sqrt{\varepsilon_k(f_H^{\text{NL}})} = n_*, \quad \text{Re}\sqrt{\varepsilon_{\text{eff}}^{\text{ISO}}(f_H^{\text{ISO}})} = n_*. \quad (\text{A32})$$

Then, under the low-loss hypothesis (A5),

$$f_H^{\text{NL}} < f_H^{\text{ISO}}. \quad (\text{A33})$$

Proof. By Lemma A5, evaluated at $f_H = f_H^{\text{ISO}}$,

$$n_* = \text{Re}\sqrt{\varepsilon_{\text{eff}}^{\text{ISO}}(f_H^{\text{ISO}})} < \text{Re}\sqrt{\varepsilon_k(f_H^{\text{ISO}})} = F_{\text{NL}}(f_H^{\text{ISO}}).$$

By definition of f_H^{NL} ,

$$n_* = F_{\text{NL}}(f_H^{\text{NL}}).$$

Therefore

$$F_{\text{NL}}(f_H^{\text{NL}}) < F_{\text{NL}}(f_H^{\text{ISO}}).$$

Since F_{NL} is strictly increasing by Lemma A2, the inequality of the arguments follows:

$$f_H^{\text{NL}} < f_H^{\text{ISO}}.$$

■

Final remark: Optical origin of the high-index saving. The ordered nanolaminate uses the Wiener upper optical response to minimize the amount of high-index phase required to reach a prescribed real refractive index. Since TiO_2 is the mechanically lossier constituent in the material system considered here, the strict inequality

$$f_H^{\text{NL}} < f_H^{\text{ISO}}$$

provides the optical-topological mechanism behind the reduction in high-index material content used in the main-text dissipation comparison.

APPENDIX B: ALGEBRAIC CERTIFICATION OF THE BOUNDARY SEPARATION

We begin with a general boundary criterion that extends the argument of Appendix A beyond the explicit low-loss estimate. It shows that the desired separation follows once a suitable boundary maximum is certified to be strictly negative.

Proposition B1. Certified boundary criterion beyond the explicit low-loss bound. Fix the constituent permittivities ε_L , ε_H and a volume fraction $0 < f_H < 1$. Define

$$M(f_H) := \max_{\varepsilon \in \partial\mathcal{L}(f_H)} \text{Re}\sqrt{\varepsilon} - \text{Re}\sqrt{\varepsilon_k(f_H)}.$$

If $M(f_H) < 0$ for all f_H in the interval $I = (0, 1)$, then

$$\operatorname{Re}\sqrt{\varepsilon_{\text{eff}}^{\text{ISO}}} < \operatorname{Re}\sqrt{\varepsilon_k(f_H)}$$

for every admissible isotropic effective permittivity $\varepsilon_{\text{eff}}^{\text{ISO}} \in \mathcal{L}(f_H)$ and every $f_H \in I$.

Proof. The function

$$\Phi(z) = \operatorname{Re}\sqrt{z}$$

is harmonic in the right half-plane ($\operatorname{Re}[z] > 0$). Since the Bergman-Milton lens $\mathcal{L}(f_H)$ is compact and lies in this half-plane for passive dielectrics in the regime considered here, the maximum principle gives

$$\max_{\varepsilon \in \mathcal{L}(f_H)} \operatorname{Re}\sqrt{\varepsilon} = \max_{\varepsilon \in \partial\mathcal{L}(f_H)} \operatorname{Re}\sqrt{\varepsilon}.$$

Thus, if $M(f_H) < 0$, every point of the admissible lens satisfies

$$\operatorname{Re}\sqrt{\varepsilon} < \operatorname{Re}\sqrt{\varepsilon_k(f_H)}.$$

Remark B2. Computer-assisted certification. The criterion above is independent of the explicit sufficient condition $\eta_{\max} < 1/4$. Since the Bergman-Milton boundary is the union of two fractional-linear arcs, the boundary maximum $M(f_H)$ can be reduced to one-dimensional analytic problems on compact intervals. This makes the criterion suitable for certified computer-assisted verification. Such a verification may be carried out either via rigorous numerical enclosures based on interval analysis [61,62], or by exact real-algebraic quantifier elimination after rewriting the boundary-separation condition as the nonintersection of algebraic curves. The latter approach is used below for the $\text{SiO}_2/\text{TiO}_2$ optical constants considered in this work. ■

