

Dirac mode localization in QCD near the crossover temperature

Matteo Giordano^{1,*}, Tamás G. Kovács^{1,2,†} and Ferenc Pittler^{3,‡}

¹*Institute of Physics and Astronomy, ELTE Eötvös Loránd University,
Pázmány Péter sétány 1/A, H-1117, Budapest, Hungary*

²*Institute for Nuclear Research (ATOMKI), H-4026 Debrecen, Bem tér 18/c, Hungary*

³*Computation-based Science and Technology Research Center,
The Cyprus Institute, 20 Kavafi Str., Nicosia 2121, Cyprus*



(Received 20 February 2026; accepted 13 May 2026; published 11 June 2026)

We study the localization properties of the low-lying Dirac eigenmodes in QCD near the crossover temperature, using stout-smear staggered fermions and Symanzik-improved gauge action on the lattice. On $N_t = 8$ lattices we find that localized low modes, absent at low temperature, appear at a temperature T_{loc} in the range $150 \text{ MeV} \leq T_{\text{loc}} \leq 160 \text{ MeV}$, well within the chiral crossover range as determined from the chiral condensate and from the light-quark susceptibility. Since with our choice of lattice action neither the chiral transition region nor the renormalized mobility edges change significantly above $N_t = 8$, our conclusion that T_{loc} is in the chiral crossover region is expected to remain valid in the continuum limit.

DOI: [10.1103/ttpq-mvpc](https://doi.org/10.1103/ttpq-mvpc)

I. INTRODUCTION

The finite-temperature transition in QCD and the properties of the high-temperature phase are of central importance for a number of physical phenomena, ranging from heavy-ion collisions to the physics of the early Universe, and have been the subject of intense research since the early days of QCD. Using nonperturbative lattice methods, the nature of the transition at vanishing chemical potential has been identified as an analytic crossover [1–6]. Doubts may still remain due to the “bad” chiral properties of the staggered fermion discretization [7–10] used in these studies, and investigations using a chiral fermion discretization are ongoing [11]. On the other hand, the microscopic degrees of freedom and the mechanism behind the transition are still somewhat mysterious. Several authors have also proposed the existence of an intermediate phase between the low-temperature, hadronic phase, and a truly quark-gluon-plasma phase appearing at higher temperatures [12–18]. Although based on very different arguments, these proposals qualitatively agree on the temperature range of interest.

Of course, distinguishing phases is affected by ambiguities in the presence of an analytic crossover, as there is

no sharply defined temperature where the thermodynamic properties of the system change in a singular manner. Similarly, order parameters associated with the symmetries of QCD that are relevant to the transition, namely chiral symmetry and $\mathbb{Z}_3$ center symmetry, cannot sharply distinguish the two sides of the crossover. These symmetries are exact respectively in the chiral (massless quark) limit and in the quenched (static quark) limit of QCD, but are only approximate at the physical point, and so the behavior of the various order parameters, although qualitatively different in the two phases, cannot be used to unambiguously associate a critical temperature with the transition.

Even though there are no thermodynamic observables or order parameters sharply distinguishing the low and high temperature phases of QCD, one may still be able to identify observables of a different nature that display a singular behavior at a well-defined temperature, allowing for an unambiguous separation between the hadronic and the plasma phase. The study of these quantities can still lead to useful insights, even though their connection to physically observable properties may not be direct. This is the idea, for example, behind the study of center-vortex or monopole condensation as a signature of the onset of confinement [16,17]. Unfortunately, these approaches are still affected by theoretical ambiguities due to the construction used to reveal vortices and monopoles not being gauge invariant.

An appealing possibility to characterize unambiguously and in a gauge-invariant way the low- and high-temperature phases of QCD are the localization properties of the low Dirac modes. There is by now a large amount of evidence showing that these localization properties change across the

* Contact author: giordano@bodri.elte.hu

† Contact author: tamas.gyorgy.kovacs@ttk.elte.hu

‡ Contact author: f.pittler@cyi.ac.cy

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

finite-temperature transition in a wide variety of gauge theories, including QCD [19–46] (see Refs. [47,48] for a review). While only delocalized low modes are present at low temperature, at high temperature modes become localized in a symmetric spectral range around the origin, up to critical points in the spectrum known as “mobility edges;” modes in the bulk of the spectrum beyond these points remain delocalized.

Eigenmode localization is a well-known phenomenon in condensed-matter systems with disorder, intensely studied since Anderson’s seminal paper [49]. This phenomenon has important consequences for the transport properties of metals with impurities and other disordered systems (see, e.g., the reviews [50–53]). Although its physical significance in the context of gauge theories has not been fully elucidated yet (see, however, Refs. [54,55]), it has been argued theoretically [26,54,55] and demonstrated numerically [26,39,43] that localization is not just a lattice artifact, but a physical feature of (Euclidean) gauge theories in the continuum limit. Since either there are localized low modes or there are not, their appearance is associated with a well-defined critical temperature, although one corresponding to some “geometric” rather than thermodynamic transition. (A different geometric approach to the transition is discussed in Ref. [56].)

In theories with a genuine finite-temperature deconfining phase transition, localized low modes appear exactly at the deconfinement critical temperature [31,32,34–40,42,45]. This indicates a close connection between low-mode localization and deconfinement, and parallels the connection between the density of low modes and chiral symmetry breaking embodied by the Banks–Casher relation [57]. Even though a full quantitative characterization of this connection is still lacking, the basic mechanism is well understood [58]: it is the ordering of the Polyakov loop, which opens a pseudogap of low spectral density at the low end of the Dirac spectrum, and allows the localization of the low modes on suitable gauge-field fluctuations [40,48,58–61].

The Polyakov loop is an exact order parameter for confinement in pure SU(3) gauge theory, but only an approximate one for real-world QCD: at low temperature there is only limited ordering induced by the presence of dynamical quarks, while at high temperature this ordering gets stronger. While qualitatively clear, a quantitative distinction between the two is obviously ambiguous. In a sense, low-mode localization gives a definite answer to the question of how ordered the Polyakov loop has to be, to be identified as ordered and to indicate that the system is deconfined: enough to lead to the appearance of localized low modes.

The discussion above concerns the well-established mobility edge separating the localized low modes from the delocalized modes in the bulk of the spectrum. It is worth noting that theoretical arguments [62–64] suggest the

presence of another mobility edge in the low Dirac spectrum, much closer to (but not exactly at) the origin. The motivation behind this proposal is the observation of a power-law singular near-zero peak in the spectral density, both in quenched and in full QCD [11,14,15,32,38,65–78], originating from the mixing of the approximate zero modes associated with isolated instantons and anti-instantons [11,32,65–67,69,72–74,76,79]. Such a peak provides a viable mechanism for the effective breaking of the anomalous $U(1)_A$ symmetry in the chiral limit in the symmetric phase [79]. However, for $U(1)_A$ to remain effectively broken, the constraints imposed on the spectrum by chiral symmetry restoration require that near-zero modes be not persistently localized all the way to the chiral limit [62–64]. In the presence of a singular spectral peak this most likely requires the presence of another mobility edge closer to zero, and that the lowest modes be delocalized at nonzero quark mass.¹ Numerical evidence [75] supports the existence of this mobility edge; however, this is so close to the origin that its direct study is rather challenging. Moreover, the topological origin of the peak makes its detection even more challenging when using staggered fermions.

Note that the emergence of this power-law singular near-zero peak in the spectral density is special to QCD and other theories with nontrivial topological properties. As is well known, the Dirac spectrum of free fermions at finite temperature is bounded from below by the Matsubara frequency. The interaction with the gauge field is expected to make this sharp lower bound fuzzy, but a strong suppression of the spectral density is still expected at the low end of the spectrum at high temperature. The appearance of this Matsubara pseudogap makes it possible for the topology-related near-zero eigenvalues to form a sharp peak at zero, separated from the bulk of the spectrum. In this way, the appearance of the power-law peak and of the lowest mobility edge (if confirmed) are then also closely connected to the ordering of the Polyakov loop and to deconfinement.²

Whether or not this other mobility edge very close to zero is confirmed, it is the appearance of the mobility edge farther up in the spectrum that signals the appearance of localized low modes, separating them from the delocalized bulk modes, and marking the transition to the high-temperature phase. In this paper we will then focus exclusively on this mobility edge. Through its connection to the Polyakov loop, a characterization of the QCD transition in geometric terms is not only possible, but also well motivated. One

¹This is different from the proposal of Refs. [80,81], according to which a mobility edge appears exactly at zero when the system transitions to the “IR phase” discussed in Refs. [14,15].

