

Higher derivative corrections to Kerr-AdS black hole thermodynamics

Wei Guo^{*,†}, Xiyao Guo[†], Xin Lan[‡], Hongbao Zhang^{§,||}, and Wei Zhang^{||}

School of Physics and Astronomy, Beijing Normal University, Beijing 100875, China and Key Laboratory of Multiscale Spin Physics, Ministry of Education, Beijing Normal University, Beijing 100875, China



(Received 27 March 2025; accepted 27 August 2025; published 16 September 2025)

Instead of the much more involved covariant counterterm method, we apply the well-justified background subtraction method to calculate the first-order corrections to Kerr-AdS black hole thermodynamics induced by the higher derivative terms up to the cubic of Riemann tensor, where the computation is further simplified by the decomposition trick for the bulk action. The validity of our results is further substantiated by examining the corrections induced by the Gauss-Bonnet term. Moreover, by comparing our results with those obtained via the Arnowitt-Deser-Misner and Wald formulas in Lorentzian signature, we can extract some generic information about the first-order corrected black hole solution induced by each higher derivative term.

DOI: [10.1103/v12x-l2gb](https://doi.org/10.1103/v12x-l2gb)

I. INTRODUCTION

In the Euclidean approach to quantum gravity, there are two available methods in calculating the Gibbs free energy of stationary black holes and the related thermodynamic quantities. One is called the background subtraction method proposed by Hawking and his companions [1–5], and the other is called the covariant counterterm method inspired by AdS/CFT correspondence [6–8]. Both methods are believed to have their respective disadvantages. The covariant counterterm method generically needs notoriously complicated terms added on the boundary on top of the generalized Gibbons-Hawking-York term for a general diffeomorphism covariant theory of gravity, while the background subtraction method is believed to be rather restrictive in the sense that the induced metric of the stationary black holes on the boundary cannot always be isometrically embedded into the reference spacetime [7,8]. However, by applying the covariant phase space formalism in an innovative manner, it has been shown most recently in [9] that the criterion for the applicability of the background subtraction method does not require the aforementioned isometric embedding at all; instead, the background

subtraction method turns out to be as applicable as the covariant counterterm method indeed. Speaking specifically, the applicability of the background subtraction method is well established not only for Einstein's gravity, but also for its higher derivative corrections in both asymptotically flat and anti-de Sitter (AdS) spacetimes. Accordingly, the much simpler background subtraction method emerges naturally as the favorable one to be employed in the ongoing effort for the investigation of the higher derivative corrections to black hole thermodynamics, whereby we can not only obtain some universal information about the UV behavior of quantum gravity in the context of effective field theory, but also gain some insight into the dynamics of the boundary quantum systems at a finite coupling and a finite N through the lens of AdS/CFT correspondence. With this in mind, in this paper, we shall apply the background subtraction method to calculate the first-order corrections to the four-dimensional Kerr-AdS black hole thermodynamics by the higher derivative terms up to the cubic of Riemann tensor, which, to our knowledge, remains unexplored, although much effort has recently been made in the investigation of the higher derivative corrections to thermodynamic quantities for various black holes [10–27]. To achieve this, we shall also facilitate our calculation by adopting the strategy developed in [27], with which we can extract some generic features about the first-order corrected black hole solution, although we are not required to solve it explicitly at all.

The structure of this paper is organized in the following way. In the next section, we shall provide a brief review of the applicability of the background subtraction method to asymptotically AdS spacetimes, where we also work with a different coordinate system from that in [9] to demonstrate that the criterion for the applicability of the background subtraction method is satisfied by Einstein's gravity as it

*Contact author: guow@mail.bnu.edu.cn

†Contact author: xiaoguo@mail.bnu.edu.cn

‡Contact author: xinlan@mail.bnu.edu.cn

§Contact author: hongbaozhang@bnu.edu.cn

||Contact author: w.zhang@mail.bnu.edu.cn

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should be the case. Then, in Sec. III, we shall apply the viable background subtraction method to calculate the first-order corrected Gibbs free energy as well as the associated entropy, angular momentum, and mass in the grand canonical ensemble. By comparing the resulting angular momentum and mass with those obtained via the Arnowitt-Deser-Misner (ADM) formula as well as the resulting entropy with that obtained via the Wald formula, we further extract some generic information about the full first-order corrected black hole solution. Moreover, we substantiate the validity of our result by examining the corrections induced by the Gauss-Bonnet term. We shall conclude our paper with some discussions in the last section.

But before proceeding, we summarize what will be accomplished in this work that is novel as follows. First, compared to the coordinate system used in [9], we adopt a more convenient one to justify the applicability of the background subtraction method to Einstein's gravity in asymptotically AdS spacetimes. Second, with the much simpler background subtraction method, we are the first one to achieve the first-order corrections to the four-dimensional Kerr-AdS black hole thermodynamics by the higher derivative terms. Last but not least, simply by comparing our results with those obtained from the ADM and Wald formulas, we find that one can also succeed in extracting some generic information about the full first-order corrected black hole solution by each higher derivative term without solving it out.

II. APPLICABILITY OF BACKGROUND SUBTRACTION METHOD TO ASYMPTOTICALLY ADS SPACETIMES

In this section, we shall first review the criterion for the applicability of the background subtraction method in $F(R_{abcd})$ gravity. For details, please refer to [9]. Then by taking another different coordinate system from that in [9], we shall show that the criterion is satisfied by Einstein's gravity in asymptotically AdS spacetimes, which can be regarded as a double check of the claim made in [9]. We conclude this section by recapping the main statement in [9] regarding the applicability of the background subtraction method to the higher derivative gravity as the corrections to Einstein's gravity within the framework of effective field theory.

