

Revisiting Cherenkov radiation in anisotropic chiral matter: Exact calculation reveals threshold-free emission

R. Martínez von Dossow^{*} and L. F. Urrutia[†]

*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,
04510 Ciudad de México, México*



(Received 5 February 2026; accepted 9 June 2026; published 17 July 2026)

We explore Cherenkov radiation in anisotropic chiral matter within the framework of Carroll-Field-Jackiw electrodynamics, where the axion angle exhibits a linear dependence on position. By deriving closed-form expressions for the polarization modes of electromagnetic fields in cylindrical coordinates and the space-frequency domain, we solve the modified Maxwell's equations. To enforce causality, we impose outgoing wave boundary conditions at a cylindrical surface at infinity, which yields the dispersion relations. Our analysis uncovers the specific angles and frequency ranges that allow for zero, one, or two Cherenkov cones. We also obtain the spectral energy distribution of the radiation in all cases. Notably, one sector of the model exhibits a novel phenomenon: Cherenkov radiation can be generated by slowly moving charges without a threshold, but only within a specific frequency range. This behavior is not observed in standard materials. Using our exact calculations, we also investigate the reliability of an approximate method previously proposed based on the calculation of the Green's function for the system.

DOI: [10.1103/v6sr-w2tr](https://doi.org/10.1103/v6sr-w2tr)

I. INTRODUCTION

Cherenkov radiation (CHR) has played a fundamental role in physics since its experimental discovery [1,2] and subsequent theoretical confirmation [3], with applications spanning Cherenkov detectors [4–6], light sources [7–9], and more recently medical imaging [10–12], and photodynamic therapy [13–15]. Research in this area remains vibrant, fitting within the broader theme of radiation engineered via structured environments. Key settings for current exploration include two-dimensional materials, metamaterials, photonic crystals, and external fields. For a detailed review, see, for example Refs. [16,17].

The Cherenkov threshold, requiring the charge velocity v to exceed the light velocity in the medium, usually requires high-energy particles, often only available in high-energy accelerators within the GeV energy range. However, in medical applications the energy range is much lower, typically on the order of MeV. To overcome this limitation, a significant research effort has focused on generating Cherenkov radiation with low-speed charges, effectively creating threshold-free Cherenkov radiation.

Another topic of recent interest has been the study of reversed Cherenkov radiation (RCHR) [18–20], a concept theoretically proposed in Ref. [21] using materials with negative refractive index, also known as left-handed materials. The experimental realization of such metamaterials [22] has enabled several observations of RCHR [23–26]. Challenging earlier assumptions that negative refractive index materials were required, recent studies have shown that RCHR can also occur in natural materials with positive refractive indices. A key step was taken in Ref. [27], which demonstrated the existence of RCHR when a charge is incident perpendicularly to the interface between vacuum and a topological insulator (a magnetoelectric medium described by axion electrodynamics). Additionally, Ref. [28] showed that RCHR can emerge in a positive-index isotropic slab with optical gain, and Ref. [29] investigated RCHR via Fizeau-Fresnel drag.

On the other hand, the discovery of new electromagnetic media like chiral matter (i.e., topological insulators and Weyl semimetals, for example), whose description of the electromagnetic response requires additional terms in Maxwell's equations has prompted investigations of CHR within these materials [27,30–32]. This modified electrodynamics follows under the name of axion electrodynamics (a subset of which is also known as Carroll-Field-Jackiw electrodynamics) and amounts to adding the term

$$S_{\text{axion}} = \frac{e^2}{32\pi^2} \int d^4x \theta(x) \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \quad (1)$$

^{*}Contact author: r.martinezvd@gmail.com

[†]Contact author: urrutia@nucleares.unam.mx

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

to the standard Maxwell's action. Here $\theta(x)$ is the non-dynamical axion angle, a parameter characterizing the media in the same footing as the permittivity ϵ and the permeability μ . For example, the choice of $\theta(x)$ as a piecewise constant function, with quantized values, describes the response of topological insulators, while the choice $\theta(x) = b_\mu x^\mu = b_0 t - \mathbf{b} \cdot \mathbf{x}$ characterizes Weyl semimetals (WSM's). The corresponding parameters are dictated by the microscopic structure of the materials and are valid within a restricted sector of the Brillouin zone.

The axion contribution appearing in Eq. (1), also known as the Carroll-Field-Jackiw (CFJ) term, belongs to the photon sector of the Standard Model extension (SME) [33,34], an effective field-theoretic framework in which all Lorentz and *CPT*-violating operators compatible with gauge invariance and the remaining symmetries of the Standard Model are systematically incorporated into the Lagrangian. These contributions may also be induced radiatively from Lorentz-violating operators in the fermion sector of the SME [35,36]. Axion electrodynamics has also been studied in different physical settings. In high-energy physics, it has been used to describe Lorentz-symmetry-breaking extensions of electrodynamics and related quantum field theory models [37–45]. In condensed matter physics, it has been applied to describe the electromagnetic response of magnetoelectric and topological materials [27,30,31,46–50]. Within this framework, electromagnetic processes forbidden in conventional Maxwell electrodynamics can become allowed. A notable example is vacuum Cherenkov radiation [39,40], which has been investigated using quantum-field-theoretic methods in the fermion sector of the SME, both for minimal [43] and nonminimal [51] operators, as well as through effective descriptions based directly on the CFJ term. In particular, Refs. [32,52] include a detailed investigation of CHR in WSM's, dealing with the isotropic case ($b_0 \neq 0, \mathbf{b} = 0$). In these references the authors have clarified many subtle points previously discussed in the literature, many of them reaching to contradictory conclusions. For instance, the existence of a “vacuum” CHR was firmly established, along with the gauge invariance of the total radiated energy. The expressions for the spectral energy distribution for each polarization were correctly derived, showing positivity, and the work confirmed that the nonrelativistic approximation yielded no radiation, among other achievements. Another key point in these references is the clear benefit of using cylindrical coordinates, suggested by the problem's symmetry and employed in the original theoretical explanation of the radiation [3]. Furthermore, the method used can be directly applied to the complementary case ($b_0 = 0, \mathbf{b} \neq 0$) when $\mathbf{b}$ is parallel to the constant velocity $\mathbf{v}$ of the charge producing the CHR, as elaborated in this manuscript. This differs from the more common use of spherical coordinates, as implemented in Ref. [30] for the latter case.

The present work significantly advances the state of CHR research in particular, building upon and surpassing

the related previous work in Ref. [30]. Notwithstanding, the scope in this reference was broader because it aimed to calculate the full Green's function for an arbitrary axially symmetric source in materials described by anisotropic chiral electrodynamics, with the subsequent application to solve CHR only as a particularly interesting case. A key benefit of having this Green's function is that it enables the exact or numerical calculation of electromagnetic fields for arbitrary axially symmetric sources in these materials. However, the exact calculation of the Green's function (GF) proved more complicated than anticipated. Even the stationary phase approximation could not be determined exactly, necessitating approximations to proceed with the analytic calculation. In summary, the Cherenkov data reported in Ref. [30] is approximate. Nevertheless, we can obtain exact analytical results using the approach successfully employed in Refs. [32,52], as we demonstrate in this work. In addition to achieving this important step, we compare these exact results with the approximation method reported in Ref. [30]. By doing so, we can assess the validity range of the approximation in Ref. [30] for this specific case, providing insight into the reliability of the approximate Green's function in more complex scenarios.

The paper is organized as follows. In Sec. II we solve Maxwell's equations for the system with $b_0 = 0, \mathbf{b} \parallel \mathbf{v}$, finding exact analytical expressions for electromagnetic fields in the whole space. The details are included in Appendixes A and B. In Sec. III the dispersion relation for CHR is obtained by looking at the far field approximation and demanding outgoing cylindrical waves at infinity such that causality is preserved. We find the explicit conditions for radiation in each polarization mode $\nu = \pm$. Section IV deals with the calculation of the Cherenkov angles Θ_ν for each polarization in terms of the corresponding $\cos \Theta_\nu$. Demanding $|\cos \Theta_\nu| \leq 1$ we recover the radiation conditions obtained in the previous section. An alternative way to find the CHR conditions is demanding the velocity of the charge to be larger than the phase velocity v_ν^{ph} of the light in the medium. This condition is considered in Sec. V finding agreement with previous results in the manuscript. This analysis rests heavily on the property that $v_\nu^{ph}(\omega)$ is a monotonic function of ω , which is proved in Appendix C.

In Sec. VI we calculate the Poynting vector in the radiation approximation and obtain explicit expressions for the spectral distribution of the total radiated energy in each polarization mode. The positivity of the spectral energy distribution is made manifest in the Appendix D and the cancellation of the crossed terms in the polarization arising in the calculation of the total energy flux is shown analytically in the Appendix E. In Sec. VII we illustrate some relevant consequences of our findings including different plots with an explicit choice of parameters. Highlighted features of the anisotropic case are compared with the isotropic scenario and are presented in Sec. VIII. Also, preliminary consequences due to $\mathbf{v}$ not being parallel

to $\mathbf{b}$ are examined by adding a small component of $\mathbf{b}$ in the x direction. The comparison of the exact results obtained in this manuscript with those obtained using the approximation method developed in Ref. [30] is reported in Sec. IX. The paper closes with a summary and conclusions in Sec. X.

II. MAXWELL'S EQUATIONS

Adding the contribution (1), with $b_0 = 0$, to the standard Maxwell's action we obtain the modified equations

$$\nabla \cdot \epsilon \mathcal{E} = 4\pi\bar{\rho} - \mathbf{b} \cdot \mathcal{B}, \quad \nabla \cdot \mathcal{B} = 0, \quad (2)$$

$$\nabla \times \mathcal{B} = \frac{4\pi}{c} \mathbf{J} + \frac{\epsilon}{c} \partial_t \mathcal{E} + \mathbf{b} \times \mathcal{E}, \quad \nabla \times \mathcal{E} = -\frac{1}{c} \partial_t \mathcal{B}, \quad (3)$$

which govern the electromagnetic properties of the medium. We work in Gaussian units and $\mathcal{E}$ is the electric field, $\mathcal{B}$ is the magnetic field, $\mathbf{b}$ is a parameter characterizing the chiral medium, ϵ is the permittivity denoted frequently by the index of refraction $n = \sqrt{\epsilon}$ and c is the speed of light in vacuum. We consider a charge q moving with constant velocity $\mathbf{v} = v\hat{z}$, which is parallel to the parameter $\mathbf{b}$ characterizing the medium, i.e., we have $\mathbf{b} = b\hat{z}$. In cylindrical coordinates ρ, ϕ, z the charge density $\bar{\rho}$ is

$$\bar{\rho}(\rho, z, t) = \frac{q}{2\pi\rho} \delta(\rho) \delta(z - vt), \quad (4)$$

with the corresponding current $\mathbf{J} = \mathbf{v}\bar{\rho}$. We find it convenient to work in the frequency space with the convention $f(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} f(t)$. This yields the time-Fourier transform $\bar{\rho}(\mathbf{x}, \omega) = \frac{q}{c} e^{i\frac{\omega}{v}z} \delta(x) \delta(y)$. The appearance of the factor $e^{i\frac{\omega}{v}z}$ in all the sources of Maxwell equations in the space-frequency domain motivates one to implement the separation of variables

$$\mathcal{V}_i(\mathbf{x}, \omega) = e^{ikz} V_i(x, y, \omega) = e^{ikz} V_i(\rho, \omega), \quad k = \frac{\omega}{v} \quad (5)$$

for all vector components, where the cylindrical symmetry around the z axis implies that the fields are independent of the azimuthal angle ϕ . Then, the derivatives in our space are

$$\partial_z = ik, \quad \partial_\phi = 0, \quad \partial_t = -i\omega, \quad (6)$$

and we have the convenient factorization

$$\mathcal{E}(\rho, z, \omega) = \mathbf{E}(\rho, \omega) e^{ikz}, \quad \mathcal{B}(\rho, z, \omega) = \mathbf{B}(\rho, \omega) e^{ikz}. \quad (7)$$

The coupled system of Maxwell's equations for the electric field outside the sources, i.e., in the region where $\bar{\rho} = 0$ and $\mathbf{J} = 0$, are

$$\rho^2 \partial_\rho^2 E_\phi + \rho \partial_\rho E_\phi - (1 + \alpha^2 \rho^2) E_\phi = -\frac{i b \omega}{c} \rho^2 E_\rho, \quad (8)$$

$$\rho^2 \partial_\rho^2 E_\rho + \rho \partial_\rho E_\rho - (1 + \alpha^2 \rho^2) E_\rho = \frac{i c b k^2}{n^2 \omega} \rho^2 E_\phi, \quad (9)$$

together with

$$E_z = \frac{i}{k\rho} \partial_\rho(\rho E_\rho) + \frac{c b}{n^2 \omega k \rho} \partial_\rho(\rho E_\phi). \quad (10)$$

Here

$$\alpha^2 = \frac{b^2}{n^2} + k^2 - \frac{n^2 \omega^2}{c^2} = \alpha^2 + \frac{b^2}{n^2}. \quad (11)$$

Observe that in the left-hand side of Eq. (8) we have the operator corresponding to a modified Bessel equation of order one acting upon each component of the electric field. This motivates us to propose the *Ansatz*

$$E_\phi = X K_1(Q\rho), \quad E_\rho = Y K_1(Q\rho), \quad (12)$$

where $K_1(Q\rho)$ is the corresponding modified Bessel function which decays exponentially at $\rho \rightarrow \infty$ and diverges at $\rho = 0$, as required by the presence of sources there. Substituting in Eqs. (8) and (9) we obtain the condition

$$\begin{bmatrix} Q^2 - \alpha^2 & \frac{i b \omega}{c} \\ -\frac{i c b k^2}{n^2 \omega} & Q^2 - \alpha^2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (13)$$

Demanding the determinant of the matrix in (13) is zero we obtain the dispersion relation

$$(Q^2 - \alpha^2)(Q^2 - \alpha^2) - \frac{b^2 k^2}{n^2} = 0, \quad (14)$$

with two solutions

$$Q_\nu^2 = \frac{\omega^2}{v^2} - \frac{\omega^2 n^2}{c^2} + \frac{b^2}{2n^2} \left(1 + \nu \sqrt{1 + 4 \frac{\omega^2 n^2}{b^2 v^2}} \right), \quad \nu = \pm. \quad (15)$$

In this way the fields are written as

$$E_\phi = \sum_\nu X_\nu K_1(Q_\nu \rho), \quad E_\rho = \sum_\nu Y_\nu K_1(Q_\nu \rho). \quad (16)$$

The zero determinant condition in Eq. (13) also provides the following relation between the coefficients of the expansion X_ν and Y_ν :

$$X_\nu = i \Omega_\nu Y_\nu, \quad \Omega_\nu = -\frac{b \omega}{c(Q_\nu^2 - \alpha^2)}. \quad (17)$$

The component E_z is determined by Eq. (10). At this stage the electric field is determined in terms of the coefficients $Y_\pm$. On the other hand, Faraday's law $\mathbf{B} = \frac{ic}{\omega} \nabla \times \mathbf{E}$ yields the magnetic field in analogous form, with the full result

$$E_\rho = \sum_\nu Y_\nu K_1(Q_\nu \rho), \quad E_\phi = i \sum_\nu \Omega_\nu Y_\nu K_1(Q_\nu \rho), \quad E_z = -\frac{i}{k} \sum_\nu \left(1 + \frac{cb}{n^2 \omega} \Omega_\nu\right) Q_\nu Y_\nu K_0(Q_\nu \rho), \quad (18)$$

$$B_\rho = -i \frac{c}{v} \sum_\nu \Omega_\nu Y_\nu K_1(Q_\nu \rho), \quad B_\phi = \frac{c}{k\omega} \sum_\nu \left(k^2 - Q_\nu^2 - \frac{cb}{n^2 \omega} \Omega_\nu Q_\nu^2\right) Y_\nu K_1(Q_\nu \rho),$$

$$B_z = -\frac{c}{\omega} \sum_\nu \Omega_\nu Y_\nu Q_\nu K_0(Q_\nu \rho). \quad (19)$$

The final expressions for the electromagnetic fields require finding the coefficients $Y_\pm$, which are obtained through the boundary conditions at $\rho \rightarrow 0$ via the standard application of the Gauss pillbox and the appropriate Amperian circuit. The results are

$$Y_+ = \frac{q}{vn^2} \left(1 + \frac{1}{\sqrt{1 + 4 \frac{n^2 \omega^2}{b^2 v^2}}}\right) Q_+, \quad Y_- = \frac{q}{vn^2} \left(1 - \frac{1}{\sqrt{1 + 4 \frac{n^2 \omega^2}{b^2 v^2}}}\right) Q_-, \quad (20)$$

with the details given in Appendix B. In this way, the electromagnetic fields (18) and (19) are fully determined. Just by looking at the far field approximation on the electromagnetic fields (18) and (19) in the causal limit we determine the final expression for the dispersion relation, the emission angle, and the phase velocity of the outgoing wave describing Cherenkov radiation.

III. THE CONDITION ON Q_ν FOR CHERENKOV RADIATION

The electromagnetic fields are proportional to the modified Bessel functions $K_1(Q_\nu \rho)$ or $K_0(Q_\nu \rho)$. Radiation arises in the limit $\rho \rightarrow \infty$, where these functions behave like

$$\lim_{\rho \rightarrow \infty} K_{(1,0)}(Q_\nu \rho) = \sqrt{\frac{\pi}{2Q_\nu \rho}} e^{-Q_\nu \rho}. \quad (21)$$

Causal radiation demands outgoing waves at $\rho \rightarrow \infty$ which requires the choice

$$Q_\nu = -i Q_\nu \quad (22)$$

with $Q_\nu > 0$. From the dispersion relation (15) we obtain

$$Q_\nu = \pm \sqrt{\frac{\omega^2}{v^2} - \frac{\omega^2 n^2}{c^2} + \frac{b^2}{2n^2} \left(1 + \nu \sqrt{1 + 4 \frac{\omega^2 n^2}{b^2 v^2}}\right)}. \quad (23)$$

Selecting the minus sign and rewriting the term inside the square bracket as

$$Q_\nu = -\sqrt{-\left(-\frac{\omega^2}{v^2} + \frac{\omega^2 n^2}{c^2} - \frac{b^2}{2n^2} \left(1 + \nu \sqrt{1 + 4 \frac{\omega^2 n^2}{b^2 v^2}}\right)\right)}, \quad (24)$$

we identify $Q_\nu = \sqrt{q_\nu}$, which implies

$$Q_\nu^2 = q_\nu = -\frac{\omega^2}{v^2} + \frac{\omega^2 n^2}{c^2} - \frac{b^2}{2n^2} \left(1 + \nu \sqrt{1 + 4 \frac{\omega^2 n^2}{b^2 v^2}}\right) > 0, \quad (25)$$

which gives the condition for Cherenkov radiation in the medium. This condition can be presented as

$$2n^2 \omega^2 \eta > b^2 \left(1 + \nu \sqrt{1 + 4 \frac{\omega^2 n^2}{b^2 v^2}}\right), \quad \eta \equiv \left(\frac{n^2}{c^2} - \frac{1}{v^2}\right). \quad (26)$$

Observe that

$$b^2 < b^2 \sqrt{1 + 4 \frac{\omega^2 n^2}{b^2 v^2}}. \quad (27)$$

Then, when $\nu = +$ the right-hand side of Eq. (26) is always positive while it is always negative when $\nu = -$. Next we discuss the different possibilities that arise according to the sign of η .