²Based on the results of Refs. [37,40,45], one expects yet another mobility edge in the ultraviolet region of the spectrum, but independently of the confining properties of the theory, so that such a mobility edge should have no bearing on the transition to the quark-gluon plasma phase.

may then go as far as identifying the “localization temperature” where localized low modes appear, T_{loc} , as the point where the system deconfines. Admittedly, at the present stage this would offer only some intriguing clue, rather than a full understanding of the approximate chiral-symmetry restoration and of the deconfinement of quarks and gluons taking place at the crossover. Nonetheless, the fact that both deconfinement and chiral symmetry restoration are related to the behavior of low Dirac modes provides a connection between the two phenomena, and could help in understanding their relation, and why they take place in the same temperature range.

Existing studies of localization in QCD [26,41] lead one to expect, by extrapolating the temperature dependence of the mobility edge, that the localization temperature should not be far from the crossover temperature $T_c \approx 155$ MeV, defined in terms of the chiral susceptibility or of the quark entropy [1–6], in line with the discussion above. The localization temperatures obtained via these extrapolations are indeed in the correct ballpark, although covering a rather wide range from $T_{\text{loc}} \approx 130$ –140 MeV [41] to $T_{\text{loc}} \approx 170$ MeV [26]. Surprisingly, however, a direct measurement of the localization temperature has not been performed. Such a measurement is particularly interesting in light of the fact that both chiral and confinement observables lead to the same, or at least very close pseudocritical temperatures. Finding a localization temperature close to these two would be a strong indication that restoration of chiral symmetry and spontaneous breaking of center symmetry in QCD (in the loose sense warranted by their approximate nature) originate in the same microscopic mechanism, and that this is reflected in the behavior of the low Dirac modes.

It is precisely the purpose of this work to carry out such a direct measurement. In this paper we determine the localization temperature in QCD using 2 + 1 flavors of rooted staggered fermions on the lattice. In Sec. II we briefly review localization in disordered systems, focusing on the eigenmodes of the staggered operator. In Sec. III we provide details on our numerical simulations, and present our results. In Sec. IV we summarize our results and draw our conclusions.

II. LOCALIZATION OF DIRAC EIGENMODES

We are interested in the localization properties of Dirac eigenmodes in QCD, discretized on a lattice using staggered fermions [7–10]. In this Section we briefly review how these properties are studied, referring the interested reader to the reviews [47,48] for further details.

A. Lattice QCD with rooted staggered fermions

We discretize QCD on a hypercubic lattice of lattice spacing a . Lattice sites are labeled by coordinates ax_μ , $\mu = 1, \dots, 4$, with $0 \leq x_{1,2,3} \leq N_s - 1$ the spatial

coordinates and $0 \leq x_4 \leq N_t - 1$ the time coordinate in lattice units. Both N_s and N_t are taken to be even integers. The lattice volume in lattice units is denoted by $V = N_s^3$. Link variables $U_\mu(x) \in \text{SU}(3)$ corresponding to the gauge fields are associated with the lattice edges $(x, x + \hat{\mu})$, where $\hat{\mu}$ is the unit lattice vector in direction μ . Periodic boundary conditions, both in the temporal and in the spatial directions, are imposed on $U_\mu(x)$. The temperature of the system equals the inverse of the temporal size in physical units, $T = 1/(aN_t)$.

The staggered lattice Dirac operator for fermions transforming in the fundamental representation reads

$$(aD^{\text{stag}}[U])_{xc,x'c'} = \frac{1}{2} \sum_{\mu=1}^4 \eta_\mu(x) [(U_\mu(x))_{cc'} \delta_{x+\hat{\mu},x'} - (U_\mu(x-\hat{\mu}))_{cc'}^\dagger \delta_{x-\hat{\mu},x'}], \quad (1)$$

where U denotes the link variables collectively, $\eta_\mu(x) = (-1)^{\sum_{a<\mu} x_a}$ are the usual staggered phases, and boundary conditions periodic in space and antiperiodic in time are understood in the Kronecker deltas. The spacetime indices, x, x' , run over the lattice sites, while color indices take the values $c, c' = 1, 2, 3$. The dependence on U is mostly omitted in the following. The staggered operator is anti-Hermitian and so has purely imaginary eigenvalues,

$$D^{\text{stag}}\psi_n = i\lambda_n\psi_n, \quad \lambda_n \in \mathbb{R}, \quad (2)$$

where the components of the eigenvectors ψ_n are denoted with $(\psi_n(x))_c$. Since $\{\varepsilon, D^{\text{stag}}\} = 0$, where $\varepsilon_{xc,x'c'} = (-1)^{\sum_{a=1}^4 x_a} \delta_{x,x'} \delta_{c,c'}$, one has $D^{\text{stag}}\varepsilon\psi_n = -i\lambda_n\varepsilon\psi_n$ and so the spectrum is symmetric about zero.

Expectation values in the sense of the lattice QCD path integral with rooted staggered fermions [82–84] read as follows for observables depending only on the gauge fields,

$$\begin{aligned} \langle O \rangle &= Z^{-1} \int DU e^{-S(U)} M(U, m) O(U), \\ Z &= \int DU e^{-S(U)} M(U, m), \\ M(U, m) &= \prod_f [\det(D^{\text{stag}}[U] + m_f)^{\frac{1}{4}}], \end{aligned} \quad (3)$$

where DU is the product over the lattice edges of the Haar measures associated with the link variables, the product over f runs over the flavors of dynamical quarks with m_f the corresponding bare quark masses, and $S(U)$ is a suitable discretization of the continuum Yang-Mills action. To improve the discretization and obtain results closer to the continuum limit, suitably smeared gauge fields, $U_\mu^{(s)}(x)$, are often used in the fermionic determinant, M , as well as for certain observables, amounting to the replacements

$M(U, m) \rightarrow M(U^{(s)}, m)$ and $O(U) \rightarrow O(U^{(s)})$ in Eq. (3). Details on smearing and on the gauge action are given in Sec. III.

B. Localization and spectral statistics

Formally, $(-i$ times) the staggered operator in a gauge-field background is exactly like a disordered Hamiltonian, with purely off-diagonal disorder provided by the link variables. Eigenmode localization is a common phenomenon in systems of this type [85–92].

The localization properties of the eigenmodes in a given spectral region can be seen in the scaling with volume of the inverse participation ratio (IPR),

$$\text{IPR}_n = \sum_x \|\psi_n(x)\|^4, \quad (4)$$

averaged over modes in the spectral region of interest and over gauge configurations. Here $\|\psi_n(x)\|^2 = \sum_{c=1}^3 |(\psi_n(x))_c|^2$ is the local eigenvector magnitude and the normalization $\sum_x \|\psi_n(x)\|^2 = 1$ is understood. The IPR averaged locally in the spectrum is obtained as

$$\overline{\text{IPR}}(\lambda; N_s) = \frac{\langle \sum_n \delta(\lambda - \lambda_n) \text{IPR}_n \rangle}{\varrho(\lambda; N_s)}, \quad (5)$$

where

$$\varrho(\lambda; N_s) = \left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle \quad (6)$$

is the (unnormalized) spectral density. Here and below only the dependence on the spectral region and on the lattice spatial size are shown. For the system under consideration there are additional dependences on the temporal size, N_t , and on the lattice spacing, a , that play no role in the present discussion and are therefore omitted for notational simplicity. In the present context, the expectation value $\langle \dots \rangle$ defined in Eq. (3) corresponds to averaging over disorder realizations.

For modes delocalized over the whole lattice one has $\|\psi_n(x)\|^2 \sim 1/(N_t V)$ everywhere, so $\overline{\text{IPR}} \sim 1/V \rightarrow 0$ in the large-volume limit in the delocalized regime of the spectrum. For modes localized in a finite spatial region of volume V_0 one has instead $\|\psi_n(x)\|^2 \sim 1/(N_t V_0)$ inside this region and negligible outside, so $\overline{\text{IPR}} \sim 1/V_0$ remains finite in the large-volume limit in the localized regime of the spectrum. Since for the staggered eigenmodes the local eigenvector magnitude and so the IPR are the same for ψ_n and $\varepsilon\psi_n$, their localization properties are symmetric about the origin.