1. Computer-aided verification

We now apply the real-algebraic route to the specific $\text{SiO}_2/\text{TiO}_2$ optical constants used throughout the paper. The goal is to certify directly, without using the explicit low-loss estimate of Appendix A, that the Bergman-Milton boundary does not intersect the level curve

$$\operatorname{Re}\sqrt{z} = \operatorname{Re}\sqrt{\varepsilon_k(f_H)}$$

for any $0 < f_H < 1$. This provides a material-specific certification of the boundary separation

$$\operatorname{Re}\sqrt{\varepsilon_b} < \operatorname{Re}\sqrt{\varepsilon_k(f_H)}, \quad \varepsilon_b \in \partial\mathcal{L}(f_H).$$

Let $z = x + iy$ and $\varepsilon_k(f_H) = x_k + iy_k$. By the polar identity,

$$2(\operatorname{Re}\sqrt{z})^2 = |z| + x.$$

Therefore the level curve

$$\operatorname{Re}\sqrt{z} = \operatorname{Re}\sqrt{\varepsilon_k(f_H)}$$

is equivalently given by

$$|z| + x = |\varepsilon_k| + x_k.$$

Introducing $c := |\varepsilon_k| + x_k > 0$, one obtains the algebraic relation

$$c^2 - 2x_k c - y_k^2 = 0,$$

and the level curve becomes the parabola

$$y^2 - c^2 + 2cx = 0.$$

Each Bergman-Milton boundary branch has the rational form

$$\varepsilon_b(t) = \varepsilon_k(f_H) - \frac{f_L f_H \Delta^2}{D_B(t)},$$

where $t = u$ on the LH branch and $t = v$ on the HL branch. For each branch $B \in \{LH, HL\}$ the denominator $D_B(t)$ is affine in t , with

$$D_{LH}(t) = 3[t\varepsilon_L + (1-t)\varepsilon_H], \quad D_{HL}(t) = 3[t\varepsilon_H + (1-t)\varepsilon_L].$$

Writing

$$D_B(t) = D_r(t) + iD_i(t), \quad Q_B(t) := D_r(t)^2 + D_i(t)^2,$$

and

$$f_L f_H \Delta^2 = A_r + iA_i,$$

one obtains

$$\varepsilon_b(t) = x_b(t) + iy_b(t),$$

with

$$x_b(t) = \frac{N_x(t)}{Q_B(t)}, \quad y_b(t) = \frac{N_y(t)}{Q_B(t)},$$

where

$$N_x(t) = x_k Q_B(t) - [A_r D_r(t) + A_i D_i(t)],$$

and

Algorithm 1. Algebraic certification of boundary separation.

-
-
- ▷ Phase 1: Setup constituent permittivities
1. Set material constants $n_L, \kappa_L, n_H, \kappa_H$ as exact rational numbers.
 2. Compute $\varepsilon'_i = n_i^2 - \kappa_i^2$ and $\varepsilon''_i = 2n_i\kappa_i$ for $i \in \{L, H\}$.
 3. Define $f_L = 1 - f_H$, $\Delta = \varepsilon_H - \varepsilon_L$, $\varepsilon_k = f_L\varepsilon_L + f_H\varepsilon_H$.
 4. Set $x_k = \text{Re}\varepsilon_k$, $y_k = \text{Im}\varepsilon_k$, $A = f_L f_H \Delta^2 = A_r + iA_i$.
 - ▷ Phase 2: Test each Bergman-Milton boundary branch
 5. For each branch $B \in \{LH, HL\}$ do
 - ▷ Bergman-Milton arc: $\varepsilon_b(t) = \varepsilon_k - f_L f_H \Delta^2 / D_B(t)$.
 6. Define the affine denominator $D_B(t) = D_r(t) + iD_i(t)$, where:

$$D_{LH}(t) = 3[t\varepsilon_L + (1-t)\varepsilon_H], t \in I_{LH}(f_H) = [\frac{f_H}{3}, 1 - \frac{f_L}{3}],$$

$$D_{HL}(t) = 3[t\varepsilon_H + (1-t)\varepsilon_L], t \in I_{HL}(f_H) = [\frac{f_L}{3}, 1 - \frac{f_H}{3}].$$
 7. Set $Q_B(t) = D_r(t)^2 + D_i(t)^2$.
 8. Compute the $\varepsilon_b(t)$ numerator polynomials:

$$N_x(t) = x_k Q_B(t) - [A_r D_r(t) + A_i D_i(t)],$$

$$N_y(t) = y_k Q_B(t) - [A_i D_r(t) - A_r D_i(t)].$$
 9. Construct the separation polynomial:

$$P_B(t, c, f_H) = N_y(t)^2 - c^2 Q_B(t)^2 + 2c N_x(t) Q_B(t).$$
 10. Define the level-curve constraint $C(c, f_H) := c^2 - 2x_k c - y_k^2 = 0, c > 0$.
 11. Decide over $\mathbb{R}$ the existential formula:

$$\exists f_H, t, c: 0 < f_H < 1, t \in I_B(f_H), C(c, f_H) = 0, P_B(t, c, f_H) \geq 0.$$
 12. Record the output of the quantifier-elimination routine.
 13. Compute the resultant:

$$R_B(t, f_H) = \text{Res}_c(P_B, C).$$
 14. Decide whether $R_B(t, f_H) = 0$ admits a real solution with $0 < f_H < 1$ and $t \in I_B(f_H)$.
 15. End for
 - ▷ Phase 3: Conclude
 16. Conclude boundary separation if both decision problems return False for both branches $B \in \{LH, HL\}$.
-
-

$$N_y(t) = y_k Q_B(t) - [A_i D_r(t) - A_r D_i(t)].$$

After clearing the positive denominator $Q_B(t)^2$, the condition that a boundary point lies on or beyond the level curve, namely,

$$\text{Re}\sqrt{\varepsilon_b(t)} \geq \text{Re}\sqrt{\varepsilon_k(f_H)},$$

is $P_B(t, c, f_H) \geq 0$, where

$$P_B(t, c, f_H) = N_y(t)^2 - c^2 Q_B(t)^2 + 2c N_x(t) Q_B(t).$$

Indeed,

$$\frac{P_B(t, c, f_H)}{Q_B(t)^2} = y_b(t)^2 - c^2 + 2cx_b(t),$$

so $P_B(t, c, f_H) \geq 0$ is equivalent to

$$\text{Re}\sqrt{\varepsilon_b(t)} \geq \text{Re}\sqrt{\varepsilon_k(f_H)}.$$

The algebraic certification consists of deciding, for each boundary branch, the real existential formula

$$\exists f_H, t, c: 0 < f_H < 1, \quad t \in I_B(f_H), \quad c > 0,$$

$$C(c, f_H) = c^2 - 2x_k c - y_k^2 = 0, \quad P_B(t, c, f_H) \geq 0.$$

Here I_B can be one of the following closed intervals:

$$I_{LH}(f_H) = \left[\frac{f_H}{3}, 1 - \frac{f_L}{3}\right], \quad I_{HL}(f_H) = \left[\frac{f_L}{3}, 1 - \frac{f_H}{3}\right].$$

Using exact rational input values for complex refractive indices

$$n_L = \frac{145}{100}, \quad \kappa_L = \frac{1}{10^8}, \quad n_H = \frac{247}{100}, \quad \kappa_H = \frac{1}{1000},$$

with

$$\varepsilon'_i = n_i^2 - \kappa_i^2, \quad \varepsilon''_i = 2n_i\kappa_i,$$

this real-algebraic decision problem was evaluated using Wolfram Mathematica by real-algebraic quantifier elimination. For both the LH and HL branches, the existential formula returned `False`. Hence no point of either Bergman-Milton boundary branch satisfies

$$\text{Re}\sqrt{\varepsilon_b} \geq \text{Re}\sqrt{\varepsilon_k(f_H)}.$$

Therefore,

$$\text{Re}\sqrt{\varepsilon_b} < \text{Re}\sqrt{\varepsilon_k(f_H)}$$

for every $0 < f_H < 1$ and every $\varepsilon_b \in \partial\mathcal{L}(f_H)$.