A. The case $\eta > 0$

Here we have $v > c/n$ and the inequality (26) can be satisfied for both $\nu = +$ and $\nu = -$. However, for $\nu = -$, Eq. (26) is automatically satisfied for any value of ω , and radiation is therefore emitted at all frequencies. In the complementary case $\nu = +$ there is a restriction on ω which we now elucidate. Writing (26) as

$$2n^2 \omega^2 \eta - b^2 > \nu b^2 \sqrt{1 + 4 \frac{\omega^2 n^2}{b^2 v^2}}, \quad (28)$$

and squaring yields the condition $\omega^2 > \frac{b^2 \beta^2}{\eta^2 v^2}$, which is finally presented as

$$\omega > cb \frac{\beta^2}{(n^2 \beta^2 - 1)} \equiv \omega_{C+}, \quad (29)$$

since all involved terms are positive. Summarizing, when $v > c/n$ ($\eta > 0$), the polarization $\nu = -$ radiates for arbitrary frequency, while radiation with polarization $\nu = +$ is present only for frequencies larger than ω_{C+} in Eq. (29).

B. The case $\eta < 0$

This corresponds to $v < c/n$ and the polarization $\nu = +$ is forbidden for all frequencies. The polarization $\nu = -$ present a cutoff frequency which we determine in a similar way to the previous case. The result is

$$\omega < cb \frac{\beta^2}{1 - n^2 \beta^2} \equiv \omega_{C-}. \quad (30)$$

Summarizing, when $v < c/n$ radiation with polarization $\nu = +$ is always absent while polarization $\nu = -$ occurs provided $\omega < \omega_{C-}$. This opens a frequency window where threshold-free CHR is allowed.

IV. THE CHERENKOV ANGLES Θ_ν

They are determined from the long-distance behavior of the fields where they behave like

$$e^{iQ_\nu \rho + i\frac{\omega}{v} z} = e^{i\mathbf{k}_\nu \cdot \mathbf{x}}, \quad (31)$$

defining the wave vectors

$$\mathbf{k}_\nu = Q_\nu \hat{\rho} + \frac{\omega}{v} \hat{z}. \quad (32)$$

For each polarization, Eq. (31) describes a plane wave front propagating in the direction $\mathbf{k}_\nu$, making an angle Θ_ν with the direction of motion of the charge such that

$$\cos \Theta_\nu = \frac{\mathbf{k}_\nu \cdot \hat{z}}{|\mathbf{k}_\nu| |\hat{z}|}, \quad (33)$$

with $\mathbf{k}_\nu \cdot \hat{z} = \frac{\omega}{v}$ and

$$|\mathbf{k}_\nu| = \sqrt{Q_\nu^2 + \frac{\omega^2}{v^2}} = \sqrt{\frac{\omega^2 n^2}{c^2} - \frac{b^2}{2n^2} \left(1 + \nu \sqrt{1 + 4 \frac{\omega^2 n^2}{v^2 b^2}} \right)}. \quad (34)$$

Here we have substituted the value of Q_ν from Eq. (25) and the final expression for the Cherenkov angle θ_ν is

$$\cos \Theta_\nu = \frac{\omega}{v \sqrt{\frac{\omega^2 n^2}{c^2} - \frac{b^2}{2n^2} \left(1 + \nu \sqrt{1 + 4 \frac{\omega^2 n^2}{v^2 b^2}} \right)}}. \quad (35)$$

Observe that $\cos \Theta_\nu > 0$.

Besides the condition (25) we must also require the radicand in the big square root of Eq. (35) to be positive. In the high-velocity regime, $v > c/n$, the $\nu = -$ mode always satisfies this condition. For $\nu = +$ we have to solve the inequality for ω finding the following condition:

$$\omega > cb \frac{\sqrt{1 + n^2 \beta^2}}{n^3 \beta}. \quad (36)$$

There is the additional condition $\cos^2 \Theta_\nu < 1$, which yields the same Eq. (25) that guaranties the reality of Q_ν . Recalling that when $v > c/n$, Eq. (25) imposes $\omega > bv\beta/(n^2 \beta^2 - 1)$ for $\nu = +$ we have two restrictions upon ω in this case. However, since

$$bv \frac{\beta}{n^2 \beta^2 - 1} > bc \frac{\sqrt{1 + n^2 \beta^2}}{n^3 \beta}, \quad (37)$$

when $n\beta > 1$, it is sufficient to consider only

$$\omega > bc \frac{\beta^2}{n^2 \beta^2 - 1} \equiv \omega_{C+}, \quad (38)$$

which is the same condition obtained in Eq. (29).

In the low-velocity regime, $\beta < 1/n$, the $\nu = +$ mode never emits radiation, whereas we find that the $\nu = -$ mode radiates only within a finite frequency range, $0 < \omega < cb \frac{\beta^2}{1 - n^2 \beta^2} \equiv \omega_{C-}$, confirming the threshold-free frequency window found in the previous section.

V. THE PHASE VELOCITY

An alternative way of identifying Cherenkov radiation is when the charge velocity is larger than the phase velocity $v^{ph}(\omega)$ of light in the medium. In our case we have $v^{ph}(\omega) = \omega/|\mathbf{k}_\nu|$ such that

$$v^{ph}(\omega) = \frac{\omega}{\sqrt{\frac{\omega^2 n^2}{c^2} - \frac{b^2}{2n^2} \left(1 + \nu \sqrt{1 + 4 \frac{\omega^2 n^2}{v^2 b^2}} \right)}}. \quad (39)$$

The analysis of the different situations stemming from the perspective of the phase velocity condition is greatly simplified by recalling that $v^{ph}(\omega)$ is a monotonically increasing (decreasing) function of ω for $\nu = -$, ($\nu = +$), respectively. This property is shown in Appendix C. Next we examine each choice of the polarization.

A. The polarization $\nu = -$

The corresponding phase velocity $v_-^{ph}(\omega)$ exists for all frequencies $\omega \geq 0$. It is given by Eq. (39) and is an increasing function of frequency. Let us consider its values at the low- and high-frequency limits. In the low-frequency limit one finds

$$\lim_{\omega \rightarrow 0} v_-^{ph} = \frac{v}{\sqrt{1 + n^2 \beta^2}} < v, \quad (40)$$

since the denominator is larger than one. Therefore, independently of the value of n and of the charge velocity, the phase velocity at $\omega \rightarrow 0$ is always smaller than the charge velocity. As a consequence, the inequality $v > v_-^{ph}(\omega)$ is always satisfied in this limit. In the opposite case, for the large frequency limit we have

$$\lim_{\omega \rightarrow \infty} v_-^{ph} = \frac{c}{n}, \quad (41)$$

which coincides with the standard phase velocity.

Next we distinguish the two cases $v > c/n$ and $v < c/n$ recalling that $v_-^{ph}(\omega)$ is a strictly increasing function of the frequency. Consequently, for charge velocities $v > c/n$, one has $v_-^{ph}(\omega) < v$ for all frequencies and the $\nu = -1$ mode always contributes to Cherenkov radiation. In contrast, for $v < c/n$ we never approach the upper limit (41) and the condition $v > v_-^{ph}(\omega)$ is satisfied only within a finite frequency interval,

$$0 < \omega < \omega_{C-}, \quad \omega_{C-} = cb \frac{\beta^2}{1 - n^2 \beta^2}, \quad (42)$$

beyond which the phase velocity exceeds the charge velocity and radiation is no longer emitted. However, observe that this last sector allows threshold-free Cherenkov radiation, validating our previous results. In particular, in the “chiral vacuum,” i.e., the case $n = 1$, the condition $v < c/n$ is always satisfied for any subluminal charge velocity. As a result, threshold-free Cherenkov radiation occurs for the $\nu = -$ mode within a finite frequency interval, which increases with the particle velocity. This highlights a qualitative difference between chiral matter and the “chiral vacuum,” where Cherenkov emission at high particle velocities without a threshold is allowed since the refractive index is $n = 1$.

B. The polarization $\nu = +$

The corresponding phase velocity $v_+^{ph}(\omega)$ is a decreasing function of the frequency and exists only for frequencies larger than a minimum value,

$$\omega > \omega_{\min} = bc \frac{\sqrt{1 + n^2 \beta^2}}{n^3 \beta}. \quad (43)$$

At this lower bound the phase velocity diverges,

$$\lim_{\omega \rightarrow \omega_{\min}} v_+^{ph}(\omega) = \infty, \quad (44)$$

so that $v_+^{ph}(\omega)$ always exceeds the charge velocity in the vicinity of $\omega_{\min}$. In the high-frequency limit one finds again

$$\lim_{\omega \rightarrow \infty} v_+^{ph} = \frac{c}{n}. \quad (45)$$

Consequently, for charge velocities $v < c/n$ one has $v < v_+^{ph}(\omega)$ for all $\omega > \omega_{\min}$ and the $\nu = +1$ mode does not contribute to Cherenkov radiation. In contrast, for $v > c/n$, there exists a critical frequency $\omega_{C+} > \omega_{\min}$ such that the condition $v > v_+^{ph}(\omega)$ is satisfied only when

$$\omega > \omega_{C+}, \quad \omega_{C+} = cb \frac{\beta^2}{(n^2 \beta^2 - 1)}, \quad (46)$$

allowing the emission of Cherenkov radiation in the $\nu = +1$ mode.

Both the requirement that q_ν be positive and the conditions for the existence of the Cherenkov angle, together with the constraint that the charge velocity exceed the phase velocity, lead to identical radiation criteria: for high velocities ($\beta > 1/n$) the $\nu = -$ mode always radiates, while the $\nu = +$ mode contributes only for frequencies $\omega > \omega_{C+}$; in contrast, for low velocities ($\beta < 1/n$) the $\nu = +$ mode never radiates, whereas the $\nu = -$ mode emits radiation only in the frequency interval $0 < \omega < \omega_{C-}$, corresponding to the threshold-free sector.

VI. THE RADIATION REGIME

Contrary to the isotropic case where the Poynting vector contains contributions from the electromagnetic potentials here we have the standard expression

$$\mathbf{S} = \frac{c}{4\pi} (\mathcal{E} \times \mathcal{B}), \quad (47)$$

which allows the calculation of the total radiated energy across a cylindrical surface at infinity according to

$$E = 2\pi \lim_{\rho \rightarrow \infty} \rho \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dz S_\rho, \quad (48)$$

where S_ρ is the component of the Poynting vector in the direction $\hat{\rho}$ with

$$S_\rho = \frac{c}{4\pi} (\mathcal{E}_\phi \mathcal{B}_z - \mathcal{E}_z \mathcal{B}_\phi). \quad (49)$$

Going to the space-frequency domain we read the spectral distribution of the total radiated energy per unit length $\mathcal{E}$ (SED) as

$$\mathcal{E} = \frac{d^2 E}{d\omega dz} = \lim_{\rho \rightarrow \infty} \rho \frac{c}{2\pi} \text{Re}[E_\phi^* B_z - E_z^* B_\phi]. \quad (50)$$

We have to evaluate the electromagnetic fields in the limit $Q_{\lambda}\rho \gg 1$ where

$$K_{(1,0)}(-iQ_{\nu}\rho) = \sqrt{\frac{\pi}{-2iQ_{\nu}\rho}} e^{iQ_{\nu}\rho}. \quad (51)$$

The required products are

$$\lim_{\rho \rightarrow \infty} K_1(-iQ_{\mu}\rho) K_0^*(-iQ_{\nu}\rho), \quad (52)$$

which reduce to

$$\begin{aligned} & \text{Re}[\lim_{\rho \rightarrow \infty} K_1^*(-iQ_{\mu}\rho) K_0(-iQ_{\nu}\rho)] \\ &= \begin{cases} \frac{\pi}{2\rho Q_{\nu}}, & \mu = \nu, \\ \frac{\pi}{2\rho \sqrt{Q_{\mu}Q_{\nu}}} \cos[(Q_{\mu} - Q_{\nu})\rho], & \mu \neq \nu. \end{cases} \end{aligned} \quad (53)$$

Substituting the corresponding expressions in (18) and (19) for the fields entering (50) we obtain two sets of contributions according to the choices of polarization.

When $\mu = \nu$ the results are

$$E_{\phi}^* B_z = \frac{\pi c}{2\omega\rho} \sum_{\nu} Y_{\nu}^* Y_{\nu} \Omega_{\nu}^2, \quad (54)$$

$$\begin{aligned} E_z^* B_{\phi} &= -\frac{\pi c}{2k^2\omega\rho} \sum_{\nu} Y_{\nu}^* Y_{\nu} \left(1 + \frac{cb}{n^2\omega} \Omega_{\nu}\right) \\ &\times \left(k^2 + Q_{\nu}^2 + \frac{cb}{n^2\omega} \Omega_{\nu} Q_{\nu}^2\right), \end{aligned} \quad (55)$$

yielding

$$\begin{aligned} \mathcal{E} &= \frac{c^2}{4\omega} \sum_{\nu} Y_{\nu}^* Y_{\nu} \left[\Omega_{\nu}^2 + \frac{1}{k^2} \left(1 + \frac{cb}{n^2\omega} \Omega_{\nu}\right) \right. \\ &\times \left. \left(k^2 + Q_{\nu}^2 + \frac{cb}{n^2\omega} \Omega_{\nu} Q_{\nu}^2\right) \right]. \end{aligned} \quad (56)$$

The total spectral energy distribution can be split into $\mathcal{E} = \mathcal{E}_+ + \mathcal{E}_-$ such that

$$\begin{aligned} \mathcal{E}_{\nu} &= \frac{q^2\omega}{2c^2} \left[\left(1 - \frac{c^2}{n^2v^2}\right) - \nu \left(1 + \frac{c^2}{n^2v^2}\right) \frac{1}{\sqrt{1 + 4\frac{n^2\omega^2}{b^2v^2}}} \right], \\ &\equiv \frac{q^2\omega}{c} \Lambda_{\nu} \end{aligned} \quad (57)$$

which provides the contribution of each individual Cherenkov cone. In Appendix D we show that the SED can be nicely written as

$$\mathcal{E}_{\nu} = \frac{q^2\omega}{2n^2v^2} \left(1 - \nu \frac{1}{\sqrt{1 + 4\frac{n^2\omega^2}{b^2v^2}}}\right) \tan^2 \Theta_{\nu}, \quad (58)$$

clearly stating that $\mathcal{E}_{\nu} \geq 0$, since the term in round brackets is always positive. When both modes radiate, the total energy distribution is

$$\mathcal{E} = \frac{q^2\omega}{c^2} \left(1 - \frac{1}{n^2\beta^2}\right). \quad (59)$$

When $\nu \neq \mu$ we have the following crossed terms:

$$\begin{aligned} & \mathcal{E}_{(+,-)} + \mathcal{E}_{(-,+)} \\ &= \frac{c^2}{4\omega} Y_+^* Y_- \left[\Omega_+ \Omega_- Q_- + \Omega_- \Omega_+ Q_+ \right. \\ &\quad + \frac{1}{k^2} \left(1 + \frac{cb}{n^2\omega} \Omega_+\right) Q_+ \left(k^2 + Q_-^2 + \frac{cb}{n^2\omega} \Omega_- Q_-^2\right) \\ &\quad + \frac{1}{k^2} \left(1 + \frac{cb}{n^2\omega} \Omega_-\right) Q_- \left(k^2 + Q_+^2 + \frac{cb}{n^2\omega} \Omega_+ Q_+^2\right) \Big] \\ &\quad \times \frac{\cos((Q_+ - Q_-)\rho)}{\sqrt{Q_+ Q_-}}, \end{aligned} \quad (60)$$

where we have implemented the equality $Y_+^* Y_- = Y_-^* Y_+$ stemming from the reality of the coefficients. Calculating the square bracket using *Mathematica* we obtain a null result, which is verified analytically in Appendix E. In other words, there is no mixing between the polarizations + and - in the radiation output.

Before closing this section we make some remarks concerning the polarization $\nu = \pm$ in this system. Since $\nabla \cdot \mathcal{E} \neq 0$ in the radiation region, the electric field is not orthogonal to the wave vector $\mathbf{k}_{\nu}$ and a longitudinal component appears. Following the standard convention, the polarization is defined by the orientation of the electric field in the plane perpendicular to $\mathbf{k}_{\nu}$. In other words we must consider $\mathcal{E}_{\nu T} = \mathcal{E}_{\nu} - (\hat{\mathbf{k}}_{\nu} \cdot \mathcal{E}_{\nu}) \hat{\mathbf{k}}_{\nu}$, which lies on this plane and identify the polarization there. From the radiation approximation for the electric field together with Eq. (32) we have

$$\begin{aligned} \mathcal{E}_{\nu} &= e^{i\mathbf{k}_{\nu} \cdot \mathbf{x}} Y_{\nu} \left(\sqrt{\frac{\pi}{-2iQ_{\nu}\rho}} \right) \\ &\times \left\{ \hat{\rho} + i\Omega_{\nu} \hat{\phi} - \frac{Q_{\nu}}{k} \left(1 + \frac{cb}{n^2\omega} \Omega_{\nu}\right) \hat{z} \right\}, \\ \hat{\mathbf{k}}_{\nu} &= \frac{1}{\sqrt{Q_{\nu}^2 + k^2}} (Q_{\nu} \hat{\rho} + k \hat{z}). \end{aligned} \quad (61)$$

The quantities Ω_{ν} and Q_{ν} are real with expressions given in Eqs. (17) and (25), respectively. The general expression for the transverse field is not very illuminating, and we choose

a specific simple case to investigate the polarization. We take $n\beta > 1$ and $\nu = -$, where CHR always exists with no restriction over the parameters, together with the limit of very large b , i. e. $b/k \gg 1$. Then, our starting point is

$$\Omega_\nu = \beta \frac{b}{k}, \quad Q_\nu = kn\beta, \quad \bar{\mathcal{E}}_\nu = \hat{\rho} + i\beta \frac{b}{k} \hat{\phi} - \frac{\beta b^2}{n k^2} \hat{z}. \quad (62)$$

Since only the relative phases among the components of the electric field matter, we consider just the vector in braces in Eq. (61), denoted by $\bar{\mathcal{E}}_\nu$. The resulting transverse field is

$$\bar{\mathcal{E}}_{\nu T} = \frac{\beta^2}{(\beta^2 n^2 + 1)} \frac{b^2}{k^2} \hat{\rho} + i\beta \frac{b}{k} \hat{\phi} + \frac{n\beta^3}{(\beta^2 n^2 + 1)} \frac{b^2}{k^2} \hat{z}. \quad (63)$$

Next we choose the coordinate system in that plane as spanned by the orthogonal vectors $\mathbf{V}_1 = i\beta \frac{b}{k} \hat{\phi}$ and $\mathbf{V}_2 = \bar{\mathcal{E}}_{\nu T} - \mathbf{V}_1 = \frac{\beta^2}{(\beta^2 n^2 + 1)} \frac{b^2}{k^2} \hat{\rho} + \frac{n\beta^3}{(\beta^2 n^2 + 1)} \frac{b^2}{k^2} \hat{z}$, yielding the magnitudes

$$|\mathbf{V}_1| = \beta \frac{b}{k}, \quad |\mathbf{V}_2| = \frac{\beta^2}{\sqrt{\beta^2 n^2 + 1}} \frac{b^2}{k^2}. \quad (64)$$

Since $|\mathbf{V}_1| \neq |\mathbf{V}_2|$ we conclude that we have elliptical polarization, which we take as a general feature in our setup.