Let us start from the following Lagrangian form for $F(R_{abcd})$ gravity:

$$\mathbf{L} = \epsilon F(R_{abcd}, g_{ab}) \quad (1)$$

with ϵ the spacetime volume and F an arbitrary function of the Riemann tensor R_{abcd} and the metric g_{ab} . Its variation further gives rise to

$$\delta \mathbf{L} = \epsilon E_g^{ab} \delta g_{ab} + d\Theta. \quad (2)$$

Here,

$$E_g^{ab} = \frac{1}{2} g^{ab} F + \frac{1}{2} \frac{\partial F}{\partial g_{ab}} + 2 \nabla_c \nabla_d \psi^{c(ab)d} \quad (3)$$

with $E_g^{ab} = 0$ the equation of motion, and $\Theta = \theta \cdot \epsilon$ is the bulk symplectic potential with

$$\theta^a = 2(\nabla_d \psi^{bdca} \delta g_{bc} - \psi^{bdca} \nabla_d \delta g_{bc}), \quad (4)$$

where ψ^{abcd} is defined as the derivative of F with respect to R_{abcd} by assuming it is independent of the metric, namely, $\psi^{abcd} \equiv \frac{\partial F}{\partial R_{abcd}}$.

With the outward-pointing normal vector and the induced metric of a timelike boundary Γ denoted, respectively, as n_a and h_{ab} , the variation of the metric can be expressed as follows:

$$\delta g^{ab}|_\Gamma = -2\delta a n^a n^b - \delta A^a n^b - \delta A^b n^a + \delta h^{ab} \quad (5)$$

with $n_a \delta A^a = 0$, where we have required Γ remain fixed under variation such that $\delta n_a = \delta a n_a$. Accordingly, Θ can be rewritten in the following form:

$$\Theta|_\Gamma = -\delta \mathbf{B} + d\mathbf{C} + \mathbf{F}. \quad (6)$$

Here,

$$\begin{aligned} \mathbf{B} &= 4\Psi_{ab} K^{ab} \hat{\epsilon}, & \mathbf{C} &= \omega \cdot \hat{\epsilon}, \\ \mathbf{F} &= \hat{\epsilon}(T_{hbc} \delta h^{bc} + T_{\Psi bc} \delta \Psi^{bc}) \end{aligned} \quad (7)$$

with K_{ab} the extrinsic curvature, $\hat{\epsilon}$ the induced volume defined as $\epsilon = \mathbf{n} \wedge \hat{\epsilon}$, $\Psi_{ab} = \psi_{acbd} n^c n^d$, and

$$\begin{aligned} \omega^a &= -2\Psi^a_b \delta A^b + 2h^{ae} \psi_{ecdb} n^d \delta h^{bc}, \\ T_{hbc} &= -2\Psi_{de} K^{de} h_{bc} + 2n^a \nabla^e \psi_{deaf} h^d_{(b} h^f_{c)} \\ &\quad - 2\Psi_{a(b} K^a_{c)} - 2D^a (h_a^e h_{(c}^f \psi_{|ef|b)} n^d), \\ T_{\Psi bc} &= 4K_{bc}, \end{aligned} \quad (8)$$

where D_a denotes the covariant derivative on Γ .

Now let Σ be a spacelike hypersurface emanating from the bifurcation surface $\mathcal{B}$ of a stationary black hole and terminating at Γ with their intersection denoted by $\mathcal{S}$. Then, one can show that if and only if

$$\int_{\mathcal{S}} \xi \cdot [\mathbf{F}] = 0, \quad \text{as } \mathcal{S} \rightarrow \text{spatial infinity}, \quad (9)$$

with the square bracket representing the difference from the reference spacetime as $[\mathbf{F}] \equiv \mathbf{F} - \mathbf{F}^0$, the first law of black hole thermodynamics holds, i.e.,

$$T\delta S = \delta[H_\xi] \quad (10)$$

with the Hawking temperature $T = \frac{\kappa}{2\pi}$. Here, ξ is the Killing vector field normal to the black hole event horizon and tangential to Γ . The black hole entropy is given by

$$S = \beta \int_B \mathbf{Q}_\xi \quad (11)$$

with $\beta = \frac{1}{T}$ the period along the Euclidean time $\tau = it$ and the Noether charge

$$\mathbf{Q}_\xi = -\psi^{cadb} \nabla_{[d} \xi_{b]} \epsilon_{ca...} \quad (12)$$

The Hamiltonian conjugate to ξ can be expressed as

$$H_\xi = \int_S q_\xi \cdot \hat{\epsilon}, \quad (13)$$

where

$$q_\xi^a = \mathcal{T}^a_{\ c} \xi^c \quad (14)$$

with the generalized Brown-York boundary energy-momentum tensor $\mathcal{T}^a_{\ c} = -(2T^a_{\ h} \delta^h_c + 2\Psi^{ab} T_{\Psi cb})$.

The condition (9) together with the finiteness condition for $[H_\xi]$ gives rise to the criterion for the applicability of the background subtraction method. If this criterion is fulfilled, the Gibbs free energy of the stationary black hole can be expressed as

$$\beta G \equiv \beta([H_\xi] - TS) = [I_E]. \quad (15)$$

Here, the Euclidean action I_E is given by

$$I_E = \int_M \mathbf{L}_E + \int_\infty \mathbf{B}_E \quad (16)$$

with $\mathbf{L}_E = -i\mathbf{L}$ and $\mathbf{B}_E = -i\mathbf{B}$.

