

Degenerate vortices and worldline instantons in three-dimensional gauge theories

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In this paper we continue the study of particlelike topological solitons with degenerate masses and their mixing due to worldline instantons. Previously, this phenomenon was studied in $(1+1)$ -dimensional setups. Here we take a step further and consider degenerate vortices in $2+1$ dimensions. We find that, while classically such vortices may be degenerate, they generally mix and split at the quantum level. Supersymmetry protects BPS-saturated vortices only when the number of supercharges in the bulk is large enough.

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I. INTRODUCTION

Whenever a theory has multiple vacua that are degenerate in energy at the classical level, it typically also admits quasiclassical field configurations of nontrivial topology, interpolating between these vacua. A familiar example comes from $(0+1)$ -dimensional quantum mechanics (QM) with a double-well potential: in the purely bosonic system, instantons generate a nonzero overlap between the two perturbative ground states, while in the supersymmetric counterpart the tunneling amplitude is suppressed. If the tunneling is not suppressed, the would-be degenerate ground states become split in energy.

In higher dimensions, it becomes possible to have multiple solitons (kinks, etc.) that are degenerate in energy. As such, they may mix on the quantum level—an effect that is known in condensed matter [1–3], and that recently gained attention in the particle physics community [4,5].

While the previous field-theoretic studies focused on $1+1$ D theories, here we take a step further and consider the $2+1$ D setup. In this case, one can consider codimension-1 and codimension-2 solitons. The former are the domain lines (walls); however, they are not the main focus of this paper. Here, we concentrate on vortices, which fall into the latter category above.

We cover several examples where the vortices have internal features. This can occur in both Abelian and

non-Abelian gauge theories. Classically, the presence of the internal structure (like discrete moduli) leads to having several species of vortices with the same mass and topological charge. On the quantum level, the mass degeneracy is generically lifted. This can happen due to worldline instantons, which are the instantons in the effective quantum mechanics living on the vortex worldline.

The instanton tunneling amplitude can be suppressed in supersymmetric quantum mechanics, e.g. in the case with a double-well potential. Therefore, if the vortex is Bogomol’nyi-Prasad-Sommerfield (BPS) saturated in a theory with large enough supersymmetry in the bulk, we can expect that the degeneracy will be protected. We find that with eight supercharges the vortices stay degenerate, while for lesser supersymmetry their degeneracy may be lifted.

We begin this paper by discussing some basic 3D models in Sec. II. We focus on the cases with $\mathcal{N} = 1$ supersymmetry and their bosonic counterparts. The issue of BPS-saturated vortices is also discussed there. After that, we turn to the main subject of this paper, namely, degenerate vortices, starting with Abelian examples in Sec. III. In Sec. IV we continue with non-Abelian gauge theories, which can support vortices with large degeneracy. Our conclusions are summarized in Sec. V. The Appendix provides some details on $\mathcal{N} = 1$ supersymmetry in three spacetime dimensions.

II. MODELS WITH $\mathcal{N} = 1$ SUPERSYMMETRY

We are going to start here by reviewing some basic setups for three-dimensional models with and without supersymmetry. These models (and their generalizations) will be studied in more detail below. Additional details on three-dimensional supersymmetry are reviewed in the Appendix.

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A. Scalar model and its gauging

The minimal example is an $\mathcal{N} = 1$ supersymmetric Wess-Zumino type model (two supercharges). The Lagrangian for such a model can be written as

$$S = \frac{1}{2} \int d^3x \{ \partial_\mu \varphi^a \partial^\mu \varphi^a + \bar{\psi}^a i \gamma^\mu \partial_\mu \psi^a - [\partial_a \mathcal{W}(\varphi^c)]^2 - (\partial_a \partial_b \mathcal{W}(\varphi^c)) \bar{\psi}^a \psi^b \}, \quad (2.1)$$

where φ^a are real scalar fields, a is a flavor index, and $\mathcal{W}(\varphi^a)$ is the superpotential depending on all the φ^a fields. Below we will discuss an example with

$$\mathcal{W}(\varphi^a) = \frac{m}{2} (\varphi^a \varphi^a) - \frac{g}{8} (\varphi^a \varphi^a)^2. \quad (2.2)$$

Finally, $\bar{\psi} = \psi^\dagger \gamma^0 = \psi \gamma^0$ and ψ is a Majorana two-component spinor. This model has two real supercharges. In 3D there is no chirality; therefore, no chiral subspace can be constructed.

When the number of flavors in (2.1) is even, one can define a $U(1)$ symmetry, which then can be gauged. Of course, the superpotential must be invariant with respect to the symmetry transformations. For example, in the case with two flavors, we can combine two real fields into one complex field ϕ and two Majorana spinors into a complex two-component spinor Ψ ,

$$\phi = \frac{1}{\sqrt{2}} (\varphi^1 + i\varphi^2), \quad \Psi = \frac{1}{\sqrt{2}} (\psi^1 + i\psi^2). \quad (2.3)$$

There is a natural global $U(1)$ symmetry acting on these complex fields. Gauging this symmetry brings up the gauge field (photon) A_μ which in 3D carries one physical component, together with its superpartner—a neutral Majorana field λ (photino). The Lagrangian of 3D $\mathcal{N} = 1$ supersymmetric quantum electrodynamics (SQED) in the Wess-Zumino gauge takes the form

$$\begin{aligned} \mathcal{L}_{\text{3DSQED}} = & \frac{1}{e^2} \left(-\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{i}{2} \lambda^\alpha \partial_{\alpha\beta} \lambda^\beta \right) \\ & + (\mathcal{D}_\mu \phi)^\dagger \mathcal{D}^\mu \phi + i((\Psi^\alpha)^\dagger (\mathcal{D})_{\alpha\beta} \Psi^\beta) - U(\phi, \phi^\dagger) \\ & + [\lambda \gamma^0 \Psi \phi^\dagger + \text{H.c.}] - \frac{i}{2} \frac{\partial^2 \mathcal{W}}{\partial \phi \partial \phi^\dagger} \Psi^\dagger \gamma^0 \Psi, \end{aligned} \quad (2.4)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \mathcal{D}_\mu = \partial_\mu - A_\mu, \quad \partial_{\alpha\beta} = (\gamma^0 \gamma^\mu)_{\alpha\beta} \partial_\mu, \quad (2.5)$$

and the potential

$$U = (\phi \phi^\dagger) [m - g(\phi \phi^\dagger)]^2 \quad (2.6)$$

follows directly from the superpotential (2.2). Note that this potential is sextic, which is renormalizable in $2 + 1\text{D}$.

The theory (2.4) can be obtained from the 4D $\mathcal{N} = 1$ SQED (four supercharges) upon dimensional reduction and discarding the fields that are absent in 3D $\mathcal{N} = 1$ SQED¹ or, alternatively, constructed directly in 3D; see the Appendix.

The model under consideration has two vacua with $\langle \phi \rangle$ given by

$$\phi_{*\text{sym}} = 0; \quad (2.7a)$$

$$\phi_{*\text{as}} = v e^{i\alpha}, \quad v = \sqrt{\frac{m}{g}}. \quad (2.7b)$$

These vacua are degenerate in energy, at least at the classical level.

The first vacuum, Eq. (2.7a), is the normal, or symmetric, or nonsuperconducting phase. It is invariant under the $U(1)$ phase rotations. The photon in this phase is massless, while the complex scalar has mass m .

The second vacuum, Eq. (2.7b), is the asymmetric, or superconducting, or Higgs phase. Strictly speaking, it is a whole manifold, but all its points are gauge equivalent to each other. The $U(1)$ symmetry is realized nonlinearly; alternatively, one can say it is spontaneously broken. A standard redistribution of degrees of freedom occurs in this vacuum. The phase of the ϕ field will be eaten up by the photon (thus acquiring a longitudinal degree of freedom and a mass $M_{\text{ph}} \sim ev$). The physical Higgs field with mass

$$m_{|\phi|} = 2m = 2gv^2 \quad (2.8)$$

remains the lower component of the *real* Higgs superfield. The vacuum expectation value (VEV) v is large at weak coupling, i.e., $g \ll 1$. This paves the way to study, in

¹Equation (2.4) may be compared with Ref. [6] where vortices in a 3D reduction of 4D SQED with four supercharges were considered (see Eq. (7) in [6] and Sect. 10.6.9 in [7]). In 4D the photino field λ is described by a four-component Majorana spinor, while in 3D $\mathcal{N} = 1$ SQED the photino is a two-component Majorana spinor. In passing from 4D to 3D with two supercharges we discard the A_2 component of the gauge field—it is absent in 3D $\mathcal{N} = 1$ (it becomes a scalar field N in the notation of a [6]). Finally, all tilded fields present in Eq. (7) of [6] disappear. Note that in $\mathcal{N} = 1$ 3D SQED there is no Fayet-Iliopoulos term and its D -term companions. It is also important to emphasize that in $\mathcal{N} = 1$ SQED in three dimensions no bona fide chiral superpotential exists because there are no chiral subspaces. Instead, we have a superpotential (A45). Our definition of 4D $\rightarrow$ 3D reduction is as follows: out of four coordinates we drop y and each vector loses its y component, see Ref. [7]. In colloquial language, the corresponding $\mathcal{Z}_\mu$ [see Eq. (A49)] is often referred to as central charge, although it is not quite accurate because $\mathcal{Z}_\mu$ does not commute with the generators of rotations. A more accurate name in this case is the brane charge suggested by Seiberg and Komargodski [8,9], see also Appendix A 5 here.

addition to vortices in the Higgs phase, also domain lines separating the two phases.

In three dimensions, vortices are particles with finite masses. Below we will see that, in some modified versions of this theory, the Higgsed regime exhibits a number of mass-degenerate vortices. When supersymmetry is absent, these vortices mix through instantons. In 3D this is a new phenomenon similar to that discussed in 2D in [4,5]. Supersymmetrizing the model, one can force the instanton contribution to vanish because of the fermion zero modes.

The model under consideration exhibits other interesting phenomena, such as the erasure of vortices crossing the domain line from the Higgsed to the un-Higgsed phase [10] and nontrivial effects associated with the inclusion of the Chern-Simons term. These two topics will be considered in a forthcoming publication [11].