As a complementary geometric certification, the auxiliary variable c was formally eliminated by computing the Sylvester resultant [63,64] of the two polynomials:

$$R_B(t, f_H) = \text{Res}_c(P_B(t, c, f_H), C(c, f_H) = c^2 - 2x_k c - y_k^2).$$

The corresponding real-root problem,

$$R_B(t, f_H) = 0, \quad 0 < f_H < 1, \quad t \in I_B(f_H),$$

was also decided by quantifier elimination and returned `False` for both branches. This class of problems is deterministically solvable via the cylindrical algebraic decomposition algorithm, originally introduced by Collins [65] and extensively developed for modern real algebraic geometry [63].

This confirms algebraically that the two Bergman-Milton arcs do not intersect the level parabola.

2. Pseudocode of the algebraic certification

The Algorithm 1 below certifies algebraically that no point of either Bergman-Milton boundary arc can attain a real refractive index equal to or greater than that of the Wiener upper bound, for any volume fraction $f_H \in (0, 1)$.

Two auxiliary objects drive the construction. The *level-curve constraint*

$$C(c, f_H) := c^2 - 2x_k c - y_k^2 = 0 \quad \text{with} \quad c > 0$$

is the polynomial encoding of the scalar $c := |\varepsilon_k| + x_k$, which by the polar identity $2(\text{Re}\sqrt{z})^2 = |z| + \text{Re}(z)$ parametrizes the level curve $\text{Re}\sqrt{z} = \text{Re}\sqrt{\varepsilon_k(f_H)}$ in the complex ε -plane. The square-root definition of c is nonpolynomial and therefore unsuitable for quantifier elimination; the constraint $C = 0$ replaces it with an equivalent algebraic relation.

The *separation polynomial*

$$P_B(t, c, f_H) := N_y(t)^2 - c^2 Q(t)^2 + 2c N_x(t) Q(t)$$

is obtained by clearing the denominator $Q(t)^2 > 0$ from the parabolic inequality $y_b(t)^2 - c^2 + 2cx_b(t) \geq 0$, which is itself the algebraic form of the condition $\text{Re}\sqrt{\varepsilon_b(t)} \geq \text{Re}\sqrt{\varepsilon_k(f_H)}$. Hence $P_B \geq 0$ signifies that the boundary point $\varepsilon_b(t)$ violates the separation, i.e., the isotropic admissible set reaches the Wiener upper bound. The certification succeeds if the existential formula $\exists f_H, t, c: C = 0 \wedge P_B \geq 0$ is decided `False` for both branches by quantifier elimination, establishing that the separation is strict everywhere.

-
- [1] <https://www.virgo-gw.eu/>.
 - [2] F. Acernese *et al.* (Virgo Collaboration), Advanced Virgo: A second-generation interferometric gravitational wave detector, *Classical Quantum Gravity* **32**, 024001 (2015).
 - [3] <https://www.ligo.caltech.edu/>.
 - [4] J. Aasi *et al.* (LIGO Scientific Collaboration), Advanced LIGO, *Classical Quantum Gravity* **32**, 074001 (2015).
 - [5] <https://gwcenter.icrr.u-tokyo.ac.jp/en/>.
 - [6] T. Akutsu *et al.* (KAGRA Collaboration), Overview of KAGRA: Detector design and construction history, *Prog. Theor. Exp. Phys.* **2021**, 05A101 (2021).
 - [7] G. M. Harry *et al.*, Thermal noise in interferometric gravitational wave detectors due to dielectric optical coatings, *Classical Quantum Gravity* **19**, 897 (2002).
 - [8] R. Flaminio, J. Franc, C. Michel, N. Morgado, L. Pinard, and B. Sassolas, A study of coating mechanical and optical losses in view of reducing mirror thermal noise in gravitational wave detectors, *Classical Quantum Gravity* **27**, 084030 (2010).
 - [9] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Observation of gravitational waves from a binary black hole merger, *Phys. Rev. Lett.* **116**, 061102 (2016).
 - [10] G. M. Harry *et al.*, Titania-doped tantala/silica coatings for gravitational-wave detection, *Classical Quantum Gravity* **24**, 405 (2007).
 - [11] G. I. McGhee *et al.*, Reduced thermal noise in gravitational wave detectors from titania-doped silica coatings, *Phys. Rev. Lett.* **131**, 171401 (2023).
 - [12] M. Fazio, I. W. Martin, P. Hill, M. Ben Yaala, M. Chicoine, C. Clark, N. Demos, M. M. Fejer, D. Gibson, S. Gras, J. Hough, A. Markosyan, G. McGhee, S. Rowan, J. Smith, F. Schiettekatte, S. Tait, G. Vajente, and S. Reid, Low mechanical loss titania-doped silica coatings for gravitational wave detectors, [arXiv:2502.07679](https://arxiv.org/abs/2502.07679).
 - [13] G. Vajente, L. Yang, A. Davenport, M. Fazio, A. Ananyeva, L. Zhang, G. Billingsley, K. Prasai, A. Markosyan, R. Bassiri, M. M. Fejer, M. Chicoine, F. Schiettekatte, and C. S. Menoni, Low mechanical loss TiO₂-doped GeO₂ coatings for reduced thermal noise in gravitational wave interferometers, *Phys. Rev. Lett.* **127**, 071101 (2021).
 - [14] V. Pierro, V. Fiumara, F. Chiadini, V. Granata, O. Durante, J. Neilson, C. Di Giorgio, R. Fittipaldi, G. Carapella, F. Bobba, M. Principe, and I. M. Pinto, Ternary quarter wavelength coatings for gravitational wave detector mirrors: Design optimization via exhaustive search, *Phys. Rev. Res.* **3**, 023172 (2021).
 - [15] V. Pierro *et al.*, Thermal noise reduction in ternary optical coatings: From Ti:GeO₂-based ternary systems to high index materials, *Phys. Rev. D* **112**, 102006 (2025).