Next, let us demonstrate that the above criterion is satisfied by Einstein's gravity, whose Lagrangian form reads

$$\mathbf{L}_{eh} = \frac{1}{16\pi} \epsilon \left(R + \frac{6}{l^2} \right). \quad (17)$$

Hence, it is easy to show $\psi_{abcd} = \frac{1}{32\pi} (g_{ac}g_{bd} - g_{ad}g_{bc})$ and $\Psi_{ab} = \frac{1}{32\pi} h_{ab}$. Accordingly, we obtain

$$\begin{aligned} \mathbf{B} &= \frac{K}{8\pi} \hat{\epsilon}, & \mathbf{C} &= -\frac{1}{16\pi} \not{A} \cdot \hat{\epsilon}, \\ \mathbf{F} &= -\frac{1}{2} T_{bc} \delta h^{bc} \hat{\epsilon}, & q_\xi^a &= T^a_{\ c} \xi^c, \end{aligned} \quad (18)$$

where $T_{bc} = -\frac{1}{8\pi} (K_{bc} - K h_{bc})$ is the familiar Brown-York boundary energy-momentum tensor. On the other hand, the corresponding Kerr-AdS black hole solution reads

$$\begin{aligned} ds^2 &= -W \left(1 + \frac{r^2}{l^2} \right) dt^2 + \frac{2m}{U} \left(W dt - \frac{a \sin^2 \theta d\phi}{\Xi} \right)^2 \\ &\quad + \frac{U}{V-2m} dr^2 + \frac{rU}{W\Xi} d\theta^2 + \frac{r^2 + a^2}{\Xi} \sin^2 \theta d\phi^2 \end{aligned} \quad (19)$$

in the Boyer-Lindquist coordinates, where

$$\begin{aligned} W &= \frac{\sin^2 \theta}{\Xi} + \cos^2 \theta, & \Xi &= 1 - \frac{a^2}{l^2}, \\ U &= \frac{r^2 + a^2 \cos^2 \theta}{r}, & V &= \frac{1}{r} \left(1 + \frac{r^2}{l^2} \right) (r^2 + a^2). \end{aligned} \quad (20)$$

As alluded to in the beginning of this section, the coordinate system we are taking here is different from that in [9]. The main advantage of this coordinate system is that the corresponding Killing vector field $\frac{\partial}{\partial t}$ is nonrotating at infinity. With the Killing vector field normal to the black hole event horizon taking the form $\xi = \frac{\partial}{\partial t} + \left(\frac{a\Xi}{r_+^2 + a^2} + \frac{a}{l^2} \right) \frac{\partial}{\partial \phi}$, we can obtain the black hole temperature and angular velocity as follows:

$$T = \frac{r_+(1 + \frac{a^2}{l^2} + 3\frac{r_+^2}{l^2} - \frac{a^2}{r_+^2})}{4\pi(r_+^2 + a^2)}, \quad \Omega = \frac{a(1 + \frac{r_+^2}{l^2})}{r_+^2 + a^2}, \quad (21)$$

where r_+ denotes the black hole event horizon radius, satisfying $V(r_+) - 2m = 0$.

To proceed, we shall choose the timelike boundary Γ to be the surface of $r = \bar{r}$ and make the following coordinate transformation:

$$t \rightarrow \frac{\sqrt{V(\bar{r}) - 2m}}{\sqrt{V(\bar{r})}} t \quad (22)$$

for the solution with $m = 0$, which is actually the pure AdS solution in the Boyer-Lindquist coordinates and will serve as our reference spacetime. After such a rescaling of the time coordinate, the metric of our reference spacetime reads

$$\begin{aligned} ds^2 &= -W \frac{V(\bar{r}) - 2m}{V(\bar{r})} \left(1 + \frac{r^2}{l^2} \right) dt^2 + \frac{U}{V} dr^2 + \frac{rU}{W\Xi} d\theta^2 \\ &\quad + \frac{r^2 + a^2}{\Xi} \sin^2 \theta d\phi^2, \end{aligned} \quad (23)$$

whereby one can choose

$$\begin{aligned}
e^0 &\equiv \sqrt{W \frac{V(\bar{r}) - 2m}{V(\bar{r})} \left(1 + \frac{r^2}{l^2}\right)} dt, & e^1 &\equiv \sqrt{\frac{U}{V}} dr, \\
e^2 &\equiv \sqrt{\frac{r^2 + a^2 \cos^2 \theta}{W \Xi}} d\theta, & e^3 &\equiv \sqrt{\frac{r^2 + a^2}{\Xi}} \sin \theta d\phi
\end{aligned} \tag{24}$$

as its orthonormal basis. Accordingly, the nonvanishing basis components of the induced metric and extrinsic curvature at Γ can be written as

$$\begin{aligned}
h_{00} &= -1 + \frac{2ma^2 \sin^2 \theta}{\Xi \bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^5}\right), & h_{03} &= -\frac{2mal \sin \theta \sqrt{W}}{\sqrt{\Xi} \bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^5}\right), & h_{22} &= 1, & h_{33} &= 1 + \frac{2ma^2 \sin^2 \theta}{\Xi \bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^5}\right), \\
K_{00} &= -\frac{1}{l} + \frac{1}{2l\bar{r}^2} (l^2 - a^2 \sin^2 \theta) + \frac{m[-2l^2 + a^2(1 + \cos^2 \theta)]}{l\Xi \bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), & K_{03} &= \frac{ma \sin \theta \sqrt{W}}{\sqrt{\Xi} \bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \\
K_{22} &= \frac{1}{l} + \frac{a^2 + l^2 - 3a^2 \cos^2 \theta}{2l\bar{r}^2} - \frac{ml}{\bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), & K_{33} &= \frac{1}{l} + \frac{l^2 - a^2(1 + \cos^2 \theta)}{2l\bar{r}^2} - \frac{mlW}{\bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right),
\end{aligned} \tag{25}$$