VII. NUMERICAL ESTIMATIONS

A representative Weyl semimetal realizing $b_0 = 0$ and $b_z \neq 0$ is EuCd_2As_2 , where the separation of the Weyl nodes occurs along the k_z direction. This behavior was predicted by density-functional-theory (DFT) calculations and experimentally verified by angle-resolved photoemission spectroscopy (ARPES) yielding

$$b_z \simeq 0.03 \times \frac{2\pi}{\bar{c}} \simeq 2.6 \times 10^8 \text{ m}^{-1}, \quad (65)$$

where $\bar{c} = 0.729 \text{ nm}$ is the lattice constant along the crystallographic $\bar{c}$ axis [53]. Optical and magneto-optical studies indicate that the refractive index of EuCd_2As_2 takes values of order unity over a finite range of frequencies [54]. We therefore adopt the representative constant value $n = 2$ for simplicity and to facilitate a direct comparison with the results reported in Ref. [30].

As a matter of notation, we designate the regimes $\beta > 1/n$ ($\beta < 1/n$) as the high (low) velocity sectors, respectively.

With the above material parameters in mind, we set $n = 2$, $b = 2.6 \times 10^8 \text{ m}^{-1}$, and introduce the dimensionless frequency $\bar{\omega} = \omega/(bc)$, where $bc = 323 \text{ eV}$. In the following, we analyze these two velocity sectors separately.

A. The high-velocity sector

We consider $\beta = 0.75 > 1/n = 0.5$. In this regime, the $\nu = -$ mode always radiates, whereas the $\nu = +$ mode radiates only for dimensionless frequencies above

$$\bar{\omega}_{C+} = \frac{\beta^2}{n^2 \beta^2 - 1} = 0.45. \quad (66)$$

This behavior is illustrated in Fig. 1. In the left panel, we plot the cosine of the Cherenkov angle as a function of the dimensionless frequency $\bar{\omega}$. For the $\nu = +$ mode, the formal expression for the Cherenkov angle yields $\cos \Theta > 1$ in the interval $0 < \bar{\omega} < \bar{\omega}_{C+}$, implying that no real Cherenkov angle exists and, consequently, no radiation is emitted in this frequency range. For $\bar{\omega} > \bar{\omega}_{C+}$, $\cos \Theta < 1$ and the Cherenkov angle becomes well defined, allowing radiation to occur. In contrast, the $\nu = -$ mode satisfies $\cos \Theta < 1$ for all frequencies, and therefore radiates over the entire range.

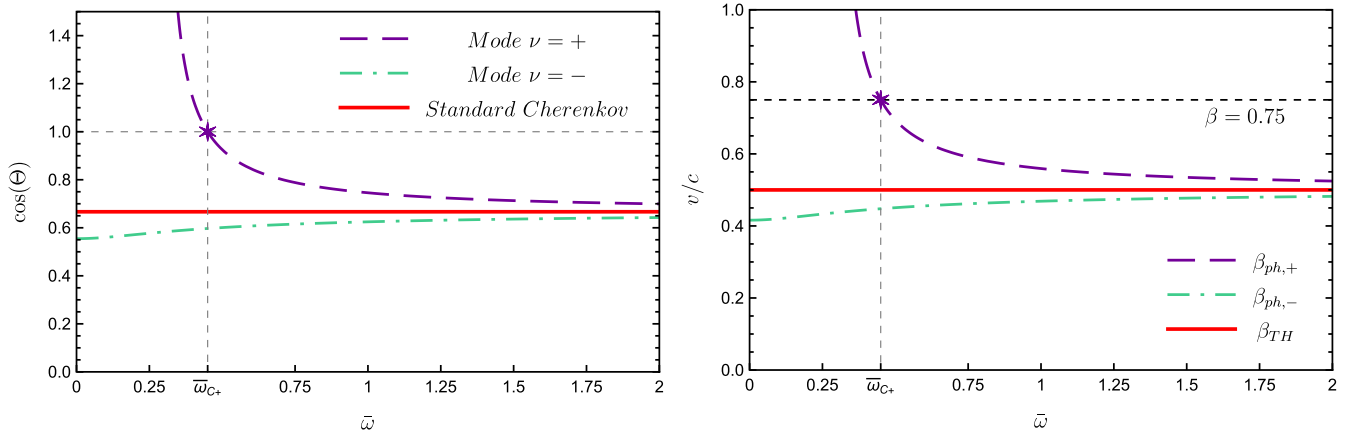


FIG. 1. Left: plot of $\cos \Theta$ for the $\nu = \pm$ modes as a function of the dimensionless frequency $\bar{\omega}$. Right: plot of the phase velocity for each mode as a function of $\bar{\omega}$. The parameters are $n = 2$, $b = 2.6 \times 10^8 \text{ m}^{-1}$, and $\beta = 0.75$.

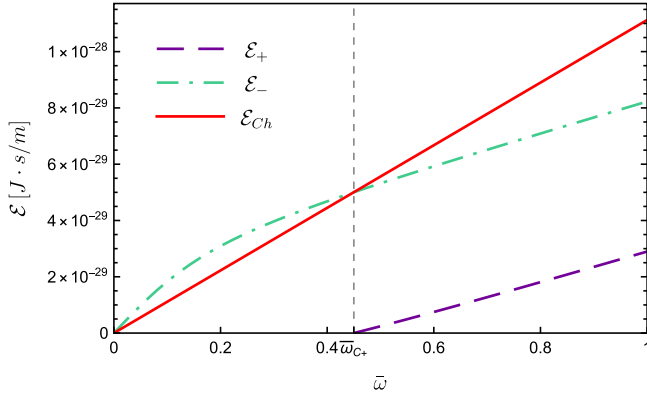


FIG. 2. Spectral energy density in the high-velocity sector as a function of the dimensionless frequency $\bar{\omega}$. The contributions from the $\nu = +$ and $\nu = -$ modes are shown, together with the standard Cherenkov result. The parameters are $n = 2$, $b = 2.6 \times 10^8 \text{ m}^{-1}$, and $\beta = 0.75$.

The right panel displays the corresponding phase velocities for both modes. In the interval $0 < \bar{\omega} < \bar{\omega}_{C+}$, the phase velocity of the $\nu = +$ mode exceeds the charge velocity ($\beta = 0.75$), consistently with the absence of a real Cherenkov angle and the lack of radiation. For $\bar{\omega} > \bar{\omega}_{C+}$, the phase velocity drops below the charge velocity, in agreement with the onset of Cherenkov emission. The phase velocity of the $\nu = -$ mode remains below the charge velocity for all frequencies, fully consistent with its radiative behavior in this sector.

Figure 2 shows the SED in the high-velocity sector. As expected from the analysis of the Cherenkov angle and the phase velocity, the $\nu = -$ mode contributes over the entire frequency range, indicating that this mode always emits radiation. In contrast, the $\nu = +$ mode contributes only for $\bar{\omega} > \bar{\omega}_{C+}$, because for $\bar{\omega} < \bar{\omega}_{C+}$ the Cherenkov angle is not

real and the phase velocity is larger than the charge velocity. The standard Cherenkov result is shown for comparison. For $\bar{\omega} > \bar{\omega}_{C+}$, the sum of the contributions of each mode satisfies $\mathcal{E}_+ + \mathcal{E}_- = \mathcal{E}_{Ch}$, as indicated in Fig. 2.

B. The low-velocity sector

We consider $\beta = 0.2 < 1/n = 0.5$. In this case, the $\nu = +$ mode does not radiate, whereas the $\nu = -$ mode radiates only within a finite frequency interval,

$$0 < \bar{\omega} < \bar{\omega}_{C-}, \quad \bar{\omega}_{C-} = \frac{\beta^2}{1 - n^2\beta^2} = 0.048, \quad (67)$$

giving rise to threshold-free CHR.

This behavior is illustrated in Fig. 3. In the left panel, we plot the cosine of the Cherenkov angle as a function of the dimensionless frequency $\bar{\omega}$. For the $\nu = +$ mode, the expression for $\cos \Theta$ remains larger than unity over the entire frequency range, implying that no real Cherenkov angle exists and radiation is therefore absent. Consistently, the corresponding curve lies outside the plotted range. In contrast, the $\nu = -$ mode yields $\cos \Theta < 1$ only in the interval $0 < \bar{\omega} < \bar{\omega}_{C-}$, so the Cherenkov angle is real and radiation can occur within this frequency window indicating the emission of threshold-free CHR. For $\bar{\omega} > \bar{\omega}_{C-}$, $\cos \Theta$ exceeds unity and the Cherenkov angle is no longer defined.

The right panel in Fig. 3 shows the corresponding phase velocities. For the $\nu = +$ mode, the phase velocity remains larger than the charge velocity for all frequencies, fully consistent with the absence of radiation. For the $\nu = -$ mode, the phase velocity is smaller than the charge velocity only in the interval $0 < \bar{\omega} < \bar{\omega}_{C-}$, in agreement with the existence of a real Cherenkov angle and the occurrence of

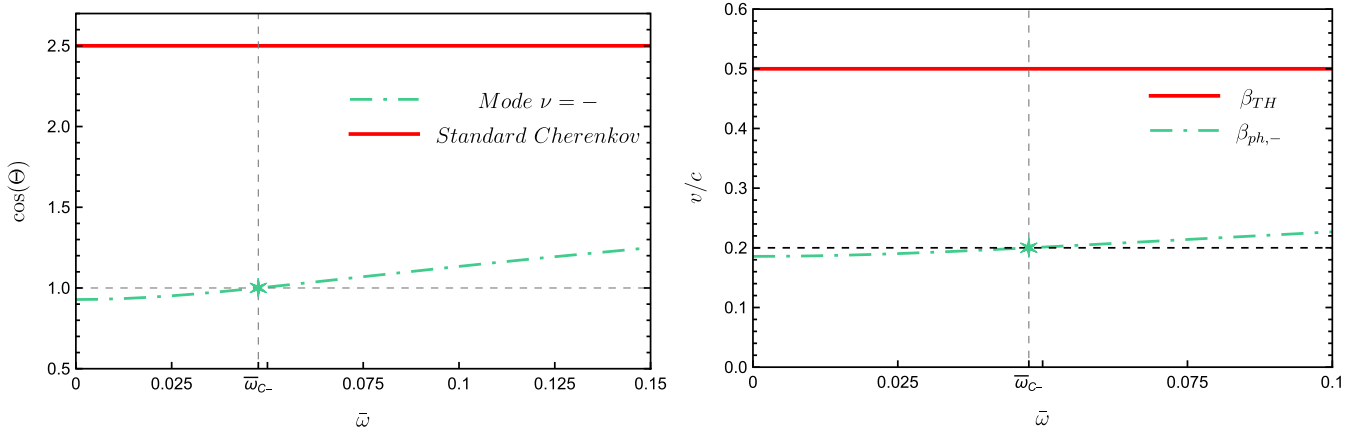


FIG. 3. Left: plot of $\cos \Theta$ for the $\nu = -$ mode as a function of the dimensionless frequency $\bar{\omega}$. Right: plot of the phase velocity for the $\nu = -$ mode as a function of $\bar{\omega}$. The parameters are $n = 2$, $b = 2.6 \times 10^8 \text{ m}^{-1}$, and $\beta = 0.2$. In both cases the $\nu = +$ mode lies outside the plotted range. The solid red line would represent the standard Cherenkov case, which is certainly not allowed. Threshold-free CHR can be appreciated in both panels.

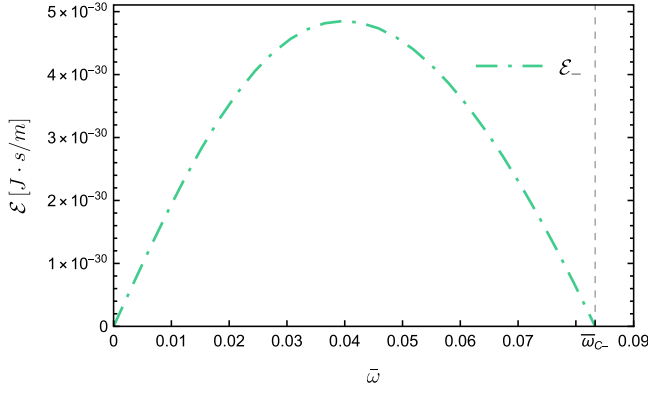


FIG. 4. Spectral energy density in the low-velocity sector as a function of the dimensionless frequency $\bar{\omega}$. Only the contribution from the $\nu = -$ mode is present, and radiation occurs in the interval $0 < \bar{\omega} < \bar{\omega}_{C-}$. The parameters are $n = 2$, $b = 2.6 \times 10^8 \text{ m}^{-1}$, and $\beta = 0.2$. This SED corresponds entirely to threshold-free CHR.

threshold-free CHR. Outside this interval, the phase velocity exceeds the charge velocity, and radiation ceases.

Figure 4, shows that in the low-velocity sector the spectral energy density is entirely due to the $\nu = -$ mode and that it is nonvanishing only in the finite frequency interval $0 < \bar{\omega} < \bar{\omega}_{C-}$, in agreement with the analysis of the Cherenkov angle and the phase velocity.

To complement the spectral analysis above and provide an estimate of the photon yield, we evaluate the photon extraction efficiency per unit length for the channel $\nu = -$, defined in Refs. [55,56]

$$\tilde{\eta}_{-} = \frac{d\eta_{-}}{dz} = \frac{\mathcal{E}_{-}}{\hbar\omega E_{\text{ch}}}, \quad (68)$$

where $E_{\text{ch}} = m_{\text{ch}}c^2(\gamma_L - 1)$ is the kinetic energy of the charge and γ_L is the standard Lorentz factor. In the present case this quantity takes the form

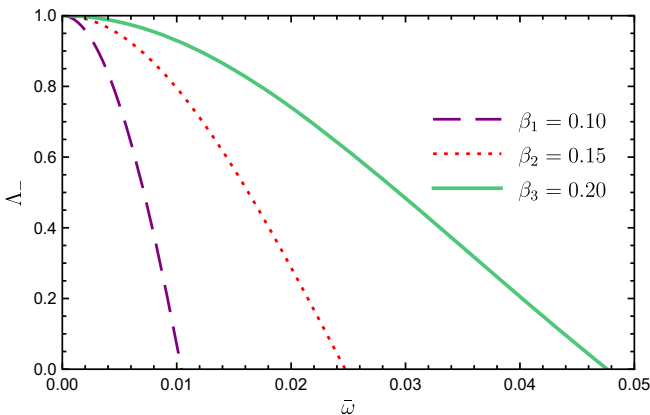


FIG. 5. The function $\Lambda_{-}(\omega)$ for the charge velocities $\beta_1 = 0.10$, $\beta_2 = 0.15$ and $\beta_3 = 0.20$, with $n = 2$ and $b_z \simeq 2.6 \times 10^8 \text{ m}^{-1}$.

TABLE I. The amplification factor $1/(\gamma_L - 1)$ for different charge velocities, together with the corresponding maximum allowed frequency ω_{C-} and the resulting value of $\langle \tilde{\eta}_{-} \rangle$ given by $\tilde{\eta}_{-}$ calculated at $\Lambda_{-} = 1$.

β	0.10	0.15	0.20
$1/(\gamma_L - 1)$	198.5	87.4	48.5
ω_{C-} [eV]	3.23	8.08	15.5
$\langle \tilde{\eta}_{-} \rangle$ [Watt m] $^{-1}$	5.9×10^4	2.6×10^4	1.4×10^4

$$\tilde{\eta}_{-} = f \frac{1}{(\gamma_L - 1)} \Lambda_{-}, \quad f = \frac{q^2}{c^4 \hbar m_{\text{ch}}}, \quad (69)$$

with Λ_{ν} defined in Eq. (57). For an electron, the constant factor evaluates to $f = 296.8 \text{ [Watt m]}^{-1}$. The behavior of $\Lambda_{-}(\omega)$ for the velocities considered in this work is shown in Fig. 5, and the resulting values of $\langle \tilde{\eta}_{-} \rangle$ are collected in Table I.

A direct comparison with the isotropic case [32,52] reveals a substantial advantage of the anisotropic configuration. While the extraction efficiencies $\langle \tilde{\eta}_{-} \rangle$ are of the same order of magnitude in both cases ($\sim 10^4 \text{ [Watt m]}^{-1}$), the allowed frequency windows ω_{C-} in the present work lie in the eV range, compared to the meV range reported in Refs. [32,52], representing an enhancement of roughly 3 orders of magnitude. Such a broadening of the emission bandwidth, without a corresponding reduction in extraction efficiency, suggests that the anisotropic medium could significantly improve the prospects for experimental detection of threshold-free CHR.

Previous research supports that the ranges we find for threshold-free CHR in the low velocity sector are experimentally accessible. For example, Ref. [9] reports detection of photons with frequencies in the range $\omega = 1.4\text{--}2.5 \text{ eV}$ arising from electrons with $\beta = 0.03\text{--}0.07$. More energetic electrons in the range $\beta = 0.53\text{--}0.70$ yield radiation with $\omega = 2.08\text{--}2.30 \text{ eV}$, as detailed in Ref. [57].

C. The “chiral vacuum” case

We now turn to the “chiral vacuum” case, i.e., $n = 1$. In this situation, the conventional Cherenkov threshold corresponds to $\beta = 1$, implying that no CHR is allowed in the standard scenario, since no charge can propagate faster than the speed of light. However, in the present framework the $\nu = -$ mode leads to threshold-free CHR within a finite frequency interval for any subluminal charge velocity. This behavior is illustrated in the following figures. In Fig. 6 we show that a Cherenkov angle exists and that the charge velocity exceeds the phase velocity in the interval $0 < \omega < \omega_{C-}$. The corresponding spectral energy density is displayed in Fig. 7.