B. Vortices in gauged 3D model, Higgs regime

As explained in [7], Sec. 11.1.2.2, in three-dimensional $\mathcal{N} = 1$ supersymmetric QED (SQED) there is no central charge corresponding to vortices. This implies that the vortices cannot be BPS saturated. However, we will see later that they still can be approximately BPS with a judiciously chosen superpotential.

Recall that the model (2.4) has two vacua, one symmetric at $\phi_{\text{sym}} = 0$, and another asymmetric with $\phi_{\text{as}} = ve^{i\alpha}$. Let us have a closer look at the second vacuum.

First, we will study the possibility of the Bogomol'ny completion for the vortex equation. For static field configurations, the energy functional takes the form (in the gauge $A_0 = 0$)

$$\mathcal{E}[\vec{A}(\vec{x}), \phi(\vec{x})] = \int dx_1 dx_2 \left[\frac{1}{4e^2} F_{ij} F_{ij} + |\mathcal{D}_i \phi|^2 + U(\phi) \right], \quad (2.9)$$

where $U(\phi)$ is presented in Eq. (2.6). It is rather obvious that with this superpotential the BPS completion is impossible.

Thus, to find the vortex solution we have to solve a second-order differential equation. The relevant ansatz and the boundary conditions are the same as in Sec. 3.1.3 of [7] [here, (r, α) are the polar coordinates on the 2D space]:

$$\phi(r, \alpha) = v f_\phi(r) e^{i\alpha}, \quad A_i(r, \alpha) = -\varepsilon_{ij} \frac{x_j}{r^2} [1 - f_A(r)]; \quad (2.10)$$

$$\begin{aligned} f_\phi(0) &= 0, & f_\phi(\infty) &= 1, \\ f_A(0) &= 1, & f_A(\infty) &= 0. \end{aligned} \quad (2.11)$$

However, the profile functions are different from the BPS case.

Since the BPS completion is impossible, both supercharges are broken on the vortex solution. Then on such a vortex we will have two fermion zero modes. Can we change the superpotential in such a way that the BPS saturation becomes possible, at least at the classical level? The answer is yes, provided we add to the theory one extra scalar field h , neutral under the gauge $U(1)$, and choose the superpotential as follows,

$$\mathcal{W}_h = gh(\Phi^\dagger \Phi - v^2), \quad (2.12)$$

implying

$$U = \frac{1}{2} g^2 (v^2 - \phi^\dagger \phi)^2 + g^2 h^2 \phi^\dagger \phi \quad (2.13)$$

with the ground state at $|\phi_{\text{vac}}| = v$, $h = 0$ and $\dim g = [m^{1/2}]$. Then in the classical solution we can put $h = 0$, arriving at basically the same classical first-order equations in the bosonic sector as in Sec. 11.5.1 of [7]² (cf. also Sec. 3.1.3 there),

$$\begin{aligned} B - g^2(|\phi|^2 - v^2) &= 0, \\ (\mathcal{D}_x + \mathcal{D}_z)\phi &= 0. \end{aligned} \quad (2.14)$$

The bosonic profile solution is also the same as in the BPS case, only supplemented by $h = 0$ for the extra scalar field that we introduced here. The difference from the BPS case is going to show up in the zero modes and quantum corrections.

From the action (2.4) we obtain a pair of coupled equations of motion for the fermion fields,

$$\begin{aligned} i\partial_{\alpha\beta} \lambda^\beta + e^2(\Psi^\dagger + \Psi)_\alpha &= 0, \\ i\mathcal{D}_{\alpha\beta} \Psi^\beta + \phi \lambda_\alpha + \frac{\partial^2 \mathcal{W}}{\partial \phi \partial \phi^\dagger} \Psi_\alpha &= 0. \end{aligned} \quad (2.15)$$

On the vortex background, these equations describe the fermion zero modes.

To conclude this section, let us note that the theory described here has a domain line (wall) solution separating the two phases in Eqs. (2.7a) and (2.7b); see Appendix A 5. It would be interesting to study the degenerate “domain wall” vortices, which would be the analogs of the objects studied in [12] in the context of 4D gauge theories.

III. DEGENERATE VORTICES INSIDE AN ABELIAN SUPERCONDUCTOR

In this section we come to the main topic of this study, namely, degenerate vortices. We are going to start with the

²The emergence of the term $g^2(v^2 - \phi^\dagger \phi)^2$ in the potential in Sec. 11.5.1 of [7] is due to the Fayet-Iliopoulos ξ term. In our case it cannot be introduced; we have to generate it from F terms.

case of the $U(1)$ gauge theory. Vortices inside a non-Abelian superconductor will be described in the next section.

A. $\mathbb{Z}_2$ vortices

Here we extend the analysis presented in [5] and study another example—perhaps more realistic for a condensed matter system. This example is motivated by the study of degenerate vortices in gauged sigma models [13,14]. In the setup of [13,14], the sigma model target space imposes a rigid constraint on the fields, and worldline instantons are *not possible*. We will relax the rigid constraint, introducing a penalizing term in the potential, which will make the instanton-induced tunneling possible. A similar model was studied in [15] (in a somewhat different notation) and the vortex solutions were constructed. Here, we go one step further and discuss a possible mixing of these vortices.

1. The model

We consider a $2+1$ D Abelian theory with the $U(1)$ gauge group, a complex scalar field ϕ of charge 1, and a real scalar χ (charge 0). The Lagrangian $\mathcal{L}$ consists of two parts [16–18],

$$\mathcal{L} = \mathcal{L}_{\text{QED}} + \mathcal{L}_\chi. \quad (3.1)$$

The first piece is the standard scalar QED, for now without SUSY:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4e^2} F_{\mu\nu}^2 + |\mathcal{D}^\mu \phi|^2 - V(\phi), \quad (3.2)$$

$$V = e^2(|\phi|^2 - v^2)^2, \quad (3.3)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \mathcal{D}_\mu \phi = (\partial_\mu - iA_\mu)\phi. \quad (3.4)$$

The second piece describes an extra real scalar field χ coupled to the QED sector in a special way,

$$\mathcal{L}_\chi = \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - U(\chi, \phi), \quad (3.5)$$

$$U = \gamma[(-\mu + |\phi|^2)\chi^2 + \beta\chi^4], \quad (3.6)$$

where χ is a real scalar field, γ is a (positive) coupling constant, and for simplicity we will assume $\beta \gtrsim 1$. The parameters μ and v are also real and positive. In order to obtain desired degenerate vortices, we impose the following constraints³ on the couplings:

³Couplings in the inequalities (3.7) should not be vastly different; however, a factor of 2 or so between them is acceptable. This will guarantee that $m_\chi \sim m_\phi$, cf. Eq. (3.10).

$$\mu \lesssim v^2, \quad e^2 \lesssim \gamma. \quad (3.7)$$

Note the similarity of the two potentials in Eq. (2.13) on one hand, and in Eqs. (3.3) and (3.6) on the other hand.

The nonvanishing vacuum expectation value of ϕ , $|\phi|_{\text{vac}} \neq 0$, Higgses the gauge field. In the vacuum

$$|\phi|_{\text{vac}}^2 = v^2, \quad \chi = 0. \quad (3.8)$$

The Lagrangian (3.1) has a $\mathbb{Z}_2$ symmetry acting as

$$\chi \rightarrow -\chi, \quad (3.9)$$

and it is unbroken in the vacuum (3.8). The expectation value of ϕ and Eq. (3.7) imply that χ is stable, no vacuum condensate of χ develops, and the global $\mathbb{Z}_2$ symmetry remains unbroken. The masses of the photon, physical Higgs, and the neutral scalar boson χ are

$$m_A = m_\phi = \sqrt{2}ev, \quad m_\chi = \sqrt{2\gamma}\sqrt{v^2 - \mu}. \quad (3.10)$$

2. Vortices

The minimal winding-one vortex in this theory has the following structure. Asymptotically far from the vortex core, we have

$$\phi = ve^{i\alpha}, \quad \chi = 0, \quad B = 0. \quad (3.11)$$

where $B = F_{12}$ is the magnetic field. Here, α is the polar angle on the spatial plane, while B is the magnetic field. Inside the vortex core, however,

$$\phi \approx 0, \quad B \neq 0. \quad (3.12)$$

From the potential (3.6) one can see that the mass term for χ turns negative at the vortex core. Therefore, $\chi \neq 0$ and is double valued there.

Thus, one can explicitly see the appearance of a degeneracy: inside the vortex, the field χ is determined only up to a sign. This means that a vortex spontaneously breaks the $\mathbb{Z}_2$ symmetry (3.9). Now recall that the vortices in $2+1$ D, being localized in space, are particlelike states (in much the same way as kinks in 2D, see Ref. [5]). Therefore, they may or may not tunnel into each other by virtue of quantum-mechanical instantons on the vortex worldlines. The mixing depends on whether the instanton action is finite or infinite.

Let us study the vortex properties in more detail. For a short while let us forget about the Lagrangian $\mathcal{L}_\chi$ in (3.5), i.e. we set $\gamma = 0$. To obtain a (minimal) vortex one must find the field ϕ at large distances from the origin, $\phi \rightarrow ve^{i\alpha}$, where α is the polar angle. By the Poincaré-Hopf theorem, this implies in turn that ϕ must vanish in (at least) one point of the spatial plane. Conventionally, we consider the

minimal case where ϕ has a single zero, and we choose the coordinate system so that this zero is positioned at the origin.

The solution to the corresponding equations of motion is discussed in detail in [7]. In fact, the solution turns out BPS saturated, $\phi = \phi_{\text{crit vort}}$, with the unit value of the magnetic flux, so our “prevortex” mass can be found without calculations, just from the boundary conditions,

$$M_{\text{vort}0} = 2\pi v^2. \quad (3.13)$$

Now, let us reinstate the χ field and consider the coupling $\gamma > 0$. Since the ϕ field still has to vanish at the vortex center, χ has a nonzero VEV there, as was mentioned above. This VEV can be found from the potential (3.6):

$$(\chi^2)_{\text{vort core}} \approx \frac{\mu}{2\beta}, \quad (\chi)_{\text{vort core}} \approx \pm \sqrt{\frac{\mu}{2\beta}}, \quad (3.14)$$

see Fig. 1. An “effective mass” inside the core is

$$m_{\chi, \text{vort core}} \sim \sqrt{\mu\gamma}. \quad (3.15)$$

We have enough parameters to guarantee that the latter is close to m_χ in Eq. (3.10) which, in turn, is close to $m_A = m_\phi$.