whereby we can further obtain

$$K = \frac{3}{l} + \frac{a^2 + l^2 - 5a^2 \cos^2 \theta}{2l\bar{r}^2} + \frac{l^4 - 2a^2 l^2 (1 + 3 \cos^2 \theta) + a^4 (1 - 6 \cos^2 \theta + 21 \cos^4 \theta)}{8l\bar{r}^4} - \frac{ml(l^2 + a^2 \sin^2 \theta)}{\bar{r}^5} + \mathcal{O}\left(\frac{1}{\bar{r}^6}\right) \tag{26}$$

and the nonvanishing basis components of the Brown-York tensor of the Kerr metric

$$\begin{aligned}
T_{00} &= -\frac{1}{4\pi l} - \frac{l^2 - 2a^2 \cos^2 \theta}{8\pi l\bar{r}^2} + \frac{ml[2 + \frac{a^2}{l^2}(5 - 7 \cos^2 \theta)]}{8\pi \Xi \bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), & T_{03} &= \frac{7ma \sin \theta \sqrt{W}}{8\sqrt{\Xi} \pi \bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \\
T_{22} &= \frac{1}{4\pi l} - \frac{a^2 \cos^2 \theta}{8\pi l\bar{r}^2} + \frac{ml}{8\pi \bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), & T_{33} &= \frac{1}{4\pi l} - \frac{a^2(2 \cos^2 \theta - 1)}{8\pi l\bar{r}^2} + \frac{ml[1 + \frac{a^2}{l^2}(6 - 7 \cos^2 \theta)]}{8\pi \Xi \bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right)
\end{aligned} \tag{27}$$

with $m = 0$ corresponding to the result for the reference spacetime. Note that $\sqrt{|h|} = \frac{\sqrt{(V-2m)Ur} \sin \theta}{\Xi}$ is the same for the Kerr-AdS black hole and reference spacetime with $\frac{\bar{r}^3 \sin \theta}{l\Xi}$ as the leading-order term, but the induced metric of the Kerr-AdS black hole at Γ is different from that of the reference spacetime. Luckily, this does not destroy the applicability of the background subtraction method according to our criterion. To be more specific, we have the induced metrics

$$\begin{aligned}
h'_{00} &= -1 - \frac{2a\delta a \sin^2 \theta}{l^2 \Xi^2 W} + \frac{1}{\bar{r}^3} \left[\frac{2ma^2 \sin^2 \theta + 2W \Xi l^2 \delta m}{\Xi} + \frac{4ma\delta a \sin^2 \theta (l^2 - a^2 \cos^2 \theta)}{W l^2 \Xi^2} \right] + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \\
h'_{03} &= -\frac{1}{\bar{r}^3} \left[\frac{2(m + \delta m)al \sin \theta \sqrt{W}}{\sqrt{\Xi}} + \frac{2m\delta a \sin^2 \theta (l^4 + 3a^2 l^2 \sin^2 \theta - a^4 \cos^2 \theta)}{l^3 \Xi^2 \sin \theta \sqrt{W \Xi}} \right] + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \\
h'_{22} &= 1 + \frac{2a\delta a \cos^2 \theta}{l^2 W \Xi} + \frac{2a\delta a \cos^2 \theta}{\bar{r}^2} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \\
h'_{33} &= 1 + \frac{2a\delta a}{l^2 \Xi} + \frac{2a\delta a}{\bar{r}^2} + \frac{1}{\bar{r}^3} \left[\frac{2(m + \delta m)a^2 \sin^2 \theta}{\Xi} + \frac{4ma\delta a (l^2 + a^2) \sin^2 \theta}{l^2 \Xi^2} \right] + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \\
h'^0_{00} &= -1 - \frac{2a\delta a \sin^2 \theta}{l^2 \Xi^2 W} + \frac{2\delta m l^2}{\bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), & h'^0_{03} &= 0, \\
h'^0_{22} &= 1 + \frac{2a\delta a \cos^2 \theta}{l^2 W \Xi} + \frac{2a\delta a \cos^2 \theta}{\bar{r}^2} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), & h'^0_{33} &= 1 + \frac{2a\delta a}{l^2 \Xi} + \frac{2a\delta a}{\bar{r}^2} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right)
\end{aligned} \tag{28}$$

for the first-order perturbed Kerr-AdS black hole and reference spacetime. By $\delta h^{ab} = -h^{ac}h^{bd}\delta h_{cd}$, one can further obtain

$$\begin{aligned}
\delta h^{00} &= \frac{2a\delta a \sin^2\theta}{l^2\Xi^2 W} - \frac{1}{\bar{r}^3} \left[2\delta m W l^2 + \frac{4ma\delta a \sin^2\theta}{W\Xi^2} \right] + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \\
\delta h^{03} &= \frac{1}{\bar{r}^3} \left[\frac{2a\delta m l \sin\theta \sqrt{W}}{\sqrt{\Xi}} + \frac{2m\delta a \sin\theta [l^4\Xi - 3a^4\cos^2\theta + a^2l^2(7-5\cos^2\theta)]}{l^3\Xi\sqrt{W\Xi}} \right] + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \\
\delta h^{22} &= -\frac{2a\delta a \cos^2\theta}{l^2 W \Xi} - \frac{2a\delta a \cos^2\theta}{\bar{r}^2} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \\
\delta h^{33} &= -\frac{2a\delta a}{l^2\Xi} - \frac{2a\delta a}{\bar{r}^2} - \frac{1}{\bar{r}^3} \left[\frac{2a^2\delta m \sin^2\theta}{\Xi} + \frac{4ma\delta a \sin^2\theta}{\Xi} \right] + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \\
\delta h^{000} &= \frac{2a\delta a \sin^2\theta}{l^2\Xi^2 W} - \frac{2\delta m l^2}{\bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \quad \delta h^{003} = 0, \\
\delta h^{022} &= -\frac{2a\delta a \cos^2\theta}{l^2 W \Xi} - \frac{2a\delta a \cos^2\theta}{\bar{r}^2} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \quad \delta h^{033} = -\frac{2a\delta a}{l^2\Xi} - \frac{2a\delta a}{\bar{r}^2} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right).
\end{aligned} \tag{29}$$