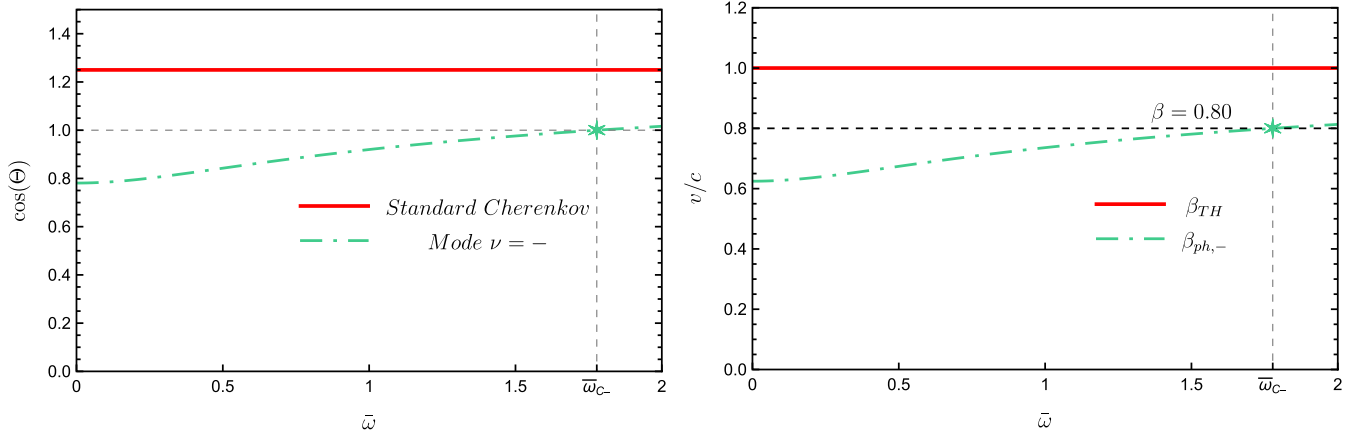


FIG. 6. Left: plot of $\cos \Theta$ for the $\nu = -$ mode as a function of the dimensionless frequency $\bar{\omega}$. Right: plot of the phase velocity for the $\nu = -$ mode as a function of $\bar{\omega}$. The parameters are $n = 1$, $b = 2.6 \times 10^8 \text{ m}^{-1}$, and $\beta = 0.8$. Threshold-free CHR can be appreciated in both panels. The $\nu = +$ mode lies outside the plotted range. The solid red line would represent the Standard Cherenkov case, which is not allowed.

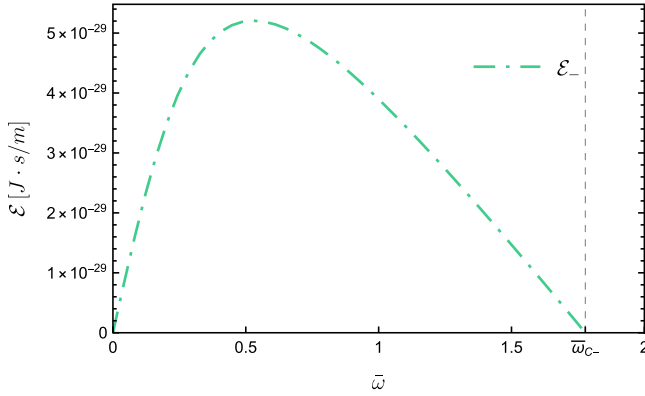


FIG. 7. Spectral energy density in the “chiral vacuum” as a function of the dimensionless frequency $\bar{\omega}$. Only the contribution from the $\nu = -$ mode is present, and radiation occurs in a finite frequency interval $0 < \bar{\omega} < \bar{\omega}_{C-}$. The parameters are $n = 1$, $b = 2.6 \times 10^8 \text{ m}^{-1}$, and $\beta = 0.8$. This SED corresponds entirely to threshold-free CHR.

VIII. MAIN CHARACTERISTICS OF THE ANISOTROPIC CASE

We highlight some features of the anisotropic case by comparing with the previously considered isotropic case. As explained in the Introduction, both cases correspond to distinct sectors of CFJ electrodynamics. While previous references [32,52] dealt with a purely temporal background vector, $b^\mu = (\sigma/c, \mathbf{0})$, named as the σ case, the present work studies a purely spatial configuration, $b^\mu = (0, \mathbf{b})$, with $\mathbf{b} \parallel \mathbf{v}$, to be called the b case. This leads to quantitative different radiative properties even though both cases may look rather similar due to the choice $\mathbf{b} \parallel \mathbf{v}$, which still preserves axial symmetry. In particular, the anisotropic configuration modifies the radiative kinematics in a non-trivial way since the phase velocity acquires an explicit

dependence on the particle velocity, which is absent in the isotropic case. To clearly highlight this difference let us focus on the phase velocity v_ν^{ph} written in terms of the frequency

$$[v_\nu^{ph}]_b = \frac{\omega}{\sqrt{\frac{n^2 \omega^2}{c^2} - \frac{b^2}{2n^2} \left(1 + \nu \sqrt{1 + \frac{4n^2 \omega^2}{b^2} \frac{1}{v^2}}\right)}}, \quad (70)$$

$$[v_\nu^{ph}]_\sigma = \frac{\omega}{\sqrt{\frac{n^2 \omega^2}{c^2} + \frac{\sigma^2}{2c^2} \left(1 - \nu \sqrt{1 + \frac{4n^2 \omega^2}{\sigma^2} \frac{c^2}{v^2}}\right)}}, \quad (71)$$

where we label with a subindex b or σ the corresponding case. The comparison sharpens if we introduce $b_{\text{eff}} = \sigma/c$ in Eq. (70) yielding

$$[v_\nu^{ph}]_{b_{\text{eff}}} = \frac{\omega}{\sqrt{\frac{n^2 \omega^2}{c^2} - \frac{1}{n^2} \frac{\sigma^2}{2c^2} \left(1 + \nu \sqrt{1 + \frac{4n^2 \omega^2 c^2}{\sigma^2} \frac{c^2}{v^2}}\right)}}. \quad (72)$$

Comparing Eqs. (71) and (72) we identify the relevant factor c^2/v^2 appearing inside the square root multiplying ν in the b case, which is absent in the σ case. This factor modifies the widths of the low- and high-velocity frequency windows in each polarization of one case with respect to the other.

TABLE II. The radiation pattern of the high ($v > c/n$) and low ($v < c/n$) velocity sectors (HVS and LVS, respectively) for each polarization mode $\nu = \pm 1$.

	$\nu = -1$	$\nu = +1$
HVS	✓	$\omega > \omega_{C+}$
LVS	$\omega < \omega_{C-}$	×

TABLE III. The frequency thresholds ω_{C+} and ω_{C-} for the high and low velocity sectors, respectively.

	ω_{C+}	ω_{C-}
σ case	$\sigma \frac{\beta}{n^2\beta^2-1}$	$\sigma \frac{\beta}{1-n^2\beta^2}$
b case	$bc \frac{\beta^2}{n^2\beta^2-1}$	$bc \frac{\beta^2}{1-n^2\beta^2}$

Let us recall that the radiation pattern for the anisotropic case, which is summarized in Table II. We observe that the same arrangement is valid in the isotropic case, with appropriate values for the frequency thresholds $\omega_{C\pm}$. The resulting values for these thresholds are shown in Table III.

Assuming $\sigma_{\text{eff}} = cb$ would be of the same order of σ , we can say that the thresholds for the frequency windows in the b case are suppressed by a factor β with respect to the σ case. In other words, for the b case the high velocity window opens, while the low velocity window closes, with respect to the σ case. However, we warn the reader that this interpretation is not valid in our case since we have $\sigma_{\text{eff}} = 323$ eV versus $\sigma = 0.05$ eV. Let us emphasize that the introduction of σ_{eff} , which is useful in the above comparison, by no means indicates that the full solution in each case can be related by this substitution.

Another important consequence in the b case is the polarization structure of the emitted radiation. The isotropic case is naturally described in terms of circular polarization modes, while the anisotropic configuration gives rise to elliptically polarized radiation.

Finally, beyond improving the analytical treatment previously available in the literature, the exact solution obtained here for the b case reveals the threshold-free emission sector that remained inaccessible within the previous approximate treatment of the Green's function.

Before closing this section, we want to comment on the importance of considering the fully anisotropic case where $\mathbf{v}$ and $\mathbf{b}$ form an arbitrary angle, since it will be very difficult in practice to achieve perfect alignment between both vectors. As a preliminary approximation to the problem, which is beyond the scope of this work, let us consider the case when $\mathbf{b}$ also has a component δ in the x direction, such that $\delta \ll b$. Starting from Ampere's law, and after the substitution of Faraday's relation in Eq. (3) we can find the modified dispersion relation in Cartesian coordinates assuming the standard plane wave $e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$ in the radiation zone. Expanding to first order in δ we obtain

$$\Omega^6 - 2\Omega^4\mathbf{k}^2 + \Omega^2(\mathbf{k}^2)^2 - k_0^2b^2(\Omega^2 - \mathbf{k}^2 + k^2) - 2k_0^2b\delta k_x k = 0, \quad (73)$$

with the notation

$$\Omega = \frac{\omega n}{c}, \quad k_z = k = \frac{\omega}{v}, \quad k_0 = \frac{\omega}{c}. \quad (74)$$

In the following all unbarred quantities denote those calculated in the unperturbed case ($\delta = 0$). It is convenient to introduce the perpendicular momentum $\bar{\mathbf{Q}} = \bar{Q}\hat{\rho}$ such that $\mathbf{k}^2 = \bar{Q}^2 + k^2$. We verify that in the limit $\delta = 0$ the usual expressions for Q_ν^2 in Eq. (25) are recovered from Eq. (73). To get additional information regarding the effects of anisotropy we solve Eq. (73) to first order in δ setting $\bar{Q}_\nu = Q_\nu + \delta Q_{1\nu}$ and writing $\bar{\mathbf{k}}_\nu = (\bar{Q}_\nu \cos \phi, \bar{Q}_\nu \sin \phi, k)$. Here we explicitly break axial symmetry introducing the angle ϕ defining the direction of k_x in the plane perpendicular to $\mathbf{v}$. The result is

$$\bar{Q}_\nu = Q_\nu \left(1 - (\nu \cos \phi) \frac{\delta}{b} \frac{k}{Q_\nu} \frac{1}{\sqrt{1 + \frac{4n^2k^2}{b^2}}} \right). \quad (75)$$

We assume also $k/Q_\nu < 1$ such that the round bracket in the above equation is always positive. From expression (75) we obtain the anisotropic phase velocity $\bar{v}_\nu^{ph} = \omega/|\bar{\mathbf{k}}_\nu|$

$$\bar{v}_\nu^{ph} = v_{0\nu}^{ph} \left(1 + (\nu \cos \phi) \left(\frac{\delta}{b} \right) \frac{(\frac{Q_\nu}{k})}{(1 + (\frac{Q_\nu}{k})^2)} \frac{1}{\sqrt{1 + \frac{4n^2k^2}{b^2}}} \right). \quad (76)$$

The corrections in $\bar{Q}_\nu$ and $\bar{v}_\nu^{ph}$ are very small being proportional to $\delta/b \ll 1$, since the remaining factors are all less than one. The anisotropy of the solution is manifest in the ϕ dependence of the phase velocity, which also produces Cherenkov angles depending on the direction of observation.

Let us consider now the new frequency thresholds $\bar{\omega}_C$, which are given by the condition $\bar{Q}(\bar{\omega}_C) = 0$, for each polarization mode, according to Sec. III. We omit the index ν in the following when no confusion arises. Since we are dealing with small corrections we set $\bar{\omega}_C = \omega_C + \Delta$, searching for Δ to lowest order.

An important observation is required at this point. Demanding that $\bar{Q}_\nu$ is real enforces the previous conditions for radiation $\omega < \omega_{C-}$ and $\omega > \omega_{C+}$, otherwise Q_ν would be imaginary. Then we expect that possible modifications to the frequency threshold could arise only in the ranges $\bar{\omega}_{C-} < \omega_{C-}$ and $\bar{\omega}_{C+} > \omega_{C+}$. To investigate these possibilities we need to solve

$$0 = \bar{Q}(\omega_C + \Delta) = Q(\omega_C + \Delta) - \delta \frac{k}{b} \nu \frac{\cos \phi}{\sqrt{1 + 4 \frac{n^2\omega_C^2}{b^2v^2}}}, \quad (77)$$

where we evaluate $\omega = \omega_C$ in the last term since it is already proportional to δ . Also we recall that $Q(\omega_C) = 0$ provide the unperturbed thresholds. Being a square root,

$\mathcal{Q}(\omega)$ has an algebraic branch point at $\omega = \omega_C$, meaning that its expansion around this point will be in powers of $\Delta^{1/2}$ instead of Δ . Then, up to factors in the expansion, we obtain $\Delta \sim \delta^2$, which is beyond the linear approximation we have considered. In other words, we find no corrections to the frequency thresholds at this stage.

IX. COMPARISON WITH THE APPROXIMATION IN REF. [30]

To facilitate this comparison we collect the required expressions from Ref. [30] in Appendix F.

A. The Cherenkov angles

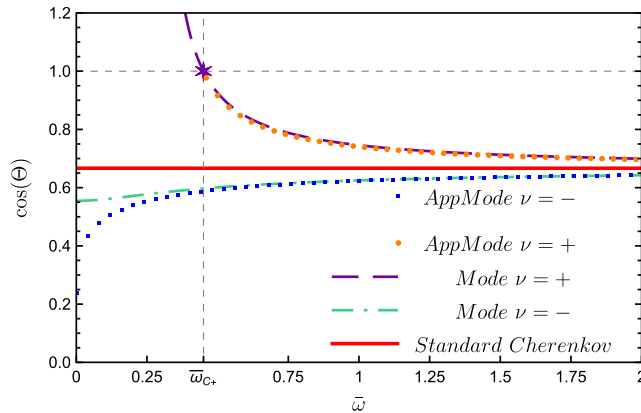
The starting point is the equation

$$\cos \theta \left[\sin^2 \theta + \cos^2 \theta \sqrt{1 - \nu \xi \sec \theta} \right] = \frac{1}{n\beta}, \quad (78)$$

where the dimensionless frequency-dependent chirality parameter

$$\xi = \frac{cb}{\omega n^2} \equiv \frac{1}{\bar{\omega} n^2}, \quad (79)$$

is defined. Equation (78) determines the Cherenkov angle for each polarization ν in the approximation of Ref. [30]. We compare the expressions resulting from this equation with our exact result (35) for the Cherenkov angle in two cases: (i) analytically in the limit $\xi \ll 1$, i.e. $b \ll \omega n^2/c < \omega n^2/v$, and (ii) numerically in Fig. 8 for $\cos \Theta_\nu$ as a function of the frequency. The choice of parameters is $n = 2$ and $b = 2.6 \times 10^8 \text{ m}^{-1}$, and the left panel is for $\beta = 0.75$ (high velocities) while the right panel corresponds to $\beta = 0.2$ (low velocities).



(i) Let us begin with the small b expansion. To first order in b we have, from Eq. (F2),

$$\tilde{C}_\nu(\Theta) = 1 - \nu \frac{cb}{2\omega n^2} \cos \Theta. \quad (80)$$

Substituting in (78) and solving for $\cos \Theta$ yields

$$\cos \Theta = \nu \frac{\omega n^2}{cb} \left(1 + \lambda \sqrt{1 - 2\nu \frac{cb}{\omega n^3 \beta}} \right), \quad (81)$$

with $\lambda = \pm 1$ labeling the two solutions of the quadratic equation. After a further expansion in b we obtain

$$\cos \Theta = \frac{\omega n^2 (1 + \lambda)}{\nu cb} - \lambda \frac{c}{nv} - \lambda \nu \frac{c^3 b}{2n^4 v^2 \omega}. \quad (82)$$

Since we must recover the finite value of the standard Cherenkov angle in the limit $b = 0$ we have to choose the solution for $\lambda = -1$ to cancel the divergent term going like $1/b$. This leaves us with

$$\cos \Theta = \frac{1}{n\beta} + \nu \frac{1}{2n^4 \beta^2 \bar{\omega}}. \quad (83)$$

Now let us consider Eq. (35) which gives the exact expression for $\cos \Theta$. Here small b means $b \ll n\omega/v < n^2\omega/v$, which follows within the same range as the expansion in Eq. (80). Performing the expansion to first order in b we find exactly the result (83). From Eq. (80), the above expansion can be viewed as a high-frequency approximation, which works really well.

(ii) The numerical solution of Eq. (78) for $\cos \Theta_\nu$, which defines the AppModes, allows us to explore the whole frequency spectrum with the results shown in Fig. 8 for each polarization. The parameters are $n = 2$, $b = 2.6 \times 10^8 \text{ m}^{-1}$

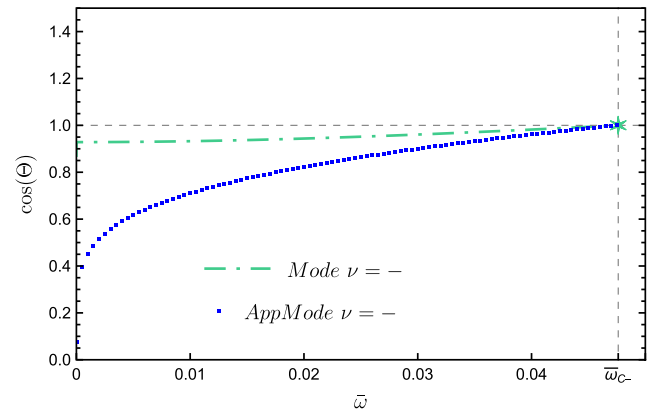


FIG. 8. Cosine of the Cherenkov angle for the $\nu = \pm$ modes as a function of the dimensionless parameter $\bar{\omega}$. The dot-dashed green line and the long-dashed purple line correspond to the exact analytical expressions (35) for the $\nu = -$ and $\nu = +$ modes, respectively. The solid red line represents the standard Cherenkov result. The blue squares and the orange circles correspond to the numerical solutions of Eq. (78) for the cosine of the Cherenkov angle associated with the $\nu = -$ and $\nu = +$ modes, respectively. The parameters are $n = 2$ and $b = 2.6 \times 10^8 \text{ m}^{-1}$. Left panel: $\beta = 0.75$. Right panel: $\beta = 0.2$.

and the standard Cherenkov result $1/(n\beta)$ is added for comparison. The left panel is for $\beta = 0.75$ and shows the two polarization modes: $\nu = +$ (exact result: purple long-dashed line; approximate result: orange circles) and $\nu = -$ (exact result: dot-dashed green line; approximate result: blue squares). The line conventions in the right panel are the same as in the left panel. However, the right panel is for the low velocity case with $n\beta = 0.4$, such that the standard Cherenkov is forbidden ($\cos \Theta = 2.5$) and only the mode $\nu = -$ radiates provided a small frequency window opens, characterized by an upper threshold $\bar{\omega}_{0-}$, shown in the figure for the exact case. The approximate and exact results exhibit approximately equal frequency windows, although significant differences in the Cherenkov angle arise at low frequencies.

B. The spectral energy distribution

In an analogous way to the previous section, the comparison between the exact SED $\mathcal{E}_\nu$ and the approximate SED $\mathcal{E}_{A,\nu}$ as a function of frequency proceeds in two steps. (i) We consider a linear expansion in b and (ii) subsequently we deal with an exact numerical evaluation of both quantities. In both cases, our starting point is the exact expression (57) for $\mathcal{E}$, together with Eq. (F4) for the approximate SED $\mathcal{E}_A$. In the latter case we make use of the auxiliary functions (F5), (F6), (F7), and (F8).