To estimate the actual vortex mass (with the field χ included), observe that the expectation value of χ in the core, see Eq. (3.14), diminishes the prevortex mass (3.13) by some fraction of $2\pi v^2$, namely,

$$M_{\text{vort}0} \rightarrow M_{\text{vort}} = M_{\text{vort}0}(1 - \kappa), \quad 0 < \kappa < 1, \quad (3.16)$$

where κ is a dimensionless parameter which is smaller than 1, but not significantly smaller. In this estimate we take into account all constraints on various parameters introduced above, so that the masses $m_A = m_\phi \sim m_\chi$, which in

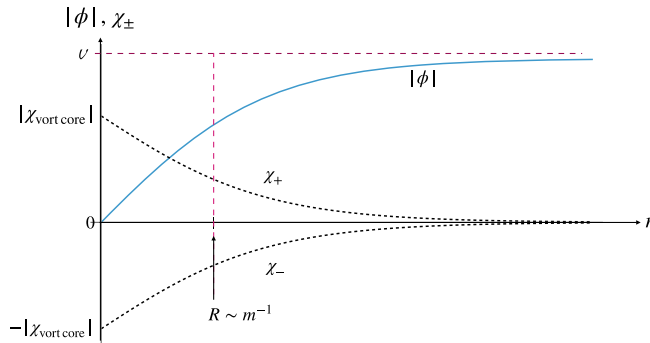


FIG. 1. Schematic vortex profile functions in the model (3.1). Here r is the radial coordinate, R is the size of the vortex core. The existence of two profile functions, $\chi_\pm$, demonstrates the spontaneous $\mathbb{Z}_2$ symmetry breaking on the vortices.

turn implies that the size of the vortex core

$$R \sim m_\phi^{-1}. \quad (3.17)$$

3. Tunneling between degenerate vortices in Euclidean time

In 3D models, the vortices of the Abrikosov type can be viewed as particles. From the standpoint of topology, they are ground states in the sector with the unit winding of ϕ which is disconnected from sectors with other windings, for instance, from the topologically trivial sector, or sectors with windings 2, 3, etc.

Recall that, in our model, the global $\mathbb{Z}_2$ symmetry (3.9) is spontaneously broken in the vortex core, and $(\chi)_{\text{vort core}}$ is double valued [see (3.14)]. Therefore, even if we arrange the topology to be the winding-1 sector, two distinct degenerate vortices coexist in this model. Let us call them V_+ and V_- . Classically, they are exactly degenerate because they are related to each other by the $\mathbb{Z}_2$ symmetry.

Now we will discuss whether V_+ and V_- can mix with each other quantum-mechanically. The mixing is realized through instantons, interpolating (in the Euclidean time τ) between

$$\{\phi, \chi\}_{\text{inst}} = \begin{cases} V_- & \text{at } \tau \rightarrow -\infty \\ V_+ & \text{at } \tau \rightarrow \infty \end{cases}. \quad (3.18)$$

If the instanton action is finite, $S_{\text{inst}} < \infty$, the tunneling obviously does happen in the model under consideration.

The instanton profile can of course be found numerically. We leave this exercise for future work, limiting ourselves to some quantitative arguments.

First, the field profiles on the instanton depend on the Euclidean time τ . As τ evolves from $-\infty$ to $+\infty$, the instanton profile must continuously evolve from V_- to V_+ without changing the winding of ϕ and the corresponding asymptotic behavior of the gauge field.

The potentials V and U in Eqs. (3.3) and (3.6) are nonsingular. The evolution of ϕ is expected to be small and, in any case, the asymptotic behavior of ϕ and A_μ stays the same. Therefore, the dominant contribution in the instanton action S_{inst} comes from the evolution of χ .

In this evolution, the energy functional reaches its maximum at $\chi = 0$. The corresponding field configuration was referred to as the “prevortex” above. Now we see that the prevortex is unstable and is nothing other than the sphaleron. Its energy is given in Eq. (3.13). Thus, the barrier separating V_- and V_+ has the height $\kappa M_{\text{vort}0}$, cf. Eq. (3.16). The instanton is localized in the Euclidean time τ with the characteristic size of $\Delta\tau \sim m^{-1}$. As a result,

$$S_{\text{inst}} \sim \Delta\tau \kappa M_{\text{vort}0} \sim 2\pi \frac{m_\phi}{e^2}. \quad (3.19)$$

Below we will provide a method for deriving this formula.

The quasiclassical approximation we rely on is valid under the condition

$$\frac{v}{e} \gg 1. \quad (3.20)$$

Under this assumption, the instanton action (3.19) is large, and the tunneling amplitude is in the controlled regime.

4. Effective quantum mechanics for vortex tunneling

To illustrate how Eq. (3.19) can be derived, let us consider an effective quantum-mechanical model that describes the τ evolution from $-\infty$ to $+\infty$. This quantum mechanics is supported on the worldline of the vortex.

We will represent each interpolation trajectory (field profile) by a certain τ -dependent parameter; for this purpose, we can choose the value of the χ field at the vortex center, $\chi(r=0)$. Let us denote

$$\chi(r=0) \equiv \eta, \quad (3.21)$$

$$\eta(\tau \rightarrow -\infty) = -(\mu/2\beta)^{1/2}, \quad \eta(\tau \rightarrow \infty) = (\mu/2\beta)^{1/2}, \quad (3.22)$$

see Eq. (3.14). The model we suggest is the standard two-hump potential

$$\mathcal{H} = \frac{1}{2} \frac{\partial^2}{\partial \eta^2} + \lambda(\eta^2 - \eta_0^2)^2, \quad (3.23)$$

see Fig. 2. This model follows from the discussion above as a phenomenological model. The parameter λ will be fixed shortly, see Eq. (3.25). The model in Eq. (3.23) is quite primitive; however, it captures all qualitative features of the instanton representing the tunneling phenomenon.

Let us pause here to make two important clarifications.

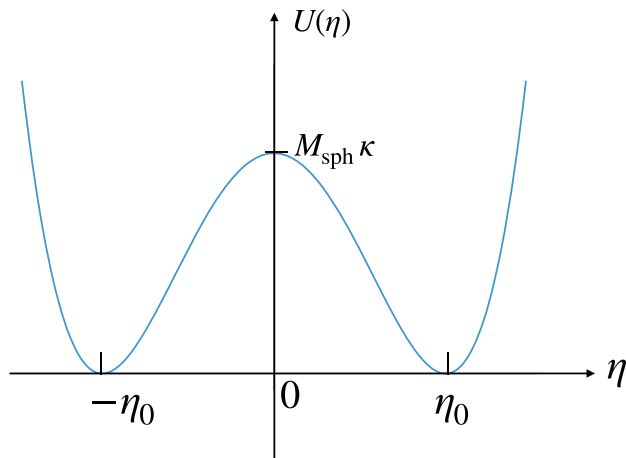


FIG. 2. Effective quantum mechanics for the instanton tunneling from V_- to V_+ , see Eqs. (3.23) and (3.24).

First of all, let us comment on the validity of the worldline instanton approximation. The full instanton in Eq. (3.18), strictly speaking, involves interpolation of all the fields in the theory. Both the magnetic field profile B and the complex scalar ϕ are deformed when χ evolves in Euclidean time τ .

However, at any χ , the boundary conditions on the other fields stay the same. The magnetic field B vanishes at the spatial infinity, and the total flux is fixed. The scalar ϕ vanishes in the vortex center, while away from the vortex it approaches the same VEV $|\phi| = v$ at all times, and its winding stays the same.

Therefore, in the instanton background, the fields ϕ and B are not deformed significantly. On the other hand, the field χ changes drastically—its boundary condition at the origin changes on the instanton Eq. (3.18). This boundary condition plays the role of the order parameter.

This justifies using the $\chi(0) \equiv \eta$ as the order parameter for the effective description. The phenomenological Hamiltonian (3.23) is expressed in terms of this η ; however, this does not mean that the dynamics of ϕ and B is ignored—the dynamics of these fields affects the value of the effective coupling λ in Eq. (3.23). In principle, this coupling can be computed microscopically (e.g. by numerical methods). In this paper we focus only on some qualitative aspects of the vortex dynamics, so we do not determine this coupling in a precise way, constraining ourselves only with an estimate in Eq. (3.25) below.

The second point that we wanted to clarify concerns the dimensionality of the parameters discussed below. All quantities we use in the original 3D theory have dimensions inherent to 3D field theory, while in Eq. (3.23) we deal with one-dimensional quantum mechanics. To avoid the discrepancy, we will measure all relevant quantities in the natural units of mass, i.e., in units of ev . For simplicity, all numerical factors will be omitted in the remainder of this subsection. In these units,

$$\eta_0 = v \rightarrow e^{-1}, \quad M_{\text{vort}} = v^2 \rightarrow e^{-2}, \quad m_\chi \sim m_\phi \sim m = 1. \quad (3.24)$$

In the topological sector, M_{vort} is the background value of the energy, which is there even at $\tau \rightarrow \pm\infty$. We can subtract this value, much in the same way as was done in [5]. The top of the potential hump in (3.23) corresponds to $M_{\text{sphaleron}} = 2\pi v^2$ [cf. Eq. (3.13)]; comparing with (3.23), we see that

$$\lambda = e^2. \quad (3.25)$$

From the above data, the instanton action in the effective quantum mechanics can be easily obtained. It is given by

$$S_{\text{inst}} = m_\chi^3 / \lambda = \frac{1}{e^2} \rightarrow \frac{m}{e^2}, \quad (3.26)$$

which is dimensionless (as required) and coincides with Eq. (3.19).

Thus, we can see that the mixing of the vortices does happen in the bosonic model. While the model with the potential (3.6) cannot be directly supersymmetrized, one can obtain a close $\mathcal{N} = 1$ cousin by deforming the supersymmetric model in Eq. (2.12) as

$$\mathcal{W} = \alpha_1 h(\Phi^\dagger \Phi - v^2) + \alpha_2 h^2 + \alpha_3 h^3, \quad (3.27)$$

where $\alpha_{1,2,3}$ are parameters. However, we do not expect that this supersymmetry would protect the vortex degeneracy. We postpone the discussion of supersymmetric models to Sec. IV C.

