

Explicit bounds on the spectrum of 6D $\mathcal{N} = (1, 0)$ supergravity

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We propose a novel strategy to derive explicit and uniform upper bounds on the particle spectrum of six-dimensional gravitational theories with minimal supersymmetry, focusing initially on the tensor sector. The strategy is motivated by considerations of F-theory compactifications on elliptic Calabi-Yau threefolds. However, it admits a clear bottom-up interpretation and is, thus, applicable to general supergravity theories modulo certain physical conjectures. At the heart of the strategy are two key structures, most natural in birational geometry: One concerns the singularity of the natural pairs on the base manifolds, and the other the fibration generically exhibited by the bases. Put physically, the former structure keeps the effective theories from decompactifying, and the latter ensures the (generic) presence of a heterotic string. We sketch our bounding strategy and present, for an illustration, the explicit bounds thereby derived on the tensor spectrum, with the technical details relegated to a companion paper [C. Birkar and S. J. Lee, [arXiv:2507.06317](https://arxiv.org/abs/2507.06317)].

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I. INTRODUCTION

The quantum nature of gravity has proven to impose general consistency constraints on the structure of effective theories. The precise origin of such constraints is often-times difficult to clarify via the field-theoretic description of the effective physics alone. It is, therefore, of utmost interest to delineate the *landscape*, the set of effective theories with a consistent embedding into quantum gravity, whose complement, the *swampland*, is an infinite set of inconsistent theories [1].

As it turns out, many of the apparently consistent effective theories end up being placed in the swampland, and, hence, the notion of finiteness is naturally brought to the scene of the landscape. To be precise, the finiteness notion here concerns the discrete data of the effective theories, with the prototypical example being the particle spectrum, which counts light degrees of freedom.

The general constraints commonly imposed on every consistent theory in the landscape are often proposed as a conjecture based on bottom-up intuitions. For a

complementary approach, it is, thus, important to analyze the individual conjectures from top-down perspectives, notably via string theory. In particular, one can pursue microscopic understanding of the conjectured constraints by revealing appropriate universal features of the internal compactification geometry, which strings are allowed to probe. In this context, enumerative geometry of Calabi-Yau varieties has served as a versatile toolkit for quantitative analyses, while the role of birational geometry has not been as extensively studied by now.

The aim of this note is to propose a novel strategy in birational geometry, by which one can derive explicit and uniform upper bounds on the supermultiplet spectrum of six-dimensional (6D) gravitational theories with $\mathcal{N} = (1, 0)$ supersymmetry. The strategy is initially developed for the string vacua obtained via geometric compactifications of F-theory on an elliptic Calabi-Yau threefold. However, it is naturally interpreted purely in physical terms, indicating that the resulting bounds also constrain general effective theories modulo certain physical conjectures, without relying *a priori* on the geometric or the string-theoretic origin of the theories.

While the finiteness of elliptic Calabi-Yau threefolds (or, equivalently, that of the geometric vacua of F-theory in six dimensions) was established decades ago [2],¹ no explicit

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¹To be precise, what was proven in [2] is that elliptic Calabi-Yau threefolds form a bounded family modulo flops. The boundedness of topological types has been addressed more recently in [3]; see also [4] for some of the F-theoretic implications.

bounds on their topological types and, in particular, on their Hodge numbers have so far been calculated. On the other hand, the strategy proposed in this note allows one to bound by concrete numbers some of the topological quantities as well as their physical counterparts and manifests the origin of the bounds obtained. For an illustration, we will apply the strategy to bound the Picard number of the base twofolds, thereby uniformly bounding the number T of the 6D tensor multiplets. Such an explicit bound on the tensor sector will then serve as a stepping stone to the associated bounds on the other sectors of the theory.

This note is organized as follows. In Sec. II, upon reviewing the basics of 6D $\mathcal{N} = (1, 0)$ supergravity, we set the geometric arena up for the analysis of 6D F-theory and introduce the notion of pairs as well as their singularity types. Next, in Sec. III, we overview our bounding strategy and argue for explicit bounds on T , notably,

$$T \leq 567. \quad (1)$$

Here, the bound (1) is by no means optimal, given that its derivation leaves much room for improvement [5], and should rather be viewed as a proof of principle. Finally, in Sec. IV, we close with further discussion of physics, as well as comments on applications and generalizations.

II. BACKGROUND AND REVIEW

The field contents of a 6D $\mathcal{N} = (1, 0)$ supergravity theory are organized into supermultiplets of three types beyond the unique gravity multiplet. These are vector, tensor, and hyper multiplets, whose numbers we denote, respectively, as V , T , and H .

The (charged) hypermultiplets are the matter fields transforming under the gauge algebra $\mathfrak{g}$ carried by the vector multiplets; for later purposes, we decompose $\mathfrak{g}$ into the Abelian and the non-Abelian parts:

$$\mathfrak{g} = (\oplus_{\kappa} \mathfrak{u}(1)_{\kappa}) \oplus (\oplus_i \mathfrak{g}_i), \quad (2)$$

where $\mathfrak{g}_i$ are simple Lie algebras. The gravitational and the gauge anomalies of the theory are characterized by the $\mathfrak{so}(1, T)$ vectors known as the anomaly coefficients:

$$a, b_{\kappa}, b_i \in \mathbb{R}^{1, T}. \quad (3)$$

Here, the T directions of $\mathbb{R}^{1, T}$ with a negative norm are associated to as many tensor multiplets, each containing one antichiral tensor along with a real scalar. Then, for the anomaly cancellation, various inner products of (3) are subject to certain consistency conditions, which, however, do not bound the spectrum by themselves [6–8], rendering the set of anomaly-free theories infinite.