(i) In this case we choose to quantify the error in the approximation defining

$$\Delta_\nu = \left| \frac{\mathcal{E}_\nu - \mathcal{E}_{A,\nu}}{\mathcal{E}_\nu} \right|. \quad (84)$$

The linear expansion in b follows directly from the corresponding expressions and we use the explicit result (83) for the solution of the angle Θ_ν in the linear approximation. We obtain

$$\mathcal{E}_\nu = \frac{q^2 \omega}{2c^2} \left(1 - \frac{1}{n^2 \beta^2} \right) - \nu \frac{q^2 b}{4v} \frac{(1 + n^2 \beta^2)}{n^3}, \quad (85)$$

$$\mathcal{E}_{A,\nu} = \frac{q^2 \omega}{2c^2} \left(1 - \frac{1}{n^2 \beta^2} \right) - \nu \frac{q^2 b}{4v} \frac{(3 - n^2 \beta^2)}{\beta^2 n^5}, \quad (86)$$

which yields

$$\Delta_\nu = \left(\frac{bc}{\omega} \right) \frac{1}{n^2} \frac{(3 + n^2 \beta^2)}{2n\beta}. \quad (87)$$

Given a specific maximum allowed error Δ_0 , we can trust the linear approximation for values of the dimensionless parameter bc/ω such that

$$\frac{bc}{\omega} < n^2 \Delta_0 \frac{2n\beta}{3 + n^2 \beta^2} < \frac{n^2}{\sqrt{3}} \Delta_0. \quad (88)$$

since the maximum value of the function $2x/(3 + x^2)$ is $1/\sqrt{3}$ for $x = \sqrt{3}$.

(ii) The numerical calculation extends the range of frequencies we can probe. However, a complication arises since, unlike $\mathcal{E}$ in Eq. (57), the approximate $\mathcal{E}_A$ must be evaluated at a specific Cherenkov angle, which is implicitly defined by Eq. (78) for a given frequency, velocity, and refraction index. Then, this requires the intermediate step of calculating this angle Θ_ν from Eq. (78) and subsequently substituting its value in the expression for $\mathcal{E}_{A,\nu}$.

In this case, we choose to present the results in terms of suitable dimensionless ratios. As a first reference, we normalize the spectral energy density with respect to the standard Cherenkov result,

$$\mathcal{R}_\nu \equiv \frac{\mathcal{E}_\nu}{\mathcal{E}_{Ch}}, \quad \mathcal{R}_{A,\nu} \equiv \frac{\mathcal{E}_{A,\nu}}{\mathcal{E}_{Ch}}, \quad (89)$$

where $\mathcal{E}_\nu$ denotes the exact spectral energy distribution derived in this paper, and

$$\mathcal{E}_{Ch} = \frac{q^2 \omega}{c^2} \left(1 - \frac{1}{n^2 \beta^2} \right) \quad (90)$$

is the standard SED for Cherenkov radiation of a charge q moving with velocity v in a medium with refraction index n . In addition, in order to assess the accuracy of the approximate expression $\mathcal{E}_{A,\nu}$, we also consider the ratio between the approximate and the exact SEDs defined as

$$\mathcal{S}_\nu \equiv \frac{\mathcal{E}_{A,\nu}}{\mathcal{E}_\nu}. \quad (91)$$

We first focus on the high-velocity regime $\beta > 1/n$, where radiation is allowed over an unbounded frequency range. The numerical results in this regime are summarized in Figs. 9 and 10. In Fig. 9, we display the ratios $\mathcal{R}_\nu$ and $\mathcal{R}_{A,\nu}$ as a function of the frequency. This figure illustrates the overall behavior of the spectrum for both polarization modes, including the presence of the cutoff ω_{C+} for the $\nu = +$ mode and the tendency of all curves to approach the same asymptotic value at high frequencies. Complementarily, Fig. 10 shows the ratio $\mathcal{S}_\nu = \mathcal{E}_{A,\nu}/\mathcal{E}_\nu$, allowing us to assess the accuracy of the approximate SED with respect to the exact result derived in this work.

As shown in Figs. 9 and 10, the quality of the approximation becomes apparent when comparing the different panels corresponding to distinct charge velocities. In particular, the agreement between the approximate and exact results improves for increasing frequency and as the velocity approaches the standard Cherenkov threshold $\beta = 1/n$. Together, these figures provide a global characterization of the spectrum in the high-velocity sector and motivate the analytical analysis of the extreme-frequency limits, which we discuss below.

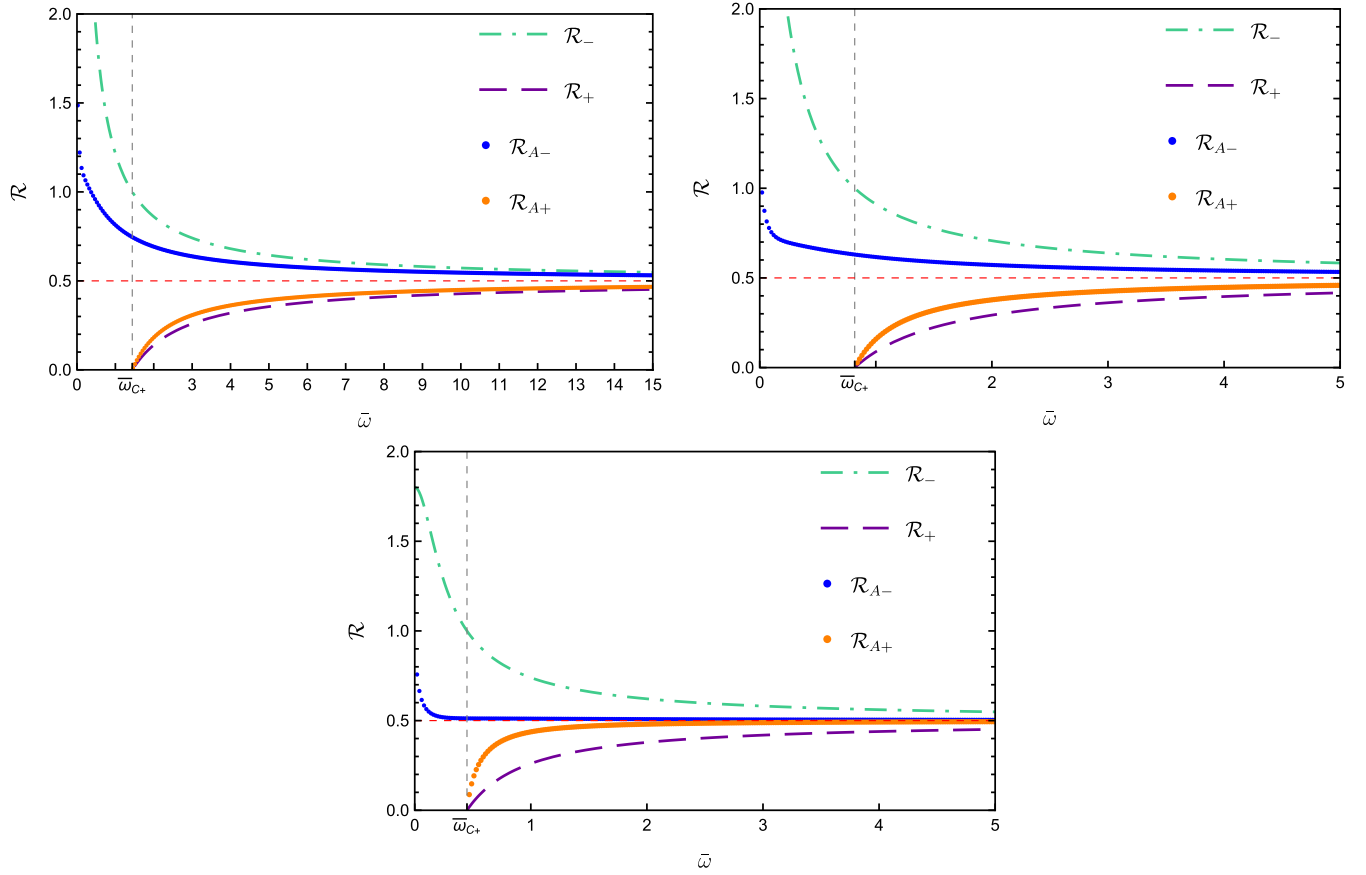


FIG. 9. The parameters are $n = 2$ and $b = 2.6 \times 10^8 \text{ m}^{-1}$. The ratios with respect to the usual Cherenkov SED, $\mathcal{R}_\nu = \mathcal{E}_\nu / \mathcal{E}_{\text{Ch}}$ and $\mathcal{R}_{A\nu} = \mathcal{E}_{A,\nu} / \mathcal{E}_{\text{Ch}}$, for the exact and approximate results, respectively, are shown as functions of the dimensionless frequency $\bar{\omega} = \omega / (cb)$ for $\beta = 0.55$ (left), $\beta = 0.60$ (right), and $\beta = 0.75$ (bottom). As the charge velocity approaches the Cherenkov threshold velocity $\beta = 1/n$, a better agreement between the approximate and exact results is observed.

We now turn to this analysis. Even though here we are dealing with a numerical calculation, we can take an analytical approach in the extreme cases where $\omega \rightarrow \infty$ and $\omega \rightarrow 0$. In this whole range, the $\nu = -$ mode radiates from $\omega = 0$ up to arbitrarily large frequencies, while the $\nu = +$ mode is only allowed above the cutoff ω_{C+} and extends also to arbitrarily large frequencies. Consequently, the high-frequency limit $\omega \rightarrow \infty$ applies to both polarizations.

In this case we take $\xi \rightarrow 0$, and the required expressions in Appendix F can be readily evaluated, mainly because the implicit condition $\cos \theta \tilde{C}_\nu(\theta) = 1/(n\beta)$, determining each Cherenkov angle Θ_ν , can be solved straightforwardly for both polarizations. This yields $\mathcal{R}_\nu = 1/2$ in the high frequency limit, as shown by the plots in Fig. 9.

In contrast, the low-frequency limit $\omega \rightarrow 0$ is only meaningful for the $\nu = -$ mode and it is analyzed next. A naive way to implement this limit is to take $\xi \rightarrow \infty$. However, we show that this option yields unphysical results. With this choice, we expect to find an approximate solution for the implicit Eq. (78) by restricting ourselves to

the polarization $\nu = -$, with Cherenkov angle denoted by Θ . In fact, for very large ξ , we expect

$$\cos^{5/2} \Theta \xi^{1/2} = \frac{1}{n\beta}, \quad \cos \Theta = \left(\frac{1}{n\beta \xi^{1/2}} \right)^{2/5} \quad (92)$$

to be a good approximate solution of Eq. (78). The correctness of this approximation hinges on the possibility that we can neglect $\sin^2 \Theta$ in front of $\cos^{3/2} \Theta \xi^{1/2}$ in Eq. (78). We examine this situation introducing the new variables $C = (n\beta)^{1/5}$ and $Y = \xi^{1/5}$, in terms of which we write

$$\sin^2 \Theta = C^4 - \frac{1}{Y^2}, \quad \cos^{3/2} \Theta \xi^{1/2} = CY, \quad (93)$$

after using Eq. (92). The point Y_0 at which both competing quantities are equal is given by the equation

$$C^4 - \frac{1}{Y_0^2} = CY_0. \quad (94)$$

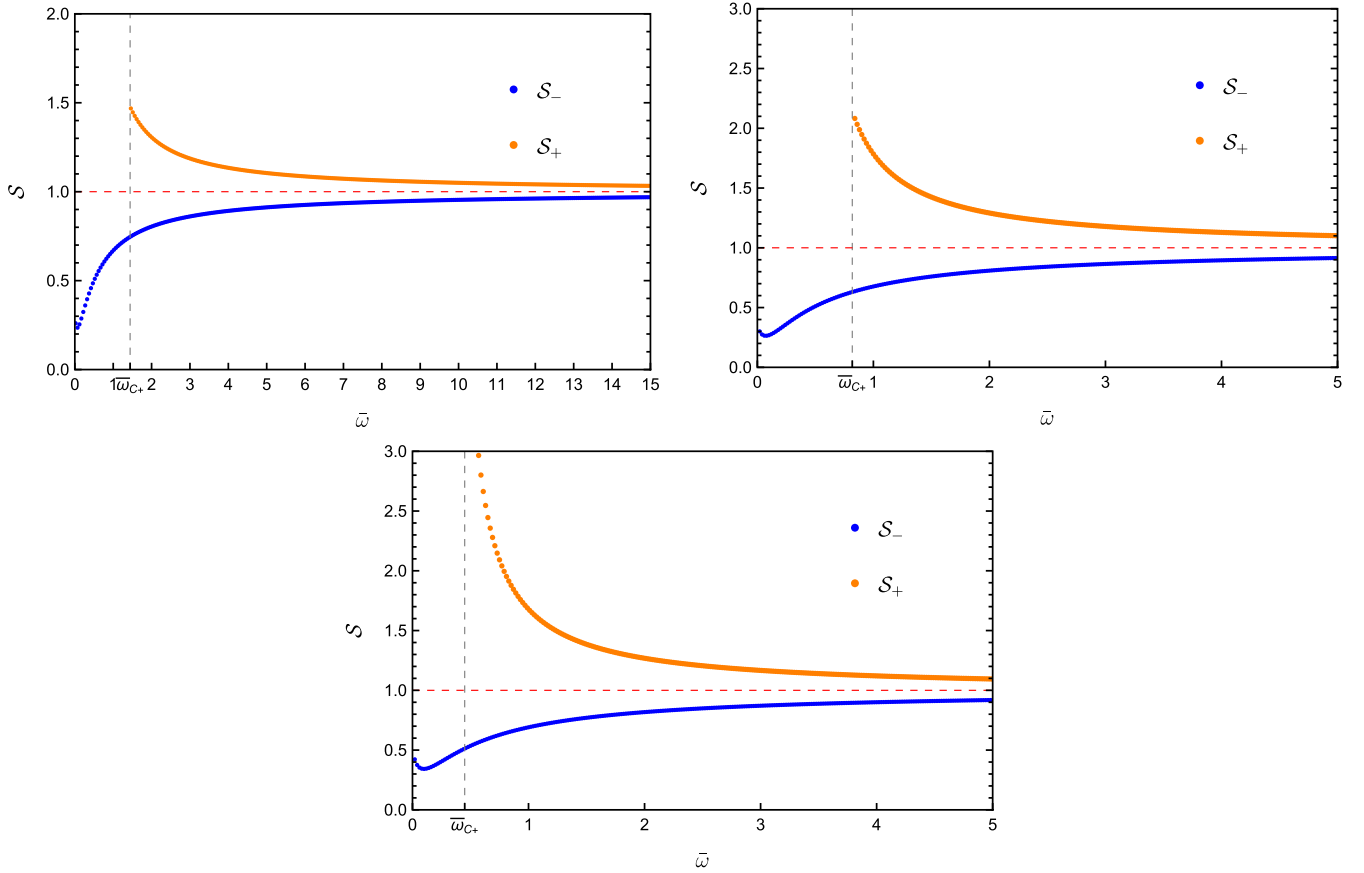


FIG. 10. The parameters are $n = 2$ and $b = 2.6 \times 10^8 \text{ m}^{-1}$. We show the ratio between the approximate and exact spectral energy distributions, $S_v = \mathcal{E}_{A,v}/\mathcal{E}_v$, as a function of the dimensionless frequency $\bar{\omega}$, for $\beta = 0.55$ (left), $\beta = 0.60$ (right), and $\beta = 0.75$ (bottom). As the frequency increases, the approximate and exact results converge, with $S_v \rightarrow 1$. In addition, lower velocities yield a smaller overall deviation between both results, although deviations are still present at low frequencies.

Since we are looking for large Y we can solve the above equation neglecting the rapidly decaying term $1/Y^2$ yielding

$$Y_0 = C^3. \quad (95)$$

We verify the following properties:

$$Y < Y_0, \quad \sin^2 \Theta > \cos^{3/2} \Theta \xi^{1/2}, \quad (96)$$

$$Y > Y_0, \quad \sin^2 \Theta < \cos^{3/2} \Theta \xi^{1/2}. \quad (97)$$

Let us now assume that we are in the region $Y > Y_0$ where the limit $\xi \rightarrow \infty$ is valid and explore its consequences. Without going into the detailed approximations performed in the related functions in Appendix F we obtain

$$\begin{aligned} \mathcal{R}_{A-} = & \frac{n^2 \beta^2}{(n^2 \beta^2 - 1)} \frac{\sin^2 \Theta}{(2 + \sin^2 \Theta)} \frac{\xi^{1/2}}{\cos^{1/2} \Theta} \\ & \times \left[\frac{\sin^2 \Theta - \frac{1}{2} \cos^{3/2} \Theta \xi^{1/2}}{\sin^2 \Theta + \frac{5}{2} \cos^{3/2} \Theta \xi^{1/2}} \right]. \end{aligned} \quad (98)$$

This equation can be simplified by considering the following properties: (i) For very large ξ , the Cherenkov angle Θ approaches $\pi/2$, allowing us to approximate $\sin \Theta \approx 1$. (ii) Using $\sin^2 \Theta$ from the approximate Eq. (78) and the expression for $\cos \Theta$ in Eq. (92), we obtain $\cos^{3/2} \Theta \xi^{1/2} = \xi^{1/5} / (n\beta)^{3/5} \gg 1$. The final result is

$$\mathcal{R}_{A-} = -\frac{1}{15} \frac{n^2 \beta^2}{(n^2 \beta^2 - 1)} (n\beta)^{1/5} \xi^{3/5}. \quad (99)$$

This expression has two drawbacks: it is negative and diverges as $\xi \rightarrow \infty$. This means we cannot trust this approximation for very large ξ 's (i.e., very small frequencies). This pathological behavior is also evident in the numerical results. In Fig. 11, we show a magnified view of the low-frequency region of the ratio $\mathcal{R}_{A-}$, which makes explicit the divergence and sign change of the approximate result as $\omega \rightarrow 0$.

These problematic features arise from the additional approximations introduced in Ref. [30] to obtain an analytical solution of the stationary phase equation. In particular, the condition $b \ll k_{\parallel} < k_0$ implies $\omega \gg bc$.

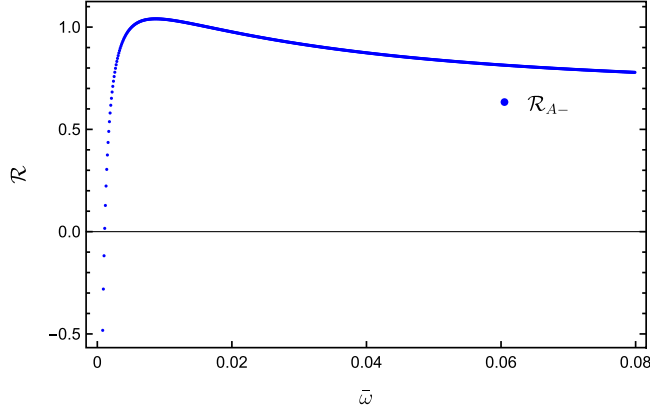


FIG. 11. The parameters are $n = 2$, $b = 2.6 \times 10^8 \text{ m}^{-1}$, and $\beta = 0.6$. We show a magnified view of the low-frequency region of the ratio between the approximate spectral energy distribution of the $\nu = -$ mode and the usual Cherenkov SED, $\mathcal{R}_{A-} = \mathcal{E}_{A,-}/\mathcal{E}_{\text{Ch}}$, as a function of the dimensionless frequency $\bar{\omega}$. This representation highlights the behavior of the approximate result in the limit $\omega \rightarrow 0$.