Instead of studying the IPR directly, it is convenient to exploit the connection between the localization properties of the eigenvectors in a given spectral region and the

statistical properties of the corresponding eigenvalues [93]. For a general disordered system, in a spectral region where modes are delocalized the spectral statistics are the same as those of a dense random matrix, and are determined by the statistical properties of the Gaussian ensemble of random matrix theory (RMT) [94–96] in the same symmetry class as the Hamiltonian of interest. For the staggered operator in the background of SU(3) gauge fields for fermions in the fundamental representation, this is the unitary class [96].³ In a spectral region where modes are localized the eigenvalues obey instead Poissonian statistics [94,95].

To uncover these universal features of the spectrum, one should remove the typical scale of the local eigenvalue spacings, which is specific to the given system. This is done by unfolding the spectrum, i.e., by performing the monotonic mapping $\lambda_i \rightarrow x_i$ defined by

$$x_i = \int_{\lambda_{\min}}^{\lambda_i} d\lambda \varrho(\lambda; N_s), \quad (7)$$

where $\lambda_{\min}$ is the lowest end of the spectrum, and ϱ is given in Eq. (6). By construction, the unfolded spectrum has unit spectral density throughout the spectrum, i.e., $\varrho(\lambda; N_s) \frac{d\lambda}{dx} = 1$ identically. The statistical properties of the unfolded spectrum are universal [94–96], determined only by the symmetry class of the system and by the localization properties of the eigenmodes, and so one can see how these properties change along the spectrum by studying how the spectral statistics change.

A particularly convenient approach is to study the probability distribution of the unfolded level spacings, $s_i = x_{i+1} - x_i$, locally in the spectrum. This probability distribution is known exactly both for RMT and Poisson statistics, and one can compare estimates in various spectral regions of the system of interest with the corresponding predictions. In the RMT case a closed expression is not available, but a good approximation is provided by the so-called Wigner surmise, which for the unitary class reads

$$p_{\text{RMT}}(s) = A s^2 e^{-B s^2}, \quad (8)$$

with $A = \frac{32}{\pi^2}$ and $B = \frac{4}{\pi}$ determined by the normalization $\int_0^\infty ds p(s) = 1$ and by the property $\int_0^\infty ds p(s) s = 1$ that holds in the infinite-volume limit. In the Poisson case one finds instead the exponential distribution,

$$p_{\text{Poisson}}(s) = e^{-s}. \quad (9)$$

One can then compute the unfolded level spacing distribution locally in the spectrum,

³More precisely, this operator belongs to the chiral unitary class. However, chiral classes have the same bulk statistical properties as the corresponding nonchiral classes, so the distinction is unimportant for our purposes.

$$p(s; \lambda; N_s) = \frac{\langle \sum_n \delta(\lambda - \lambda_n) \delta(s - s_n) \rangle}{\varrho(\lambda; N_s)}, \quad (10)$$

extract a convenient feature of the distribution, and monitor how it changes as λ moves along the spectrum.

A particularly simple choice is the integrated unfolded level spacing distribution [97],

$$\begin{aligned} I_{s_0}(\lambda; N_s) &= \int_0^{s_0} ds p(s; \lambda; N_s) \\ &= \frac{\langle \sum_n \delta(\lambda - \lambda_n) \theta(s_0 - s_n) \rangle}{\varrho(\lambda; N_s)}, \end{aligned} \quad (11)$$

where $\theta(s)$ is the Heaviside function. To maximize the difference between the RMT and the Poisson predictions, in Eq. (11) one chooses $s_0 \simeq 0.508$, finding

$$I_{s_0}^{(\text{RMT})} \simeq 0.117, \quad I_{s_0}^{(\text{Poisson})} \simeq 0.398. \quad (12)$$

As the volume of the system increases, the unfolded level spacing distribution tends to the RMT form in the delocalized regime of the spectrum, and to the Poisson form in the localized regime. At a mobility edge, i.e., a point in the spectrum separating localized and delocalized modes, the spectral properties are scale invariant [53]. One can exploit this to determine the mobility edge by means of a finite-size scaling analysis of I_{s_0} , or of other features of the distribution [97].

The spectral statistics at the mobility edge are neither of RMT nor of Poisson type, but are governed by some critical statistics that is expected to be universal. For QCD the universality class is the same as for the three-dimensional unitary Anderson model, and it was shown in Ref. [28] that the critical statistics of the two models match precisely. For theoretical arguments backing the universality of the critical statistics and for numerical studies verifying it explicitly, see the discussion in Ref. [28] and references therein. The critical value of I_{s_0} for the unitary class was determined in Ref. [27] by a finite-size scaling study of the mobility edge in QCD, and reads $I_{s_0}^{(\text{crit})} = 0.1966(25)$. This can be used to efficiently estimate the position of the mobility edge for systems in the unitary class using finite-volume results, as the point where I_{s_0} takes its critical value, without the need of a full-fledged finite-size scaling analysis. As a matter of fact, any value intermediate between those corresponding to RMT or Poisson statistics can be used to give a finite-volume estimate of the mobility edge, which eventually converges to the same position in the thermodynamic limit, since I_{s_0} converges to a step function in that limit. Using the critical value is advantageous as it reduces the finite-volume systematic effects.

C. Continuum limit: Taste symmetry and renormalization

An important source of systematic effects in the study of the spectral statistics of the staggered operator is the approximate taste symmetry of staggered fermions, which manifests in the spectrum through the formation of nearly degenerate multiplets of eigenvalues (doublets at first, then quartets) as one gets closer to the continuum limit. This near degeneracy deforms the unfolded level spacing distribution [26,98], as it favors level spacings smaller than the average spacing in the spectral region of interest, as long as the multiplets do not overlap; see Ref. [43] for a detailed discussion. This, however, is only a technical complication, which one could avoid by studying the localization properties of the eigenmodes directly by looking at how the IPR scales with the system size. However, here we still made the choice of using the spectral statistics for determining the mobility edge, as it is numerically more efficient.

Using the spectral statistics reduces the numerical cost required to identify the mobility edge when the usual relation between statistical properties of the eigenvalues and localization properties of the eigenmodes applies. In that case, the mobility edge can be extracted rather accurately even using a single lattice volume. As shown in Ref. [43], the deformation of the spectral statistics due to the taste multiplets can be avoided by working with sufficiently large volumes, so that the would-be multiplets overlap and the effects of the approximate taste symmetry get washed out in the spectrum. The statistical properties of the spacings are then unaffected by taste symmetry, and one recovers the usual universal behavior corresponding to the localization properties of the eigenmodes. This procedure has been justified in detail in Ref. [43], where it was argued that the estimates of the mobility edge in the thermodynamic and continuum limit should not depend on the order in which these limits are taken.

As the lattice spacing tends to zero, the spectrum must be multiplicatively renormalized like the quark masses, in order for spectral observables to have a finite continuum limit [99,100]. This applies in particular to the mobility edge [54,55], and so the mobility edge in units of any bare quark mass, $\frac{\lambda_c}{m_q}$, is a renormalization-group invariant quantity [26,55]. In Ref. [43] it was shown numerically that this ratio has a nonzero continuum limit in QCD above the crossover temperature.

III. NUMERICAL RESULTS

A. Simulation details

We simulated QCD with $2 + 1$ flavors of quarks at physical quark masses using rooted staggered fermions. We used the tree-level Symanzik improved Wilson gauge action [101–104], and applied two steps of stout smearing [105] with $\rho = 0.15$ to the gauge fields used in the fermion determinant M (see Ref. [106] for details), as well as in the

TABLE I. Lattice ensembles used for the present study. The columns are the temperature (T), the temporal (N_t), and spatial (N_s) linear extension of the lattice, the number of configurations (N_{conf}) in each ensemble and the number of computed eigenvalues (N_{ev}) per configuration. In the last column we report our results for λ_c/m_{ud} .