Let us now consider the 6D supergravity theories with a geometric realization in F-theory, each described by a genus-one fibered Calabi-Yau threefold $\hat{Y}$:

$$\hat{\pi}: \hat{Y} \rightarrow Z, \quad (4)$$

whose base twofold Z serves as the internal manifold. Such a fibration has an associated Weierstrass description²

$$\pi: Y \rightarrow Z, \quad (5)$$

which we focus on in this note. Specifically, the Weierstrass model in scrutiny is of the form

$$y^2 = x^3 + Fx + G, \quad (6)$$

where (F, G) is a pair of sections,

$$F \in H^0(Z, \mathcal{O}_Z(-4K_Z)), \quad G \in H^0(Z, \mathcal{O}_Z(-6K_Z)), \quad (7)$$

with K_Z denoting the canonical divisor of Z . Then, the singular fibers of π are supported over the zero loci of the discriminant section:

$$\Delta := 4F^3 + 27G^2 \in H^0(Z, \mathcal{O}_Z(-12K_Z)), \quad (8)$$

whose irreducible components Δ_i wrapped by 7-branes carry the algebras $\mathfrak{g}_i$, as encoded in the vanishing orders:

$$\text{ord}_{\Delta_i}(F, G, \Delta), \quad (9)$$

via the Kodaira-Néron classification.

On the other hand, in algebraic geometry, one can naturally associate two divisors with the fibration π : One is the *discriminant divisor* Λ characterizing the singularities, whether from the base, the fiber, or the total space, and the other is the *moduli divisor* M defined by its divisor class $[M] = -[K_Z] - [\Lambda]$ with a certain positivity property. With the base Z smooth, the combined divisor

$$B := \Lambda + M \quad (10)$$

carries the same information³ as the divisor Δ in (8).

²If $\hat{\pi}$ does not have a section, we consider the Jacobian variety.

³To be precise, the discriminant divisor Λ agrees with $\frac{1}{12}\Delta$ over the irreducible components Δ_i supporting singular fibers *not* of the Kodaira type I_1 , except that the components supporting $I_{n>1}$ and $I_{n>0}^*$ fibers, if any, lead to discrepancy; note that those exceptional loci arise only via a further fine-tuning of the Weierstrass model. On the other hand, the moduli divisor M is numerically equivalent to $\frac{1}{12}R$ for some effective integral divisor R , which, modulo the caveat stated, has the same divisor class as the I_1 locus.

Specifically, the discriminant divisor is determined as a linear combination of the irreducible divisors:

$$\Lambda = \sum (1 - t_C)C, \quad (11)$$

where t_C are the lc threshold of π^*C defined as

$$t_C := \sup\{s \mid (Y, s\pi^*C) \text{ is lc over general } z \in C\}. \quad (12)$$

Here, lc refers to the singularity type known as *log canonical*, which we now turn to, restricting for simplicity to the current geometric setup.

We start by introducing the notion of a *pair*, (Z, B) , which combines the smooth twofold Z with an effective $\mathbb{Q}$ -divisor B such that $K_Z + B$ is $\mathbb{Q}$ -Cartier; to clearly define singularities, we assume that B has coefficients in $[0, 1]$. For a log resolution $\phi: W \rightarrow Z$ of (Z, B) , we write

$$K_W + B_W = \phi^*(K_Z + B), \quad (13)$$

where B_W is uniquely determined, with the canonical divisor K_W of W chosen such that $\phi_*K_W = K_Z$. Then, we say that the pair (Z, B) is *log canonical* (lc) if each coefficient of B_W is not bigger than 1 and *ϵ -log canonical* (ϵ -lc) if not bigger than $1 - \epsilon$, where $\epsilon \geq 0$.

III. OVERVIEW OF THE STRATEGY

In this section, we sketch the birational strategy for deriving an upper bound on the tensor count T , which is related to the Picard number $h^{1,1}(Z)$ as

$$T = h^{1,1}(Z) - 1. \quad (14)$$

Mathematically, the goal is, thus, to bound $h^{1,1}(Z)$ from above. For readability, here we present the key ideas of the strategy, as well as some of the major claims, and proceed to an illustrative calculation of explicit bounds. Rigorous proofs, along with technical details, are left to the forthcoming paper [5].

A. Key to the boundedness of T

Two geometric structures lie at the heart of the bounding strategy, each bearing a clear physical interpretation. Let us elaborate on them in turn.

1. $\frac{1}{6}$ lc singularity: General structure of (Z, B)

The first key structure is found from the natural pair:

$$(Z, B := \Lambda + M), \quad (15)$$

where Z is the base twofold and the divisor Λ , along with M , encodes the fibral degenerations, as discussed in Sec. II. Then, the pair carries the singularity of a special kind:

$$(Z, B) \text{ is } \frac{1}{6}\text{-lc}. \quad (16)$$

As observed in [5], the structure (16) is inherent in the fibration (5) and has many interesting consequences, one of which is that every irreducible curve C of Z obeys

$$C \cdot C \geq -12. \quad (17)$$

For a physical interpretation of (16) and, in particular, of (17), we note that any curve C with $\text{coeff}_C(B) > \frac{5}{6}$ or $C \cdot C < -12$, if present in Z , is necessarily subject to

$$\text{ord}_C(F, G, \Delta) \geq (4, 6, 12), \quad (18)$$

indicating that the fibers over C are not of a Kodaira type, often referred to as nonminimal fibers. Such nonminimal fibers over a codimension-one locus in the base obstruct crepant resolutions of the Weierstrass model. Nevertheless, the model can be birationally modified in a systematic manner to be free of nonminimal fibers, as addressed first for the elliptic K3 surfaces [9] and later on for the elliptic Calabi-Yau threefolds [10,11]; the fate of the codimension-one nonminimal fibers is thereby identified as the affinization of the gauge algebras and, in turn, as decompactifications [12,13].