In contrast, threshold-free Cherenkov radiation emerges in the frequency interval $0 < \omega < \omega_{C-}$. For small velocities, $\omega_{C-} \simeq bc\beta^2 \ll bc$, showing that this phenomenon occurs in the opposite regime. Therefore, the low-frequency discrepancies discussed above should be interpreted as a consequence of applying the high-frequency approximation outside its domain of validity rather than as a fundamental limitation of the stationary phase approximation itself.

For comparison we write the exact ratio when $\omega \rightarrow 0$. From Eq. (57) we obtain

$$\mathcal{R}_{-} = \frac{n^2 \beta^2}{n^2 \beta^2 - 1}, \quad (100)$$

which is perfectly finite and positive.

Since the approximation is unreliable for low frequency values, we estimate a lower limit above which it becomes more trustworthy. To this end we forbid the unphysical

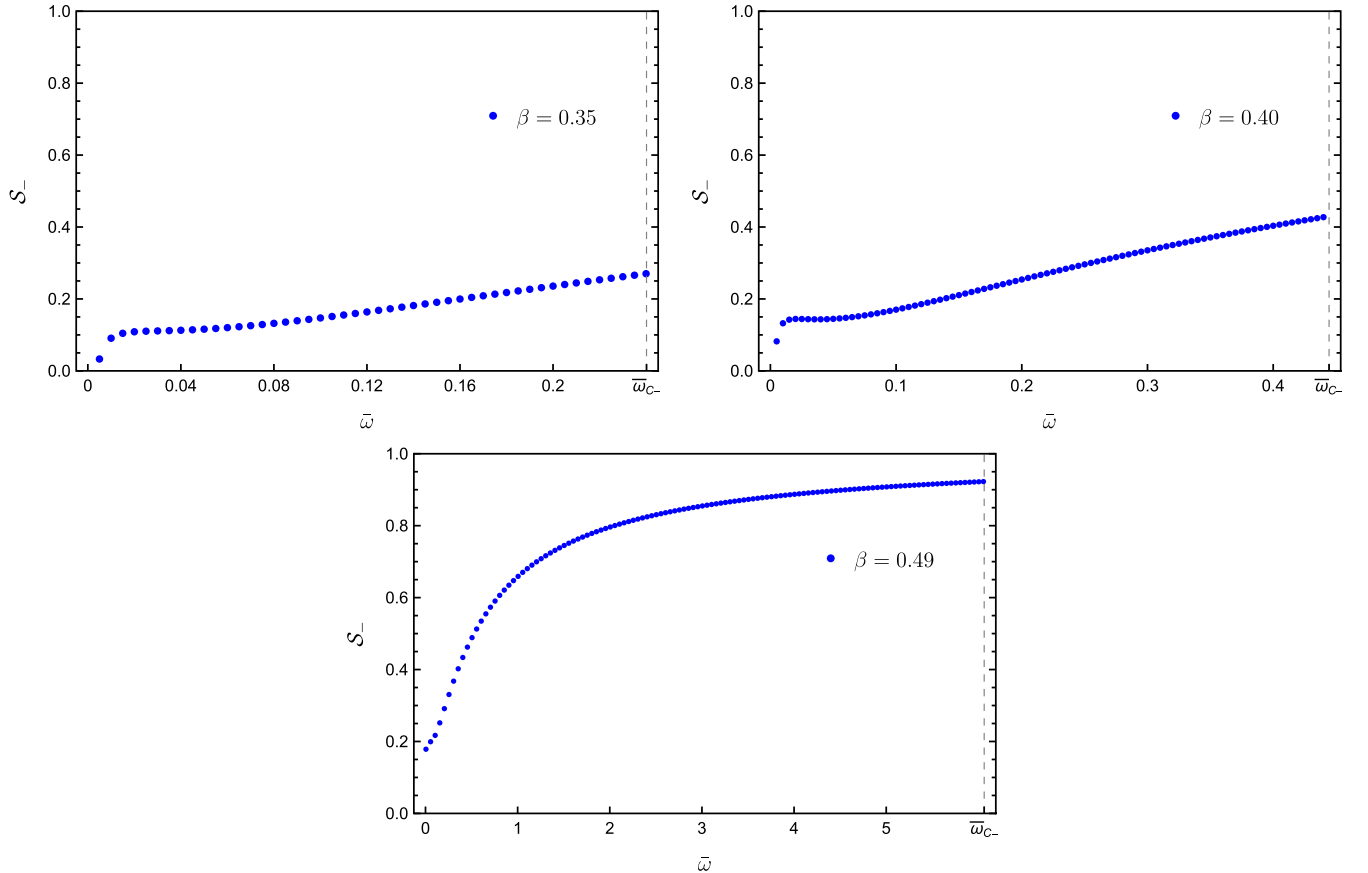


FIG. 12. The parameters are $n = 2$ and $b = 2.6 \times 10^8 \text{ m}^{-1}$. We show the ratio between the approximate and exact spectral energy distributions for the $\nu = -$ mode, $\mathcal{S}_{-} = \mathcal{E}_{A,-}/\mathcal{E}_{-}$, as a function of the dimensionless frequency $\bar{\omega}$, for $\beta = 0.35$ (left), $\beta = 0.40$ (right), and $\beta = 0.49$ (bottom). As the charge velocity increases and approaches the Cherenkov threshold $\beta = 1/n$, the agreement between the approximate and exact results systematically improves.

region determined by Eq. (97) by demanding, for example, $Y < Y_0/2$, in order to be inside the allowed region but far away from the equality point. Here the approximate solution found so far in Eq. (92) is not valid any more, and the chosen inequality sets a lower limit $\bar{\omega}_0$ above which we can start trusting the results of Ref. [30]. We find

$$\bar{\omega}_0 \equiv \frac{32}{n^2(n\beta)^3} < \bar{\omega}. \quad (101)$$

For the cases in Fig. (9) with $n = 2$ and $\beta = 0.55, 0.60, 0.75$. We find $\bar{\omega}_0 = 6.01, 4.63, 2.37$, respectively.

Considering the complementary scenario, we examine now the low-velocity regime $\beta < 1/n$, where radiation is emitted solely in the $\nu = -$ polarization, while the $\nu = +$ mode is forbidden. Moreover, since no standard Cherenkov radiation exists below the threshold $\beta = 1/n$, a comparison with the usual Cherenkov spectral energy distribution is no longer meaningful in this sector. For this reason, we restrict the analysis to the ratio between the approximate and exact spectral energy distributions for the $\nu = -$ mode $\mathcal{S}_-$ defined in Eq. (91). The numerical results are displayed in Fig. 12, where $\mathcal{S}_-$ is plotted as a function of the dimensionless frequency for representative values of the charge velocity. As the frequency increases, the ratio approaches unity, indicating a convergence between the approximate and exact results. In addition, the agreement systematically improves as the charge velocity increases and approaches the Cherenkov threshold $\beta = 1/n$ from below, consistently with the behavior observed in the different panels of the figure.

X. SUMMARY AND CONCLUSIONS

We study Cherenkov radiation (CHR) emitted by a charge q moving with constant velocity $\mathbf{v} = v\hat{\mathbf{z}}$ through anisotropic chiral matter characterized by an electric permittivity $\epsilon \geq 1$ and magnetic permeability $\mu = 1$. The electromagnetic response of the system is governed by a restricted version of axion electrodynamics, with an axion angle given by $\theta(x) = -\mathbf{b} \cdot \mathbf{x}$, where $\mathbf{b} = b\hat{\mathbf{z}}$ is aligned with the direction of motion of the charge. In this modified electrodynamics, the Poynting vector does not acquire additional contributions with respect to the standard case, in contrast to the isotropic situation ($b_0 \neq 0$, $\mathbf{b} = 0$) [32,52]. As a consequence, there is no ambiguity regarding either the positive definiteness or the gauge invariance of the radiated energy.

We start by solving the modified Maxwell equations exactly in cylindrical coordinates and in the space-frequency domain, obtaining closed analytical expressions for the electric and magnetic fields in terms of modified Bessel functions. The integration constants are fixed by solving Gauss's law and Ampere's law together with the boundary conditions at $\rho \rightarrow 0$. From these solutions, we derive the associated dispersion relation, which reveals the existence

of two distinct polarization modes, labeled by $\nu = \pm$. In a specific example we show that the polarization is elliptical, which we take as the general feature of our setup.

From the exact solutions we analyze the conditions under which they describe physical radiation. These involve the quantity Q_ν , which controls the radial behavior of the electromagnetic fields for each polarization mode. In the asymptotic region $\rho \rightarrow \infty$, the fields become proportional to $\exp(-Q_\nu \rho)$. In particular, radiation is possible only when the quantity Q_ν is purely imaginary, which we write as $Q_\nu = -iQ_\nu$, with Q_ν real. Under this condition, the fields correspond to outgoing oscillatory cylindrical waves, which transport energy to infinity and therefore describe physical radiation. If Q_ν is imaginary, the fields are evanescent and no radiation is produced.

Requiring Q_ν to be real leads to explicit frequency restrictions for each polarization mode. While in standard Cherenkov radiation emission is only possible for charge velocities above the threshold velocity $\beta_{TH} = 1/n$, threshold-free radiation has been recently discovered in chiral matter [32,52] and is also present in this case. This motivates us to analyze both the conventional high-velocity regime as well as the existence of radiation at low charge velocities.

In the high-velocity regime, $\beta > 1/n$, the quantity Q_- is real for all frequencies, and radiation is therefore always allowed in the $\nu = -$ mode. In contrast, Q_+ becomes real only above a cutoff frequency ω_{C+} , implying that the $\nu = +$ mode radiates only for $\omega > \omega_{C+}$. In the low-velocity regime, $\beta < 1/n$, the quantity Q_+ remains imaginary for all frequencies, and the corresponding polarization is completely suppressed. Nevertheless, radiation is still allowed in the $\nu = -$ mode within a finite low-frequency window

$$0 < \omega < \omega_{C-}, \quad (102)$$

where Q_- is real, giving rise to the novel feature of threshold-free CHR. Outside this interval, Q_- becomes imaginary and radiation is forbidden.

The analysis of the Cherenkov angles associated with each polarization mode showed that in the present configuration the quantity $\cos \Theta_\nu$ is always positive, so the existence of a Cherenkov angle is determined solely by the condition $\cos \Theta_\nu \leq 1$. This requirement leads exactly to the same restrictions obtained from demanding that Q_ν be real, thereby establishing a direct equivalence between the geometrical criterion based on the Cherenkov angle and the condition for radiation derived from the asymptotic behavior of the electromagnetic fields.

In the high-velocity regime the Cherenkov angle associated with the $\nu = -$ mode exists for all frequencies, while the $\nu = +$ mode develops a Cherenkov angle only above the cutoff frequency ω_{C+} . As a consequence, for $\omega > \omega_{C+}$ both polarization modes radiate and two distinct Cherenkov cones are formed, whereas for $0 < \omega < \omega_{C+}$ only a single

Cherenkov cone associated with the $\nu = -$ mode is present. In contrast, in the low-velocity regime the $\nu = +$ mode is completely suppressed, and a single Cherenkov cone associated with the $\nu = -$ mode exists only within the finite low-frequency window $0 < \omega < \omega_{C-}$ which is a consistent manifestation of threshold free CHR.

We also examine the radiation conditions from the perspective of the phase velocity associated with each polarization mode. We show that the phase velocity of the $\nu = -$ mode is a monotonically increasing function of the frequency, while the phase velocity of the $\nu = +$ mode decreases monotonically. In this way, the determination of their asymptotic limits allows for a clear physical interpretation of the radiation criteria, recalling that for radiation to occur the phase velocity of the wave must be smaller than the charge velocity. In the high-velocity regime the phase velocity of the $\nu = -$ mode remains smaller than the charge velocity for all frequencies, ensuring radiation in this sector. In contrast, the phase velocity of the $\nu = +$ mode becomes smaller than the charge velocity only above a cutoff frequency ω_{C+} , so that this mode contributes to radiation exclusively for $\omega > \omega_{C+}$. In the low-velocity regime the phase velocity of the $\nu = +$ mode always exceeds the charge velocity and therefore forbids radiation, whereas the $\nu = -$ mode allows radiation only within a finite low-frequency interval $0 < \omega < \omega_{C-}$. The phase-velocity analysis therefore provides an independent and physically transparent confirmation of the radiation criteria derived throughout this work. These results agree with the conditions obtained from the reality of Q_ν together with the existence of the Cherenkov angle, thus providing a definitive proof of threshold-free CHR.

Next we turn to the determination of the spectral energy distribution $\mathcal{E}_\nu$ for each Cherenkov cone and polarization. The calculation is carried out using the standard Poynting vector, since, as discussed previously, this is not modified in the present anisotropic chiral configuration, in contrast with the isotropic case. We show that all crossed terms involving products of electromagnetic fields belonging to different polarization modes cancel identically, so that the total SED decomposes into an independent sum of the contributions from each mode. As a result, we obtain closed analytical expressions for the SED associated with each polarization. A notable feature of these expressions is that they can be written in a manifestly positive-definite form when expressed in terms of the corresponding Cherenkov angle, ensuring the physical consistency of the result. When both polarization modes are allowed, their individual SEDs add up to reproduce exactly the standard Cherenkov spectrum, even though each mode radiates at a different Cherenkov angle. This additive behavior is not unique to the anisotropic configuration considered here, but it is also present in the isotropic case, where the total radiation also arises from independent polarization contributions [32,52].

The numerical estimations in Sec. VII corroborate the analytical radiation conditions, confirming the existence of the predicted frequency and velocity regimes for Cherenkov emission. These results are presented in Figs. 1–4. An extreme case of threshold-free CHR is the so-called “vacuum CHR,” corresponding to the limit $n \rightarrow 1$ and highlighted in Figs. 6 and 7.

Next, we perform a detailed comparison between our exact results and those obtained from the approximation introduced in Ref. [30]. Concerning the Cherenkov angle, we show that a linear expansion in the chiral parameter b of the approximation reproduces exactly the corresponding expansion of the exact result in the high-frequency regime. This agreement confirms that the approximate treatment in Ref. [30] correctly captures the angular properties of the radiation in the limit $b \ll \omega n^2/c$, as shown in Fig. 8.

However, regarding the SED, the comparison reveals a more nuanced picture. In the high velocity regime we plot the relative values of the SEDs $\mathcal{R}_\nu$ and $\mathcal{R}_{A\nu}$, defined with respect to the standard Cherenkov emission (Fig. 9), as well as the corresponding ratios $\mathcal{S}_\nu$, defined with respect to the exact result for each mode (Fig. 10). While the approximate and exact ratios converge at high frequencies, and their agreement improves as the particle velocity approaches the Cherenkov threshold $\beta = 1/n$, important deviations arise at low frequencies. Focusing now on this limit we realize that only the $\nu = -$ mode is physically meaningful since the $\nu = +$ mode radiates exclusively above the cutoff frequency ω_{C+} . In this sector, we show that the approximate ratio $\mathcal{R}_{A-}$ develops an unphysical behavior, becoming divergent and negative as shown in Fig. 11, whereas the exact result $\mathcal{R}_-$ remains finite and positive. This discrepancy signals a clear breakdown of the approximation in the low-frequency domain of the $\nu = -$ mode. In the low velocity regime the standard Cherenkov radiation is forbidden so that the only meaningful comparison for the $\nu = -$ mode is via the ratio $\mathcal{S}_-$, plotted in Fig. 12 and showing again a substantial discrepancy with the exact result in the low frequency sector.

A further analysis allows us to identify a lower frequency bound $\tilde{\omega}_0$ in Eq. (101), above which the approximation becomes more reliable, by excluding the region where the naive assumptions to solve required equations in the case $\omega \rightarrow 0$ cease to hold. The numerical results confirm that, both above and below the Cherenkov threshold, the accuracy of the approximation improves systematically as the charge velocity approaches $\beta = 1/n$, while deviations persist at sufficiently low frequencies.

Overall, this comparison delineates the precise domain of validity of the approximation of Ref. [30]: although it correctly reproduces the Cherenkov angles and captures the high-frequency behavior of the spectral energy distribution, it fails in the low-frequency sector of the radiating $\nu = -$ mode, becoming more reliable for frequencies larger than $\tilde{\omega}_0$.

Evidently, the exact solution for Cherenkov radiation in anisotropic chiral matter presented here surpasses the approximation in Ref. [30] in terms of accuracy and reliability. However, the primary focus of this reference resulted in deriving an approximate Green's function for the general problem of anisotropic chiral matter, which can be applied to arbitrary sources where an exact solution is unattainable. Thus, the comparison of the exact and approximate solution in the case of Cherenkov radiation must be taken only as an indication of the range of parameters where we can trust the approximate Green's function. Let us emphasize that the discrepancies detected in Ref. [30] should not be interpreted as a fundamental limitation of the stationary phase approximation in the presence of chiral-induced dispersion, but rather as a consequence of the validity regime of the approximation employed there to analytically solve the stationary phase equation.

We close with some general remarks regarding this work. An important outcome arises in the threshold-free emission sector. While the obtained photon extraction efficiencies remain comparable to those found previously in isotropic chiral matter, the corresponding frequency windows become substantially broader, shifting from the meV range to the eV range for representative parameters. This behavior extends the experimentally accessible spectral range of low-velocity, threshold-free Cherenkov radiation, bringing it within reach of modern optical detectors [9,57]. At the same time, these estimates should be interpreted within the effective description developed in this work. In particular, we assume a constant refractive index n , which is a useful simplification to isolate the role of the chiral response. In realistic Weyl semimetals, the refractive index is frequency dependent, $n = n(\omega)$, which may quantitatively modify the frequency windows obtained here. Furthermore, an experimental realization in a Weyl semimetal would require combining a realistic Weyl semimetal platform with controlled charged-particle propagation in the geometry considered here, which represents an additional experimental challenge. For these reasons, the photon extraction estimate presented here should be understood as an indicator of detectability within the effective model rather than as a quantitative prediction for a specific material.

A further consequence is that the anisotropic configuration leads to qualitatively distinct physical effects compared to isotropic chiral matter. While both systems exhibit threshold-free Cherenkov radiation, the anisotropic configuration modifies the radiative kinematics in a nontrivial way. In particular, the phase velocity acquires an explicit dependence on the charge velocity, and the frequency windows determining the existence or suppression of radiation become substantially modified. In addition, while isotropic chiral matter naturally leads to circular

polarization modes, the anisotropic configuration gives rise to elliptically polarized radiation.

A preliminary discussion of the case where $\mathbf{b}$ is not parallel to $\mathbf{v}$ is included by considering $\mathbf{b} = (\delta, 0, b)$ with $\delta \ll b$. We solve the dispersion relation to first order in δ/b finding that axial symmetry is explicitly broken yielding a dependence of the observables on the axial angle of the emitted momentum. No corrections on the threshold frequencies are found at linear order in δ/b .