Then, it follows from a straightforward computation that $\int_S \xi \cdot [\mathbf{F}]$ is proportional to $\int_{-1}^1 dx(1-3x^2) = 0$ with $x = \cos\theta$, which guarantees the validity of the first law of black hole thermodynamics. On the other hand, we have

$$T'_t = \frac{1}{4\pi l} + \frac{l^2 - 2a^2\cos^2\theta}{8\pi l \bar{r}^2} - \frac{ml[2 + \frac{a^2}{l^2}(1-3\cos^2\theta)]}{8\pi\Xi\bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right), \quad T'_\phi = \frac{3ma l \sin^2\theta}{8\pi\Xi\bar{r}^3} + \mathcal{O}\left(\frac{1}{\bar{r}^4}\right) \tag{30}$$

with $m = 0$ corresponding to the result for the reference spacetime, whereby the finiteness of $[H_\xi]$ is apparent. In particular, we have the familiar ADM mass and angular momentum as follows:

$$M \equiv [H_{\frac{\partial}{\partial t}}] = \frac{m}{\Xi^2}, \quad J \equiv -[H_{\frac{\partial}{\partial \phi}}] = Ma, \tag{31}$$

which is finite. Thus, the background subtraction method is applicable to Einstein's gravity in asymptotically AdS spacetime.

Finally, as detailed in [9], the criterion for the applicability of the background subtraction method is also satisfied by the higher derivative gravity as the correction to Einstein's gravity within the framework of effective field theory. Moreover, the corresponding Gibbs free energy receives the contribution only from the bulk term, as the same as that for Einstein's gravity. However, there are two additional effects from the higher derivative terms. First, suppose that the pure AdS is still the solution to the aforementioned higher derivative gravity theory and λ is a small constant; then, Eq. (2) evaluated on such a pure AdS solution with $\delta g_{ab} = \lambda g_{ab}$ gives rise to

$$\delta \mathbf{L} = 0, \tag{32}$$

where we have used $d\Theta = 0$ due to the fact that the pure AdS is a space of constant curvature. Note that both the traceless part of Ricci tensor $\mathcal{R}_{ab}$ and Weyl tensor C_{abcd} vanish for the pure AdS; thus, the above equation implies

that the four-dimensional AdS radius will get corrected only by the $\frac{\epsilon}{16\pi}\epsilon R^n$ term with $n > 2$ as follows:

$$R + \frac{12}{l^2} + \epsilon(2-n)R^n = 0. \tag{33}$$

Second, the expression for the ADM mass and angular momentum will get corrected only by the higher derivative terms $\frac{\epsilon}{16\pi}\epsilon R^n$ and $\frac{\epsilon}{16\pi}\epsilon R^n C^2$ with $C^2 \equiv C_{abcd}C^{abcd}$. It turns out that the corrections to the ADM mass and angular momentum are equally proportional to the expression for Einstein's gravity with the proportional constant $\epsilon n R^{n-1}$ for the former higher derivative correction and $\frac{\epsilon \Delta R^n}{l_e^2}$ for the latter one, where l_e denotes the corrected AdS radius.

III. HIGHER DERIVATIVE CORRECTIONS TO KERR-ADS BLACK HOLE THERMODYNAMICS

With the preparation made in the previous section, we shall apply the background subtraction method to calculate the first-order corrections to Kerr-AdS black hole thermodynamics by the higher derivative terms. To be more precise, we shall consider the Einstein-Hilbert term (17) corrected by the following high derivative terms¹:

¹Here, we have ignored all the other quadratic and cubic terms, because their corrections vanish due to the fact that the involved Kerr-AdS metric has $\mathcal{R}_{ab} = 0$ as well as an orientation reversing isometry $\theta \rightarrow \pi - \theta$.

$$\mathbf{L}_{hd} = \frac{1}{16\pi} \epsilon (\epsilon_1 L^2 R^2 + \epsilon_2 L^2 C^2 + \epsilon_3 L^4 R^3 + \epsilon_4 L^4 R C^2 + \epsilon_5 L^4 C_{abcd} C^{cdef} C^{ab}_{ef}), \quad (34)$$

where L as some UV length scale has been inserted to ensure ϵ_i dimensionless. First, according to Eq. (33), the AdS radius gets shifted as follows:

$$\frac{12}{l_e^2} = \frac{12}{l^2} + \epsilon_3 L^4 \left(\frac{12}{l^2} \right)^3 = -R. \quad (35)$$

Then, we follow the trick devised in [27] to decompose the full bulk action as follows:

$$I' = \int_M (\mathbf{L}'_{eh} + \mathbf{L}'_{hd}) \quad (36)$$

with

$$\mathbf{L}'_{eh} = \frac{1}{16\pi} \epsilon \left(R + \frac{6}{l_e^2} \right), \quad \mathbf{L}'_{hd} = \frac{1}{16\pi} \epsilon \left(\frac{6}{l^2} - \frac{6}{l_e^2} \right) + \mathbf{L}_{hd}. \quad (37)$$