- [16] S. Chao, R. De Salvo, I. M. Pinto, P. Addesso, V. Pierro, H.-W. Pan, L.-C. Kuo, S.-J. Song, and J. Wang, Nanometer composites for low noise optical coatings: Status and perspectives, in *LIGO-Virgo Collaboration Meeting, Hannover, Germany [LIGO Document No. G1301061]* (2013) [Online]. Available: <https://dcc.ligo.org/LIGO-G1301061> and <https://tds.virgo-gw.eu/?r=26397>
- [17] S. Chao, H. Pan, L. Kuo, V. Pierro, I. M. Pinto, M. Principe, and R. DeSalvo, NM-layered glassy oxide composites for 3rd generation interferometric gravitational wave detectors, in *Optical Interference Coatings 2016, OSA Technical Digest (online)* (Optica Publishing Group, Washington, DC, 2016), paper MB.10.
- [18] S. Chao, L.-C. Kuo, H.-W. Pan, C. Cheng, I. M. Pinto, V. Pierro, M. Principe, and R. DeSalvo, NM-layered composite coating materials: Status and TBD, in *LVC Meeting, Pasadena, CA [LIGO Document No. G1600665]* (2016) [Online]. Available: <https://dcc.ligo.org/LIGO-G1600665>.
- [19] I. M. Pinto *et al.*, Nanolayered optical coatings for gravitational wave detectors, LIGO Document No. G1902307, Optics Working Group, 2019 [Online]. Available: <https://dcc.ligo.org/LIGO-G1902307>.
- [20] J. Neilson *et al.*, Optimizing nanostructure deposition process for optical applications, *Micro Nano Lett.* **19**, e12186 (2024).
- [21] V. Granata, V. Pierro, I. M. Pinto, L. Mereni, M. Granata, L. Silenzi, F. Travasso, A. Trapananti, M. Magnozzi, and M. Canepa, First mechanical loss measurements on nanolayers using GeNS, LIGO Document No. G2401236, Virgo Week, Pisa, 2024 [Online]. Available: <https://dcc.ligo.org/LIGO-G2401236> and <https://tds.virgo-gw.eu/?r=23524>
- [22] O. Durante *et al.*, Toward the optimization of SiO₂ and TiO₂-based metamaterials: Morphological, structural, and optical characterization, *Opt. Mater.* **157**, 116038 (2024).
- [23] O. Durante *et al.*, Role of substrate and TiO₂ content in TiO₂:Ta₂O₅ coatings for gravitational wave detectors, *Classical Quantum Gravity* **41**, 025005 (2024).
- [24] O. Durante, V. Granata, J. Neilson, G. Carapella, F. Chiadini, R. DeSalvo, R. De Simone, V. Fiumara, V. Pierro, I. M. Pinto, A. Vecchione, R. Fittipaldi, F. Bobba, and C. Di Giorgio, Investigation of crystallization in nanolayered TiO₂-based superlattices, *Surf. Interfaces* **41**, 103309 (2023).
- [25] I. M. Pinto *et al.*, Nanolayered materials, M-ary optimized coatings, & direct TN measurements—A progress report, in LVK Meeting, LSU [LIGO Document G2400526] (2024) [Online]. Available: <https://dcc.ligo.org/LIGO-G2400526> and <https://tds.virgo-gw.eu/?r=26395>.
- [26] O. Durante *et al.*, Role of oxygen vacancies in the structural phase transformations of granular TiO₂ thin films, *Surf. Interfaces* **37**, 102698 (2023).
- [27] O. Durante *et al.*, Emergence and evolution of crystallization in TiO₂ thin films: A structural and morphological study, *Nanomater. Nanotechnol.* **11**, 1409 (2021).
- [28] I. M. Pinto, Nanolayered silica/alumina composites, in *Virtual LVK Meeting [LIGO Document No. G2001499]* (2020) [Online]. Available: <https://dcc.ligo.org/LIGO-G2001499> and <https://tds.virgo-gw.eu/?r=19962>.
- [29] Einstein Telescope official website, <https://www.et-gw.eu/>