In the current setup, the singularity structure (16) can, thus, be interpreted physically as the obvious statement that we consider only supergravity theories in six dimensions, not in higher dimensions.

2. Rational fibration: Generic structure of Z

The other key structure is the $\mathbb{P}^1$ fibration exhibited by the generic base twofolds. To be precise, the allowed base twofolds Z are classified into three types [14]: the complex projective plane, the Enriques surface, and the Hirzebruch surfaces $\mathbb{F}_{a \leq 12}$ as well as their blowups. As the first two types have $h^{1,1}(Z) = 1$ and 10, which are fairly small, however, we may assume the last type in bounding $h^{1,1}(Z)$. We, thus, consider an r -fold blowup of $\mathbb{F}_a$:

$$Z = \text{Bl}^r(\mathbb{F}_a), \quad r \geq 0, \quad (19)$$

with $h^{1,1}(Z) = 2 + r$, and seek after an upper bound on r . In this case, Z is a fibration:

$$p: Z \rightarrow \mathbb{P}_b^1, \quad (20)$$

whose generic fiber

$$f = p^{-1}(x), \quad x \in \mathbb{P}_b^1, \quad (21)$$

is a smooth rational curve.

While the singularity (16) of (Z, B) concerns the 7-brane physics, the fibration (20) of Z is more relevant to the 3-brane physics. Notably, the D3-brane wrapped on f leads to the effective *heterotic* string, which, at infinite distance in the tensor moduli space, serves as the weakly coupled

tensionless critical string [15,16]. While we are not particularly interested in the asymptotic physics here, the mere presence of a heterotic string, despite being tensionful at finite distance, turns out to play a pivotal role in bounding the spectrum.

To fully exploit the fibration structure (20), we make a dichotomy between the horizontal and the vertical parts of the divisor B :

$$B = B^h + B^v. \quad (22)$$

Here, the components of B^h intersect nontrivially with f , while those of B^v do not. Put physically, a 7-brane stack is horizontal if it is “visible” to the heterotic string and vertical otherwise.

B. The strategy and the rationale

We begin by blowing Z down to $Z_0 = \mathbb{F}_{a \leq 12}$:

$$Z \xrightarrow{\varphi} Z_0, \quad (23)$$

after which we repeatedly blow Z_0 up as much as possible:

$$Z_r \xrightarrow{\varphi_{r-1}} Z_{r-1} \rightarrow \cdots \rightarrow Z_1 \xrightarrow{\varphi_0} Z_0, \quad (24)$$

where, by abuse of notation, the same symbol r as in (19) has been used. Here, each blowup φ_i replaces a point $z_i \in Z_i$ by an exceptional divisor $E_{i+1} \subset Z_{i+1}$. We then show that Z_r maps to the original base Z , which would imply, in particular, that

$$h^{1,1}(Z) \leq h^{1,1}(Z_r) = 2 + r. \quad (25)$$

In order to find a uniform upper bound on r , we rely heavily on the $\frac{1}{6}$ -lc-ness of the pair (Z, B) . Importantly, this singularity structure is preserved under the birational transformations involved. In other words, each of the divisors B_i of Z_i defined by

$$B_0 = \varphi_*(B), \quad (26)$$

$$K_{Z_i} + B_i = (\varphi_0 \circ \cdots \circ \varphi_{i-1})^*(K_{Z_0} + B_0) \quad (27)$$

forms a $\frac{1}{6}$ -lc pair (Z_i, B_i) with B_i effective. Here, each B_i also splits as

$$B_i = B_i^h + B_i^v, \quad (28)$$

with respect to the fibration

$$p_i: Z_i \rightarrow \mathbb{P}_b^1, \quad (29)$$

induced from the $\mathbb{P}^1$ fibration p_0 of the Hirzebruch surface Z_0 as

$$p_i = p_0 \circ \varphi_0 \circ \cdots \circ \varphi_{i-1}. \quad (30)$$

Then, building essentially upon the two key structures (16) and (20), our strategy not only manifests the finiteness of the blowup chain (24), but also results in an explicit upper bound on r and, hence, on $h^{1,1}(Z)$ via (25). In the following, for a glimpse of how such a bound could possibly arise, we present without proofs some of the major claims on the pairs (Z_i, B_i) , along with their consequences; a detailed account of the claims and their rigorous proofs, as well as a comprehensive description of the bounding strategy, can be found in [5].