In light of the analysis detailed in the previous section, the first-order corrected Gibbs free energy can be obtained by evaluating the corresponding bulk $[I'_E]$ on the Kerr-AdS metric and pure AdS metric with l_e as the AdS radius. The underlying reason is the same as documented in [10] for the asymptotically flat case and can be recapitulated as follows. First, as pointed out at the end of the previous section, there is no contribution to the Gibbs free energy from the associated boundary terms. In addition, according to Eqs. (2) and (6), the contribution to the Gibbs free energy from $\mathbf{L}'_{eh}$ evaluated on the full first-order corrected solution is the same as that evaluated on the aforementioned Kerr-AdS metric, because their difference comes solely from the boundary term $\int_\infty [\mathbf{F}_E] = \int_\infty d\tau \wedge \frac{\partial}{\partial \tau} \cdot \mathbf{F}_E = \beta \int_S \xi \cdot [\mathbf{F}] = 0$ with $\mathbf{F}_E = -i\mathbf{F}$. Last but not least, similarly, the contribution to the first-order corrected Gibbs free energy from $\mathbf{L}'_{hd}$ evaluated at the full first-order corrected solution is also the same as that evaluated on the aforementioned Kerr-AdS metric, because their difference, determined by the first term in Eq. (2), starts apparently from the second order of ϵ_i , where one ϵ_i comes from the coefficients in $\mathbf{L}'_{hd}$ and the other ϵ_i comes from the variation of the metric induced by the higher derivative terms.

With this in mind, we first calculate the bulk contribution to $[I'_E]$ from $\mathbf{L}'_{eh}$. As such, we have

$$\int_{\text{BH}} \mathbf{L}'_{ehE} = \frac{\beta}{16\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_{r_+}^{\bar{r}} dr \frac{6}{l_e^2} \sqrt{|g|} = \frac{\beta(\bar{r} - r_+)(\bar{r}^2 + r_+ \bar{r} + r_+^2 + a^2)}{2l_e^2 \Xi} \quad (38)$$

and

$$\int_{\text{AdS}} \mathbf{L}'_{ehE} = \frac{\beta}{16\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_0^{\bar{r}} dr \frac{6}{l_e^2} \sqrt{|g|} = \frac{\beta}{2l_e^2 \Xi} (\bar{r}^3 + a^2 \bar{r} - m l_e^2) + \mathcal{O}\left(\frac{1}{\bar{r}}\right), \quad (39)$$

where $\sqrt{|g|}_{\text{BH}} = \frac{rU \sin \theta}{\Xi}$ and $\sqrt{|g|}_{\text{AdS}} = \frac{\sqrt{V(\bar{r})-2m}}{\sqrt{V(\bar{r})}} \frac{rU \sin \theta}{\Xi}$ are used. Then, with the background subtraction method, the resulting contribution to $[I'_E]$ is obtained as

$$-\frac{\beta(r_+^3 + a^2 r_+ - m l_e^2)}{2l_e^2 \Xi} \quad (40)$$

by taking the limit $\bar{r} \rightarrow \infty$. Next, by the same token and with

$$C^2 = \frac{48m^2(r^6 - 15a^2 r^4 x^2 + 15a^4 r^2 x^4 - a^6 x^6)}{(r^2 + a^2 x^2)^6},$$

$$C_{abcd} C^{cdef} C^{ab}_{ef} = \frac{96m^3 r(r^8 - 36a^2 r^6 x^2 + 126a^4 r^4 x^4 - 84a^6 r^2 x^6 + 9a^8 x^8)}{(r^2 + a^2 x^2)^9} \quad (41)$$

for the Kerr-AdS metric, one can further obtain the corresponding contribution to $[I'_E]$ from $\mathbf{L}'_{hd}$. Accordingly, we end up with the following first-order corrected Gibbs free energy:

$$G \simeq -\frac{r_+^3 + a^2 r_+ - m l_e^2}{2 l_e^2 \Xi} + \left(12\epsilon_1 \frac{L^2}{l_e^2} - 216\epsilon_3 \frac{L^4}{l_e^4}\right) \frac{r_+^3 + a^2 r_+ - m l_e^2}{l_e^2 \Xi} - \left(4\epsilon_2 - 48\epsilon_4 \frac{L^2}{l_e^2}\right) \frac{L^2 m^2 r_+ (r_+^2 - a^2)}{\Xi (r_+^2 + a^2)^3} - 4\epsilon_5 \frac{L^4 m^3 (7r_+^6 - 35a^2 r_+^4 + 21a^4 r_+^2 - a^6)}{7\Xi (r_+^2 + a^2)^6}, \quad (42)$$

where $\simeq$ means that the equation holds at the first order of ϵ_i .

To obtain the corresponding higher derivative corrections to other thermodynamic quantities from the above Gibbs free energy in the grand canonical ensemble, we should view m as well as r_+ and a as functions of T and Ω implicitly through Eq. (21), whereby one can get the following partial derivatives:

$$\begin{aligned} \frac{\partial r_+}{\partial T} &= \frac{4\pi r_+^2 l_e^2 (r_+^2 - a^2)}{4r_+^4 - (r_+^2 + l_e^2)(r_+^2 + a^2)}, & \frac{\partial r_+}{\partial \Omega} &= \frac{4a l_e^2 r_+^3}{4r_+^4 - (r_+^2 + l_e^2)(r_+^2 + a^2)}, \\ \frac{\partial a}{\partial T} &= \frac{8\pi a l_e^4 \Xi r_+^3}{[4r_+^4 - (r_+^2 + l_e^2)(r_+^2 + a^2)](r_+^2 + l_e^2)}, & \frac{\partial a}{\partial \Omega} &= \frac{3r_+^6 + (8a^2 - l_e^2)r_+^4 + a^2(a^2 + 4l_e^2)r_+^2 + a^4 l_e^2}{[4r_+^4 - (r_+^2 + l_e^2)(r_+^2 + a^2)](1 + \frac{r_+^2}{l_e^2})}. \end{aligned} \quad (43)$$