T (MeV)	N_t	N_s	N_{conf}	N_{ev}	λ_c/m_{ud}
150.15	8	48	400	48	
150.15	8	64	423	52	
150.15	8	80	316	160	
154.87	8	48	6856	68	
154.87	8	64	9952	52	
154.87	8	80	1944	160	
157.85	8	48	6528	48	0.243(47)
157.85	8	64	10016	52	0.233(32)
157.85	8	80	5000	100	0.171(36)
159.90	8	48	31192	48	0.486(45)
159.90	8	64	12456	52	0.412(44)
159.90	8	80	2440	100	0.45(12)
163.08	8	48	27768	48	0.945(46)
163.08	8	64	10784	52	0.855(40)
163.08	8	80	2440	150	0.838(92)
165.00	6	48	50080	50	2.085(95)
165.27	8	32	18832	60	1.54(21)
165.27	8	48	26912	100	1.270(77)
165.27	8	64	9632	48	1.232(43)
165.55	10	40	2608	48	1.58(82)
165.55	10	60	9712	50	1.166(69)
165.55	10	80	5992	48	1.155(63)
167.52	8	48	35672	48	1.67(18)
183.77	8	48	19056	120	6.7(1.5)

determination of the staggered spectrum. We generated ensembles at several temperatures in order to locate the critical point for localization, i.e., the temperature T_{loc} where localized low modes appear. To assess finite-volume effects we carried out simulations at three aspect ratios, $N_s/N_t = 6, 8, 10$, for most of the temperatures. We checked for discretization effects carrying out simulations at three different lattice spacings, corresponding to $N_t = 6, 8, 10$, at $T = 165$ MeV. The ensembles employed in this work are listed in Table I.

In this study the temperature is varied by changing the gauge coupling β , while the continuum limit is approached by varying the temporal lattice extent. Since changing β changes the lattice spacing, the bare light and strange quark masses, m_{ud} and $m_s = 28.15m_{ud}$, must be tuned to remain on the line of constant physics. Details about scale setting and the determination of the line of constant physics can be found in Ref. [106].

For every configuration we obtained the low-lying spectrum of the staggered operator (for the smeared gauge fields) using the Krylov-Schur algorithm [107]. For every

ensemble we computed sufficiently many low-lying positive eigenvalues to be able to identify the mobility edge λ_c in the bulk, if present at the given temperature. The number of eigenvalues $N_{\text{ev}}(T, N_s, a)$ computed for different lattice spatial sizes at the same temperature was chosen in order to keep the explored spectral range approximately fixed (see Table I for details).

B. Determination of the mobility edge

We implemented the procedure outlined above in Sec. II as follows. For each of our ensembles, we unfolded the spectrum by ranking the eigenvalues of all the available configurations and replacing them by their rank divided by the number of configurations in the ensemble. This makes the unfolded spectral density unity by construction, and the mean level spacing should also be unity in the large volume limit. To obtain the various quantities locally in the spectrum we divided the spectrum into small disjoint bins and averaged observables over modes in each bin as well as over gauge configurations in the ensemble.

At the low end of the spectrum we observe a small deviation of the mean unfolded level spacing from unity. As can be seen in the lower panels of Fig. 1, this is a finite volume effect, due to the sparseness of the spectrum in that region. This did not affect the determination of the mobility edge: as shown in the upper panels of Fig. 1, we found consistent results for λ_c on our largest volumes, indicating that finite-volume effects were under control.

For each ensemble we estimated the position of the mobility edge as the solution of the equation

$$I_{s_0}(\lambda_c(N_s); N_s) = I_{s_0}^{(\text{crit})}. \quad (13)$$

This is obtained by performing a correlated linear fit of our numerical data for I_{s_0} near the crossing point, using three points near the mobility edge. For each ensemble we performed six such correlated fits, using either two points below and one above or two points above and one below the putative mobility edge, and using three different bin sizes for the local spectral estimates. Following Ref. [108], we then obtained our final estimate for λ_c by a model average of the estimates $(\lambda_c)_i$ obtained with our six fits, weighting each fit with

$$p_i \propto e^{-\chi^2/2 + N_{\text{data}} - N_{\text{param}}}, \quad (14)$$

where N_{data} is the number of data points in the fit, N_{param} is the number of fit parameters, and χ^2 is the total sum of residuals in the fit. We estimated the uncertainty as

$$\text{error}(\lambda_c)^2 = \sum_i (\sigma_i^2 + (\lambda_c)_i^2) p_i - \left[\sum_i (\lambda_c)_i p_i \right]^2, \quad (15)$$

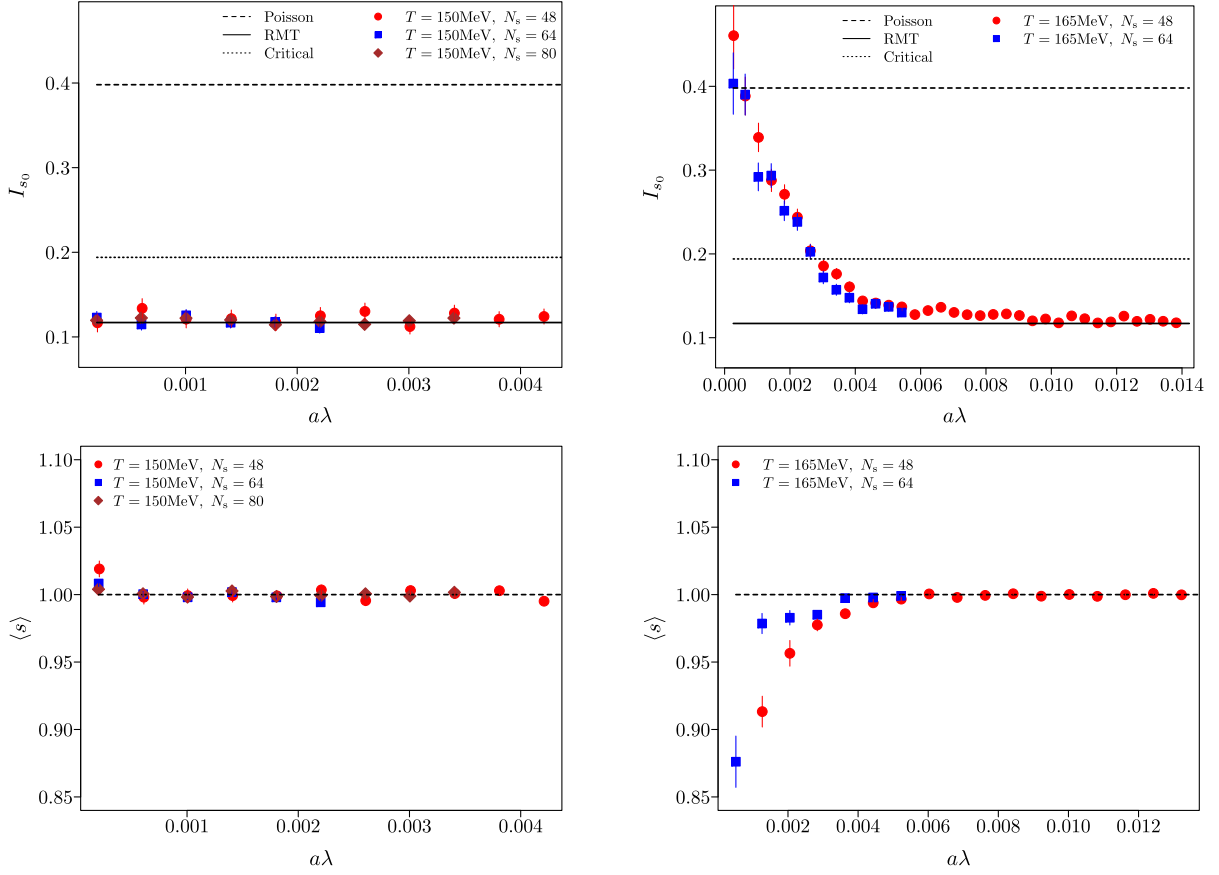


FIG. 1. Upper panels: Integrated unfolded level spacing distribution, I_{s_0} [see Eq. (11)], below (left) and above (right) the pseudocritical transition temperature. The values corresponding to Poisson, RMT, and critical statistics are also shown. Lower panels: First moment of the level spacing distribution, $\langle s \rangle = \int_0^\infty ds sp(s; \lambda; N_s)$, below (left panel) and above (right panel) the pseudocritical transition temperature. Data in all panels correspond to $N_t = 8$ simulations.

where σ_i is the statistical uncertainty on $(\lambda_c)_i$ from the linear fit.