ACKNOWLEDGMENTS

R. M. v. D. and L. F. U. have been partially supported by DGAPA-UNAM Project No. AG100224. L. F. U. also acknowledges support from project SECIHTI (México) CBF-2025-I-1862. R. M. v. D. was supported by UNAM Posdoctoral Program (POSDOC).

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: THE COUPLED MAXWELL'S EQUATIONS FOR THE ELECTRIC FIELD

We start from Eqs. (2) and (3) outside the sources ($\bar{\rho} = 0$ and $\mathbf{J} = 0$) and we work in the space-frequency domain. Taking the curl of the equation for Faraday's law and substituting $\nabla \times \mathcal{B}$ through Ampere's law we obtain

$$\nabla \times (\nabla \times \mathcal{E}) = \frac{n^2}{c^2} \omega^2 \mathcal{E} + \frac{i\omega}{c} \mathbf{b} \times \mathcal{E}. \quad (\text{A1})$$

Recalling the splitting $\mathcal{E}(\rho, z, \omega) = \mathbf{E}(\rho, \omega) e^{ikz}$ we now focus on the cylindrical components of $\mathbf{E}(\rho, \omega)$ choosing the z axis in the direction of $\mathbf{b}$ such that $\mathbf{b} = b\hat{z}$. From Eq. (A1) we find

$$\partial_\rho E_z = \frac{i}{k} \left(k^2 - \frac{n^2 \omega^2}{c^2} \right) E_\rho - \frac{b\omega}{ck} E_\phi, \quad (\text{A2})$$

$$\partial_\rho \left(\frac{1}{\rho} \partial_\rho (\rho E_\phi) \right) = \left(k^2 - n^2 \frac{\omega^2}{c^2} \right) E_\phi - \frac{ib\omega}{c} E_\rho, \quad (\text{A3})$$

$$\frac{1}{\rho} \partial_\rho (ik\rho E_\rho - \rho \partial_\rho E_z) = \frac{n^2 \omega^2}{c^2} E_z, \quad (\text{A4})$$

for the components ρ, ϕ, z , respectively.

Gauss's law provides

$$\partial_\rho E_z = \frac{i}{k} \partial_\rho \left[\frac{1}{\rho} \partial_\rho (\rho E_\rho) \right] + \frac{cb}{n^2 \omega k} \partial_\rho \left[\frac{1}{\rho} \partial_\rho (\rho E_\phi) \right], \quad (\text{A5})$$

which results in

$$\partial_\rho E_z = \frac{i}{k} \partial_\rho \left[\frac{1}{\rho} \partial_\rho (\rho E_\rho) \right] + \frac{cb}{n^2 \omega k} \left[\left(k^2 - n^2 \frac{\omega^2}{c^2} \right) E_\phi - \frac{i b \omega}{c} E_\rho \right], \quad (\text{A6})$$

after substituting Eq. (A3). An additional substitution in (A2) yields

$$\partial_\rho \left[\frac{1}{\rho} \partial_\rho (\rho E_\rho) \right] - \alpha^2 E_\rho = \frac{i c b k^2}{n^2 \omega} \rho^2 E_\phi, \quad (\text{A7})$$

with the notation

$$\alpha^2 = \frac{b^2}{n^2} + k^2 - \frac{n^2 \omega^2}{c^2}. \quad (\text{A8})$$

Making explicit the derivatives with respect to ρ in (A7) and rearranging we obtain

$$\rho^2 \partial_\rho^2 E_\rho + \rho \partial_\rho E_\rho - (1 + \alpha^2 \rho^2) E_\rho = \frac{i c b k^2}{n^2 \omega} \rho^2 E_\phi, \quad (\text{A9})$$

which yields the Eq. (9) in the main text. A similar calculation in (A3) produces

$$\rho^2 \partial_\rho^2 E_\phi + \rho \partial_\rho E_\phi - (1 + \alpha^2 \rho^2) E_\phi = -\frac{i b \omega}{c} \rho^2 E_\rho, \quad (\text{A10})$$

with the introduction of

$$\alpha^2 = k^2 - n^2 \frac{\omega^2}{c^2}, \quad (\text{A11})$$

thus recovering Eq. (8) of the manuscript.

The coupled equations (A9) and (A10) for E_ρ and E_ϕ are solved by the *Ansätze*

$$E_\phi = X K_1(Q\rho), \quad E_\rho = Y K_1(Q\rho), \quad (\text{A12})$$

which yields the condition (13). From here, we obtain the dispersion relations (15) for two polarization modes $\nu = \pm$, together with the relation (17) between the coefficients X_ν and Y_ν . Once E_ρ and E_ϕ are determined, the component E_z follows from Eqs. (A4) and (A5) yielding

$$E_z = -\frac{i}{k} \sum_\nu \left(1 + \frac{cb}{n^2 \omega} \Omega_\nu \right) Q_\nu Y_\nu K_0(Q_\nu \rho). \quad (\text{A13})$$

The magnetic field follows from Faraday's law, which produces

$$B_\rho = -\frac{ck}{\omega} E_\phi = -i \frac{c}{v} \sum_\nu \Omega_\nu Y_\nu K_1(Q_\nu \rho), \quad (\text{A14})$$

$$B_\phi = \frac{i\omega}{c} (\partial_z E_\rho - \partial_\rho E_z) = \frac{c}{k\omega} \sum_\nu \left(k^2 - Q_\nu^2 - \frac{cb}{n^2 \omega} \Omega_\nu Q_\nu^2 \right) Y_\nu K_1(Q_\nu \rho), \quad (\text{A15})$$

$$B_z = -\frac{ic}{\omega \rho} \partial_\rho (\rho E_\phi) = -\frac{c}{\omega} \sum_\nu \Omega_\nu Y_\nu Q_\nu K_0(Q_\nu \rho). \quad (\text{A16})$$

In this section we have made extensive use of the following properties of the derivatives of the modified Bessel functions:

$$\partial_z K_0(z) = -K_1(z), \quad \frac{1}{z} \partial_z (z K_1(z)) = -K_0(z). \quad (\text{A17})$$

Summarizing, we recover Eqs. (18) and (19) of the manuscript.

APPENDIX B: THE BOUNDARY CONDITIONS AT $\rho \rightarrow 0$

Now we determine the remaining coefficients Y_+ and Y_- using Gauss's law and Ampere's law in the limit $\rho \rightarrow 0$ using standard methods. Recalling Gauss's law

$$\epsilon \oint_{\partial V} \mathcal{E} \cdot \hat{n} dS = \int_V (4\pi \bar{\rho} - \mathbf{b} \cdot \mathcal{B}) d^3x, \quad (\text{B1})$$

together with the factorization

$$\mathcal{E}(\rho, z, \omega) = \mathbf{E}(\rho, \omega) e^{ikz}, \quad \mathcal{B}(\rho, z, \omega) = \mathbf{B}(\rho, \omega) e^{ikz}, \quad k = \omega/v, \quad (\text{B2})$$

we take V as the interior of a cylinder of radius $R \rightarrow 0$, with axis parallel to the charge velocity in the region $-L < z < +L$. The boundary ∂V consists of the cylinder mantle (M) plus the left circle (LC) and the right circle (RC) with exterior normals parallel to the z axis,

$$-\int_{LC} \mathcal{E}_z \rho d\rho d\phi + \int_{RC} \mathcal{E}_z \rho d\rho d\phi + \int_M \mathcal{E}_\rho R d\phi dz = \frac{1}{\epsilon} \left(\frac{4\pi q}{v} - 2\pi b \int d\rho \rho B_z \right) \int dz e^{i\frac{\omega}{v}z}. \quad (\text{B3})$$

Let us consider first the integral over each circular plate. Introducing E_z from Eq. (A13) we have

$$\begin{aligned} \int \mathcal{E}_z \rho d\rho d\phi &= \int E_z e^{i\frac{\omega}{v}z} \rho d\rho d\phi = \frac{v}{\omega} e^{i\frac{\omega}{v}z} \sum_{\nu} X_{\nu} Q_{\nu} \left(\Omega_{\nu} - \frac{cb}{n^2\omega} \right) \int_0^R K_0(Q_{\nu}\rho) \rho d\rho, \\ &= \frac{v}{\omega} e^{i\frac{\omega}{v}z} \sum_{\nu} X_{\nu} Q_{\nu} \left(\Omega_{\nu} - \frac{cb}{n^2\omega} \right) \left[\frac{1 - Q_{\nu} R K_1(Q_{\nu} R)}{Q_{\nu}^2} \right]. \end{aligned} \quad (\text{B4})$$

Recalling

$$\lim_{R \rightarrow 0} K_1(Q_{\nu} R) = \frac{1}{Q_{\nu} R}, \quad (\text{B5})$$

we conclude that the square bracket in (B4) is zero, yielding a null result for the contribution from the cylinder caps. Analogously, since B_z depends on $K_0(Q_{\nu}\rho)$ the related integral in Eq. (B3) is also zero. Then, Gauss's law reduces to

$$\frac{4\pi q}{v\epsilon} \int dz e^{i\frac{\omega}{v}z} = \lim_{R \rightarrow 0} \left(R \int E_{\rho} e^{i\frac{\omega}{v}z} d\phi dz \right) = 2\pi \lim_{R \rightarrow 0} \sum_{\nu} Y_{\nu} (R K_1(Q_{\nu} R)) \int dz e^{i\frac{\omega}{v}z}. \quad (\text{B6})$$

Factoring out the integral over dz we obtain a first relation

$$\frac{Y_+}{Q_+} + \frac{Y_-}{Q_-} = \frac{2q}{vn^2}. \quad (\text{B7})$$

To apply Ampere's law we consider the surface $\mathcal{S}$ as a circle of radius $R \rightarrow 0$ whose normal vector $\hat{\mathbf{k}}$ is parallel to the direction of motion of the charge with center at $\rho = 0$. The boundary $\partial\mathcal{S}$ is then a circumference of radius R and we get the relation

$$\oint_{\partial\mathcal{S}} \mathbf{B} \cdot d\mathbf{l} = \frac{4\pi}{c} \int_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{k}} dS + \frac{\epsilon}{c} \int_{\mathcal{S}} \partial_t \mathcal{E} \cdot \hat{\mathbf{k}} dS + \int_{\mathcal{S}} (\mathbf{b} \times \mathcal{E}) \cdot \hat{\mathbf{k}} dS. \quad (\text{B8})$$

The last term in the above equation is zero since $\mathbf{b} = b\hat{\mathbf{k}}$. In our coordinates we have

$$d\mathbf{l} = R d\phi \hat{\phi}, \quad dS = \rho d\rho d\phi, \quad (\text{B9})$$

yielding

$$\oint_{\partial\mathcal{S}} \mathcal{B}_{\phi} R d\phi = \frac{4\pi}{c} \int_{\mathcal{S}} \frac{q}{2\pi\rho} \delta(\rho) e^{i\frac{\omega}{v}z} \rho d\rho d\phi - i\omega \frac{\epsilon}{c} \int_{\mathcal{S}} \mathcal{E}_z \rho d\rho d\phi, \quad (\text{B10})$$

where the last integral vanishes in the $R \rightarrow 0$ limit, as explicitly shown in Eq. (B4), leaving us with

$$e^{i\frac{\omega}{v}z} 2\pi R B_{\phi} = \frac{4\pi q}{c} e^{i\frac{\omega}{v}z}, \quad (\text{B11})$$

after substituting $\mathcal{B}_{\phi} = e^{i\frac{\omega}{v}z} B_{\phi}$. Inserting the expression for B_{ϕ} in (19), taking the limit $R \rightarrow 0$ according to (B5), and separating the sum into each polarization, we obtain the second relation

$$\left(k^2 - Q_+^2 - \frac{cb}{n^2\omega} \Omega_+ Q_+^2 \right) \frac{Y_+}{Q_+} + \left(k^2 - Q_-^2 - \frac{cb}{n^2\omega} \Omega_- Q_-^2 \right) \frac{Y_-}{Q_-} = \frac{2kq\omega}{c^2}. \quad (\text{B12})$$

From the system of equations (B7) and (B12) we conclude that

$$Y_{\nu} = \frac{q}{vn^2} \left(1 + \nu \frac{1}{\sqrt{1 + 4 \frac{n^2\omega^2}{b^2 v^2}}} \right) Q_{\nu}, \quad (\text{B13})$$

thus completely fixing the solutions for the electromagnetic fields.

APPENDIX C: $v_\nu^{ph}(\omega)$ AS A MONOTONIC FUNCTION OF ω

To explore the monotonicity property of $v_\nu^{ph}(\omega)$ we start from Eq. (39) and take the square, defining

$$(v_\nu^{ph}(\omega))^2 = f_\nu(\omega) = \frac{\omega^2}{A\omega^2 - B(1 + \nu\sqrt{1 + C\omega^2})} \quad (C1)$$

with

$$A = \frac{n^2}{c}, \quad B = \frac{b^2}{2n^2}, \quad C = \frac{4n^2}{b^2v^2}. \quad (C2)$$

To determine the monotonic character of $v_\nu^{ph}(\omega)$ is enough to look at the sign of the derivative $f'_\nu(\omega)$, since $v_\nu^{ph} > 0$ in its domain. Calculating the derivative we present it as

$$f'_\nu(\omega) = \frac{\omega BH_\nu(\omega)}{g_\nu(\omega)^2}, \quad g_\nu(\omega) > 0. \quad (C3)$$

Recalling that $g_\nu(\omega) = A\omega^2 - B(1 + \nu\sqrt{1 + C\omega^2}) > 0$ is the condition (35) for the existence of radiation, which we always assume, we find that the sign of $f'_\nu(\omega)$ is fixed by the function $H_\nu(\omega)$

$$H_\nu(\omega) = \nu \frac{C\omega^2}{\sqrt{1 + C\omega^2}} - 2(1 + \nu\sqrt{1 + C\omega^2}). \quad (C4)$$

Redefining $x = C\omega^2$ and $s = \sqrt{1 + x} > 0$, we get

$$H_\nu(\omega) = \frac{\nu x - 2s - 2\nu s^2}{s} = \frac{-\nu(1 + \nu\sqrt{1 + x})^2}{\sqrt{1 + x}}. \quad (C5)$$

In other words $v_\nu^{ph}(\omega)$ is monotonically increasing (decreasing) for $\nu = -$ ($\nu = +$), respectively.

APPENDIX D: THE POSITIVITY OF THE SPECTRAL ENERGY DENSITIES

We show that the SED $\mathcal{E}_\nu$ can be written in a manifest positive-definite form. To simplify notation we introduce the dimensionless parameter

$$\Sigma = \frac{n^2\omega^2}{b^2v^2} > 0. \quad (D1)$$

Let us start from the expression (57) for the SEDs

$$\mathcal{E}_\nu = \frac{q^2\omega}{2c^2} \left[\left(1 - \frac{1}{n^2\beta^2}\right) - \nu \left(1 + \frac{1}{n^2\beta^2}\right) \frac{1}{\sqrt{1 + 4\Sigma}} \right], \quad (D2)$$

which we factorize as

$$\mathcal{E}_\nu = \frac{q^2\omega}{2c^2} \left(1 - \nu \frac{1}{\sqrt{1 + 4\Sigma}}\right) \left(1 - \frac{c^2}{n^2v^2} \frac{\sqrt{1 + 4\Sigma} + \nu}{\sqrt{1 + 4\Sigma} - \nu}\right). \quad (D3)$$

Amplifying the fraction in the right-hand side by $(\sqrt{1 + 4\Sigma} + \nu)$, expanding the numerator $(\sqrt{1 + 4\Sigma} + \nu)^2$ and collecting terms we find

$$\mathcal{E}_\nu = \frac{q^2\omega}{2n^2v^2} \left(1 - \nu \frac{1}{\sqrt{1 + 4\Sigma}}\right) \times \left(\frac{v^2}{\omega^2} \left(\frac{n^2\omega^2}{c^2} - \frac{b^2}{2n^2} - \nu \frac{b^2}{2n^2} \sqrt{1 + 4\Sigma}\right) - 1\right). \quad (D4)$$

Next, from Eq. (35), we recognize that the first term in the last bracket of the right-hand side of the above equation is just $1/\cos^2 \Theta_\nu$ yielding the final result

$$\mathcal{E}_\nu = \frac{q^2\omega}{2n^2v^2} \left(1 - \nu \frac{1}{\sqrt{1 + 4\Sigma}}\right) \tan^2 \Theta_\nu. \quad (D5)$$

Since $1/\sqrt{1 + 4\Sigma}$ is always less than 1 we conclude that $\mathcal{E}_\nu$ is manifestly positive definite.

APPENDIX E: THE CROSSED TERMS IN THE SPECTRAL ENERGY DISTRIBUTION

We prove that the terms $\mathcal{E}_{(+,-)}$ and $\mathcal{E}_{(-,+)}$ in the SED are zero. Let us start from Eq. (60), which we write as

$$\mathcal{E}_{(+,-)} + \mathcal{E}_{(-,+)} = \frac{c^2}{4\omega} Y_+^* Y_- (Q_+ F_+ + Q_- F_-) \times \frac{\cos((Q_+ - Q_-)\rho)}{\sqrt{Q_+ Q_-}}, \quad (E1)$$

and identify the factors F_+ and F_- . Even though the expression (E1) from which we start is written in terms of Q_ν we find it more convenient to carry the calculation using $Q_\nu = iQ_\nu$. We concentrate on

$$F_+ = \Omega_- \Omega_+ + \frac{1}{k^2} \left(1 + \frac{cb}{n^2\omega} \Omega_+\right) \times \left(k^2 - \left(1 + \frac{cb}{n^2\omega} \Omega_-\right) Q_-^2\right). \quad (E2)$$

and recall that

$$\Omega_+ = -\frac{b\omega}{c(Q_+^2 - \alpha^2)}, \quad \Omega_- = -\frac{b\omega}{c(Q_-^2 - \alpha^2)} \quad (E3)$$

$$\alpha = \frac{\omega^2}{v^2} - \frac{n^2\omega^2}{c^2}.$$

From the expression (23) we obtain

$$Q_\nu^2 - \alpha^2 = \frac{b^2}{2n^2} \left(1 + \nu \sqrt{1 + 4 \frac{\omega^2 n^2}{v^2 b^2}}\right), \quad (E4)$$

which enter in the denominators of $\Omega_\pm$. Then it is a simple matter to show that

$$\Omega_+ \Omega_- = -n^2\beta^2, \quad \Omega_+ + \Omega_- = \beta \frac{bv}{\omega}, \quad (E5)$$

where we used the product

$$(Q_+^2 - \alpha^2)(Q_-^2 - \alpha^2) = -\frac{b^2\omega^2}{n^2v^2}. \quad (\text{E6})$$

Now it is convenient to present F_+ as

$$F_+ = \Omega_- \Omega_+ + \left(1 + \frac{cb}{n^2\omega} \Omega_+\right) - \frac{1}{k^2} \left(1 + \frac{cb}{n^2\omega} (\Omega_+ + \Omega_-) + \frac{c^2b^2}{n^4\omega^2} \Omega_+ \Omega_-\right) Q_-^2. \quad (\text{E7})$$

Substituting the values of $\Omega_+ \Omega_-$ and $(\Omega_+ + \Omega_-)$ from (E5) we obtain a cancellation of the second and third term in the last round bracket of the above equation, yielding

$$F_+ = -n^2\beta^2 + 1 + \frac{cb}{n^2\omega} \Omega_+ - \frac{1}{k^2} Q_-^2. \quad (\text{E8})$$

Substituting the expression for Q_-^2 and rearranging we have

$$\frac{\omega^2}{v^2} F_+ = \left(\frac{cb}{n^2} \frac{\omega}{v^2} \Omega_+ - \frac{b^2}{2n^2} \left(1 - \sqrt{1 + \frac{4\omega^2 n^2}{v^2 b^2}}\right) \right). \quad (\text{E9})$$

Now we identify the last square bracket in (E9) as $-(Q_-^2 - \alpha^2)$. Substituting also Ω_+ we find

$$\begin{aligned} -\frac{\omega^2}{v^2} F_+ &= \left(\frac{b^2\omega^2}{n^2v^2} \frac{1}{(Q_+^2 - \alpha^2)} + (Q_-^2 - \alpha^2) \right) \\ &= \frac{\frac{b^2\omega^2}{n^2v^2} + (Q_+^2 - \alpha^2)(Q_-^2 - \alpha^2)}{(Q_+^2 - \alpha^2)}, \end{aligned} \quad (\text{E10})$$

which yields zero from the relation (E6). One can show that $F_- = 0$ along similar lines, thus obtaining no mixing between the polarizations $+$ and $-$ in the SED.