B. $\mathbb{CP}(1)$ vortices

Before passing to the study of non-Abelian superconductors in the next section, we want to mention another example. In this setup, non-Abelian degrees of freedom can emerge in a $U(1)$ superconductor.

This example was essentially worked out in [13,14]. Here we will briefly review it using an alternative language and give some additional comments with regard to instantons in a topologically nontrivial sector.

Let us start with the so-called n -field model, which is often referred to as $O(3)$ or $\mathbb{CP}(1) = SU(2)/U(1)$ in 3D. The target space is a two-dimensional sphere S_2 which can be parametrized by a unit vector

$$\vec{n} = \{n^1, n^2, n^3\}, \quad \vec{n}^2 = 1. \quad (3.28)$$

The Euclidean action of the $O(3)$ model is given by

$$A = \int d^3x \frac{1}{2g^2} \partial_\mu \vec{n} \partial^\mu \vec{n}. \quad (3.29)$$

In fact, this model has two independent real degrees of freedom which can be presented in geometric form as a complex field endowed with the Fubini-Study metric. In three dimensions, it has a localized static soliton (a particle) formally coinciding with the Belavin-Polyakov instanton [19]. However, this instanton is not what we are interested in.

To get to our main point, let us first deform (3.29) in two ways. We introduce the so-called “real mass” term

$$\mathcal{L}_m = -m^2(n^3)^2, \quad \text{in Euclidean} \quad \mathcal{L}_m = m^2(n^3)^2. \quad (3.30)$$

This term explicitly breaks $O(3)$ symmetry down to $U(1)$. The energy is minimized at $n^3 = 0$. If we introduce a complex field

$$\mathcal{S} = n^1 + in^2, \quad (3.31)$$

then due to the constraint (3.28) the vacuum field configuration takes the form

$$\mathcal{S}\bar{\mathcal{S}} = |\mathcal{S}|^2 = 1, \quad (n^3)_{\text{vac}} = 0, \quad \mathcal{S}_{\text{vac}} = e^{i\alpha}. \quad (3.32)$$

Deformation (3.30) is different from the setup of Refs. [13,14] where certain potentials are used. The real mass term allows for generalization to $\mathcal{N} = 2$ supersymmetry. Note that the “standard” way of adding the twisted mass term [7] would change the sign in front of m^2 in (3.30); the model then would have just two vacua, the north and the south poles.

The next step is to gauge the $U(1)$ symmetry. The corresponding Lagrangian can be written as

$$\begin{aligned} \mathcal{L}_{\text{gauged}} = & -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \\ & + \frac{1}{2g^2} [\mathcal{D}_\mu \mathcal{S}^\dagger \mathcal{D}^\mu \mathcal{S} + \partial_\mu n^3 \partial^\mu n^3 - m^2 (n^3)^2] \\ \mathcal{D}_\mu \mathcal{S} \equiv & (\partial_\mu - A_\mu) \mathcal{S}. \end{aligned} \quad (3.33)$$

The $U(1)$ gauging makes the physical vacuum state unique, i.e. α in (3.32) can be put to zero by a gauge rotation. The photon eats up the phase of the $\mathcal{S}$ field, acquiring the mass

$$m_\gamma = \frac{e}{g}. \quad (3.34)$$

The mass of the physical Higgs particle is

$$M_H = \frac{m}{\sqrt{2}}. \quad (3.35)$$

We assume they are of the same order of magnitude, $m_\gamma \sim M_H$.

Now, let us verify that two distinct static vortices exist in this model (in the quasiclassical approximation $eg \ll 1$).

To build a vortex, we must wind $\mathcal{S}$ at large distances from the vortex center, i.e. $\mathcal{S} = e^{i\alpha(x)}$ at $r \gg m^{-1}$, where α is now the polar angle, and the radius $r = (x_i x_i)^{1/2}$. To guarantee that the energy is finite, at $r \rightarrow 0$ we require $|\mathcal{S}| \rightarrow 0$. Then $(n^3)^2 \rightarrow 1$ and $n^3 \rightarrow \pm 1$. Thus, the $\mathbb{Z}_2$ symmetry

$$n^3 \rightarrow -n^3 \quad (3.36)$$

is spontaneously broken in the vortex core, giving rise to two distinct mass-degenerate vortex solutions, much in the same way as in Sec. III A above. The vortex mass $M_{\text{vortex}} \sim 1/g^2 \gg m_\gamma$ in the quasiclassical limit $eg \ll 1$.

The issue of the interpolating instanton is more subtle. We introduce the Euclidean time τ which varies between $-\infty$ and $+\infty$. The instanton is localized in a region around the origin of the Euclidean space, with the characteristic size $\sim m_\gamma^{-1} \times m_\gamma^{-1}$. It must interpolate between $n^3 = -1$ and

$n^3 = +1$ vortices in such a way that the corresponding action is finite; otherwise, there is no tunneling. The subtlety occurs near the sphaleron trajectory (at $\tau = 0$), a situation similar to that discussed in [5] (see Sec. 4.4.2 there).

The sphaleron trajectory is as follows:

$$(n_3)_{\text{sph}} = 0, \quad \mathcal{S} = e^{ia(x)}, \quad (3.37)$$

i.e. $\mathcal{S}$ winds. As a result, M_{sph} is infinite, because the energy density has a singularity $1/r^2$ near the origin, and the integral

$$\int d^2x \frac{1}{r^2} \quad (3.38)$$

is logarithmically divergent. The sphaleron does not exist on its own.

However, this singularity is integrable in the instanton action, which is determined by $\int d^3x$. Thus, the action is finite, and the two distinct vortices described above mix through tunneling.

C. Stability of excited vortices

To conclude, let us make a brief remark on the following curiosity. In our weakly coupled setup the bulk gap is large, while the vortex splitting is exponentially small.

This implies that, if we prepare a vortex in an excited state, it can just stay there and be stable. This happens because there is no suitable decay channel. If such vortex were to decay into the ground-state vortex, it would have to emit some light particle into the bulk, which is impossible when the bulk gap is sufficiently large.

IV. DEGENERATE VORTICES IN A NON-ABELIAN SUPERCONDUCTOR

In this section, we move to non-Abelian models, where vortices with higher multiplicity can be easily constructed.

A. Classical $\mathbb{Z}_N$ vortices in a non-Abelian superconductor

We start with a model motivated by studies of $\mathcal{N} = 2$ supersymmetric QCD (SQCD) in $3 + 1\text{D}$; see, e.g., [20,21] for a review. For now, we will be discussing only the bosonic version of this model. Later, we will discuss what happens when we actually add the corresponding fermionic fields and restore supersymmetry.

This is a relativistic field theory in $2 + 1$ dimensions. However, relativism is actually not important for most of the present discussion, as we will focus on static field configurations.

1. Model 1

We start with a $U(N)$ gauge theory coupled to scalar matter,

$$S = \int d^3x \left\{ \frac{1}{4g_2^2} (F_{\mu\nu}^a)^2 + \frac{1}{4g_1^2} (F_{\mu\nu})^2 + |\mathcal{D}_\mu \varphi^A|^2 + \frac{g_2^2}{2} (\bar{\varphi}_A T^a \varphi^A)^2 + \frac{g_1^2}{8} (|\varphi^A|^2 - N\xi)^2 \right\}. \quad (4.1)$$

Here, $F_{\mu\nu}$ and $F_{\mu\nu}^a$ are the Abelian and the non-Abelian parts of the $U(N)$ gauge field strength. Generators of $SU(N)$ are normalized as

$$\text{tr} T^a T^b = \frac{1}{2} \delta^{ab}. \quad (4.2)$$

The matter fields φ^A transform in the fundamental representation of the gauge group, and the covariant derivative is given by

$$\mathcal{D}_\mu = \partial_\mu - \frac{i}{2} A_\mu - i A_\mu^a T^a. \quad (4.3)$$

$A = 1, \dots, N_f$ is the flavor index; we focus on the case $N_f = N$. The action (4.1) enjoys $SU(N)_F$ global flavor symmetry. Each flavor has N color components; denoting the color index as $k = 1, \dots, N$, we can view φ_k^A as an $N \times N$ matrix.

The parameter ξ triggers condensation of $\varphi = \{\varphi_k^A\}$,

$$\langle \varphi \rangle = \sqrt{\xi} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}. \quad (4.4)$$

This VEV breaks both the color symmetry and the flavor symmetry to a diagonal (color-flavor locked) global subgroup:

$$U(N)_C \times SU(N)_F \rightarrow SU(N)_{C+F} \quad (4.5)$$

The spectrum around this vacuum is gapped, with masses of order $\sim g_{1,2} \sqrt{\xi}$.

Since the $U(1) \in U(N)$ is spontaneously broken by this VEV, this model supports conventional Abrikosov vortices. Such vortices would correspond to all diagonal components of φ winding with the same phase. However, the most elementary vortices correspond to field configurations where just one component has nontrivial winding at spatial infinity:

$$\langle \varphi \rangle_{\text{vor}} = \sqrt{\xi} \begin{pmatrix} e^{i\alpha} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad r \rightarrow \infty. \quad (4.6)$$

Here, (r, α) are the polar coordinates on the 2D space.

The winding (4.6) suggests the following ansatz for the field configuration:

$$\varphi(r, \alpha)_{\text{vor}} = \sqrt{\xi} \begin{pmatrix} \phi_2(r)^{i\alpha} & 0 & \cdots & 0 \\ 0 & \phi_1(r) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_1(r) \end{pmatrix}, \quad r \rightarrow \infty \quad (4.7)$$

and a corresponding ansatz for the gauge field. Such a field configuration further breaks the global symmetry (4.5) down to $SU(N-1) \times U(1)$. This suggests that the internal moduli space of such field configurations is

$$\mathbb{CP}(N-1) = \frac{SU(N)}{SU(N-1) \times U(1)}. \quad (4.8)$$

And indeed, one can show this quite explicitly; see Refs. [20,21] for a review.

The mass of this vortex is given by

$$M_{\text{vort}} = 2\pi\xi, \quad (4.9)$$

cf. Eq. (3.13). Naturally, (4.8) is supplemented by two translational zero modes, as for the usual Abrikosov vortex. These modes decouple and will not be important for the considerations here.