We focused our attention on temperatures around the pseudocritical temperature known from the literature. Since the transition is a crossover, the definition of T_c is ambiguous, and several methods are used in the literature to define it. The most widely used is the position of the inflection point of the renormalized chiral condensate,

$$\begin{aligned} \langle \bar{\psi} \psi \rangle_R &= -\frac{m_{ud}}{m_\pi^4} (\langle \bar{u} u \rangle_T - \langle \bar{u} u \rangle_{T=0}) \\ &= -\frac{m_{ud}}{m_\pi^4} (\langle \bar{d} d \rangle_T - \langle \bar{d} d \rangle_{T=0}), \end{aligned} \quad (16)$$

yielding $T_{pc} = 155(4)$ MeV [4]. Another possibility is to use the peak of the renormalized light-quark chiral susceptibility, normalized by $T^{2k} m_\pi^{4-2k}$, $k = 0, 1, 2$, i.e.,

$$\chi_{\bar{\psi} \psi}^{(k)} = \left(\frac{m_\pi}{T} \right)^{2k} \frac{m_{ud}^2}{m_\pi^4} \frac{\partial}{\partial m_{ud}} \left(\frac{m_\pi^4}{m_{ud}} \langle \bar{\psi} \psi \rangle_R \right), \quad (17)$$

yielding similar pseudocritical temperatures $T_{pc}^{(k)}$, namely $T_{pc}^{(0)} = 157(4)$ MeV, $T_{pc}^{(1)} = 152(4)$ MeV, and $T_{pc}^{(2)} = 146(4)$ MeV [3]. (The statistical and systematic errors reported in Refs. [3,4] have been added in quadrature.)

Our procedure for extracting the mobility edge is illustrated in Figs. 1 and 2. In Fig. 1 we show the integrated unfolded level spacing distribution, I_{s_0} [see Eq. (11)], computed locally in the spectrum, at one temperature below and one above T_c . At $T = 150$ MeV $\simeq 0.97 T_{pc}$ (Fig. 1, top left panel) one finds $I_{s_0} = I_{s_0}^{(RMT)}$ within numerical errors for the whole low-lying spectrum, and so all low modes are delocalized. At $T = 165$ MeV $\simeq 1.06 T_{pc}$ (Fig. 1, top right panel) one finds instead that while $I_{s_0} = I_{s_0}^{(RMT)}$ within errors in the bulk of the spectrum, it rises toward $I_{s_0}^{(Poisson)}$ as one gets closer to the origin, crossing the critical value $I_{s_0}^{(crit)}$ along the way. In Fig. 2 we show the six linear interpolations used to determine λ_c at $T = 165$ MeV (only the data points corresponding to one choice of spectral bin size are shown for clarity). Our results for λ_c/m_{ud} are reported in Table I.

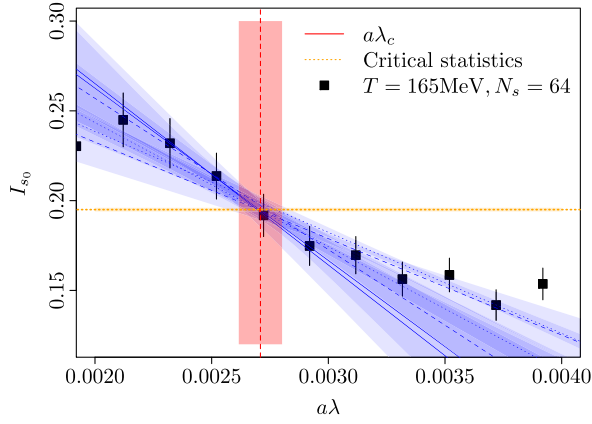


FIG. 2. Determination of the mobility edge from I_{s_0} [see Eq. (11)] at $T = 165$ MeV. Pairs of lines with the same dashed pattern correspond to the two different choices of the three points used in the linear interpolation, while different patterns correspond to different spectral bin size. Shaded areas denote the corresponding error bands.

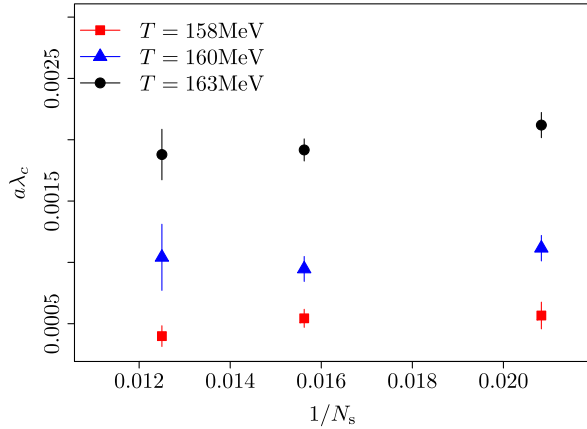


FIG. 3. Dependence of our estimate of the mobility edge on the lattice spatial size, for the three lowest temperatures above T_{pc} for the $N_t = 8$ lattices.

We carefully checked that finite-volume and finite-spacing effects were under control by estimating the mobility edge using different aspect ratios and lattice spacings. In Fig. 3 we show our estimate of the mobility edge for the three lowest temperatures above T_{pc} for spatial sizes $N_s = 48, 64, 80$ at fixed temporal size $N_t = 8$. For each temperature, our estimates using the two largest available aspect ratios are compatible within statistical errors. In Fig. 4 we show our estimate of the renormalized mobility edge at $T = 165$ MeV as a function of the lattice spacing for aspect ratios 6 and 8. While $N_t = 6$ is outside of the scaling regime, the estimates obtained using the two finest lattices and both aspect ratios are compatible with each other within statistical errors. With the lattice action used in this paper, $N_t = 8$ lattices already provide a good approximation of the relevant continuum physics [3,4].

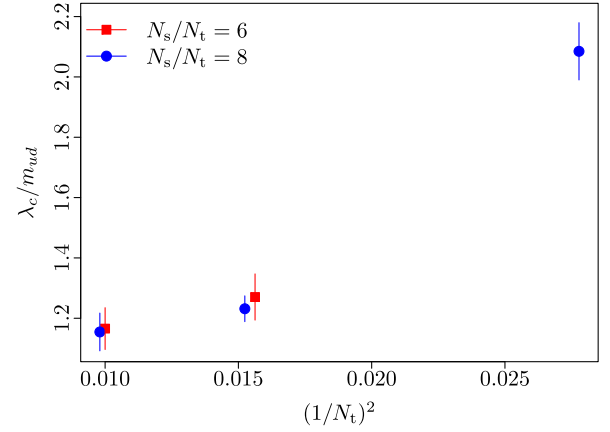


FIG. 4. Renormalized mobility edge as a function of the lattice spacing $(1/N_t = aT)$ for different aspect ratios at $T = 165$ MeV. Data points are slightly shifted horizontally to improve readability.

Moreover, λ_c/m_{ud} depends mildly on the lattice spacing [26,43], and so a mobility edge present at $N_t = 8$ is very likely surviving the continuum limit.

C. Localization temperature

The results for λ_c obtained as discussed in Sec. III B can be used to determine the localization temperature at $N_t = 8$ with two different methods.