1. Horizontal multiplicities

As it turns out, the following multiplicities:

$$h_i := \mu_{z_i} B_i^h, \quad (31)$$

which we refer to as the “horizontal” multiplicity of the respective blowup φ_i , play an important role in our bounding strategy. Let us assume without loss of generality that $\{h_i\}_{i=0}^{r-1}$ form a nonincreasing sequence. Furthermore, we may suppose

$$h_{r_1-1} \geq \frac{1}{6} > h_{r_1} \quad \text{and} \quad h_{r_2-1} > 0 = h_{r_2}, \quad (32)$$

to subdivide the blowup chain (24) into three subchains:

$$Z_r \xrightarrow{\varphi_{r-1}} Z_{r-1} \rightarrow \cdots \xrightarrow{\varphi_{r_2}} Z_{r_2} \quad (33)$$

$$\xrightarrow{\varphi_{r_2-1}} Z_{r_2-1} \rightarrow \cdots \xrightarrow{\varphi_{r_1}} Z_{r_1} \quad (34)$$

$$\xrightarrow{\varphi_{r_1-1}} Z_{r_1-1} \rightarrow \cdots \xrightarrow{\varphi_0} Z_0. \quad (35)$$

It follows that the above-defined notion of horizontal multiplicity provides intuitions behind how the total length r of the chain is constrained, which we now turn to.

2. Blowups with $h_i > 0$

To start with, the lengths of the subchains (34) and (35) are bounded via the following two claims.

Claim 1. The sum of the horizontal multiplicities obeys

$$\sum_{i=0}^{r-1} h_i \leq 28. \quad (36)$$

Claim 2. Each blowup φ_i with $h_i \in (0, \frac{1}{6})$ has, in fact, $h_i = \frac{1}{12}$. In other words, $h_i = \frac{1}{12} \quad \forall \quad i \in [r_1, r_2)$.

Claim 2 assures that the multiplicities h_i for $i < r_2$ cannot be arbitrarily small. This, once combined with Claim 1, manifests how r_2 , as well r_1 , is bounded:

$$\frac{r_1}{6} + \frac{r_2 - r_1}{12} \leq 28 \Rightarrow r_1 + r_2 \leq 336. \quad (37)$$

3. Blowups with $h_i = 0$

The next two claims below hint at the length bound for the remaining subchain (33).

Claim 3. Each blowup point $z_i \in Z_i$ with $i \geq r_1$, i.e., with $h_i < \frac{1}{6}$, lies at the intersection of a pair of curves, both vertical with respect to the fibration (29).

Claim 4. For any point $z \in Z_{r_2}$, the number, σ_z , of the blowups φ_i with $h_i = 0$ such that $(\varphi_{r_2} \circ \dots \circ \varphi_i)(E_{i+1}) = z$ is subject to

$$\sigma_z \leq 11. \quad (38)$$

Note that Z_{r_2} carries a total of r_2 points of vertical-curve intersection. Then, Claim 3 asserts that the subchain (33) can blow only such points up, over each of which at best 11 blowups may arise, according to Claim 4. We, thus, have

$$r - r_2 \leq 11r_2 \Rightarrow r \leq 12r_2. \quad (39)$$

C. Calculation of explicit bounds

Claims 1–4 not only manifest the boundedness of r , but also lead to an explicit bound as

$$r \leq 12r_2 \leq 12(r_1 + r_2) \leq 4032, \quad (40)$$

where (37) and (39) have been used. However, the derived bound of 4032 is rather conservative. In particular, the first inequality in (40) is saturated only if the subchain (33) blows up the r_2 points of vertical-curve intersection in Z_{r_2} maximally, each 11 times, and the second, if $r_1 = 0$, meaning that every blowup with $h_i > 0$, in fact, has $h_i = \frac{1}{12}$.

Indeed, as shown in [5], we may improve (40) to

$$r \leq 566 \quad (41)$$

by better constraining the number of “small- h_i ” blowups φ_i than how Claim 4 does. Specifically, to arrive at (41) we need to not only sharpen Claim 4 on σ_z , concerning the blowups with $h_i = 0$, but also constrain the number of blowups with $0 < h_i < \frac{1}{6}$ in a systematic manner. Both tasks start from looking closely into the structure of the *special fibers* of p_r in Z_r , which have more than one component. In the rest of this section, for illustrative purpose, we sketch how the former task of sharpening Claim 4 may be undertaken, relegating the detailed treatment of both tasks to [5].

To this end, let us associate with each special fiber $p_r^{-1}(x)$ over $x \in \mathbb{P}_b^1$ the multiplicity

$$\mu_x^{\text{sf}} := \mu_x \Lambda_{\mathbb{P}_b^1}, \quad (42)$$

where $\Lambda_{\mathbb{P}_b^1}$ is the discriminant divisor appearing in the canonical bundle formula

$$K_{Z_r} + B_r \equiv p_r^*(K_{\mathbb{P}_b^1} + \Lambda_{\mathbb{P}_b^1} + M_{\mathbb{P}_b^1}), \quad (43)$$

with the moduli part denoted by $M_{\mathbb{P}_b^1}$. We proceed now to some extra constraints on the coefficients in B_r of the vertical curves, which serve as a component of the special fibers with “small multiplicities.”

Claim 5. For each component of the special fiber $p_r^{-1}(x)$ over $x \in \mathbb{P}_b^1$, its coefficient b in B_r is subject to

$$b \leq \begin{cases} \frac{2}{3} & \text{if } \mu_x^{\text{sf}} < \frac{3}{4}, \\ \frac{3}{4} & \text{if } \mu_x^{\text{sf}} < \frac{5}{6}. \end{cases} \quad (44)$$

Notably, the coefficient bounds (44) are more stringent than the general bound of $\frac{5}{6}$ imposed by the $\frac{1}{6}$ -lc-ness of the pair (Z_r, B_r) . Then, they result in the following sharpened form of Claim 4, again to do with the special fibers with small μ^{sf} .