2. Model 2 (mass deformation)

As we will see below, the model (4.1) (in the purely bosonic case) does not lead to degenerate vortices, because quantum mechanics on the vortex worldline (4.8) has a unique ground state. In order to have nontrivial vortex degeneracy, we introduce a mass deformation of this model. To further simplify the model, we also take $g_1 = g_2 = g$.

For the deformation, we introduce a complex scalar field a transforming in the adjoint representation of the gauge group $U(N)$,

$$S = \int d^3x \left\{ \frac{1}{4g^2} (F_{\mu\nu}^a)^2 + \frac{1}{g^2} |D_\mu a|^2 + |\mathcal{D}_\mu \varphi^A|^2 + \frac{g^2}{2} (\bar{\varphi}_A T^a \varphi^A)^2 + \frac{g^2}{8} (|\varphi^A|^2 - N\xi)^2 + \sum_{A=1}^N |(a + M)\varphi|^2 \right\}. \quad (4.10)$$

Here, D_μ is a covariant derivative in the adjoint representation, while M is a mass matrix. We take M to be in a $\mathbb{Z}_N$ symmetric form, with the masses evenly spaced on a circle of some radius m :

$$M = \begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_N \end{pmatrix} \quad (4.11)$$

$$m_k = m e^{2\pi i k/N}, \quad m > 0. \quad (4.12)$$

The last term in (4.10) forces the adjoint scalar to have a VEV

$$\langle a \rangle = -M. \quad (4.13)$$

If the mass matrix M were proportional to an identity matrix, the VEV of a would not change the low-energy theory. However, in the present discussion the diagonal elements of M are given by (4.12). The VEV of a breaks the gauge group to an Abelian subgroup,

$$U(N) \rightarrow U(1)^N. \quad (4.14)$$

Thus, the color-flavor locked global symmetry (4.5) is also broken to $U(1)^{N-1}$.

What happens from the point of view of the vortices (4.6)? Strictly speaking, the $\mathbb{CP}(N-1)$ moduli space (4.8) gets lifted. Only N isolated points remain, corresponding to the N possible vortices where just one particular diagonal component of φ is nontrivial. These correspond to the degenerate vortices that we are interested in.

B. Worldline action and mixing of vortices

Now let us discuss the low-energy theory on the vortex world sheet in more detail. Of course, a vortex has two translational zero modes (bosonic), the same as an ordinary Abrikosov vortex. These modes are decoupled in the small-momentum limit.

On the other hand, the internal moduli space on the vortex gives rise to a nontrivial theory living on the vortex worldline. We start by writing down the low-energy theory

corresponding to the model without mass deformation; see (4.8). This situation is analogous to the study of the two-dimensional world sheet of non-Abelian strings in 4D SQCD; there, the effective theory comes out to be the $\mathbb{CP}(N-1)$ sigma model [22–25]; see Refs. [20,21,26–28] for a review. We can adapt these results to the present case, where the effective theory lives on a one-dimensional worldline. The (Euclidean) action for this theory is

$$S_{\text{vort}} = \int d\tau \left\{ |(\partial_\tau + iA_\tau)n^k|^2 + iD \left(\sum_k |n_k|^2 - \beta \right) \right\}. \quad (4.15)$$

Here, A_τ is an auxiliary gauge field, together with the real scalar D (they play the role of Lagrange multipliers). The coupling constant $\beta > 0$ is related to the coupling of the bulk $2+1$ D theory as

$$\beta \sim \frac{2\pi}{g^2}. \quad (4.16)$$

The theory (4.15) has $SU(N)$ global symmetry [there is no $U(1)$ factor because the overall phase of n_k is gauged away]. Naively, the constraint imposed by the last term in (4.15) forces n_k to have a nonzero VEV

$$\left\langle \sum_k |n_k|^2 \right\rangle = \beta, \quad (4.17)$$

thereby spontaneously breaking this $SU(N)$. However, we know that continuous global symmetries cannot be spontaneously broken in quantum mechanics. We stress that the theory (4.15), while having a nontrivial target space, lives on the one-dimensional worldline of the vortex. This means that the ground state of the effective theory in (4.15) is unique, and there is only one stable species of vortices.

To have a more interesting scenario, let us introduce the mass deformation discussed in Sec. IV A 2, explicitly breaking the global $SU(N)$ down to $\mathbb{Z}_N$. The effective action reads, in this case,

$$S_{\text{vort}} = \int d\tau \left\{ |(\partial_\tau + iA_\tau)n^k|^2 + \sum_k |\sigma - m_k|^2 |n^k|^2 + iD \left(\sum_k |n_k|^2 - \beta \right) \right\}. \quad (4.18)$$

Here, m_k are the mass parameters from the model (4.10), which are given by (4.12). The complex scalar σ is a new auxiliary field. Now, the classical minima of the model (4.18) are given by

$$\langle |n_k|^2 \rangle = \beta \delta_{kk_0}, \quad k_0 \in \mathbb{Z}_N \text{ is fixed.} \quad (4.19)$$

Classically, there are N ground states, where only one flavor of n 's has a nonzero VEV. These ground states are permuted by $\mathbb{Z}_N$ symmetry generated by shifts of the index k .

On the quantum mechanical level, these ground states may mix. Let us discuss this question in more detail.

It is known that the $(1+1)$ -dimensional version of the theory (4.18) has kinks; see Ref. [29]. Each kink interpolates between a pair of neighboring vacua (4.19).

In the present case, the model is one dimensional. The kinks mentioned above become instantons—classical solutions localized in Euclidean time τ . They lead to mixing of the N ground states.

Perturbatively, each ground state $|\psi_k^{\text{pert}}\rangle$ is localized near the corresponding minimum. The corresponding low-energy Hamiltonian is simply proportional to an identity matrix,

$$H_{\text{eff}}^{\text{pert}} \approx E_{\text{pert}}^{(0)} \mathbb{I}_{N \times N}, \quad (4.20)$$

where $E_{\text{pert}}^{(0)}$ is the perturbative part of the ground state energy in each of the N minima.

In the nonperturbative sector, instantons induce mixing between these states. This will induce a splitting of a would-be N -fold degenerate ground state into a band with a unique⁴ ground state, as we will see shortly.

To understand this splitting quantitatively, let us write down the low-energy Hamiltonian that takes the non-perturbative effects into account. The first step is computing the instanton actions. Fortunately, the $0+1$ D model at hand is closely related to a dimensional reduction of the $1+1$ D $\mathbb{CP}(N-1)$ model—or, more precisely, the reduction of the bosonic part of a supersymmetric model. In that supersymmetric model, the spectrum of kinks is known exactly [29]. As mentioned above, kinks in $1+1$ D become instantons in $0+1$ D, and the instanton action coincides with the mass of the “parent” kink. The action of an instanton interpolating from the perturbative vacuum number i to the vacuum number j is given by

$$S_{ij} = \beta |m_i - m_j|, \quad (4.21)$$

where β is the coupling, cf. Eq. (4.15). The quantity S_{ij} is smallest when $i = j \pm 1$, while the tunneling amplitudes between more distant states are suppressed, at least for $N > 3$. This allows us to write down the following low-energy Hamiltonian (leading order, with exponential accuracy):

⁴On the one-instanton level the ground state may turn out to be degenerate in some special cases, but such degeneracy is lifted by multi-instanton effects.

$$H_{\text{eff}} = \begin{pmatrix} E^{(0)} & -\Delta & 0 & \cdots & 0 & -\Delta \\ -\Delta & E^{(0)} & -\Delta & \ddots & & 0 \\ 0 & -\Delta & E^{(0)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\Delta & 0 \\ 0 & & \ddots & -\Delta & E^{(0)} & -\Delta \\ -\Delta & 0 & \cdots & 0 & -\Delta & E^{(0)} \end{pmatrix}. \quad (4.22)$$

Here, $E^{(0)}$ is the energy in a particular perturbative ground state given by the matrix element $\langle \psi_k^{\text{pert}} | H | \psi_k^{\text{pert}} \rangle$, while Δ is the absolute value of the transition amplitude $\langle \psi_k^{\text{pert}} | H | \psi_{k+1}^{\text{pert}} \rangle$, which is the same for any pair of adjacent minima,

$$\Delta \sim e^{-S_{12}} = e^{-\beta|m_k - m_{k+1}|} \sim e^{-2\beta m |\sin(\pi/N)|}. \quad (4.23)$$

Minus signs in front of Δ in (4.22) are not essential and simply reflect a convenient choice⁵ of the phases of the perturbative ground states $|\psi_k^{\text{pert}}\rangle$.

The matrix (4.22) is a real symmetric circulant matrix, whose spectrum is well known (this is basically the Bloch theorem),

$$E_k = E^{(0)} - 2\Delta \cos\left(\frac{2\pi k}{N}\right), \quad k = 1, \dots, N. \quad (4.24)$$

In either case, the ground state wave function is a symmetric combination of our basis $|\psi_k^{\text{pert}}\rangle$, with the ground state energy

$$E_{\text{ground}} \approx E^{(0)} - 2\Delta \sim E^{(0)} - 2me^{-2\beta m |\sin(\pi/N)|}. \quad (4.25)$$

This formula gives a good approximation at least when N is large enough. When N is small, the approximate form (4.22) does not work that well [the target space of $\mathbb{CP}(N-1)$ relevant at low energies becomes very different from a circle]. Still, a more careful analysis shows that the ground state is again unique [31].

From the point of view of the $2+1\text{D}$ bulk, these worldline instantons are nothing but monopole-instantons. These monopole-instantons cannot exist in the bulk by themselves—instead, they are confined to the vortex worldlines; see also [24]. The worldline then resembles a vine or a churchkhela sweet; see Fig. 3.

Let us reiterate what we observe here. We have a superconducting medium with vortices which classically appear to have N -fold degeneracy with respect to some

⁵Strictly speaking, this is true when N is even. For odd N , naively a two-fold degeneracy persists on the nonperturbative level, but in this case it will be lifted by further corrections, cf. [30]. Anyway, it is not important for the qualitative picture.