Then, the corresponding first-order corrected entropy, angular momentum, and mass can be calculated out as follows:

$$S = -\frac{\partial G}{\partial T} \simeq \frac{\pi(r_+^2 + a^2)}{\Xi} - \epsilon_1 \frac{24\pi L^2(r_+^2 + a^2)}{l_e^2 \Xi} + \epsilon_2 \frac{4\pi L^2(r_+^2 + l_e^2)}{l_e^2 \Xi} + \epsilon_3 \frac{432\pi L^4(r_+^2 + a^2)}{l_e^4 \Xi} - \epsilon_4 \frac{48\pi L^4(r_+^2 + l_e^2)}{l_e^4 \Xi} - \epsilon_5 \frac{8\pi L^4 m^2 [a^8(3l_e^2 + r_+^2) - 6a^6 r_+^2(3l_e^2 + 11r_+^4) - 28a^4 r_+^4(2l_e^2 - 7r_+^2) - 14a^2 r_+^6(l_e^2 + r_+^2) + 21r_+^8(l_e^2 - r_+^2)]}{7\Xi(r_+^2 + a^2)^5[4r_+^4 - (r_+^2 + a^2)(r_+^2 + l_e^2)]}, \quad (44)$$

$$\begin{aligned} J = -\frac{\partial G}{\partial \Omega} &\simeq \frac{ma}{\Xi^2} - \epsilon_1 \frac{24maL^2}{l_e^2 \Xi^2} + \epsilon_2 \frac{4maL^2}{l_e^2 \Xi^2} + \epsilon_3 \frac{432maL^4}{l_e^4 \Xi^2} - \epsilon_4 \frac{48maL^4}{l_e^4 \Xi^2} \\ &+ \epsilon_5 \left\{ \frac{4m^2 L^4 [-a^{11}(r_+^2 + l_e^2) + a^9 r_+^2(35l_e^2 + 43r_+^2) - 2a^7 r_+^2(9l_e^4 - 5l_e^2 r_+^2 + 18r_+^4) + 10a^5 r_+^4(l_e^4 - 3l_e^2 r_+^2 - 28r_+^4)]}{7l_e^2 \Xi^2 r_+(r_+^2 + a^2)^5[4r_+^4 - (r_+^2 + a^2)(r_+^2 + l_e^2)]} \right. \\ &\left. + \frac{4m^2 L^4 [7a^3 r_+^6(6l_e^4 + 17l_e^2 r_+^2 + 11r_+^4) + 7ar_+^8(2l_e^4 - 19l_e^2 r_+^2 + 3r_+^4)]}{7l_e^2 \Xi^2 r_+(r_+^2 + a^2)^5[4r_+^4 - (r_+^2 + a^2)(r_+^2 + l_e^2)]} \right\}, \end{aligned} \quad (45)$$

$$\begin{aligned} M = G + TS + \Omega J &\simeq \frac{m}{\Xi^2} - \epsilon_1 \frac{24mL^2}{l_e^2 \Xi^2} + \epsilon_2 \frac{4mL^2}{l_e^2 \Xi^2} + \epsilon_3 \frac{432mL^4}{l_e^4 \Xi^2} - \epsilon_4 \frac{48mL^4}{l_e^4 \Xi^2} \\ &- \epsilon_5 \left\{ \frac{4m^2 L^4 [2a^{10} l_e^2 - a^8(l_e^4 + 36l_e^2 r_+^2 + 85r_+^4) + 4a^6 r_+^2(5l_e^4 + 9l_e^2 r_+^2 + 55r_+^4) - 14a^4 r_+^4(l_e^4 + 12l_e^2 r_+^2 - 7r_+^4)]}{7l_e^2 \Xi^2 r_+(r_+^2 + a^2)^5[4r_+^4 - (r_+^2 + a^2)(r_+^2 + l_e^2)]} \right. \\ &\left. + \frac{4m^2 L^4 [14a^2 r_+^6(-2l_e^4 + 3l_e^2 r_+^2 + 2r_+^4) + 7r_+^8(l_e^4 + 4l_e^2 r_+^2 - 3r_+^4)]}{7l_e^2 \Xi^2 r_+(r_+^2 + a^2)^5[4r_+^4 - (r_+^2 + a^2)(r_+^2 + l_e^2)]} \right\}. \end{aligned} \quad (46)$$

By comparing the above corrected mass and angular momentum with those obtained through the ADM formula stated at the end of the previous section, one can know that the higher derivative terms except the one associated with ϵ_5 do not correct the parameters m and a appearing in the Kerr-AdS metric under consideration. On the other hand, according to Eq. (11), $\nabla_a \xi_b = \kappa \bar{\epsilon}_{ab}$ and $\epsilon = \bar{\epsilon} \wedge \tilde{\epsilon}$ at the bifurcation surface $\mathcal{B}$ with $\bar{\epsilon}$ the binormal to $\mathcal{B}$ and $\tilde{\epsilon}$ the induced volume at $\mathcal{B}$, the black hole entropy can be expressed by the Wald formula as [28,29]

$$S = -2\pi \int_{\mathcal{B}} \psi^{cadb} \bar{\epsilon}_{db} \bar{\epsilon}_{ca} \tilde{\epsilon}, \quad (47)$$

whereby the first-order corrected entropy can be written as follows:

$$S \simeq \frac{A(\mathcal{B})}{4} + \frac{\delta A(\mathcal{B})}{4} - 2\pi \int_{\mathcal{B}} \psi_{hd}^{cadb} \bar{\epsilon}_{db} \bar{\epsilon}_{ca} \tilde{\epsilon}. \quad (48)$$