- [30] Cosmic Explorer official website, <https://cosmicexplorer.org/>
- [31] I. M. Pinto, M. Principe, R. DeSalvo, S. Chao, H. Pan, and L. Kuo, NM-layered amorphous glassy oxide composites for 3rd generation interferometric gravitational wave detectors, in *6th ET Symposium, Lyon [LIGO Document No. G1401358]* (2014) [Online]. Available: <https://dcc-lho.ligo.org/LIGO-G1401358>.
- [32] I. M. Pinto, V. Pierro, R. DeSalvo, S. Chao, J. Wang, and H. Pan, Interdiffused coatings, in *LVC Meeting, Rome [LIGO Document No. G1200976]* (2012) [Online]. Available: <https://dcc.ligo.org/LIGO-G1200976> and <https://tds.virgo-gw.eu/?r=26396>.
- [33] Y. Levin, Internal thermal noise in the LIGO test masses: A direct approach, *Phys. Rev. D* **57**, 659 (1998).
- [34] *Optical Coatings and Thermal Noise in Precision Measurement*, edited by G. Harry, T. P. Bodiya, and R. DeSalvo (Cambridge University Press, Cambridge, England, 2012).
- [35] M. Magnozzi *et al.*, Optical properties of amorphous SiO₂ – TiO₂ multi-nanolayered coatings for 1064-nm mirror technology, *Opt. Mater.* **75**, 94 (2018).
- [36] O. Wiener, Die Theorie des Mischkörpers für das Feld der stationären Strömung. I. Die Mittelwertsätze für Kraft, Polarisation und Energie, *Abhd. Math.-Phys. Klasse Königl. Sächsischen Ges. Wiss.* **32**, 509 (1912).
- [37] M. Born and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*, 7th ed. (Cambridge University Press, Cambridge, England, 1999).
- [38] G. E. Backus, Long-wave elastic anisotropy produced by horizontal layering, *J. Geophys. Res.* **67**, 4427 (1962).
- [39] D. A. G. Bruggeman, Berechnung verschiedener physikalischer Konstanten von heterogenen Substanzen. I. Die Dielektrizitätskonstanten und Leitfähigkeiten der Mischkörper aus isotropen Substanzen, *Ann. Phys. (Berlin)* **416**, 636 (1935).
- [40] D. Polder and J. H. van Santen, The effective permeability of mixtures of solids, *Physica (Amsterdam)* **12**, 257 (1946).
- [41] R. Hill, A self-consistent mechanics of composite materials, *J. Mech. Phys. Solids* **13**, 213 (1965).
- [42] B. Budiansky, On the elastic moduli of some heterogeneous materials, *J. Mech. Phys. Solids* **13**, 223 (1965).
- [43] E. Barta, Calculation of the effective elastic moduli of a two-phase isotropic composite, *J. Compos. Mater.* **18**, 488 (1984).
- [44] D. J. Bergman, Exactly solvable microscopic geometries and rigorous bounds for the complex dielectric constant of a two-component composite material, *Phys. Rev. Lett.* **44**, 1285 (1980).
- [45] D. J. Bergman, Rigorous bounds for the complex dielectric constant of a two-component composite, *Ann. Phys. (N.Y.)* **138**, 78 (1982).
- [46] G. W. Milton, Bounds on the complex dielectric constant of a composite material, *Appl. Phys. Lett.* **37**, 300 (1980).
- [47] G. W. Milton, Bounds on the complex permittivity of a two-component composite material, *J. Appl. Phys.* **52**, 5286 (1981).
- [48] L. V. Gibiansky and G. W. Milton, On the effective viscoelastic moduli of two-phase media. I. Rigorous bounds on the complex bulk modulus, *Proc. R. Soc. A* **440**, 163 (1993).
- [49] G. W. Milton and J. G. Berryman, On the effective viscoelastic moduli of two-phase media. II. Rigorous bounds on