Claim 6. For a point $z \in Z_{r_2}$, the number σ_z of blowups φ_i with $h_i = 0$ such that $(\varphi_{r_2} \circ \dots \circ \varphi_i)(E_{i+1}) = z$ is bounded from above as

$$\sigma_z \leq \begin{cases} 3 & \text{if } \mu_x^{\text{sf}} < \frac{3}{4}, \\ 5 & \text{if } \mu_x^{\text{sf}} < \frac{5}{6}, \end{cases} \quad (45)$$

where $x = p_{r_2}(z) \in \mathbb{P}_b^1$.

Claim 6 is particularly useful in improving the conservative bound (40) on r , because Z_r cannot carry too many special fibers with big μ^{sf} . In particular, one can show the following.

Claim 7. The fibration $p_r: Z_r \rightarrow \mathbb{P}_b^1$ has at best two special fibers with $\mu_x^{\text{sf}} \geq \frac{3}{4}$, $x \in \mathbb{P}_b^1$.

Interestingly, Claim 7 is a consequence of the degree bound

$$\deg(\Lambda_{\mathbb{P}_b^1}) \leq 2, \quad (46)$$

which reflects the compactness of the base $\mathbb{P}_b^1$ of the fibration p_r .

Furthermore, one can also show that, of the total of r_2 points of vertical intersection in Z_{r_2} , only a limited number of points can lie on each individual fiber $p_{r_2}^{-1}(x)$ over a fixed $x \in \mathbb{P}_b^1$. Such a fiberwise constraint arises essentially from the following simple property.

Claim 8. For each i , the intersection of the fiber f_i of $p_i: Z_i \rightarrow \mathbb{P}_b^1$ and the horizontal divisor B_i^h is 2, i.e.,

$$f_i \cdot B_i^h = 2. \quad (47)$$

Note that (47) may be considered as a compactness criterion along the fiber of p_i , in much the same way as (46) is along the base.

Then, it turns out that a set of constraints in the spirit of Claims 5–8 eventually result in the improved bound (41), which extends also to the bounds on the Picard number and the tensor count as

$$r \leq 566 \Leftrightarrow h^{1,1}(Z) \leq 568 \Leftrightarrow T \leq 567. \quad (48)$$

While the naive bound (40) has been significantly improved in (41), we emphasize that much room is still left for even further improvement, as will be clear from the detailed derivation of the latter in [5].

Indeed, it is our belief that one may derive

$$r \leq 192 \Leftrightarrow h^{1,1}(Z) \leq 194 \Leftrightarrow T \leq 193, \quad (49)$$

which are saturated in concrete F-theory vacua [17,18] but which have remained a conjecture [19] for a long time.⁴ The strongest bounds (49) can, in fact, be established physically via bottom-up ingredients, as recently pointed out in [21], where various anomaly constraints among others play a central role in arguing for the abstract boundedness to start with.

In this regard, the virtue of our strategy is that it manifests both the abstract and the explicit boundedness, essentially from a handful of principal structures, namely, (16) and (20), which, as discussed, are interpretable purely in physical terms. While there is no doubt from the physical perspective that anomalies must cancel, their geometric incarnation has not been fully addressed.⁵ It is, thus, worth emphasizing that the way our strategy leads to the bounds (48) still leaves much room for an optimization without directly relying on the anomaly considerations.

IV. SUMMARY AND DISCUSSION

In this short note, we have overviewed a novel strategy for bounding the number T of tensor multiplets in compactifications of F-theory to six dimensions. Exploiting the strategy, we have also derived an explicit upper bound (48) on T , which uniformly applies to all 6D effective theories of F-theory.

Geometrically, the defining data for a 6D F-theory vacuum consist of a complex surface Z and an elliptic fibration $\pi: Y \rightarrow Z$, with a Calabi-Yau total space Y . Here, the singularities of Y are encoded in a certain divisor B of Z with $K_Z + B \equiv 0$, as defined in (10), so that the pair (Z, B) may naturally label the effective theory in scrutiny. For the

purpose of bounding $h^{1,1}(Z) = T + 1$, we have then focused on the two key structures:

- (1) singularity (16): (Z, B) is $\frac{1}{6}$ -lc; and
- (2) fibration (20): Z is fibered with generic $\mathbb{P}^1$ fibers.

While the former structure is found in a general F-theory vacuum, the latter is exhibited only generically, i.e., when the base surface Z is not $\mathbb{P}^2$ nor the Enriques surface; the two exceptional bases can, however, be safely ignored in view of our bounding task. We have then learned that essentially these two simple structures severely constrain the geometry of F-theory vacua in a universal manner, leading to an explicit bound on $h^{1,1}(Z)$. To make full use of both structures, we have considered a twofold split of the divisor B , the key player for the $\frac{1}{6}$ -lc-ness, into the horizontal and the vertical parts, with respect to the fibration (20).

Specifically, beginning by blowing Z down to a Hirzebruch surface $Z_0 = \mathbb{F}_a$, we have blown the latter up as much as possible, eventually arriving at Z_r , not allowed to be further blown up. Crucially, both of the key structures are preserved along the entire chain (24) of birational transformations involved, where at each stage a pair (Z_i, B_i) , as well as the twofold split of B_i , is naturally defined. The spirit of our bounding strategy is then to appropriately subdivide the blowup chain (24) into subchains, e.g., into (33)–(35), and to constrain each subchain. For the subdivision of this kind, the horizontal multiplicity h_i , as defined in (31), naturally serves as an order parameter.