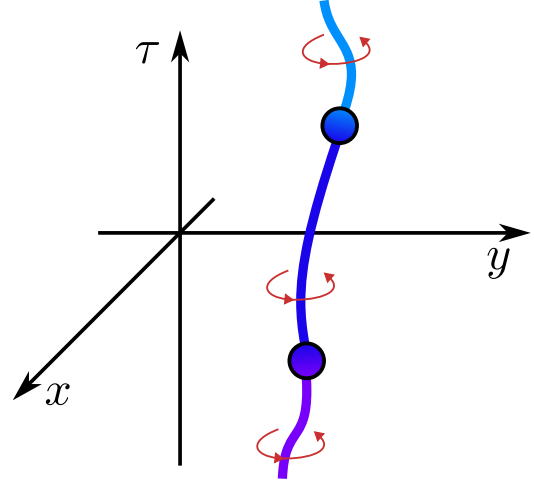


FIG. 3. Churchkhela tunneling process (schematically). The vertical axis is the Euclidean time τ . The vortex worldline has “lumps” representing the instantons. The line color represents the vortex species. Winding around every vortex is the same (shown by red curvy lines with arrows).

internal quantum numbers. However, because of the worldline instantons, this degeneracy is lifted, and the lowest-energy vortex is unique.

C. Supersymmetric case

Having discussed the vortices in bosonic theories, let us now add supersymmetry and see what changes.

1. Eight supercharges in the 3D bulk ($\mathcal{N}=4$)

Now, let us ask the question: what happens if we consider a supersymmetric version of the setup above? It often happens that, in the presence of fermions, the instantons acquire fermionic zero modes, which suppress the tunneling amplitudes between perturbative ground states.

As mentioned above, the model in Eq. (4.10) is motivated by the bosonic part of the $3+1\text{D}$ SQCD with $\mathcal{N}=2$ supersymmetry (eight supercharges). Basically, one can arrive at the action (4.10) by a dimensional reduction from $3+1\text{D}$ and setting the holonomy of the gauge field to zero (it is not turned on in the vortex background that we are interested in).

Let us now consider a reduction of the full supersymmetric SQCD by compactifying it on a circle. After the reduction, we will arrive at $2+1\text{D}$ SQCD with $\mathcal{N}=4$ supersymmetry (again, eight supercharges). The vortices discussed above become $1/2$ BPS objects. The worldline $\mathbb{CP}(N-1)$ quantum mechanics (4.18) becomes supersymmetric with four supercharges. In the context of the dimensional reduction, this QM can be viewed as descending from the world sheet effective $\mathbb{CP}(N-1)$ theory on the non-Abelian vortex in $3+1\text{D}$ SQCD; the latter has $\mathcal{N}=(2,2)$ supersymmetry.

Since the vortex configuration is determined first of all by its bosonic profiles, and these do not change when we include fermions, classically we still have degenerate vortices exchanged by the internal $\mathbb{Z}_N$ symmetry. In the purely bosonic case, these vortices mix due to worldline instantons. Let us see what happens in the supersymmetrized version of the model.

Consider the supersymmetric quantum mechanics on the vortex worldline. In principle, supersymmetry does not automatically guarantee that all classical ground states persist at the quantum level. The vacua with different fermionic structure do not mix in supersymmetric QM, but vacua with the same fermionic structure do mix. For example, this happens in a theory with one complex supercharge and a quartic superpotential; see, e.g., [32]. In general, the surviving degeneracy of ground states in quantum theory cannot exceed the number of linearly independent structures that one can form from the fermionic degrees of freedom at hand.

The $\mathbb{CP}(N-1)$ QM with four real supercharges (two complex) has $4(N-1)$ fermionic degrees of freedom, which can be enough to save all N ground states. In this case, the ground states are in one-to-one correspondence with the space of harmonic forms on $\mathbb{CP}(N-1)$ [33] (see also [32,34]), and the number of ground states is given by the Witten index, which coincides with the Euler characteristic of the target space:

$$\#_{\text{ground}} = \chi(\mathbb{CP}(N-1)) = N. \quad (4.26)$$

All ground states are described by even-degree forms. The instanton has exactly one fermionic zero mode, and because of that all transition amplitudes vanish.

The above argument for $\mathbb{CP}(N-1)$ QM and the ground state counting (4.26) turns out to be correct not only in the mass-deformed model, but also persists in the limit $m \rightarrow 0$.

We can conclude that when the $2+1\text{D}$ theory has eight supercharges, the BPS vortices remain degenerate at the quantum level.

2. Less supersymmetry ($\mathcal{N}=2$ or $\mathcal{N}=1$ in $2+1\text{D}$)

What happens if we reduce the number of supercharges in the bulk? Depending on the details, different scenarios are possible.

One can break $\mathcal{N}=4$ to $\mathcal{N}=2$ by deforming the bulk $2+1\text{D}$ theory. This can be achieved, e.g., by deforming the $3+1\text{D}$ SQCD and then dimensionally reducing to $2+1\text{D}$. We will mention here two well-studied examples, where the supersymmetry is softly broken by a mass term for the adjoint supermultiplet.

In the first example [35], the deformation in $3+1\text{D}$ keeps two supercharges on the flux tube world sheet. Two fermionic zero modes from the translational sector mix with internal $\mathbb{CP}(N-1)$ moduli, breaking the other two of the four supercharges from the undeformed case.

The supersymmetry on the world sheet turns out to be spontaneously broken, and, consequently, the Witten index is zero. However, the ground state degeneracy remains N , as the energies of all vacua become shifted from zero by the same amount. These ground states are associated with spontaneous breaking of the chiral symmetry on the world sheet, $\mathbb{Z}_{2N} \rightarrow \mathbb{Z}_2$.

When we perform a dimensional reduction, the flux tubes with $1+1\text{D}$ world sheets become vortices with $0+1\text{D}$ worldlines. Chiral symmetry disappears. Therefore, we expect that the ground state degeneracy is not protected and becomes lifted.

In the second example [36], the deformation just eliminates the world sheet supersymmetry completely. There are no massless fermions in the $\mathbb{CP}(N-1)$ internal sector. Therefore, the ground state degeneracy is lifted from the start.

Moving further, consider the case where the supersymmetry in the $2+1\text{D}$ theory reduces to $\mathcal{N}=1$ (two supercharges). In this situation, the vortex cannot be BPS. This can be seen from the fact that in $2+1\text{D}$, a vortex with winding number one always has two translational zero modes, and therefore it can be $1/2$ BPS only in a theory with four (or more) supercharges; see also the discussion in Sec. II.A of [37] or in Sec. 2.2.2 of [21]. In this case, while it may be possible that the instanton mixing is still suppressed (cf. the discussion in Sec. 3.8 of [5]), generically we cannot expect that. Rather, the most natural assumption is that the degeneracy is lifted in this case as well.

To reiterate, in the example of a non-Abelian superconductor considered here, we see degenerate vortices on the classical level. However, this degeneracy persists on the quantum level only in the case with $\mathcal{N}=4$ supersymmetry in $2+1\text{D}$ (eight supercharges), which gives four supercharges on the vortex.

When the amount of supersymmetry in the bulk is lowered, the degenerate vortices mix due to worldline instantons. As a result, the degeneracy is lifted, and there is a single lowest-energy vortex.

We expect that it should be possible to have an $\mathcal{N}=2$ model where the Witten index of the worldline quantum mechanics does not vanish (cf. [38]) and the vortices stay degenerate. We leave detailed investigation of this possibility to future work.

V. CONCLUSIONS

In this paper, we extended the study of quantum lifting of classical soliton multiplicity from $1+1$ dimensions to genuinely $(2+1)$ -dimensional settings, focusing on vortices. The central message is that whenever the classical vortex sector contains several distinct species with identical mass and topological charge, the effective $0+1\text{D}$ dynamics on the vortex worldline can typically admit Euclidean-time instantons that interpolate between these would-be distinct states, splitting the classical degeneracy.

We illustrated this mechanism first in Abelian superconductors. In the $U(1)$ model supplemented by a neutral scalar with an unbroken $\mathbb{Z}_2$ symmetry in the vacuum, the vortex core spontaneously breaks this $\mathbb{Z}_2$ and supports two classically degenerate vortices. These states can tunnel into each other through a finite-action worldline instanton, so the corresponding vortex “doublet” is not protected in the purely bosonic theory. A simple effective quantum-mechanical description captures the parametric behavior of the tunneling action, $S_{\text{inst}} \sim m/e^2$, with the splitting controlled at weak coupling. Interestingly, when the bulk gap is larger than the vortex splitting, excited vortices stay stable.

We then analyzed non-Abelian superconductors, where internal moduli are naturally present. In the $U(N)$ setup, the undeformed vortex has a $\mathbb{CP}(N-1)$ internal moduli space, while an appropriate mass deformation lifts the moduli space to N isolated classical vortices, providing an explicit realization of an N -fold classical multiplicity. The resulting worldline theory admits instantons [descended from $\mathbb{CP}(N-1)$ kinks], which generate mixing and split the N classical vacua into a band with a unique lowest-energy state in the generic bosonic case. From the bulk viewpoint, these worldline instantons can be understood as monopole-instantons confined to the vortex. Supersymmetry, if present, can still protect the vortex degeneracy.

Logically, the next step would be a discussion of degenerate monopoles in a $3+1$ D setting, where they become particlelike states. One can generalize the basic idea of Sec. III. Namely, the simplest way is to introduce an extra field that vanishes in the vacuum, but which has a nonzero VEV at the monopole core. If there is a discrete symmetry associated with this extra field, then that symmetry is spontaneously broken by the monopole, and one can obtain multiple monopole species, degenerate in energy.

Another potentially interesting avenue could be an investigation of 2D theories with $\mathcal{N} = (0, 2)$ supersymmetry. An example of this is a heterotic $\mathbb{CP}(1)$ with “matter” [e.g. a $\mathbb{CP}(1) \times \mathbb{C}$ model]. It may be possible to construct degenerate kinks that are BPS and hence conserve one supercharge. In this case, it would be interesting to understand the structure of the fermion zero modes on the kink, and also what happens to the worldline instantons and tunneling in this case.

We leave these questions for future work.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: $\mathcal{N} = 1$ SUPERSYMMETRY IN THREE DIMENSIONS

In this appendix, we are going to review some basic constructions used in 3D theories with $\mathcal{N} = 1$ supersymmetry (two supercharges). For the most part, we are going to follow [39], although our notation is somewhat modernized to be consistent with the generally accepted 4D conventions; see, e.g., [40] and Secs. 10.2-10.6 of [7].