Here, $A(\mathcal{B})$ denotes the horizon area of the Kerr-AdS black hole with the AdS radius l_e , given by

$$A(\mathcal{B}) = \int_{\mathcal{B}} \tilde{\epsilon} = \frac{4\pi(r_+^2 + a^2)}{\Xi}, \quad (49)$$

where we have used $\tilde{\epsilon} = \frac{(r_+^2 + a^2) \sin \theta}{\Xi} d\theta \wedge d\phi$ while $\delta A(\mathcal{B})$ represents the area correction due to the deviation of the full

first-order corrected black hole solution from the Kerr-AdS metric by the higher derivative terms. On the other hand, the third term in Eq. (48), evaluated at the aforementioned Kerr-AdS black hole, gives rise to the rest corrections from the higher derivative terms. In particular, by $\bar{\epsilon} = \frac{r_+^2 + a^2 \cos^2 \theta}{r_+^2 + a^2} dt \wedge dr$ and

$$\begin{aligned} \psi_{hd}^{abcd} = & \frac{1}{16\pi} \{ (2\epsilon_1 L^2 R + 3\epsilon_3 L^4 R^2 + \epsilon_4 L^4 C^2 + \epsilon_5 L^4 C^2) g^{[c} g^{d]b} + (2\epsilon_2 L^2 + 2\epsilon_4 L^4 R) C^{abcd} \\ & + 3\epsilon_5 L^4 [C^{abmn} C_{mn}{}^{cd} - (C^{mnpa} C_{mnp}{}^{[c} g^{d]b} - C^{mnpb} C_{mnp}{}^{[c} g^{d]a})] \}, \end{aligned} \quad (50)$$

we can calculate it out explicitly as follows:

$$\begin{aligned} -2\pi \int_{\mathcal{B}} \psi_{hd}^{cabd} \bar{\epsilon}_{db} \bar{\epsilon}_{ca} \tilde{\epsilon} = & -\epsilon_1 \frac{24L^2 \pi (r_+^2 + a^2)}{l_e^2 \Xi} + \epsilon_2 \frac{4\pi L^2 (r_+^2 + l_e^2)}{l_e^2 \Xi} + \epsilon_3 \frac{432\pi L^4 (r_+^2 + a^2)}{l_e^4 \Xi} - \epsilon_4 \frac{48\pi L^4 (r_+^2 + l_e^2)}{l_e^4 \Xi} \\ & + \epsilon_4 \frac{48\pi L^4 m^2 (a^4 - 10a^2 r_+^2 + 5r_+^4)}{5\Xi (r_+^2 + a^2)^4} + \epsilon_5 \frac{24\pi L^4 m^2 (a^4 - 10a^2 r_+^2 + 5r_+^4)}{5\Xi (r_+^2 + a^2)^4}. \end{aligned} \quad (51)$$

By inspection of Eqs. (44) and (51), we find that the higher derivative terms associated with ϵ_1 , ϵ_2 , and ϵ_3 do not induce the additional variation of the black hole horizon area, while the terms associated with ϵ_4 and ϵ_5 do so.

As a simple check of the validity of our result, we conclude this section by applying it to the corrections induced by the Gauss-Bonnet term

$$R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} = \frac{1}{6}R^2 - 2\mathcal{R}_{ab}\mathcal{R}^{ab} + C^2, \quad (52)$$

which is well known to be a topological term in four dimensions and does not correct the solution to Einstein's gravity. On the one hand, it follows from the remark made at the end of the previous section that the correction to the ADM mass and angular momentum from the R^2 term is precisely canceled out by that from the C^2 term. Thus, we end up with the ADM mass and angular momentum uncorrected, which is consistent with Eqs. (45) and (46). On the other hand, note that $\mathcal{R}_{ab} = 0$ when evaluated on the Kerr-AdS black hole, so the corresponding entropy correction via Eq. (51) turns out to be exactly the same as that via Eq. (44) as should be the case.

IV. CONCLUSION

With the viable background subtraction method, we have accomplished the calculation for the first-order corrected Gibbs free energy as well as the corresponding entropy, angular momentum, and mass in the grand canonical

ensemble by the higher derivative terms up to the cubic of Riemann tensor. To the best of our knowledge, this is a new result, which otherwise would be difficult to obtain by the much more involved covariant counterterm method. The validity of our result is further substantiated by examining the corrections induced by the Gauss-Bonnet term. Although our strategy does not require us to solve the full first-order corrected black hole solutions by the higher derivative terms, we can extract some generic information about the first-order corrected black hole solutions by comparing our results obtained via the Euclidean approach with those obtained via the ADM and Wald formulas in Lorentzian signature.

Among others, there are two interesting issues worthy of further investigation. One is the first-order corrections to our Kerr-AdS black hole by the higher derivative terms than the cubic of Riemann tensor. The other is to apply the background subtraction method to calculate the higher derivative corrections to thermodynamic quantities for the higher dimensional rotating black holes, which turns out to be much more sophisticated using the covariant counterterm method. We hope to report our investigation along this line somewhere else in the future.

ACKNOWLEDGMENTS

This work is partially supported by the National Key Research and Development Program of China with Grant No. 2021YFC2203001 as well as the National Natural Science Foundation of China with Grants No. 12361141825 and No. 12575047.

DATA AVAILABILITY

No data were created or analyzed in this study.

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