The proposed strategy is all the more interesting in that it is naturally interpreted in physical terms. In particular, each of the two key structures playing a pivotal role in the derivation of (48) results from a simple physical principle as follows:

- (1) absence of affine gauge algebras $\Rightarrow (Z, B)$ is $\frac{1}{6}$ -lc; and
- (2) presence of a critical string $\Rightarrow Z$ is a fibration.

First, the $\frac{1}{6}$ -lc structure is imposed by banning the affinization of gauge algebras. Already from the bottom-up perspective, any affine enhancement involves an infinite tower of massless states, indicating that the effective description of the theory breaks down. Furthermore, a quantum gravity expectation is that those states are organized into a light Kaluza-Klein tower, triggering decompactification of the external spacetime [13]; we may, thus, impose the absence of affinized algebras when constraining the theories in a fixed spacetime dimension.⁶

Next, the fibration structure follows from the presence of a (dual) critical string. A physical motivation for the latter is provided by the emergent string conjecture [27], which posits that a weakly coupled string emerges in the asymptotic boundary of the moduli space, unless the spacetime decompactifies. We may, thus, impose that all the theories

⁴Note that the authors of [19] classify the smooth *toric* base surfaces, one of which saturates (49). See also [20] for further development of the classification methodology, applicable to general smooth base surfaces under certain Hodge number constraints.

⁵Nontrivial evidence is found, e.g., in [22,23], providing partial proofs from the geometric perspective in special cases.

⁶For the earlier analyses of decompactifications via an affine enhancement, see, e.g., [12,24–26], where twice as many conserved supercharges are assumed.

in scrutiny possess a critical string, which becomes tensionless at infinite distance to serve as a predicted weakly coupled string. In the current F-theoretic setup, this implies that Z is a fibration with the generic fiber either rational or elliptic, respectively, resulting in a heterotic or a type II string. However, the latter implies that no 7-branes are present and, hence, that Z is an Enriques surface. For completeness, we must also consider the case where no moduli except for the overall volume modulus are available; no fibration structures are required in this case, for which $\mathbb{P}^2$ is singled out as the only other exceptional base surface.

The individual geometric claims 1–8 may be thought of as imposing global constraints on the conformal matter [28], thereby limiting the superconformal field theories and the little string theories that are embeddable into a quantum gravity theory. In other words, many of those constraints would have not arisen had it not been for gravity, i.e., had the internal manifold Z been noncompact. Notably, the bounding task as a geometric problem from the top-down perspective allows us to impose outright both the singularity and the fibration structures, in particular. However, we may as well argue for the derived bounds also from the bottom-up perspective, given that each of the two key structures follows from an underlying physical principle as discussed.

In this regard, it is fascinating to notice that the bottom-up principles behind are tied precisely with the two universal *infinite-distance* phenomena, as characterized by the emergent string conjecture; surprisingly, they turn out to severely constrain the *finite-distance* physics as well. The mere presence of a heterotic string leads to the useful dichotomy between the “horizontal” and the “vertical” gauge sectors, not necessarily relying on its top-down realization via geometry as a solitonic string. Furthermore, the absence of affine gauge algebras, on top of the anomaly cancellation, is an unquestioned prerequisite for the classification in [29,30] of superconformal field theories and little string theories; the outcome of such classification efforts then serves in [21] as one of the crucial ingredients for the strong bound (49) on T , along with other spectral bounds. While the bound (48) derived in this note for the purpose of illustrating our bounding strategy is somewhat weaker, we emphasize again that there is much room left for improvement without bringing in additional principles such as the anomaly cancellation. We leave it to future work to clarify the precise connection between the two approaches.

We end with further comments on applications and generalizations of our results.

A. Bounds on the other sectors

An explicit and uniform bound on T naturally serves as a starting point for bounding the other sectors of the theory, i.e., the vector and the hypermultiplet sectors, which we briefly discuss in turn.

1. Vector multiplet sector

The geometric origin of the gauge algebra $\mathfrak{g}$ in (2) is rather different for the Abelian and the non-Abelian parts. The Abelian vectors are in one-to-one correspondence with the nonzero sections of the elliptic fibration π in (5) and, hence, are counted as

$$R_0 := \text{rk}(\text{MW}(\pi)), \quad (50)$$

where $\text{MW}(\pi)$ denotes the Mordell-Weil group of rational sections. On the other hand, the non-Abelian are tied with the crepant resolution $\hat{Y} \rightarrow Y$ of the singular Weierstrass model (5) to (4).⁷ In particular, the resolution divisors are responsible for the Cartan subalgebra of each simple algebra $\mathfrak{g}_i$, so that the combined non-Abelian rank $\hat{R} := \sum_i \text{rk}(\mathfrak{g}_i)$ is given as

$$\hat{R} = h^{1,1}(\hat{Y}) - h^{1,1}(Y), \quad (51)$$

where $h^{1,1}(Y)$ represents the following sum:

$$h^{1,1}(Y) = h^{1,1}(Z) + R_0 + 1, \quad (52)$$

with R_0 defined in (50) [32].