APPENDIX F: SOME RELEVANT EQUATIONS FROM REF. [30]

To facilitate the comparison with the results reported in Ref. [30], we collect here the expressions for the Cherenkov angle and for the spectral distribution of the total radiated energy per unit length, translating the notation in Ref. [30] to the conventions used in this manuscript. These formulas correspond to an anisotropic chiral medium characterized by a constant refractive index n and a chiral parameter b .

The Cherenkov emission angle Θ_ν for each circular polarization mode $\nu = \pm$ is determined from the implicit condition

$$\cos \theta \tilde{C}_\nu(\theta) = \frac{c}{nv}, \quad (\text{F1})$$

where the angular function $\tilde{C}_\nu(\theta)$ is given by

$$\tilde{C}_\nu(\theta) = \sin^2 \theta + \cos^2 \theta \sqrt{1 - \nu \xi \sec \theta}, \quad (\text{F2})$$

and the dimensionless frequency-dependent chirality parameter is defined as

$$\xi = \frac{cb}{\omega} \frac{1}{n^2} = \frac{1}{\bar{\omega} n^2}, \quad \bar{\omega} \equiv \frac{\omega}{cb}. \quad (\text{F3})$$

Once the emission angle Θ_ν is determined, the spectral distribution of the total radiated energy per unit length and frequency is given by

$$\mathcal{E}_{A,\nu} = \frac{\omega q^2}{4c^2} \left[\frac{\sin \theta \tilde{K}_\nu(\omega, \theta)}{|\sin \theta \tilde{C}_\nu(\theta) - \cos \theta \frac{\partial \tilde{C}_\nu(\theta)}{\partial \theta}|} \right]_{\theta=\Theta_\nu}. \quad (\text{F4})$$

The factor $\tilde{K}_\nu(\omega, \theta)$ is

$$\tilde{K}_\nu(\omega, \theta) = \frac{\tilde{T}_{1,\nu}(\omega, \theta)}{\tilde{g}_\nu^2(\theta)}, \quad (\text{F5})$$

and the function $\tilde{g}_\nu(\omega, \theta)$ reads

$$\begin{aligned} \tilde{g}_\nu(\omega, \theta) &= \frac{1}{(1 - \nu \xi \sec \theta)^{1/4}} \\ &\times \sqrt{(1 - \nu \xi \sec \theta) \left(1 - \frac{\nu \xi}{2} \sec \theta\right) + \frac{\xi^2}{4} \tan^2 \theta}. \end{aligned} \quad (\text{F6})$$

The numerator $\tilde{T}_{1,\nu}$ in (F5) reads

$$\tilde{T}_{1,\nu} = \left(\tilde{p}_\nu^2 + \frac{1}{n^2\beta^2} \tan^2 \theta \right) \tilde{C}_\nu + \tilde{p}_\nu \tilde{q}_\nu \frac{\partial \tilde{C}_\nu}{\partial \theta}, \quad (\text{F7})$$

where the auxiliary functions $\tilde{p}_\nu$ and $\tilde{q}_\nu$ are

$$\begin{aligned} \tilde{p}_\nu(\omega, \theta) &= \sin \theta - \nu \xi \tan \theta + \frac{1}{n\beta} \frac{\partial \tilde{C}_\nu}{\partial \theta}, \\ \tilde{q}_\nu(\omega, \theta) &= \cos \theta - \nu \xi - \frac{c}{nv} \tilde{C}_\nu. \end{aligned} \quad (\text{F8})$$

The ratio between the approximate and the standard spectral energy distributions is

$$\mathcal{R}_{A\nu} = \frac{\mathcal{E}_{A\nu}}{\mathcal{E}_{\text{CH}}} = \frac{1}{4} \frac{1}{\left(1 - \frac{1}{n^2\beta^2}\right)} \left[\frac{\sin \theta \tilde{K}_\nu(\omega, \theta)}{|\sin \theta \tilde{C}_\nu(\theta) - \cos \theta \frac{\partial \tilde{C}_\nu(\theta)}{\partial \theta}|} \right]_{\theta=\Theta_\nu}. \quad (\text{F9})$$

- [1] P. A. Cherenkov, Visible luminescence of pure liquids under the influence of γ -radiation, *Dokl. Akad. Nauk SSSR* **2**, 451 (1934).
- [2] S. Vavilov, *Cr Acad. Sci. USSR* **2**, 457 (1934); [*Dokl. Akad. Nauk SSSR*, Vol. 2 (1934) p. 457]; reprinted in *Usp. Fiz. Nauk* **93**, 383 (1967).
- [3] I. M. Frank and I. E. Tamm, Coherent visible radiation of fast electrons passing through matter, *Compt. Rend. Acad. Sci. URSS* **14**, 109 (1937).
- [4] T. Ypsilantis and J. Seguinot, Theory of ring imaging Cherenkov counters, *Nucl. Instrum. Methods Phys. Res., Sect. A* **343**, 30 (1994).
- [5] J. J. Aubert *et al.* (E598 Collaboration), Experimental observation of a heavy particle J , *Phys. Rev. Lett.* **33**, 1404 (1974).
- [6] M. Amenomori *et al.* (Tibet ASgamma Collaboration), First detection of sub-PeV diffuse gamma rays from the galactic disk: Evidence for ubiquitous galactic cosmic rays beyond PeV energies, *Phys. Rev. Lett.* **126**, 141101 (2021).
- [7] G. Adamo, K. F. MacDonald, Y. Fu, C. Wang, D. Tsai, F. J. García de Abajo, and N. Zheludev, Light well: A tunable free-electron light source on a chip, *Phys. Rev. Lett.* **103**, 113901 (2009).
- [8] S. Liu, P. Zhang, W. Liu, S. Gong, R. Zhong, Y. Zhang, and M. Hu, Surface polariton Cherenkov light radiation source, *Phys. Rev. Lett.* **109**, 153902 (2012).
- [9] F. Liu, L. Xiao, Y. Ye, M. Wang, K. Cui, X. Feng, W. Zhang, and Y. Huang, Integrated Cherenkov radiation emitter eliminating the electron velocity threshold, *Nat. Photonics* **11**, 289 (2017).
- [10] R. Hachadorian, P. Bruza, M. Jermyn, D. Gladstone, B. Pogue, and L. Jarvis, Imaging radiation dose in breast radiotherapy by x-ray ct calibration of Cherenkov light, *Nat. Commun.* **11**, 2298 (2020).
- [11] D. A. Alexander, A. Nomezine, L. A. Jarvis, D. J. Gladstone, B. W. Pogue, and P. Bruza, Color Cherenkov imaging of clinical radiation therapy, *Light Sci. Appl.* **10**, 226 (2021).
- [12] T. M. Shaffer, E. C. Pratt, and J. Grimm, Utilizing the power of Cerenkov light with nanotechnology, *Nat. Nanotechnol.* **12**, 106 (2017).
- [13] X. Wang, L. Li, J. Li, P. Wang, J. Lang, and Y. Yang, Cherenkov luminescence in tumor diagnosis and treatment: A review, *Photonics* **9**, 390 (2022).
- [14] N. Kotagiri, G. P. Sudlow, W. J. Akers, and S. Achilefu, Breaking the depth dependency of phototherapy with cerenkov radiation and low-radiance-responsive nanophotosensitizers, *Nat. Nanotechnol.* **10**, 370 (2015).
- [15] A. Kamkaew, L. Cheng, S. Goel, H. F. Valdovinos, T. E. Barnhart, Z. Liu, and W. Cai, Cerenkov radiation induced photodynamic therapy using chlorin e6-loaded hollow mesoporous silica nanoparticles, *ACS Appl. Mater. Interfaces* **8**, 26630 (2016).
- [16] V. P. Zrelov, Cherenkov Radiation in High-Energy Physics, edited by Benny Baruch, U.S. Atomic Energy Commission, AEC-tr-7099 (Jerusalem, Israel Program for Scientific Translations; available from the U.S. Dept. of Commerce, Clearinghouse for Federal Scientific and Technical Information, Springfield, VA, 1970) translated from Russian by Y. Oren.
- [17] H. Hu, X. Lin, and Y. Luo, Free-electron radiation engineering via structured environments, *arXiv:2108.06149*.
- [18] D. Skryabin, Y. Kartashov, O. Egorov, M. Sich, J. Chana, L. Tapia Rodriguez, P. Walker, E. Clarke, B. Royall, M. Skolnick *et al.*, Backward Cherenkov radiation emitted by polariton solitons in a microcavity wire, *Nat. Commun.* **8**, 1554 (2017).
- [19] P. Genevet, D. Wintz, A. Ambrosio, A. She, R. Blanchard, and F. Capasso, Controlled steering of Cherenkov surface plasmon wakes with a one-dimensional metamaterial, *Nat. Nanotechnol.* **10**, 804 (2015).
- [20] S. N. Galyamin, A. V. Tyukhtin, A. Kanareykin, and P. Schoessow, Reversed Cherenkov-transition radiation by a charge crossing a left-handed medium boundary, *Phys. Rev. Lett.* **103**, 194802 (2009).
- [21] V. Veselago, The electrodynamics of substances with simultaneously negative values of ϵ and μ , *Usp. Fiz. Nauk* **92**, 517 (1967).
- [22] R. A. Shelby, D. R. Smith, and S. Schultz, Experimental verification of a negative index of refraction, *Science* **292**, 77 (2001).
- [23] S. Xi, H. Chen, T. Jiang, L. Ran, J. Huangfu, B.-I. Wu, . f. J. A. Kong, and M. Chen, Experimental verification of reversed Cherenkov radiation in left-handed metamaterial, *Phys. Rev. Lett.* **103**, 194801 (2009).
- [24] S. Zhang and X. Zhang, Flipping a photonic shock wave, *Physics* **2**, 91 (2009).
- [25] X. Lu, M. A. Shapiro, I. Mastovsky, R. J. Temkin, M. Conde, J. G. Power, J. Shao, E. E. Wisniewski, and C. Jing, Generation of high-power, reversed-Cherenkov wake-field radiation in a metamaterial structure, *Phys. Rev. Lett.* **122**, 014801 (2019).
- [26] Z. Duan, X. Tang, Z. Wang, Y. Zhang, X. Chen, M. Chen, and Y. Gong, Observation of the reversed Cherenkov radiation, *Nat. Commun.* **8**, 14901 (2017).
- [27] O. J. Franca, L. F. Urrutia, and O. Rodríguez-Tzompantzi, Reversed electromagnetic Vavilov-Čerenkov radiation in naturally existing magnetoelectric media, *Phys. Rev. D* **99**, 116020 (2019).
- [28] R. Chen, Z. Gong, Z. Wang, X. Xi, B. Zhang, Y. Yang, B. Zhang, I. Kaminer, H. Chen, and X. Lin, A gain route to reversed Cherenkov radiation, *Sci. Adv.* **11**, eads5113 (2025).
- [29] B. Zhang, Z. Gong, R. Chen, X. Chen, Y. Yang, H. Chen, I. Kaminer, and X. Lin, Reversed Cherenkov radiation via Fizeau–Fresnel drag, *Appl. Phys. Rev.* **12** (2025).
- [30] E. Barredo-Alamilla, L. F. Urrutia, and M. M. Ferreira, Jr., Electromagnetic radiation in chiral matter: The Cherenkov case, *Phys. Rev. D* **107**, 096024 (2023).
- [31] O. J. Franca and S. Y. Buhmann, Vavilov-Cherenkov radiation for parallel motion in three-dimensional topological insulators, *Phys. Rev. B* **110**, 195150 (2024).
- [32] R. Martínez von Dossow, E. Barredo-Alamilla, M. A. Górlach, and L. F. Urrutia, Cherenkov radiation in isotropic chiral matter: The space-frequency domain, *Phys. Lett. A* **556**, 130810 (2025).
- [33] D. Colladay and V. A. Kostelecký, *CPT* violation and the standard model, *Phys. Rev. D* **55**, 6760 (1997).

- [34] D. Colladay and V. A. Kostelecký, Lorentz-violating extension of the standard model, *Phys. Rev. D* **58**, 116002 (1998).
- [35] R. Jackiw and V. A. Kostelecký, Radiatively induced Lorentz and *CPT* violation in electrodynamics, *Phys. Rev. Lett.* **82**, 3572 (1999).
- [36] R. A. Martínez von Dossow and L. F. Urrutia, Effective electromagnetic Lagrangians in the derivative expansion method, *J. Phys. A* **58**, 145401 (2025).
- [37] C. Adam and F. R. Klinkhamer, Causality and *CPT* violation from an Abelian Chern-Simons like term, *Nucl. Phys. B* **607**, 247 (2001).
- [38] V. A. Kostelecký and A. G. M. Pickering, Vacuum photon splitting in Lorentz violating quantum electrodynamics, *Phys. Rev. Lett.* **91**, 031801 (2003).
- [39] R. Lehnert and R. Potting, Vacuum Cerenkov radiation, *Phys. Rev. Lett.* **93**, 110402 (2004).
- [40] R. Lehnert and R. Potting, The Cerenkov effect in Lorentz-violating vacua, *Phys. Rev. D* **70**, 125010 (2004); **70**, 129906(E) (2004).
- [41] C. Kaufhold and F. R. Klinkhamer, Vacuum Cherenkov radiation and photon triple-splitting in a Lorentz-noninvariant extension of quantum electrodynamics, *Nucl. Phys. B* **734**, 1 (2006).
- [42] D. Colladay, P. McDonald, and R. Potting, Cherenkov radiation with massive, *CPT*-violating photons, *Phys. Rev. D* **93**, 125007 (2016).
- [43] M. Schreck, Vacuum Cherenkov radiation for Lorentz-violating fermions, *Phys. Rev. D* **96**, 095026 (2017).
- [44] L. Lisboa-Santos, J. a. A. A. S. Reis, M. Schreck, and M. M. Ferreira, Jr., Planar electrodynamics modified by higher-derivative terms, *Phys. Rev. D* **108**, 115032 (2023).
- [45] J. O'Connor and B. Altschul, Radiation from an oscillating dipole in the presence of photon-sector *CPT* and Lorentz violation, *Phys. Rev. D* **109**, 045005 (2024).
- [46] P. D. S. Silva, M. M. Ferreira, M. Schreck, and L. F. Urrutia, Magnetic-conductivity effects on electromagnetic propagation in dispersive matter, *Phys. Rev. D* **102**, 076001 (2020).
- [47] P. D. S. Silva, L. Lisboa-Santos, M. M. Ferreira, Jr., and M. Schreck, Effects of *CPT*-odd terms of dimensions three and five on electromagnetic propagation in continuous matter, *Phys. Rev. D* **104**, 116023 (2021).
- [48] O. J. Franca and S. Y. Buhmann, Modification of transition radiation by three-dimensional topological insulators, *Phys. Rev. B* **105**, 155120 (2022).
- [49] O. J. Franca and L. F. Urrutia, Radiation from a dipole perpendicular to the interface between two planar semi-infinite magnetoelectric media, *Rev. Mex. Fis.* **68**, 060701 (2022).
- [50] P. D. S. Silva, M. J. Neves, and M. M. Ferreira, Jr., Optical properties and energy propagation in a dielectric medium supporting magnetic current, *Phys. Rev. B* **109**, 184439 (2024).
- [51] A. Y. Petrov, M. Schreck, and A. R. Vieira, Vacuum Cherenkov radiation for nonminimal dimension-5 Lorentz violation, *Eur. Phys. J. C* **86**, 30 (2026).
- [52] R. Martínez von Dossow, E. Barredo-Alamilla, M. A. Gorlach, and L. F. Urrutia, Cherenkov radiation in isotropic chiral matter: Unlocking threshold-free emission, *Phys. Rev. D* **113**, 016010 (2026).
- [53] J.-R. Soh, F. de Juan, M. G. Vergniory, N. B. M. Schröter, M. C. Rahn, D. Y. Yan, J. Jiang, M. Bristow, P. Reiss, J. N. Blandy, Y. F. Guo, Y. G. Shi, T. K. Kim, A. McCollam, S. H. Simon, Y. Chen, A. I. Coldea, and A. T. Boothroyd, Ideal Weyl semimetal induced by magnetic exchange, *Phys. Rev. B* **100**, 201102 (2019).
- [54] S. Nasrallah, D. Santos-Cottin, F. Le Mardelé, I. Mohelský, J. Wyzula, L. Akšamović, P. Sačar, J. W. H. Barrett, W. Galloway, K. Rigaux, F. Guo, M. Puppini, I. Živković, J. H. Dil, M. Novak, C. C. Homes, M. Orlita, N. Barišić, and A. Akrap, Magneto-optical response of the magnetic semiconductors EuCd_2X_2 ($x = \text{p, As, Sb}$), *Phys. Rev. B* **110**, L201201 (2024).
- [55] Z. Gong, J. Chen, R. Chen, X. Zhu, C. Wang, X. Zhang, H. Hu, Y. Yang, B. Zhang, H. Chen *et al.*, Interfacial Cherenkov radiation from ultralow-energy electrons, *Proc. Natl. Acad. Sci. U.S.A.* **120**, e2306601120 (2023).
- [56] J. Chen *et al.*, Low-velocity-favored transition radiation, *Phys. Rev. Lett.* **131**, 113002 (2023).
- [57] Y. Adiv, H. Hu, S. Tsesses, R. Dahan, K. Wang, Y. Kurman, A. Gorlach, H. Chen, X. Lin, G. Bartal, and I. Kaminer, Observation of 2d Cherenkov radiation, *Phys. Rev. X* **13**, 011002 (2023).