1. Spinors in 3D

The superspace is parametrized by coordinates x^μ and θ^α , where

$$x^\mu = \{x^1, x^2, x^3\} \equiv \{t, x, z\},$$

and the fermion directions of superspace are parametrized by two real Grassmann numbers θ^1 and θ^2 forming a Majorana spinor in 3D,

$$\theta^\alpha = \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}. \quad (\text{A1})$$

We use Greek letters from the beginning of the Greek alphabet ($\alpha, \beta, \dots$) as spinorial indices for 3D Majorana spinors, while those from the end of the alphabet ($\mu, \nu, \dots$) are used as vectorial indices.

Our convention for the metric tensor is (note that it is different from the one used in [39])

$$g^{\mu\nu} = \text{diag}\{1, -1, -1\}. \quad (\text{A2})$$

The gamma matrices satisfy the standard algebra

$$\gamma^\mu \quad \text{for } \mu = 0, 1, 3 \quad \text{and} \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (\text{A3})$$

It is convenient to choose the Majorana representation where the γ matrices are purely imaginary, and γ^0 is Hermitian while γ^1 and γ^3 are anti-Hermitian. Explicitly,

$$(\gamma^0)_{\alpha\beta} = (\sigma_2)_{\alpha\beta}, \quad (\gamma^1)_{\alpha\beta} = i(\sigma_1)_{\alpha\beta}, \quad (\gamma^3)_{\alpha\beta} = -i(\sigma_3)_{\alpha\beta}. \quad (\text{A4})$$

Dirac conjugation is defined as

$$\bar{\psi} = \psi^\dagger \gamma^0 = (i\psi_2, -i\psi_1). \quad (\text{A5})$$

Raising and lowering of spinorial indices is performed as usual—by multiplying a spinorial quantity from the left by

$$\begin{aligned} \epsilon^{\alpha\beta} &= i(\sigma_2)_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \epsilon_{\alpha\beta} &= -i(\sigma_2)_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \quad (\text{A6})$$

With this convention,

$$\varepsilon^{\alpha\beta}\varepsilon_{\beta\delta} = \delta_{\delta}^{\alpha}. \quad (\text{A7})$$

In 4D supersymmetric theories, along with the gamma matrices it is very convenient to use σ^{μ} matrices, namely (see e.g. [7]) $(\sigma^{\mu})_{\alpha\beta} = \{1, \sigma_1, \sigma_2, \sigma_3\}_{\alpha\beta}$. In three dimensions we can define the analogous tensor as follows:

$$\begin{aligned} (\Gamma^{\mu})_{\alpha\beta} &= (\gamma^0\gamma^{\mu})_{\alpha\beta} = \{1, \sigma_3, \sigma_1\}_{\alpha\beta}, \\ (\Gamma^{\mu})^{\alpha\beta} &= \varepsilon^{\alpha\delta}\varepsilon^{\beta\rho}(\Gamma^{\mu})_{\delta\rho} = \{1, -\sigma_3, -\sigma_1\}_{\alpha\beta}. \end{aligned} \quad (\text{A8})$$

All Γ^{μ} matrices defined this way are Hermitian, real, and symmetric. With their help we can convert 3D vectors into the spinor representation, where they become *symmetric* 2×2 matrices. For example, a vector $A_{\mu} = \{A^0, -\vec{A}\}$ becomes

$$A_{\alpha\beta} \equiv A_{\mu}(\Gamma^{\mu})_{\alpha\beta} = (A_0 - A^1\sigma_1 - A^2\sigma_2 - A^3\sigma_3)_{\alpha\beta}, \quad (\text{A9})$$

while the inverse transformation is

$$A^{\mu} = \frac{1}{2}A_{\alpha\beta}(\Gamma^{\mu})^{\alpha\beta}. \quad (\text{A10})$$

Other useful identities involving vectors include

$$\begin{aligned} A_{\mu}B^{\mu} &= A_0B_0 - \vec{A}\vec{B} = \frac{1}{2}A_{\alpha\beta}B^{\beta\alpha}, \\ \partial_{\alpha\beta}\partial^{\gamma\beta} &= \delta_{\alpha}^{\gamma}\partial^{\mu}\partial_{\mu}. \end{aligned} \quad (\text{A11})$$

Other useful spinorial identities include

$$\begin{aligned} (\Gamma^{\mu})_{\alpha\beta}(\Gamma^{\nu})^{\alpha\beta} &= 2g^{\mu\nu}; \\ \theta^{\alpha}\theta^{\beta} &= -\frac{1}{2}\varepsilon^{\alpha\beta}(\theta^1\theta_1 + \theta^2\theta_2) \equiv -\frac{1}{2}\varepsilon^{\alpha\beta}\theta^{\{2\}}; \\ \theta^{\{2\}} &\equiv (\theta^1\theta_1 + \theta^2\theta_2) = -2\theta^1\theta^2 = i\bar{\theta}\theta; \\ \theta^{\alpha}\theta'_{\alpha} &= \theta^{\beta}\theta_{\beta}; \quad \frac{\partial}{\partial\theta^{\alpha}}\frac{i}{2}\theta^{\{2\}} = i\varepsilon_{\alpha\beta}\theta^{\beta}. \end{aligned} \quad (\text{A12})$$

Note that in three dimensions with one real superfield, the measure $d\theta^1d\theta^2$ would be anti-Hermitian.⁶ To make it Hermitian in the superaction, we will replace it by

$$id\theta^1d\theta^2 \xrightarrow{\text{def}} d^2\theta. \quad (\text{A13})$$

Then we will normalize the Berezin integral as

$$\int d\theta^1d\theta^2\theta^1\theta^2 = \frac{i}{2}\int d^2\theta\theta^{\{2\}} = 1. \quad (\text{A14})$$

⁶Conjugation switches the order of the terms in the product, and bringing it back to $d\theta^1d\theta^2$ yields an extra minus sign.

2. Superspace and superfields

There is no chirality in the minimal 3D theory. The superspace is based on five coordinates— c -numerical (t, x^1, x^3) and two Grassmann $\theta^{1,2}$. Supercoordinate transformations are defined as

$$\begin{aligned} \mathcal{M} = \{x^{\mu}, \theta^{\alpha}\} &\xrightarrow{\text{susy}} \{x^{\mu} + \delta x^{\mu}, \theta^{\alpha} + \delta\theta^{\alpha}\}, \\ \delta\theta^{\alpha} &= \epsilon^{\alpha}, \quad \delta x_{\alpha\beta} = -i(\theta_{\alpha}\epsilon_{\beta} + \theta_{\beta}\epsilon_{\alpha}), \\ x_{\alpha\beta} &= x_{\mu}(\Gamma^{\mu})_{\alpha\beta}, \quad x^{\mu} = \frac{1}{2}x_{\alpha\beta}(\Gamma^{\mu})^{\alpha\beta}, \end{aligned} \quad (\text{A15})$$

with fermionic parameters ϵ_{α} . It is easy to see that under the above transformations the components of the *real* scalar superfield,

$$\Phi_{\text{r}} = \varphi + i\theta^{\alpha}\psi_{\alpha} + \frac{i}{2}\theta^{\{2\}}F, \quad (\text{A16})$$

transform through each other as

$$\delta\varphi = \epsilon^{\alpha}\psi_{\alpha}, \quad \delta\psi_{\alpha} = -i\epsilon^{\beta}\partial_{\alpha\beta}\varphi + \epsilon_{\alpha}F, \quad \delta F = -i\epsilon^{\beta}\partial_{\alpha\beta}\psi^{\alpha}. \quad (\text{A17})$$

Here we omit the “flavor” superscript a introduced in (2.1).

Spin-covariant derivatives on the superspace are defined as

$$D_{\alpha} = \left(i\frac{\partial}{\partial\theta^{\alpha}} + \theta^{\beta}\partial_{\alpha\beta}\right), \quad \partial_{\alpha\beta} = \frac{\partial}{\partial x^{\mu}}(\Gamma^{\mu})_{\alpha\beta}. \quad (\text{A18})$$

With this definition at hand,

$$D_{\alpha}\Phi_{\text{r}} = -\psi_{\alpha} - \theta_{\alpha}F + \theta^{\beta}(\partial_{\alpha\beta}\varphi) - \frac{i}{2}\theta^{\{2\}}\partial_{\alpha\beta}\psi^{\beta}. \quad (\text{A19})$$

Other useful expressions include

$$D_{\alpha}D_{\beta} = i\partial_{\alpha\beta} - \varepsilon_{\alpha\beta}D^{\{2\}}, \quad \{D_{\alpha}D_{\beta}\} = 2i\partial_{\alpha\beta}. \quad (\text{A20})$$

3. Wess-Zumino type model

Now we have all we need to write down the kinetic part of the action for a real matter field in superspace,

$$\begin{aligned} S_{\text{r}} &= \int d^3x d^2\theta \frac{i}{4}(D^{\alpha}\Phi_{\text{r}})^{\dagger}(D_{\alpha}\Phi_{\text{r}}) \\ &= \int d^3x \frac{1}{2}[\partial_{\mu}\varphi\partial^{\mu}\varphi + \psi^{\alpha}i\partial_{\alpha\beta}\psi^{\beta} + F^2]; \end{aligned} \quad (\text{A21})$$

see also our definition for the θ -measure (A13). The potential interaction is usually introduced through a superpotential. Because of the absence of the chiral subspace in 3D, we cannot do it in the standard way. Nevertheless, if we introduce a function $\mathcal{W}(\Phi_{\text{r}})$ and add it in the Lagrangian as

$$\mathcal{L}_{\mathcal{W}} = \int d^2\theta \mathcal{W}(\Phi_r), \quad (\text{A22})$$

some general features of the superpotential survive. For instance, the mass term is given by

$$\mathcal{W}(\Phi_r) = \frac{1}{2} m \Phi_r^2, \quad \mathcal{L}_{\text{mass}} = \frac{1}{2} m \bar{\psi} \psi + m \phi F. \quad (\text{A23})$$

Note that in the present case with Majorana fermions $\bar{\psi}\psi = 2i\psi_2\psi_1$, cf. Eq. (A12). In the case of a more general superpotential, the scalar potential and the fermion mass term are given by

$$U(\phi) = \frac{1}{2} \left(\frac{\partial \mathcal{W}}{\partial \Phi_r} \right)^2 \Big|_{\theta=0}; \quad \mathcal{L}_{\psi\text{mass}} = \frac{1}{2} \left(\frac{\partial^2 \mathcal{W}}{\partial \Phi_r^2} \right) \Big|_{\theta=0} \quad (\text{A24})$$

where we have eliminated the auxiliary field F by virtue of its equation of motion.