Let us first consider the full rank of $\mathfrak{g}$:

$$R := \text{rk}(\mathfrak{g}) = R_0 + \hat{R}, \quad (53)$$

which receives separate contributions (50) and (51), respectively, from the Abelian and the non-Abelian vectors. With the former subject to a known bound [33,34],

$$R_0 \leq 18, \quad (54)$$

here we sketch how the latter could be bounded. To this end, we observe that the Cartier divisor

$$12B =: \bar{\Delta} = \sum \bar{\delta}_i \bar{\Delta}_i, \quad (55)$$

which is essentially⁸ the zero locus of the section Δ in (8), has each coefficient $\bar{\delta}_i$ bounded. Knowing how $h^{1,1}(Z)$ is bounded, we may then also bound $\sum \bar{\delta}_i$, thereby obtaining an explicit bound on $\hat{R}$ and, hence, on R via (54).

Furthermore, given the rank bound, we may also bound the number V of vector multiplets, which amounts to the dimension of the gauge algebra $\mathfrak{g}$, splitting as

$$V := V_0 + \hat{V}, \quad (56)$$

⁷While we assume, for simplicity of presentation, that $\hat{Y}$ is smooth, $\mathbb{Q}$ -factorial terminal singularities may, in principle, remain, in which case (58) receives corrections from Milnor numbers [31]. Our bounding strategy still persists, however.

⁸See footnote 3 for the potential discrepancy.

where $V_0 = R_0$ and $\hat{V} = \sum_i \dim(\mathfrak{g}_i)$ are the dimensions of the Abelian and the non-Abelian sectors, respectively. Here, the individual dimensions $\dim(\mathfrak{g}_i)$ are bounded, given the associated rank bounds. An explicit bound, thus, arises on $\hat{V}$ and, hence, on V ; we leave the calculation of such an explicit bound for future work.

2. Hypermultiplet sector

Once explicit bounds are imposed on T and V , what remains to be bounded is the number H of hypermultiplets. We split H as

$$H = H_0 + (H - H_0), \quad (57)$$

where H_0 counts the uncharged matter fields and connects to the deformations of $\hat{Y}$ via

$$H_0 = h^{1,2}(\hat{Y}) + 1. \quad (58)$$

On the other hand, the remaining contribution $H - H_0$ from the charged fields is related to the fibral singularities over isolated points in the base Z .

Note that H_0 in (58) indicates that H is not entirely birational in nature, unlike T and V , as it involves counting of 3-cycles. In the meantime, physically, the gravitational anomaly constrains H as

$$H = 273 - 29T + V, \quad (59)$$

which, once combined with given bounds on T and V , leads to an associated explicit bound on H , too. We leave an independent geometric exploration of the bound on H for future work.

B. Generalizations

As for a most natural generalization, we may consider tackling the analogous problem of bounding $h^{1,1}(Z)$ in the case with $\dim(Z) = 3$, where Z serves as the base threefold of an elliptic Calabi-Yau variety, again with only Kodaira-type fibers. This new task is much harder, given the richer

geometric structure. Building upon recent advances in birational geometry [35–37], however, one can attempt to generalize our bounding strategy, thereby establishing a similar, explicit boundedness notion for the 4D gravitational theories with $\mathcal{N} = 1$ supersymmetry. Notably, the $\frac{1}{6}$ -lc structure may still be imposed to prohibit any affine gauge algebras. Furthermore, the fibration structure with generic $\mathbb{P}^1$ fibers persists to a great extent; see [38]. It is, thus, particularly exciting that the key ideas of our bounding strategy are still applicable to the base threefolds. We leave this as an interesting future direction of investigation.

Let us also note that the fibration π in (5) does not play any significant role in bounding $h^{1,1}(Z)$; while the two key structures (16) and (20) follow from π , in no later steps do we refer back to it. We may, thus, consider generalizing the bounding strategy to the case where (16) is replaced by ϵ -lc-ness with $\epsilon \neq \frac{1}{6}$. Obviously, there should arise an explicit bound on $h^{1,1}(Z)$ for each chosen value of ϵ . Despite that its physical interpretation would not be as clear, the task of revealing the ϵ dependence of the bound serves as an interesting geometrical problem *per se*.

Note added. Having announced part of our results [39] at the stage of completing the work, we learned of the highly relevant paper [21], where the same physics problem was tackled.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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- [1] C. Vafa, The string landscape and the swampland, [arXiv:hep-th/0509212](#).
 - [2] M. Gross, A finiteness theorem for elliptic Calabi-Yau threefolds, [Duke Math. J.](#) **74**, 271 (1993).
 - [3] S. Filipazzi, C. D. Hacon, and R. Svaldi, Boundedness of elliptic Calabi-Yau threefolds, [J. Eur. Math. Soc.](#) **27**, 3583 (2025).
 - [4] A. Grassi, Spectrum bounds in geometry, [arXiv:2304.07819](#).
 - [5] C. Birkar and S. J. Lee, A Picard rank bound for base surfaces of elliptic Calabi-Yau 3-folds, [arXiv:2507.06317](#).
 - [6] V. Kumar, D. R. Morrison, and W. Taylor, Global aspects of the space of 6D $\mathcal{N} = 1$ supergravities, [J. High Energy Phys.](#) **11** (2010) 118.
 - [7] D. S. Park and W. Taylor, Constraints on 6D supergravity theories with Abelian gauge symmetry, [J. High Energy Phys.](#) **01** (2012) 141.