- the complex shear modulus in three dimensions, *Proc. R. Soc. A* **453**, 1849 (1997).
- [50] L. V. Gibiansky, G. W. Milton, and J. G. Berryman, On the effective viscoelastic moduli of two-phase media. III. Rigorous bounds on the complex shear modulus in two dimensions, *Proc. R. Soc. A* **455**, 2117 (1999).
- [51] V. B. Braginsky, M. L. Gorodetsky, and S. P. Vyatchanin, Thermodynamical fluctuations and photo-thermal intrinsic noise in optics of beam splitters, *Phys. Lett. A* **264**, 1 (1999).
- [52] V. B. Braginsky and S. P. Vyatchanin, Thermodynamical fluctuations in optical mirror coatings, *Phys. Lett. A* **271**, 303 (2000).
- [53] M. M. Fejer, Effective medium description of multilayer coatings, LIGO Document No. LIGO-T2100186-v1, version 14, 2021, <https://dcc.ligo.org/LIGO-T2100186/public>.
- [54] A. Sihvola, *Effective Medium Theory: Principles and Applications*, 2nd ed. (Oxford University Press, New York, 2016).
- [55] W. Voigt, Ueber die Beziehung zwischen den beiden Elasticitätsconstanten isotroper Körper, *Ann. Phys. (Berlin)* **274**, 573 (1889).
- [56] G. Vajente, M. Fazio, L. Yang, A. Gupta, A. Ananyeva, G. Billingsley, and C. S. Menoni, Method for the experimental measurement of bulk and shear loss angles in amorphous thin films, *Phys. Rev. D* **101**, 042004 (2020),
- [57] M. R. Abernathy, G. M. Harry, J. Newport, H. Fair, M. Kinley-Hanlon, S. Hickey, I. Jiffar, A. Gretarsson, S. Penn, R. Bassiri, E. Gustafson, I. W. Martin, S. Rowan, and J. Hough, Bulk and shear mechanical loss of titania-doped tantala, *Phys. Lett. A* **382**, 2282 (2018),
- [58] G. W. Milton, *The Theory of Composites* (Cambridge University Press, Cambridge, England, 2002), Chaps. 13 and 27.
- [59] Z. Hashin and S. Shtrikman, A variational approach to the theory of the elastic behaviour of multiphase materials, *J. Mech. Phys. Solids* **11**, 127 (1963).
- [60] L. J. Walpole, On bounds for the overall elastic moduli of inhomogeneous systems—I, *J. Mech. Phys. Solids* **14**, 151 (1966).
- [61] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, Julia: A fresh approach to numerical computing, *SIAM Rev.* **59**, 65 (2017).
- [62] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis* (Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2009).
- [63] S. Basu, R. Pollack, and M.-F. Roy, *Algorithms in Real Algebraic Geometry*, 2nd ed., Algorithms and Computation in Mathematics Vol. 10 (Springer-Verlag, Berlin Heidelberg, 2006).
- [64] D. A. Cox, J. Little, and D. O’Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 4th ed., Undergraduate Texts in Mathematics (Springer International Publishing, New York, 2015).
- [65] G. E. Collins, Quantifier elimination for real closed fields by cylindrical algebraic decomposition, in *Automata Theory and Formal Languages (2nd GI Conference, Kaiserslautern, 1975)*, Lecture Notes in Computer Science Vol. 33 (Springer, Berlin, Heidelberg, 1975), pp. 134–183.