4. Complexification and gauging

In order to have matter that is charged under a $U(1)$ gauge group, we, quite plainly, need a $U(1)$ symmetry in the matter sector. To this end, we consider the above setup with two real scalar superfields Φ_r^a , $a = 1, 2$ defined in (A16), which we combine into one complex superfield

$$\Phi_c = \frac{1}{\sqrt{2}} (\Phi_r^1 + i\Phi_r^2) = \phi + i\theta^\alpha \Psi_\alpha + \frac{i}{2} \theta^{\{2\}} \mathcal{F}, \quad (\text{A25})$$

the same as was defined in Eq. (2.3). In what follows, we endow the complex superfield (A25) with unit electric charge.

The gauge superfield V_α can be constructed following the line of reasoning of Gates *et al.* [39]. Recall that in the nonsupersymmetric QED the photon field is transformed under the gauge transformation as

$$(A_\mu)_{\text{gt}} = A_\mu + \partial_\mu \alpha(t, x); \quad (\text{A26})$$

here, $x = (x^1, x^2)$ are the spatial coordinates, and the subscript “gt” stands for gauge transformed. This transformation can be derived, for instance, from the requirement

$$(\mathcal{D}_\mu \phi(x))_{\text{gt}} = e^{i\alpha(x)} (\mathcal{D}_\mu \phi(x)) \quad \text{if } \phi_{\text{gt}}(x) = e^{i\alpha(x)} \phi(x), \quad (\text{A27})$$

where ϕ is a matter field and $\mathcal{D}_\mu$ is the gauge covariant derivative,

$$\mathcal{D}_\mu = \partial_\mu - iA_\mu. \quad (\text{A28})$$

Next, we do the same for supergauge transformations. In analogy with the second term in (A27), we have

$$\Phi_{\text{gt}} = e^{iK} \Phi, \quad (\text{A29})$$

where K is a real scalar superfield,

$$K = \kappa + i\theta^\alpha \xi_\alpha + \frac{i}{2} \theta^{\{2\}} \chi. \quad (\text{A30})$$

The gauge covariantization starts from the spinorial derivative D_α [39],

$$D_\alpha \longrightarrow \mathcal{D}_\alpha = D_\alpha - iV_\alpha. \quad (\text{A31})$$

In analogy with (A27), we demand that

$$(\mathcal{D}_\alpha \Phi)_{\text{gt}} = e^{iK} (\mathcal{D}_\alpha \Phi). \quad (\text{A32})$$

This can be achieved provided that

$$(\mathcal{D}_\alpha)_{\text{gt}} = e^{iK} \mathcal{D}_\alpha e^{-iK}, \quad (\text{A33})$$

which in turn implies

$$(V_\alpha)_{\text{gt}} = V_\alpha + \delta V_\alpha, \quad \delta V_\alpha = D_\alpha K. \quad (\text{A34})$$

Equation (A34) prompts the form of the gauge connection V_α . Indeed, from Eqs. (A18) and (A30) we conclude that

$$D_\alpha K = -\xi_\alpha + \theta^\beta \partial_{\alpha\beta} \kappa - \theta_\alpha \chi - \frac{i}{2} \theta^{\{2\}} \partial_{\alpha\beta} \xi^\beta. \quad (\text{A35})$$

In components, V_α takes the form

$$V_\alpha = \zeta_\alpha + \theta^\beta A_{\alpha\beta} + \theta_\alpha \sigma + \frac{i}{2} \theta^{\{2\}} (\lambda_\alpha + \partial_{\alpha\beta} \zeta^\beta). \quad (\text{A36})$$

Comparing (A35) and (A36), we see that we can choose a gauge which is analogous to the Wess-Zumino gauge in 4D:

$$(V_\alpha)_{\text{WZ}} = \theta^\beta A_{\alpha\beta} + \frac{i}{2} \theta^{\{2\}} \lambda_\alpha. \quad (\text{A37})$$

In what follows, we will omit the subscript WZ.

A superfield representing the gauge field strength can be defined as

$$W_\alpha = \frac{1}{2} D^\beta \mathcal{D}_\alpha V_\beta = \lambda_\alpha + \tilde{F}_\mu (\Gamma^\mu)_{\alpha\beta} \theta^\beta + i\theta^{\{2\}} \partial_{\alpha\beta} \lambda^\beta, \quad (\text{A38})$$

where $\tilde{F}_\mu$ is the dual photon field tensor,

$$\tilde{F}_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho} F^{\nu\rho} = \varepsilon_{\mu\nu\rho} \partial^\nu A^\rho. \quad (\text{A39})$$

The super-Bianchi identity has the form

$$D^\alpha W_\alpha = 0. \quad (\text{A40})$$

This formula can be trivially verified if we use the component expansion (A38), the property $\partial^\mu \tilde{F}_\mu \equiv 0$ following from (A39), and the fact that all three Γ matrices are symmetric with respect to the spinorial indices.

a. Three-dimensional $\mathcal{N} = 1$ supersymmetric QED

In three dimensions, the photon field A_μ represents one physical degree of freedom, and so does the Majorana two-component spinor λ_α —the lowest component in the superfield (A38). The photon/photino kinetic terms in the action take the form

$$S_{\gamma\text{kin}} = \int d^3x d^2\theta \frac{1}{2e^2} W^\alpha W_\alpha, \\ \mathcal{L}_{\gamma\text{kin}} = \frac{1}{2e^2} (-\tilde{F}_\mu \tilde{F}^\mu + i\lambda^\alpha \partial_{\alpha\beta} \lambda^\beta). \quad (\text{A41})$$

The kinetic part of the matter action can be written down covariantly as

$$S_{\text{kin matter}} = \int d^3x d^2\theta (\mathcal{D}^\alpha \Phi_c^\dagger \mathcal{D}_\alpha \Phi_c), \quad (\text{A42})$$

where the covariantized spinor derivative $\mathcal{D}_\alpha \Phi_c$ is defined above Eqs. (A31) and (A37). The “superpotential” term has the standard form⁷

$$S_{\text{matter}\mathcal{W}} = \int d^3x d^2\theta \mathcal{W}(\Phi_c, \Phi_c^\dagger). \quad (\text{A43})$$

The $\theta^{\{2\}}$ component of the superfield Φ_c from Eq. (A25) (i.e. the $\mathcal{F}$ term) enters the action without derivatives and, therefore, can be eliminated by using the classical equation of motion. In this way, we derive the scalar potential U ,

$$U(\phi, \bar{\phi}) = \left. \frac{\partial \mathcal{W}}{\partial \Phi_c} \frac{\partial \mathcal{W}}{\partial \Phi_c^\dagger} \right|_{\theta=0}. \quad (\text{A44})$$

The “superpotential” $\mathcal{W}$ which will be responsible for Higgsing of the gauge field and its superpartner λ is chosen as follows. Usually, to this end, we combine quadratic and cubic matter superfields, but in the case at hand this is

⁷Similar to the discussion of the real superfields above Eq. (A22), in $\mathcal{N} = 1$ SQED no bona fide chiral superpotential exists because there are no chiral subspaces. This is the reason for the quotation marks. The “superpotential” in (A43) depends on both Φ_c and $\Phi_c^\dagger$. Nevertheless, many conventional relations are still valid.

impossible. Indeed, no cubic gauge-invariant term exists in the case at hand. Therefore, we are forced to invoke a quartic term,

$$\mathcal{W} = m \Phi_c^\dagger \Phi_c - \frac{g}{2} (\Phi_c^\dagger \Phi_c)^2, \quad (\text{A45})$$

where m and g are assumed to be real positive parameters. This yields

$$U(\phi, \bar{\phi}) = (\phi \phi^\dagger) [m - g(\phi \phi^\dagger)]^2. \quad (\text{A46})$$

Since in the case at hand the mass dimension of the superpotential is 2, and that of the superfield Φ_c is $1/2$, m has mass dimension 1 while g is dimensionless. The above expression implies that the scalar potential will have a term of sixth order in the matter field. This does not spoil renormalizability of our 3D theory. Moreover, in addition to the Higgs regime $\Phi_{\text{vac}} \neq 0$ we will also have a $\Phi_{\text{vac}} = 0$ vacuum with the un-Higgsed gauge field.

5. Central (brane) charges

The $\mathcal{N} = 1$ SUSY algebra in three dimensions $\{t, x, z\}$ is as follows:

$$\{Q_\alpha, Q_\beta\} = (\Gamma^\mu)_{\alpha\beta} (P_\mu + \mathcal{Z}_\mu), \quad [Q_\alpha, P_\mu] = [Q_\alpha, \mathcal{Z}_\mu] = 0, \\ \mu, \nu = 0, 1, 3, \quad (\text{A47})$$

where P_μ , ($\mu = 0, 1, 3$) is the energy-momentum vector.

If we introduce a topologically (i.e. nondynamically) conserved two-index tensor

$$\zeta_{\mu\nu} = \varepsilon_{\mu\nu\rho} \partial^\rho \mathcal{W}, \quad (\text{A48})$$

we find a domain line (wall) charge defined as

$$\mathcal{Z}_\mu = \int d^2x \zeta_{\mu 0}. \quad (\text{A49})$$

For a static domain line stretched along the x axis (i.e. its profile depends only on z), $\mathcal{Z}_\mu$ reduces to

$$L \mathcal{Z}_{\mu=1} = \int dz \partial_z \phi^a \partial_a \mathcal{W}, \quad \mathcal{Z}_{\mu=0 \text{ or } 3} = 0, \quad (\text{A50})$$

and $P_\mu \mathcal{Z}^\mu = 0$ as is required.

We can conclude that, while $\mathcal{N} = 1$ superalgebra in 3D admits codimension-1 BPS objects, there is no central charge for a codimension-2 state (like a vortex).

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