- [8] W. Taylor and A. P. Turner, An infinite swampland of U(1) charge spectra in 6D supergravity theories, *J. High Energy Phys.* **06** (2018) 010.
- [9] S. J. Lee and T. Weigand, Elliptic K3 surfaces at infinite complex structure and their refined Kulikov models, *J. High Energy Phys.* **09** (2022) 143.
- [10] R. Álvarez-García, S. J. Lee, and T. Weigand, Non-minimal elliptic threefolds at infinite distance. Part I. Log Calabi-Yau resolutions, *J. High Energy Phys.* **08** (2024) 240.
- [11] R. Álvarez-García, S. J. Lee, and T. Weigand (to be published).
- [12] S. J. Lee, W. Lerche, and T. Weigand, Physics of infinite complex structure limits in eight dimensions, *J. High Energy Phys.* **06** (2022) 042.
- [13] R. Álvarez-García, S. J. Lee, and T. Weigand, Non-minimal elliptic threefolds at infinite distance II: Asymptotic physics, *J. High Energy Phys.* **01** (2025) 058.
- [14] A. Grassi, On minimal models of elliptic threefolds, *Math. Ann.* **290**, 287 (1991).
- [15] S. J. Lee, W. Lerche, and T. Weigand, Tensionless strings and the weak gravity conjecture, *J. High Energy Phys.* **10** (2018) 164.
- [16] S. J. Lee, W. Lerche, and T. Weigand, A stringy test of the scalar weak gravity conjecture, *Nucl. Phys.* **B938**, 321 (2019).
- [17] P. S. Aspinwall and D. R. Morrison, Point—like instantons on K3 orbifolds, *Nucl. Phys.* **B503**, 533 (1997).
- [18] P. Candelas, E. Peralvalov, and G. Rajesh, Toric geometry and enhanced gauge symmetry of F theory / heterotic vacua, *Nucl. Phys. B* **507**, 445 (1997).
- [19] D. R. Morrison and W. Taylor, Toric bases for 6D F-theory models, *Fortschr. Phys.* **60**, 1187 (2012).
- [20] W. Taylor and Y. N. Wang, Non-toric bases for elliptic Calabi-Yau threefolds and 6D F-theory vacua, *Adv. Theor. Math. Phys.* **21**, 1063 (2017).
- [21] H. C. Kim, C. Vafa, and K. Xu, Finite landscape of 6d $N = (1, 0)$ supergravity, [arXiv:2411.19155](https://arxiv.org/abs/2411.19155).
- [22] A. Grassi and D. R. Morrison, Anomalies and the Euler characteristic of elliptic Calabi-Yau threefolds, *Commun. Number Theor. Phys.* **6**, 51 (2012).
- [23] A. Grassi and T. Weigand, On topological invariants of algebraic threefolds with ($\mathbb{Q}$ -factorial) singularities, [arXiv:1804.02424](https://arxiv.org/abs/1804.02424).
- [24] A. Font, B. Fraiman, M. Graña, C. A. Núñez, and H. P. De Freitas, Exploring the landscape of heterotic strings on T^d , *J. High Energy Phys.* **10** (2020) 194.
- [25] V. Collazuo, M. Graña, A. Herráez, and H. Parra De Freitas, Affine algebras at infinite distance limits in the heterotic string, *J. High Energy Phys.* **07** (2023) 036.
- [26] F. A. Cachazo and C. Vafa, Type I' and real algebraic geometry, [arXiv:hep-th/0001029](https://arxiv.org/abs/hep-th/0001029).
- [27] S. J. Lee, W. Lerche, and T. Weigand, Emergent strings from infinite distance limits, *J. High Energy Phys.* **02** (2022) 190.
- [28] M. Del Zotto, J. J. Heckman, A. Tomasiello, and C. Vafa, 6d conformal matter, *J. High Energy Phys.* **02** (2015) 054.
- [29] J. J. Heckman, D. R. Morrison, T. Rudelius, and C. Vafa, Atomic Classification of 6D SCFTs, *Fortschr. Phys.* **63**, 468 (2015).
- [30] L. Bhardwaj, M. Del Zotto, J. J. Heckman, D. R. Morrison, T. Rudelius, and C. Vafa, F-theory and the classification of little strings, *Phys. Rev. D* **93**, 086002 (2016); **100**, 029901(E) (2019).
- [31] P. Arras, A. Grassi, and T. Weigand, Terminal singularities, milnor numbers, and matter in F-theory, *J. Geom. Phys.* **123**, 71 (2018).
- [32] R. Wazir, Arithmetic on elliptic threefolds, *Compos. Math.* **140**, 567 (2001).
- [33] S. J. Lee and T. Weigand, Swampland bounds on the Abelian gauge sector, *Phys. Rev. D* **100**, 026015 (2019).
- [34] S. J. Lee and P. K. Oehlmann, Geometric bounds on the 1-form gauge sector, *Phys. Rev. D* **108**, 086021 (2023).
- [35] C. Birkar, Singularities of linear systems and boundedness of Fano varieties, *Ann. Math.* **193**, 347 (2021).
- [36] C. Birkar, Singularities on Fano fibrations and beyond, [arXiv:2305.18770](https://arxiv.org/abs/2305.18770).
- [37] C. Birkar, Boundedness of Fano type fibrations, *Ann. Sci. ENS* **57**, 787 (2024).
- [38] C. Birkar, G. Di Cerbo, and R. Svaldi, Boundedness of elliptic Calabi-Yau varieties with a rational section, *J. Diff. Geom.* **128**, 463 (2024).
- [39] C. Birkar, Birational geometry and interactions with physics, *2024 KIAS-APCTP Frontiers of Theoretical Physics* (2024), <https://physics.kias.re.kr/FTP2024/>.