The first method is to bracket the localization temperature in the window between the highest T where one does not find a mobility edge, and the lowest T where one does. The resolution of this method is of course limited by the available temperatures. Since this method requires only establishing the existence of a mobility edge, it is not plagued by any possible uncertainties (finite volume, interpolation of data, etc.) that affect the determination of the exact position of the mobility edge. With this method we determined that the critical temperature of localization at $N_t = 8$ is in the range $155 \text{ MeV} \leq T_{loc} \leq 158 \text{ MeV}$. To assess the reliability of this interval, we inspect in more detail how I_{s_0} changes across the spectrum for temperatures near the transition. In Fig. 5 we show I_{s_0} for $T = 155$ MeV and $T = 158$ MeV for $N_t = 8$ and the largest available volume. Close to T_{loc} one expects that the localization properties of the lowest modes fully stabilize only in a large enough volume: localized low modes right above T_{loc} have a large localization length, and delocalized low modes right below T_{loc} start being lumpy. These features of the eigenvectors change how the corresponding eigenvalues respond to a fluctuation in the gauge configuration, with stronger (respectively, weaker) correlations with the neighboring eigenvalues right above (respectively, right below) T_{loc} , leading in turn to deformations in the unfolded level spacing distribution in a finite volume. These reflect in I_{s_0} near $\lambda = 0$, which on the available volumes does not reach

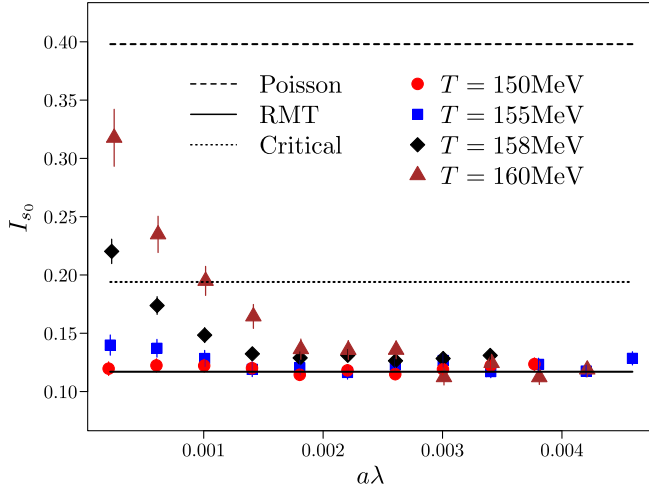


FIG. 5. Integrated unfolded level spacing distribution, I_{s_0} [see Eq. (11)], for the four temperatures closest to T_{pc} , for $N_t = 8$ and $N_s = 80$.

the Poisson value at $T = 158$ MeV, and is visibly above the RMT value at $T = 155$ MeV. However, while I_{s_0} crosses the critical value at $T = 158$ MeV, it does not at $T = 155$ MeV, showing a qualitatively different strength in the eigenvalue correlations. This supports the presence of a mobility edge at $T = 158$ MeV, and its absence at $T = 155$ MeV.

In Fig. 5 we show I_{s_0} also for the temperatures second closest to T_{pc} , namely $T = 160$ MeV and $T = 150$ MeV. Since for these temperatures there is little doubt about the presence or not of a mobility edge, we can quote as a more conservative estimate for T_{loc} the interval

$$150 \text{ MeV} \leq T_{loc} \leq 160 \text{ MeV}. \quad (18)$$

The result Eq. (18) is in good agreement with the continuum-extrapolated pseudocritical temperature $T_{pc} = 155(4)$ MeV obtained in Ref. [4] as the inflection point of the renormalized chiral condensate in units of m_π^4 , Eq. (16), and with the determination $T_{pc}^{(0)} = 157(4)$ MeV of Ref. [3] obtained from the renormalized light-quark chiral susceptibility in units of m_π^4 , Eq. (17) for $k = 0$. These estimates use fully renormalized fermionic quantities made dimensionless without including any additional temperature dependence, and seem therefore the most natural ones to compare to. Our result is also in good agreement with the pseudocritical temperature obtained from a gluonic observable, namely the static quark entropy, whose maximum is found at $T_{pc}^{(s)} = 153_{-5}^{+6.5}$ MeV in the continuum limit [6], and is also compatible with $T_{pc}^{(1)} = 152(4)$ MeV obtained from the renormalized light-quark chiral susceptibility in units of $m_\pi^2 T^2$, Eq. (17) for $k = 1$ [3].

While, strictly speaking, our results provide a bracket for T_{loc} at finite lattice spacing for $N_t = 8$, it is meaningful to

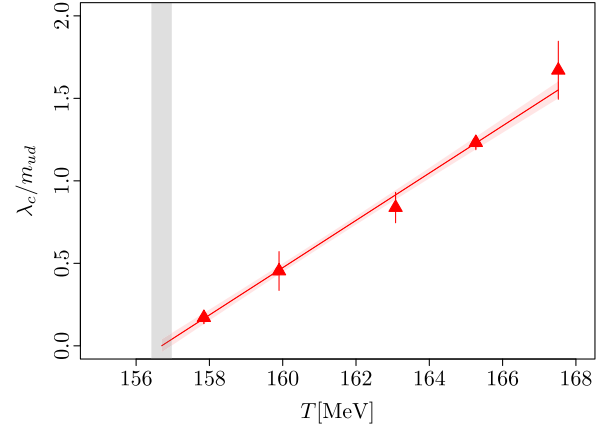


FIG. 6. Renormalized mobility edge as a function of the temperature, using $N_t = 8$ and the largest aspect ratio at each temperature. The red band indicates the result of an uncorrelated linear fit to all the plotted data ($\chi^2/\text{ndf} = 0.4$). The grey vertical band shows our estimate for T_{loc} and the corresponding uncertainty, obtained assuming a smoothly vanishing mobility edge.

compare this with continuum-extrapolated estimates of the pseudocritical temperature. In fact, for the stout improved action used in this paper, at $N_t = 8$ the pseudocritical temperatures $T_{pc}^{(j)}$ obtained from the chiral susceptibility are compatible with their continuum extrapolation [2]. Our conservative bracketing for T_{loc} then certainly includes $T_{pc}^{(0)}(N_t = 8)$ and $T_{pc}^{(1)}(N_t = 8)$, and so the localization temperature and these pseudocritical temperatures are compatible at this lattice spacing. Moreover, the detailed continuum extrapolation performed in Ref. [43] shows that λ_c/m_{ud} depends only mildly on the lattice spacing for sufficiently fine lattices ($N_t \geq 8$). This is also confirmed by our results for $T = 165$ MeV shown in Fig. 3. We do not expect then that a mobility edge present at nonzero spacing will disappear in the continuum limit. Therefore, although we cannot perform a proper continuum extrapolation, our conservative bracketing Eq. (18) should remain valid in the continuum limit, and T_{loc} in the continuum should be well within the crossover region.

Our second method for extracting T_{loc} at $N_t = 8$ is to fit the available estimates for λ_c at different temperatures and extrapolate in temperature to the point where λ_c vanishes. This method allows for a higher resolution, but is clearly affected by finite-volume systematics, finite-spacing systematics, and other systematics, as well as by the choice of fitting function. Moreover, it implicitly assumes that λ_c vanishes continuously as a function of T , and while there is no reason to expect otherwise in the case of QCD since thermodynamic properties change in an analytic fashion, it is known that the mobility edge can disappear discontinuously [36,38], even in the presence of a second-order deconfinement transition [45]. On the other hand, this method is less affected by the stronger finite-size effects near T_{loc} discussed above.

In Fig. 6 we show the results of a linear fit to our estimates of λ_c on our $N_t = 8$ lattices with the largest aspect ratio. Using all the estimates for λ_c in the range of temperatures $158 \text{ MeV} \leq T \leq 168 \text{ MeV}$, we found $T_{\text{loc}} = 156.7(3) \text{ MeV}$. We then excluded $T = 158 \text{ MeV}$ to assess the effect of this data point on the fit, finding $T'_{\text{loc}} = 157(1) \text{ MeV}$, clearly compatible with T_{loc} within errors. This is in agreement with the results found with our first method, therefore validating our assessment of the $T = 158 \text{ MeV}$ data, and suggests that λ_c goes smoothly to zero at T_{loc} . Again, while this is, strictly speaking, a statement about $N_t = 8$ lattices, it is expected to remain true also in the continuum limit.

IV. CONCLUSIONS

In the present paper we determined the temperature T_{loc} where localized eigenmodes of the Dirac operator appear at the low end of the spectrum. To this end we used lattice simulations with $N_f = 2 + 1$ flavors of staggered quarks at the physical point at several temperatures around the crossover in the range 150 to 184 MeV. To control finite volume effects we used several lattice spatial volumes, up to an aspect ratio of $N_s/N_t = 10$, finding consistent results for the largest aspect ratios. At one temperature, $T = 165 \text{ MeV}$, we also simulated at three different lattice spacings corresponding to $N_t = 6, 8, 10$. The results obtained on the two finer lattices agree within the uncertainties, so we concluded that cutoff effects are also under control.

Using the unfolded level spacing distribution, for each of our ensembles above $T = 155 \text{ MeV}$ we located the mobility edge that separates localized and delocalized modes in the bulk of the spectrum. At and below $T = 155 \text{ MeV}$ all the low eigenmodes turned out to be delocalized, and no mobility edge was found. The lowest temperature where we could detect localized modes in the spectrum and a mobility edge was $T = 158 \text{ MeV}$, so we conclude that the critical temperature for localization T_{loc} is between 155 MeV and 158 MeV at $N_t = 8$. Accounting for finite-size uncertainties, a more conservative bracket is

$150 \text{ MeV} \leq T_{\text{loc}} \leq 160 \text{ MeV}$. Since $N_t = 8$ is close to the continuum for our choice of lattice action, we expect this bracket to remain valid in the continuum limit. Extrapolation in temperature of the mobility edge values obtained above T_{loc} strongly suggests that the mobility edge goes to zero smoothly, as expected from the cross-over nature of the transition. This was found again at $N_t = 8$, but there is no reason to expect any change in the continuum limit.

Our main result is that the localization critical temperature, above which localized modes are present at the low end of the Dirac spectrum, is found well within the crossover region, as determined from the chiral condensate and from the light-quark susceptibility. This has long been suspected, but this is the first explicit demonstration of this fact. This indicates a close connection between deconfinement, chiral symmetry restoration, and localization of the low Dirac modes.

We used the open source software packages R [109] and HADRON [110] in the analysis.

ACKNOWLEDGMENTS

We thank G. Baranka for collaboration in the early stages of this work. T. G. K. and M. G. acknowledge support by NKFIH Grant No. K-147396 and excellence Grant No. TKP2021-NKTA-64. The work of F. P. was supported by the projects PulseQCD, DeNuTra, MuonHVP (EXCELLENCE/0524/0269, EXCELLENCE/524/0455, EXCELLENCE/524/0017) co-financed by the European Regional Development Fund and the Republic of Cyprus through the Research and Innovation Foundation.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

-
- [1] Y. Aoki, G. Endrődi, Z. Fodor, S. D. Katz, and K. K. Szabó, *Nature (London)* **443**, 675 (2006).
 - [2] Y. Aoki, Z. Fodor, S. D. Katz, and K. K. Szabó, *Phys. Lett. B* **643**, 46 (2006).
 - [3] Y. Aoki, S. Borsányi, S. Dürr, Z. Fodor, S. D. Katz, S. Krieg, and K. K. Szabó, *J. High Energy Phys.* **06** (2009) 088.
 - [4] S. Borsányi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabó (Wuppertal-Budapest Collaboration), *J. High Energy Phys.* **09** (2010) 073.
 - [5] A. Bazavov *et al.*, *Phys. Rev. D* **85**, 054503 (2012).
 - [6] A. Bazavov, N. Brambilla, H.-T. Ding, P. Petreczky, H. P. Schadler, A. Vairo, and J. H. Weber, *Phys. Rev. D* **93**, 114502 (2016).
 - [7] J. B. Kogut and L. Susskind, *Phys. Rev. D* **11**, 395 (1975).
 - [8] T. Banks, L. Susskind, and J. B. Kogut, *Phys. Rev. D* **13**, 1043 (1976).
 - [9] L. Susskind, *Phys. Rev. D* **16**, 3031 (1977).

- [10] T. Banks, S. Raby, L. Susskind, J. B. Kogut, D. R. T. Jones, P. N. Scharbach, and D. K. Sinclair (Cornell-Oxford-Tel Aviv-Yeshiva Collaboration), *Phys. Rev. D* **15**, 1111 (1977).
- [11] Z. Fodor, A. Yu. Kotov, T. G. Kovács, and K. K. Szabó, *Proc. Sci., QCHSC24* (2025) 057.
- [12] L. Y. Glozman, *Int. J. Mod. Phys. A* **36**, 2044031 (2021).
- [13] T. D. Cohen and L. Y. Glozman, *Eur. Phys. J. A* **60**, 171 (2024).
- [14] A. Alexandru and I. Horváth, *Phys. Rev. D* **92**, 045038 (2015).
- [15] A. Alexandru and I. Horváth, *Phys. Rev. D* **100**, 094507 (2019).
- [16] M. Cardinali, M. D'Elia, and A. Pasqui, *arXiv:2107.02745*.
- [17] J. A. Mickley, C. Allton, R. Bignell, and D. B. Leinweber, *Phys. Rev. D* **111**, 034508 (2025).
- [18] Y. Fujimoto, K. Fukushima, Y. Hidaka, and L. McLerran, *Phys. Rev. D* **112**, 074006 (2025).
- [19] M. Göckeler, P. E. L. Rakow, A. Schäfer, W. Söldner, and T. Wettig, *Phys. Rev. Lett.* **87**, 042001 (2001).
- [20] C. Gattringer, M. Göckeler, P. E. L. Rakow, S. Schaefer, and A. Schäfer, *Nucl. Phys.* **B618**, 205 (2001).
- [21] A. M. García-García and J. C. Osborn, *Nucl. Phys.* **A770**, 141 (2006).
- [22] A. M. García-García and J. C. Osborn, *Phys. Rev. D* **75**, 034503 (2007).
- [23] R. V. Gavai, S. Gupta, and R. Lacaze, *Phys. Rev. D* **77**, 114506 (2008).
- [24] T. G. Kovács, *Phys. Rev. Lett.* **104**, 031601 (2010).
- [25] T. G. Kovács and F. Pittler, *Phys. Rev. Lett.* **105**, 192001 (2010).
- [26] T. G. Kovács and F. Pittler, *Phys. Rev. D* **86**, 114515 (2012).
- [27] M. Giordano, T. G. Kovács, and F. Pittler, *Phys. Rev. Lett.* **112**, 102002 (2014).
- [28] S. M. Nishigaki, M. Giordano, T. G. Kovács, and F. Pittler, *Proc. Sci., LATTICE2013* (2014) 018 [*arXiv:1312.3286*].
- [29] L. Ujfalusi, M. Giordano, F. Pittler, T. G. Kovács, and I. Varga, *Phys. Rev. D* **92**, 094513 (2015).
- [30] G. Cossu and S. Hashimoto, *J. High Energy Phys.* **06** (2016) 056.
- [31] M. Giordano, S. D. Katz, T. G. Kovács, and F. Pittler, *J. High Energy Phys.* **02** (2017) 055.
- [32] T. G. Kovács and R. Á. Vig, *Phys. Rev. D* **97**, 014502 (2018).
- [33] L. Holicki, E.-M. Ilgenfritz, and L. von Smekal, *Proc. Sci., LATTICE2018* (2018) 180.
- [34] M. Giordano, *J. High Energy Phys.* **05** (2019) 204.
- [35] R. Á. Vig and T. G. Kovács, *Phys. Rev. D* **101**, 094511 (2020).
- [36] C. Bonati, M. Cardinali, M. D'Elia, M. Giordano, and F. Mazzioti, *Phys. Rev. D* **103**, 034506 (2021).
- [37] G. Baranka and M. Giordano, *Phys. Rev. D* **104**, 054513 (2021).
- [38] T. G. Kovács, *Proc. Sci., LATTICE2021* (2022) 238.
- [39] M. Cardinali, M. D'Elia, F. Garosi, and M. Giordano, *Phys. Rev. D* **105**, 014506 (2022).
- [40] G. Baranka and M. Giordano, *Phys. Rev. D* **106**, 094508 (2022).
- [41] R. Kehr, D. Smith, and L. von Smekal, *Phys. Rev. D* **109**, 074512 (2024).
- [42] G. Baranka and M. Giordano, *Phys. Rev. D* **108**, 114508 (2023).
- [43] C. Bonanno and M. Giordano, *Phys. Rev. D* **109**, 054510 (2024).
- [44] H. Pandey, R. Shanker, and S. Sharma, *arXiv:2407.09253*.
- [45] G. Baranka, D. Berta, and M. Giordano, *Phys. Rev. D* **111**, 074512 (2025).
- [46] C. Bonanno and M. Giordano, *Phys. Rev. D* **111**, 114515 (2025).
- [47] M. Giordano, T. G. Kovács, and F. Pittler, *Int. J. Mod. Phys. A* **29**, 1445005 (2014).
- [48] M. Giordano and T. G. Kovács, *Universe* **7**, 194 (2021).
- [49] P. W. Anderson, *Phys. Rev.* **109**, 1492 (1958).
- [50] D. J. Thouless, *Phys. Rep.* **13**, 93 (1974).
- [51] P. A. Lee and T. V. Ramakrishnan, *Rev. Mod. Phys.* **57**, 287 (1985).
- [52] B. Kramer and A. MacKinnon, *Rep. Prog. Phys.* **56**, 1469 (1993).
- [53] F. Evers and A. D. Mirlin, *Rev. Mod. Phys.* **80**, 1355 (2008).
- [54] M. Giordano, *J. Phys. A* **54**, 37LT01 (2021).
- [55] M. Giordano, *J. High Energy Phys.* **12** (2022) 103.
- [56] M. Ghanbarpour and L. von Smekal, *Phys. Rev. D* **106**, 054513 (2022).
- [57] T. Banks and A. Casher, *Nucl. Phys.* **B169**, 103 (1980).
- [58] F. Bruckmann, T. G. Kovács, and S. Schierenberg, *Phys. Rev. D* **84**, 034505 (2011).
- [59] M. Giordano, T. G. Kovács, and F. Pittler, *J. High Energy Phys.* **04** (2015) 112.
- [60] M. Giordano, T. G. Kovács, and F. Pittler, *J. High Energy Phys.* **06** (2016) 007.
- [61] M. Giordano, T. G. Kovács, and F. Pittler, *Phys. Rev. D* **95**, 074503 (2017).
- [62] M. Giordano, *Phys. Rev. D* **110**, L091504 (2024).
- [63] M. Giordano, *arXiv:2510.24381*.
- [64] M. Giordano, *arXiv:2510.24392*.
- [65] R. G. Edwards, U. M. Heller, J. E. Kiskis, and R. Narayanan, *Phys. Rev. D* **61**, 074504 (2000).
- [66] A. Bazavov, T. Bhattacharya, M. I. Buchoff, M. Cheng, N. H. Christ, H.-T. Ding, R. Gupta, P. Hegde, C. Jung, F. Karsch, Z. Lin, R. D. Mawhinney, S. Mukherjee, P. Petreczky, R. A. Soltz, P. M. Vranas, and H. Yin (HotQCD Collaboration), *Phys. Rev. D* **86**, 094503 (2012).
- [67] M. I. Buchoff *et al.*, *Phys. Rev. D* **89**, 054514 (2014).
- [68] G. Cossu, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko, H. Matsufuru, and J.-I. Noaki, *Phys. Rev. D* **87**, 114514 (2013); **88**, 019901(E) (2013).
- [69] V. Dick, F. Karsch, E. Laermann, S. Mukherjee, and S. Sharma, *Phys. Rev. D* **91**, 094504 (2015).
- [70] A. Tomiya, G. Cossu, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko, and J. Noaki, *Phys. Rev. D* **96**, 034509 (2017); **96**, 079902(A) (2017).
- [71] S. Aoki, Y. Aoki, G. Cossu, H. Fukaya, S. Hashimoto, T. Kaneko, C. Rohrhofer, and K. Suzuki (JLQCD Collaboration), *Phys. Rev. D* **103**, 074506 (2021).
- [72] H.-T. Ding, S.-T. Li, S. Mukherjee, A. Tomiya, X.-D. Wang, and Y. Zhang, *Phys. Rev. Lett.* **126**, 082001 (2021).

- [73] O. Kaczmarek, L. Mazur, and S. Sharma, *Phys. Rev. D* **104**, 094518 (2021).
- [74] R. Á. Vig and T. G. Kovács, *Phys. Rev. D* **103**, 114510 (2021).
- [75] X.-L. Meng, P. Sun, A. Alexandru, I. Horváth, K.-F. Liu, G. Wang, and Y.-B. Yang (χ QCD Collaboration and CLQCD Collaboration), *J. High Energy Phys.* **12** (2024) 101.
- [76] O. Kaczmarek, R. Shanker, and S. Sharma, *Phys. Rev. D* **108**, 094501 (2023).
- [77] A. Alexandru, C. Bonanno, M. D’Elia, and I. Horváth, *Phys. Rev. D* **110**, 074515 (2024).
- [78] S. Aoki, Y. Aoki, H. Fukaya, S. Hashimoto, I. Kanamori, T. Kaneko, Y. Nakamura, C. Rohrhofer, K. Suzuki, and D. Ward (JLQCD Collaboration), *Proc. Sci., LATTICE2023* (2024) 185.
- [79] T. G. Kovács, *Phys. Rev. Lett.* **132**, 131902 (2024).
- [80] A. Alexandru and I. Horváth, *Phys. Rev. Lett.* **127**, 052303 (2021).
- [81] A. Alexandru and I. Horváth, *Phys. Lett. B* **833**, 137370 (2022).
- [82] H. W. Hamber, E. Marinari, G. Parisi, and C. Rebbi, *Phys. Lett.* **124B**, 99 (1983).
- [83] F. Fucito and S. Solomon, *Phys. Lett.* **140B**, 387 (1984).
- [84] S. A. Gottlieb, W. Liu, R. L. Renken, R. L. Sugar, and D. Toussaint, *Phys. Rev. D* **38**, 2245 (1988).
- [85] P. D. Antoniou and E. N. Economou, *Phys. Rev. B* **16**, 3768 (1977).
- [86] E. N. Economou and P. D. Antoniou, *Solid State Commun.* **21**, 285 (1977).
- [87] D. Weaire and V. Srivastava, *Solid State Commun.* **23**, 863 (1977).
- [88] P. Cain, R. A. Römer, and M. Schreiber, *Ann. Phys. (Berlin)* **8**, 507 (1999).
- [89] P. Biswas, P. Cain, and R. A. Römer, and M. Schreiber, *Phys. Status Solidi (b)* **218**, 205 (2000).
- [90] S. N. Evangelou and D. E. Katsanos, *J. Phys. A* **36**, 3237 (2003).
- [91] A. M. García-García and E. Cuevas, *Phys. Rev. B* **74**, 113101 (2006).
- [92] T. Wang, T. Ohtsuki, and R. Shindou, *Phys. Rev. B* **104**, 014206 (2021).
- [93] B. L. Al’tshuler and B. I. Shklovskii, *Sov. Phys. JETP* **64**, 127 (1986).
- [94] M. L. Mehta, *Random Matrices*, Pure and Applied Mathematics (Academic Press, New York, 2004), 3rd ed., Vol. 142.
- [95] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, *Phys. Rep.* **299**, 189 (1998).
- [96] J. J. M. Verbaarschot and T. Wettig, *Annu. Rev. Nucl. Part. Sci.* **50**, 343 (2000).
- [97] B. I. Shklovskii, B. Shapiro, B. R. Sears, P. Lambrianides, and H. B. Shore, *Phys. Rev. B* **47**, 11487 (1993).
- [98] T. G. Kovács and F. Pittler, *Proc. Sci., LATTICE2011* (2011) 213.
- [99] L. Del Debbio, L. Giusti, M. Lüscher, R. Petronzio, and N. Tantalo, *J. High Energy Phys.* **02** (2006) 011.
- [100] L. Giusti and M. Lüscher, *J. High Energy Phys.* **03** (2009) 013.
- [101] P. Weisz, *Nucl. Phys.* **B212**, 1 (1983).
- [102] G. Curci, P. Menotti, and G. Paffuti, *Phys. Lett. B* **130**, 205 (1983); **135**, 516(E) (1984).
- [103] K. Symanzik, *Nucl. Phys.* **B226**, 187 (1983).
- [104] M. Lüscher and P. Weisz, *Commun. Math. Phys.* **97**, 59 (1985); **98**, 433(E) (1985).
- [105] C. Morningstar and M. J. Peardon, *Phys. Rev. D* **69**, 054501 (2004).
- [106] Y. Aoki, Z. Fodor, S. D. Katz, and K. K. Szabó, *J. High Energy Phys.* **01** (2006) 089.
- [107] G. W. Stewart, *SIAM J. Matrix Anal. Appl.* **23**, 601 (2002).
- [108] W. I. Jay and E. T. Neil, *Phys. Rev. D* **103**, 114502 (2021).
- [109] R Core Team, *R: A Language, and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria (2019).
- [110] B. Kostrzewa, J. Ostmeier, M. Ueding, C. Urbach, N. Schlage, M. Werner, F. Pittler, M. Fischer, and V. Lubicz, [10.5281/zenodo.17375156](https://doi.org/10.5281/zenodo.17375156) (2025).